

Delaunay Graphs Are Almost as Good as Complete Graphs*

David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit

Department of Computer Science, Princeton University, Princeton, NJ 08544, USA

Abstract. Let S be any set of N points in the plane and let $DT(S)$ be the graph of the Delaunay triangulation of S . For all points a and b of S , let $d(a, b)$ be the Euclidean distance from a to b and let $DT(a, b)$ be the length of the shortest path in $DT(S)$ from a to b . We show that there is a constant c ($\leq ((1+\sqrt{5})/2)\pi \approx 5.08$) independent of S and N such that

$$\frac{DT(a, b)}{d(a, b)} < c.$$

1. Introduction

Let $DL_i(S)$ be the Delaunay triangulation of S in the L_i norm ($i = 1, 2$). Chew [Ch] shows that there exists a constant c_1 such that the ratio of shortest distances in $DL_1(S)$ to straight line (i.e., L_2) distances is bounded above by c_1 where $c_1 = \sqrt{10} \approx 3.16228$. We extend this result here demonstrating a constant c_2 such that the ratio of shortest distances in $DL_2(S)$ to straight line distances is bounded above by $c_2 = ((1+\sqrt{5})/2)\pi \approx 5.08$. The best-known lower bound on c_2 is $\pi/2$ and is also due to Chew.

In his paper, Chew describes applications of his (and our) result to problems of motion planning, polygon visibility, and extensions of Voronoi diagrams/Delaunay triangulations. Our focus is the derivation of c_2 and potential extensions to other problems involving distances in the plane.

In what follows, we provide the definitions and lemmas necessary to prove our main result in Section 2; Section 3 contains the proofs. We conclude with some open problems.

* This research was supported in part by an AT&T Bell Laboratories Scholarship, by NSF Grants DMC-8451214, CCR87-00917, and CCR85-05517, and by a grant from the IBM Corporation.

2. The Main Result

We begin with (informal) definitions of the Voronoi diagram and the Delaunay triangulation. The *Voronoi diagram* for a set S of N points in the plane is a partition of the plane into regions, each containing exactly one point in S , such that, for each point $p \in S$, every point within its corresponding region (denoted $\text{Vor}(p)$) is closer to p than to any other point of S . The boundaries of these regions form a planar graph. The *Delaunay triangulation* of S is the straight-line dual of the Voronoi diagram for S ; that is, we connect a pair of points in S if and only if they share a Voronoi boundary. Under the standard assumption that no four points of S are cocircular, the Delaunay triangulation is indeed a triangulation [PS]; we denote its corresponding graph by $\text{DT}(S)$.

For the remainder of this section, fix points $a, b \in S$; we will construct a path in $\text{DT}(S)$ that is not too long in relation to $d(a, b)$. Assume for simplicity that a and b lie on the x -axis, with $x(a) < x(b)$ (we denote the coordinates of a point q in the plane by $x(q)$ and $y(q)$, respectively). We refer to members of S alternatively as points or vertices, and to edges of $\text{DT}(S)$ as edges or line segments, as the context indicates.

Our original idea for the path was simply to use the vertices $a = b_0, b_1, \dots, b_{m-1}, b_m = b$ corresponding to the sequence of Voronoi regions traversed by walking from a to b along the x -axis, as illustrated in Fig. 1, where $m = 4$ (in the case in which a Voronoi edge happens to lie on the x -axis somewhere between a and b , we—arbitrarily—choose that Voronoi region lying above, rather than

