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Delay-dependent stability for uncertain cellular neural networks with discrete and distribute time-varying delays

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Abstract

In this paper, the problem of stability of uncertain cellular neural networks with discrete and distribute time-varying delays is considered. Based on the Lyapunov function method and convex optimization approach, a new delay-dependent stability criterion of the system is derived in terms of LMI (linear matrix inequality). In order to solve effectively the LMI as a convex optimization problem, the interior-point algorithm is utilized in this work. A numerical example is given to show the effectiveness of our results.

Keywords: Neural networks; Delays; LMI optimization; Lyapunov method

1. Introduction

During two decades, there have been great attentions on the stability analysis of cellular neural networks (CNNs) since it has been widely applied to various systems such as pattern recognition, associative memories, signal processing, and fixed-point computation. These applications rely on the existence of the equilibrium point and require commonly need the equilibrium point of the required network to be stable. Time-delay exists in many

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applications due to the finite switching speed of amplifiers in electronic networks or finite speed for signal propagation in biological networks (see [1–3]). The delay in the desired networks influences on the stability and occasionally causes the oscillation and poor performance of the network. Therefore, there have been many efforts to the study for the stability analysis of delayed Cellular Neural Networks (DCNNs) [4–16] during the last decade.

On the other hand, it is noticed that the signal propagation is sometimes instantaneous and can be modeled with discrete delays. Also it may be distributed during a certain time period so that the distributed delays should be incorporated in the model. In other words, it is often the case that the neural network model possesses both discrete and distributed delays [7]. Recently, the stability of CNNs with discrete and distributed delays has been investigated in [13–16]. Very recently, the parametric uncertainties, which often break the stability of systems, are considered in the works [14–16]. Nevertheless, it is necessary to do more research on this topic to obtain less conservative criteria for global asymptotic stability.

In this paper, we consider the problem for global asymptotic stability of a class of DCNNs. The DCNNs tackled in the work have parametric uncertainties, discrete, and distribute time-varying delays, which is more general cases than time-invariant ones. By constructing a suitable Lyapunov–Krasovskii functionals and utilizing free weight matrices, a new stability criterion for the system is derived. The criterion is delay-dependent one which is less conservative than delay-independent one when the size of delays is small [17]. Also, the proposed stability criterion is of the form of linear matrix inequalities (LMIs) which can be solved efficiently by using the interior-point algorithms [18]. Finally, we include a numerical example to show that our results are less conservative than those of the existing ones.

Notation: $\mathscr{R}^n$ is the n-dimensional Euclidean space, $\mathscr{R}^{m\times n}$ denotes the set of $m \times n$ real matrix. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices X and Y, the notation $X > Y$ (respectively, $X \geqslant Y$) means that the matrix $X - Y$ is positive definite (respectively, nonnegative). diag$\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents the elements below the main diagonal of a symmetric matrix. $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ mean the largest and smallest eigenvalue of given square matrix, respectively.

2. Problem statements

Consider the CNNs with discrete and distributed time-varying delays by the following state equations:

$$\begin{aligned}\dot{y}(t) &= (-A + \Delta A)y(t) + (W_0 + \Delta W_0)f(y(t)) + (W_1 + \Delta W_1)f(y(t-h(t))) \\ &\quad + (W_2 + \Delta W_2)\int_{t-\tau(t)}^{t} f(y(s))\,\mathrm{d}s + b, \end{aligned} \tag{1}$$

where $y(t) = [y_1(t), \ldots, y_n(t)]^{\mathrm{T}} \in \mathscr{R}^n$ is the neuron state vector, n denotes the number of neurons in a neural network, $f(y(t)) = [f_1(y_1(t)), \ldots, f_n(y_n(t))]^{\mathrm{T}} \in \mathscr{R}^n$ denotes the activation functions, $f(y(t-h(t))) = [f_1(y_1(t-h(t))), \ldots, f_n(y_n(t-h(t)))]^{\mathrm{T}} \in \mathscr{R}^n$, $A = \mathrm{diag}\{a_i\}$ is a positive diagonal matrix, $W_0 = (w_{ij}^0)_{n\times n}$, $W_1 = (w_{ij}^1)_{n\times n}$, and $W_2 = (w_{ij}^2)_{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons,