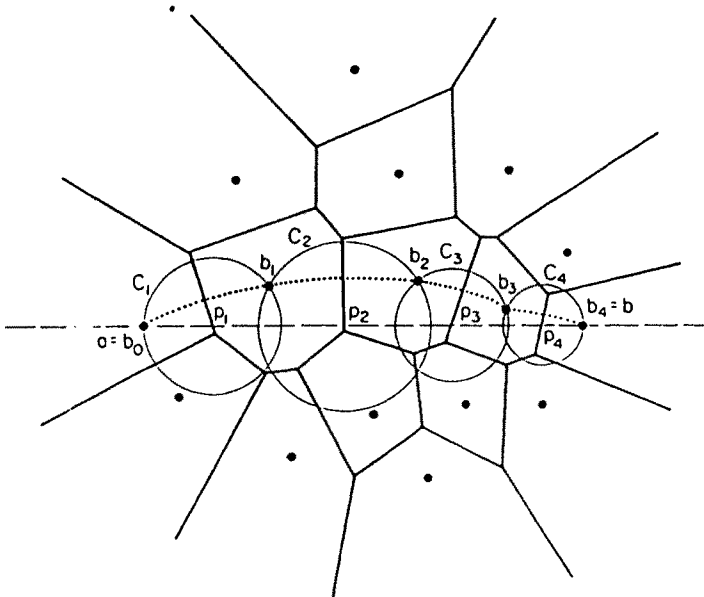


Fig. 1. The Voronoi diagram is shown in solid line, and the direct DT path between a and b in dotted line.

below, the x -axis). In general, we refer to the DT path constructed in this way between some z and z' in S as the *direct DT path* from z to z' . Let p_i denote the point on the x -axis that also lies on the boundary between $\text{Vor}(b_{i-1})$ and $\text{Vor}(b_i)$, for $i = 1, 2, \dots, m$. The definition of the Voronoi diagram immediately gives that p_i is the center of a circle C_i passing through b_{i-1} and b_i but containing no points of S in its interior.

Two simple properties of direct DT paths are:

Lemma 1. $x(b_0) \leq x(b_1) \leq \dots \leq x(b_m)$.

Lemma 2. For all i , $0 \leq i \leq m$, b_i is contained within, or on the boundary of, circle(a, b) (by which we denote the circle with a and b diametrically opposed).

Note in Fig. 1 that all the b_i happen to be in the same half-plane defined by the line connecting a and b (i.e., $y(b_i) \geq 0$ for all $0 \leq i \leq m$). In such cases, we say that the direct path between the two points is *one-sided*. One-sided paths are fortuitous for our purposes, because the ratio of the path length to the Euclidean distance is at most $\pi/2$; this is a simple consequence of Lemma 1 above and the following:

Lemma 3. Let D_1, D_2, \dots, D_k be circles all centered on the x -axis such that $D = \bigcup_{1 \leq i \leq k} D_i$ is connected. Then $\text{boundary}(D)$ has length at most $\pi \cdot (x_r - x_l)$, where x_l and x_r are the least and greatest x -coordinates of D , respectively (see Fig. 2).

Lemma 3 applies to the one-sided paths because the half of $\text{boundary}(C)$ (where C is defined as $\bigcup_{1 \leq k \leq m} C_k$) that lies above the x -axis has length at least as great as the path itself (because the b_i are monotonic in x).

The trouble with this approach is that the path is not necessarily even close to being one-sided; the path may zig-zag across the x -axis (as is illustrated in Fig. 3) $\Theta(N)$ times.

Our modified approach, then, is to try to stay above the x -axis. Should the direct path dip below the x -axis, we determine how costly the dip will be. If dipping below is not too expensive (in a sense defined below) then we follow the direct path below the x -axis and then back up. Otherwise, we construct a shortcut between the two points above the x -axis. Most of the proof consists of showing that the shortcut is not too long. The exact path we take is made more precise in the proof of the following:

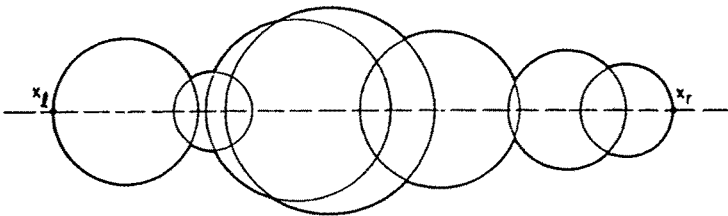


Fig. 2. Illustration for Lemma 3.

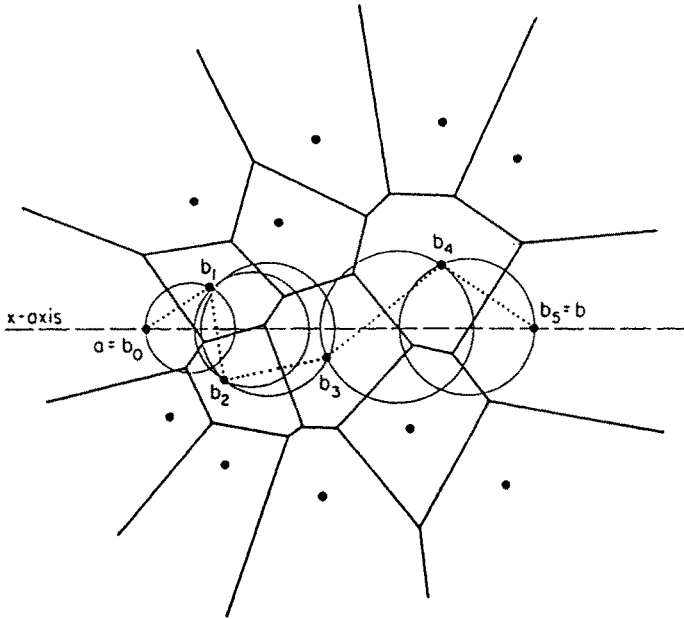


Fig. 3. A direct DT path that is *not* one-sided.

Theorem. *There exists a DT path from a to b of length $\leq ((1+\sqrt{5})/2)\pi \cdot d(a, b)$.*

Proof. We present an algorithm for constructing a DT path from $a = b_0$ to $b = b_m$, and then analyze the length of the path it produces. Assume that the path so far has brought us to some b_i such that (1) $y(b_i) \geq 0$ (initially, $i = 0$), (2) $i < m$ (meaning we are not finished), and (3) $y(b_{i+1}) < 0$. Thus the direct path would dip below the x -axis for a while after b_i . Let j be the least number greater than i such that $y(b_j) \geq 0$ (e.g., in Fig. 4, if $i = 2$ then $j = 4$). Let T denote the path along the boundary of C clockwise from b_i to b_j . Let w denote the length of the projection of T onto the x -axis (thus $w = x(b_j) - x(b_i)$). Define $h = \min\{y(q) : q \text{ lies on } T\}$. Now if $h \leq w/4$ then continue along the direct path to b_j

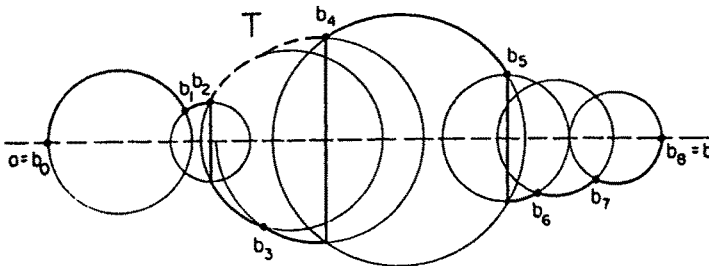


Fig. 4. An upper bound on the length of the direct DT path.

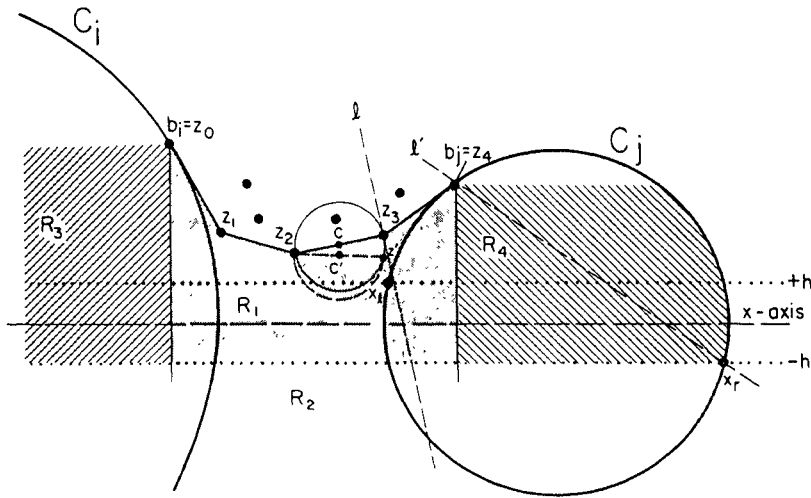


Fig. 5. The shortcut from b_i to b_j . Here $k = 2$.