$b=[b_1,\ldots,b_n]^{\mathrm{T}}$ means a constant input vector, and ΔA, ΔW_0, ΔW_1, and ΔW_2 are the uncertainties of system matrices of the form

$$[\Delta A \;\; \Delta W_0 \;\; \Delta W_1 \;\; \Delta W_2] = DF(t)[E_a \;\; E_0 \;\; E_1 \;\; E_2], \tag{2}$$

where the time-varying nonlinear function $F(t)$ satisfies

$$F^{\mathrm{T}}(t)F(t) \leqslant I, \quad \forall t \geqslant 0. \tag{3}$$

The delays, $h(t)$ and $\tau(t)$, are time-varying continuous function that satisfies

$$0 < h(t) \leqslant \bar{h}, \quad \dot{h}(t) \leqslant \mu, \quad 0 < \tau(t) \leqslant \bar{\tau}, \tag{4}$$

where $\bar{h}$, $\bar{\tau}$, and μ are positive constants.

The activation functions, $f_i(y_i(t)), i=1,\ldots,n$, are assumed to be nondecreasing, bounded and globally Lipschiz, that is,

$$0 \leqslant \frac{f_i(\xi_i)-f_j(\xi_j)}{\xi_i-\xi_j} \leqslant l_i, \quad \xi_i,\xi_j \in \mathscr{R}, \;\; \xi_i \neq \xi_j, \;\; i,j=1,\ldots,n. \tag{5}$$

Note that by using the Brouwer's fixed-point theorem [4], it can be easily proven that there exists at least one equilibrium point for Eq. (1).

For simplicity, in stability analysis of system (1), the equilibrium point $y^* = [y_1^*,\ldots,y_n^*]^{\mathrm{T}}$ is shifted to the origin by utilizing the transformation $x(\cdot)=y(\cdot)-y^*$, which leads system (1) to the following form:

$$\begin{aligned}\dot{x}(t) &= (-A+\Delta A)x(t) + (W_0+\Delta W_0)g(x(t)) + (W_1+\Delta W_1)g(x(t-h(t))) \\ &\quad + (W_2+\Delta W_2)\int_{t-\tau(t)}^{t} g(x(s))\,\mathrm{d}s,\end{aligned} \tag{6}$$

where $x(t)=[x_1(t),\ldots,x_n(t)]^{\mathrm{T}} \in \mathscr{R}^n$ is the state vector of the transformed system, $g(x(t)) = [g_1(x(t)),\ldots,g_n(x(t))]^{\mathrm{T}}$ and $g_j(x_j(t)) = f_j(x_j(t)+y_j^*) - f_j(y_j^*)$ with $g_j(0)=0 (j=1,\ldots,n)$. It is noted from (5) that $g_j(\cdot)$ satisfies the following condition:

$$0 \leqslant \frac{g_j(\xi_j)}{\xi_j} \leqslant l_j, \quad \forall \xi_j \neq 0, \;\; j=1,\ldots,n \tag{7}$$

which is equivalent to

$$g_j(\xi_j)[g_j(\xi_j)-l_j\xi_j] \leqslant 0, \quad g_j(0)=0, \;\; j=1,\ldots,n. \tag{8}$$

Here, as a mathematical tool for our analysis, the following zero equation is introduced:

$$Gx(t) - Gx(t-h(t)) - G\int_{t-h(t)}^{t} \dot{x}(s)\,\mathrm{d}s = 0.$$

Then, we can represent system (1) as

$$\dot{x}(t) = (-A + G)x(t) - Gx(t-h(t)) - G\int_{t-h(t)}^{t} \dot{x}(s)\,\mathrm{d}s + W_0 g(x(t)) + W_1 g(x(t-h(t))) + W_2 \int_{t-\tau(t)}^{t} g(x(s))\,\mathrm{d}s + Dp(t),$$

$$p(t) = F(t)q(t),$$

$$q(t) = E_a x(t) + E_0 g(x(t)) + E_1 g(x(t-h(t))) + E_2 \int_{t-\tau(t)}^{t} g(x(s))\,\mathrm{d}s, \tag{9}$$

where $p(t) \in \mathscr{R}^n$, $q(t) \in \mathscr{R}^n$, and $G \in \mathscr{R}^{n\times n}$ will be chosen later.