(i.e., use edges $b_i b_{i+1}, b_{i+1} b_{i+2}, \dots, b_{j-1} b_j$). Otherwise we take a shortcut as follows. Construct the lower convex hull $b_i = z_0, z_1, z_2, \dots, z_n = b_j$ of the set

$$\{q \in S: x(b_i) \leq x(q) \leq x(b_j) \text{ and } y(q) \geq 0 \text{ and } q \text{ lies under } b_i b_j\}$$

(see Fig. 5). Note that these convex hull edges are certainly not on the direct DT path from a to b . Now the shortcut consists of taking the direct DT path from z_k to z_{k+1} for each $0 \leq k \leq n - 1$. The key fact (proved in Section 3) is:

Lemma 4. *Let $z_k z_{k+1}$ be an edge of the lower convex hull described above. Then the direct DT path from z_k to z_{k+1} is one-sided.*

Next we analyze the length of the path produced by this algorithm. When proceeding from b_i to b_j , let t denote the length of T . If $h \leq w/4$ then let q_0 be the point of T with least y -value (see Fig. 6), let t_i denote the length of the portion of T from b_i to q_0 , and t_j the length of the portion of T from q_0 to b_j (thus $t_i + t_j = t$). Let w_i and w_j denote the lengths of the projections of those two portions of T , respectively (thus $w_i + w_j = w$). Then the path we take (i.e., no shortcuts) has length at most

$$\begin{aligned} t + 2(y(b_i) + y(b_j)) &= t + 2(2h + (y(b_i) - h) + (y(b_j) - h)) \\ &\leq t + 2\left(\frac{w}{2} + (y(b_i) - h) + (y(b_j) - h)\right) \\ &= t + 2\left(\frac{w_i}{2} + (y(b_i) - h) + \frac{w_j}{2} + (y(b_j) - h)\right) \\ &\leq t + 2\left(\frac{\sqrt{5}}{2} t_i + \frac{\sqrt{5}}{2} t_j\right) = t(1 + \sqrt{5}). \end{aligned}$$

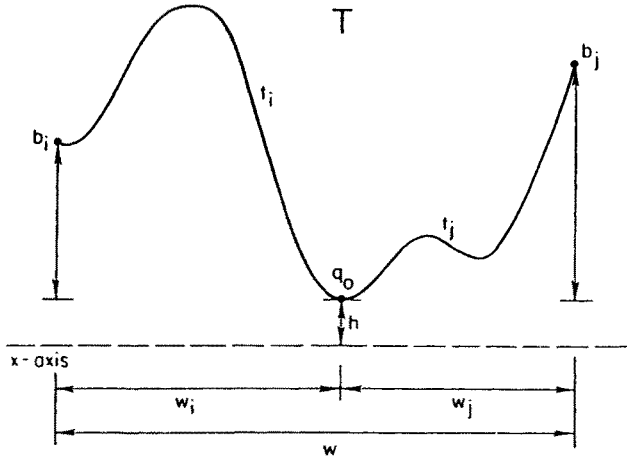


Fig. 6. Analyzing the path length when the shortcut is *not* taken.

The last inequality follows from the (easily proved) fact that

$$\frac{a}{2} + b \leq \frac{\sqrt{5}}{2} c$$

whenever a and b are the legs of a right triangle with hypotenuse c .

On the other hand, if $h > w/4$ then we take the shortcut, which has length at most

$$\sum_{k=0}^{n-1} \text{length of one-sided path from } z_k \text{ to } z_{k+1}$$

(by Lemma 4) which is $\leq \sum_{k=0}^{n-1} d(z_k, z_{k-1})\pi/2 \leq t\pi/2$ (by Lemma 3). Hence in either case, the distance we travel in getting from b_i to b_j is at most $(1+\sqrt{5})t$. Therefore summing over all such trips b_i to b_j as well as the trips (for which we travel at most t units) where the direct DT path from a to b stays completely above the x -axis, we get (by Lemma 3) a total path length of at most $d(a, b)((1+\sqrt{5})/2)\pi$. □

3. Proofs of the Lemmas

Proof of Lemma 1. The perpendicular bisector of b_i and b_{i+1} contains p_i . Point b_{i+1} lies to the right of this bisector, and b_i lies to the left; hence $x(b_i) \leq x(b_{i+1})$. □