Before deriving our main results, we state the following facts, and lemma.

Fact 1 (*Schur complement*). Given constant symmetric matrices $\Sigma_1, \Sigma_2, \Sigma_3$ where $\Sigma_1 = \Sigma_1^{\mathrm{T}}$ and $0 < \Sigma_2 = \Sigma_2^{\mathrm{T}}$, then $\Sigma_1 + \Sigma_3^{\mathrm{T}}\Sigma_2^{-1}\Sigma_3 < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^{\mathrm{T}} \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^{\mathrm{T}} & \Sigma_1 \end{bmatrix} < 0.$$

Fact 2. For any real vectors a, b and any matrix $Q > 0$ with appropriate dimensions, it follows that

$$2a^{\mathrm{T}}b \leqslant a^{\mathrm{T}}Qa + b^{\mathrm{T}}Q^{-1}b.$$

Lemma 1. *For a positive matrix $Q > 0$, any matrices $F_i (i = 1, \ldots, 8)$, and scalar $\bar{h} \geqslant 0$, the following inequality holds*:

$$-\int_{t-h(t)}^{t} \dot{x}^{\mathrm{T}}(s) Q \dot{x}(s)\,\mathrm{d}s \leqslant \zeta^{\mathrm{T}}(t)\widetilde{F}\zeta(t) + \bar{h}\zeta(t)^{\mathrm{T}}\bar{F}^{\mathrm{T}}Q^{-1}\bar{F}\zeta(t),$$

where

$$\bar{F} = [F_1 \;\; F_2 \;\; F_3 \;\; F_4 \;\; F_5 \;\; F_6 \;\; F_7 \;\; F_8], \tag{10}$$

$$\widetilde{F} = \begin{bmatrix} 0 & 0 & F_1^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\ \star & 0 & F_2^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & F_3^{\mathrm{T}} + F_3 & F_4 & F_5 & F_6 & F_7 & F_8 \\ \star & \star & \star & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & 0 \end{bmatrix}, \tag{11}$$

and

$$\zeta^{\mathrm{T}}(t) = \left[x^{\mathrm{T}}(t) \;\; x^{\mathrm{T}}(t-h(t)) \;\; \left(\int_{t-h(t)}^{t} \dot{x}(s)\,\mathrm{d}s\right)^{\mathrm{T}} \;\; \dot{x}^{\mathrm{T}}(t) \;\; g^{\mathrm{T}}(x(t)) \;\; g^{\mathrm{T}}(x(t-h(t))) \;\; \left(\int_{t-\tau(t)}^{t} g(x(s))\,\mathrm{d}s\right)^{\mathrm{T}} \;\; p^{\mathrm{T}}(t) \right]. \tag{12}$$

Proof. Utilizing Fact 2, we have

$$
\begin{aligned}
&-\int_{t-h(t)}^{t} \dot{x}^{\mathrm{T}}(s) Q \dot{x}(s)\,\mathrm{d}s \\
&\leqslant 2\left(\int_{t-h(t)}^{t} \dot{x}(s)\,\mathrm{d}s\right)^{\mathrm{T}} \bar{F}\zeta(t) + \int_{t-h(t)}^{t} \zeta^{\mathrm{T}}(t)\bar{F}^{\mathrm{T}} Q^{-1}\bar{F}\zeta(t)\,\mathrm{d}s \\
&\leqslant 2\zeta^{\mathrm{T}}(t)\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \bar{F}\zeta(t) + \bar{h}\zeta^{\mathrm{T}}(t)\bar{F}^{\mathrm{T}} Q^{-1}\bar{F}\zeta(t) \\
&= \zeta^{\mathrm{T}}(t)\widetilde{F}\zeta(t) + \bar{h}\zeta^{\mathrm{T}}(t)\bar{F}^{\mathrm{T}} Q^{-1}\bar{F}\zeta(t). \qquad \square
\end{aligned}
$$

3. Main results

In this section, we propose a new global stability criterion for CNNs with discrete and distributed time-varying delays (9).