Proof of Lemma 2. Let c denote the midpoint of segment ab ; let k be such that c lies in the Voronoi region of b_k . Then

$$d(b_0, c) \geq d(b_1, c) \geq \dots \geq d(b_k, c)$$

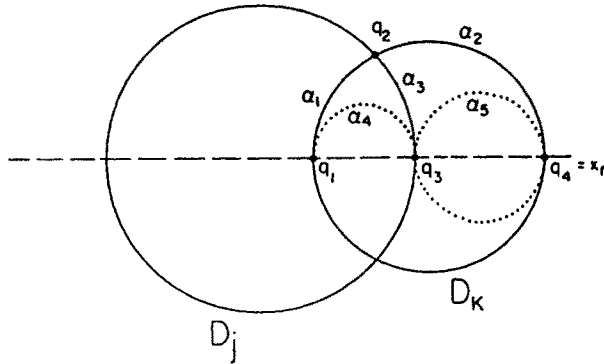


Fig. 7. Illustration for the proof of Lemma 3.

and

$$d(b_k, c) \leq d(b_{k+1}, c) \leq \dots \leq d(b_m, c). \quad \square$$

Proof of Lemma 3. By induction on k . The claim is easy if $k = 1$; so let $k \geq 2$ and assume it for $k - 1$. Let q_1 and q_4 denote the leftmost and rightmost points of D_k , respectively (see Fig. 7), and assume without loss of generality that $q_4 = x_r$. Let q_2 be the rightmost point at which D_k intersects another circle D_j (thus $j < k$); let q_3 be the rightmost point of D_j . We can assume that D_k does not entirely contain any circle D_i ($i \neq k$), since otherwise D_i would not contribute to boundary(D) and hence the induction would be trivial. Denote by α_1 (α_2) the length of the arc on circle D_k clockwise from q_1 to q_2 (resp. q_2 to q_4). Let α_3 be the length of the arc on circle D_j clockwise from q_2 to q_3 . Finally, let $\alpha_4 = (\pi/2)(x(q_3) - x(q_1))$ and let $\alpha_5 = (\pi/2)(x(q_4) - x(q_3))$. Then a simple convexity argument shows that

$$\alpha_1 + \alpha_3 \geq \alpha_4.$$

Also, we have

$$\alpha_4 + \alpha_5 = \alpha_1 + \alpha_2.$$

Hence

$$\alpha_1 + \alpha_3 + \alpha_5 \geq \alpha_4 + \alpha_5 = \alpha_1 + \alpha_2,$$

implying $\alpha_3 + \alpha_5 \geq \alpha_2$. Therefore, denoting the length of the boundary of D by $\text{bd}(D)$, we have

$$\begin{aligned} \text{bd}(D) &\leq \text{bd}\left(\text{circle}(q_3, q_4) \cup \bigcup_{1 \leq i \leq k-1} D_i\right) \\ &\leq \text{bd}(\text{circle}(q_3, q_4)) + \text{bd}\left(\bigcup_{1 \leq i \leq k-1} D_i\right) \\ &\leq \pi(x_r - x(q_3)) + \pi(x(q_3) - x_1) \quad (\text{by the inductive hypothesis}) \\ &\leq \pi(x_r - x_1). \end{aligned} \quad \square$$

Proof of Lemma 4. By Lemma 2, the direct DT path from z_k to z_{k+1} lies entirely within $\text{circle}(z_k, z_{k+1})$. We now show that there are no points of S within the lower semicircle of $\text{circle}(z_k, z_{k+1})$, so the path must be one-sided.

Let q be an arbitrary point in this lower semicircle; we must show $q \notin S$. If $x(b_i) \leq x(q) \leq x(b_j)$ and $y(q) \geq -h$ (i.e., q lies in region R_1 in Fig. 5) then we claim $q \notin S$. To see this, note that if $y(q) \geq h$ then it lies outside the lower convex hull; whereas if $-h < y(q) < h$ then q lies in the interior of $\bigcup_{i \leq k \leq j} C_k$.