Before stating our main results, let us define the matrices for simplicity in Appendix. Now, we have the following main results.

Theorem 1. *For given $\bar{h}$, μ, $\bar{\tau}$, and $L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\}$, the equilibrium point of* Eq. (1) *is globally stable for $0 < h(t) \leqslant \bar{h}$, $\dot{h}(t) \leqslant \mu$, and $0 < \tau(t) \leqslant \bar{\tau}$ if there exist positive diagonal matrices $R_1 = \mathrm{diag}\{r_1, r_2, \ldots, r_n\}$, $K_1 = \mathrm{diag}\{k_{11}, k_{12}, \ldots, k_{1n}\}$, and $K_2 = \mathrm{diag}\{k_{21}, k_{22}, \ldots, k_{2n}\}$, positive definite matrices P, $R_i (i = 2, \ldots, 5)$, H, and any matrices Y_1, F_i, J_i, M_i, N_i $(i = 1, \ldots, 8)$ satisfying the following LMI*:

$$
\begin{bmatrix} \Sigma & \bar{h}\bar{F}^{\mathrm{T}} & \bar{\tau}\bar{J}^{\mathrm{T}} & \Psi^{\mathrm{T}} H \\ \star & -\bar{h}R_4 & 0 & 0 \\ \star & \star & -\bar{\tau}R_5 & 0 \\ \star & \star & \star & -H \end{bmatrix} < 0, \tag{13}
$$

where $\Sigma, \bar{F}, \bar{J}$, and Ψ are defined in Appendix.

Proof. For positive definite matrices P, $R_1 = \mathrm{diag}\{r_1, \ldots, r_n\}$, $R_i (i = 2, \ldots, 5)$, let us consider the Lyapunov–Krasovskii functional candidate:

$$
V = V_1 + V_2 + V_3, \tag{14}
$$

where

$$V_1 = x^{\mathrm{T}}(t)Px(t),$$
$$V_2 = 2\sum_{i=1}^{n} r_i \int_0^{x_i(t)} g_i(s)\,\mathrm{d}s + \int_{t-h(t)}^{t} g^{\mathrm{T}}(x(s))R_2 g(x(s))\,\mathrm{d}s + \int_{t-h(t)}^{t} x^{\mathrm{T}}(s)R_3 x(s)\,\mathrm{d}s,$$
$$V_3 = \int_{t-\bar{h}}^{t}\int_{s}^{t} \dot{x}^{\mathrm{T}}(u)R_4\dot{x}(u)\,\mathrm{d}u\,\mathrm{d}s + \int_{t-\bar{\tau}}^{t}\int_{s}^{t} g^{\mathrm{T}}(x(u))R_5 g(x(u))\,\mathrm{d}u\,\mathrm{d}s. \tag{15}$$

From Eq. (9), differentiating V_1 leads to

$$\begin{aligned}\dot{V}_1 &= 2x^{\mathrm{T}}(t)P\dot{x}(t)\\ &= x^{\mathrm{T}}(t)(-PA - A^{\mathrm{T}}P + PG + G^{\mathrm{T}}P)x(t) - 2x^{\mathrm{T}}(t)PGx(t-h(t))\\ &\quad - 2x^{\mathrm{T}}(t)PG\int_{t-h(t)}^{t}\dot{x}(s)\,\mathrm{d}s + 2x^{\mathrm{T}}(t)PW_0 g(x(t)) + 2x^{\mathrm{T}}(t)PW_1 g(x(t-h(t)))\\ &\quad + 2x^{\mathrm{T}}(t)PW_2\int_{t-\tau(t)}^{t} g(x(s))\,\mathrm{d}s + 2x^{\mathrm{T}}(t)PDp(t).\end{aligned} \tag{16}$$