We next show that $y(q) > -h$ (that is, $q \notin R_2$). Assume without loss of generality that $y(z_k) \leq y(z_{k+1})$. Since $z_k \in S$ it must lie directly above some point of T , since the area below T and above the x -axis is contained in C and therefore contains no members of S . Therefore $y(z_k) \geq h > w/4$. Let z' be the point with coordinates $(x(z_{k+1}), y(z_k))$. Let c and c' denote the midpoints of segments $z_k z_{k+1}$ and $z_k z'$, respectively. Then $y(c') > w/4$. That $q \in \text{circle}(z_k, z')$ follows from $q \in \text{circle}(z_k, z_{k+1})$ and $y(q) \leq y(z_k) = y(z')$. Furthermore, $x(z_{k+1}) - x(z_k) \leq w$, since by extending $z_k z_{k+1}$ on both sides we encounter points on T and since T is connected (and hence the projection of T onto the x -axis is at least as long as the projection of $z_k z_{k+1}$ onto the x -axis). Therefore $\text{radius}(\text{circle}(z_k, z')) \leq w/2$. Hence

$$y(q) \geq y(c') - \text{radius}(\text{circle}(z_k, z')) > w/4 - w/2 = -w/4.$$

Note that $x(q) \geq x(b_i)$ (that is $q \notin R_3$), because of our assumption $y(z_k) \leq y(z_{k+1})$.

Finally, we assume $x(q) > x(b_j)$ (hence $q \in R_4$). We show that q lies in the interior of C_j , implying $q \notin S$. Let x_l be the leftmost point of intersection of $\text{circle } C_j$ with the line $y = h$. Let x_r be the rightmost point of intersection of C_j with the line $y = -h$. Let l denote the line that passes through z_{k+1} perpendicular to segment $z_k z_{k+1}$, and let l' be the line containing b_j and x_r . Note that both l and l' must have negative slopes. Clearly, the entire circle (z_k, z_{k+1}) lies below l and in particular so does q . We claim that this implies that q lies below l' as well. To see this, first note that our assumption $y(z_k) \leq y(z_{k+1})$ implies $y(z_{k+1}) \leq y(b_j)$, and hence line l intersects the line $x = x(b_j)$ below b_j . Therefore it suffices to show that $\text{slope}(l) \leq \text{slope}(l')$ (recall that both are negative). The monotonicity of slopes in the lower convex hull gives $\text{slope}(z_k z_{k+1}) \leq \text{slope}(x_l b_j)$. Therefore since l and l' are perpendicular to $z_k z_{k+1}$ and $x_l b_j$, respectively (the latter is because x_l and x_r are diametrically opposed on C_j), we have $\text{slope}(l) \leq \text{slope}(l')$. Thus q indeed lies below l' ; hence since q is in R_4 it must also be in C_j and therefore not in S . \square

4. Related Problems

There are many interesting problems related to that solved here. For example, Raghavan [Ra] suggests that our results extend to a special case problem in 3-space. He conjectures that if S is a set of points on the unit sphere, there is a

constant c such that

$$\frac{d_H(a, b)}{d(a, b)} < c,$$

where d_H is the distance along edges of the convex hull and d is the (three-dimensional) Euclidean distance.

The generalization of our result to arbitrary point sets in 3-space and their Delaunay graphs remains open.

In another direction, Feder and others [Fe] have shown that for each $k \geq 7$ there is a constant c such that, for each finite set S of points in the plane, there is a graph G with vertices corresponding to these points, and the following properties:

- (1) Each vertex in G has degree at most k .
- (2) $d_G(a, b)/d(a, b) < c$, where d_G is the distance along edges of G .

Extensions to the cases $k=5$ and 6 have been proposed by others. It is not difficult to show that no such constant exists for $k=2$. What is the minimum k for which such a result is possible?

Acknowledgment

The authors thank Arthur Watson for his useful comments during this research.

References

- [Ch] P. Chew, There is a planar graph almost as good as the complete graph, *Proceedings of the Second Symposium on Computational Geometry*, Yorktown Heights, NY, 1986, pp. 169-177.
- [Fe] T. Feder, personal communication, 1988.
- [PS] F. P. Preparata and M. I. Shamos, *Computational Geometry: An Introduction*, Springer-Verlag, New York, 1985.
- [Ra] P. Raghavan, personal communication, 1987.

Received October 28, 1987, and in revised form April 8, 1988.