By differentiating V_2, we have

$$\begin{aligned}\dot{V}_2 &= 2\sum_{i=1}^{n} r_i g_i(x_i(t))\dot{x}_i(t)\\ &\quad + g^{\mathrm{T}}(x(t))R_2 g(x(t)) - (1-\dot{h}(t))g^{\mathrm{T}}(x(t-h(t)))R_2 g(x(t-h(t)))\\ &\quad + x^{\mathrm{T}}(t)R_3 x(t) - (1-\dot{h}(t))x^{\mathrm{T}}(t-h(t))R_3 x(t-h(t))\\ &\leqslant 2g^{\mathrm{T}}(x(t))R_1\dot{x}(t) + g^{\mathrm{T}}(x(t))R_2 g(x(t)) - (1-\mu)g^{\mathrm{T}}(x(t-h(t)))R_2 g(x(t-h(t)))\\ &\quad + x^{\mathrm{T}}(t)R_3 x(t) - (1-\mu)x^{\mathrm{T}}(t-h(t))R_3 x(t-h(t)).\end{aligned} \tag{17}$$

The time derivatives of V_3 is obtained as

$$\begin{aligned}\dot{V}_3 &= \bar{h}\dot{x}^{\mathrm{T}}(t)R_4\dot{x}(t) - \int_{t-\bar{h}}^{t}\dot{x}^{\mathrm{T}}(s)R_4\dot{x}(s)\,\mathrm{d}s\\ &\quad + \bar{\tau}g^{\mathrm{T}}(x(t))R_5 g(x(t)) - \int_{t-\bar{\tau}}^{t} g^{\mathrm{T}}(x(s))R_5 g(x(s))(s)\,\mathrm{d}s.\end{aligned} \tag{18}$$

Here, by utilizing Lemma 1, we obtain

$$\begin{aligned}-\int_{t-\bar{h}}^{t}\dot{x}^{\mathrm{T}}(s)R_4\dot{x}(s)\,\mathrm{d}s &\leqslant -\int_{t-h(t)}^{t}\dot{x}^{\mathrm{T}}(s)R_4\dot{x}(s)\,\mathrm{d}s\\ &\leqslant \zeta^{\mathrm{T}}(t)\widetilde{F}\zeta(t) + \bar{h}\zeta^{\mathrm{T}}(t)\bar{F}^{\mathrm{T}}R_4^{-1}\bar{F}\zeta(t),\end{aligned} \tag{19}$$

and

$$\begin{aligned}-\int_{t-\bar{\tau}}^{t} g^{\mathrm{T}}(x(s))R_5 g(x(s))\,\mathrm{d}s &\leqslant -\int_{t-\tau(t)}^{t} g^{\mathrm{T}}(x(s))R_5 g(x(s))\,\mathrm{d}s\\ &\leqslant \zeta^{\mathrm{T}}(t)\widetilde{J}\zeta(t) + \bar{\tau}\zeta^{\mathrm{T}}(t)\bar{J}^{\mathrm{T}}R_5^{-1}\bar{J}\zeta(t),\end{aligned} \tag{20}$$

where $\zeta(t)$ is defined in Eq. (12), and $\bar{F}$, $\bar{J}$, $\widetilde{F}$, $\widetilde{J}$ are defined in Appendix, respectively.

Thus, we have a new upper bound of $\dot{V}_3$ as follows:

$$\dot{V}_3 \leqslant \bar{h}\dot{x}^{\mathrm{T}}(t)R_4\dot{x}(t) + \zeta^{\mathrm{T}}(t)\widetilde{F}\zeta(t) + \bar{h}\zeta^{\mathrm{T}}(t)\bar{F}^{\mathrm{T}}R_4^{-1}\bar{F}\zeta(t) + \bar{\tau}g^{\mathrm{T}}(x(t))R_5g(x(t)) + \zeta^{\mathrm{T}}(t)\widetilde{J}\zeta(t) + \bar{\tau}\zeta^{\mathrm{T}}(t)\bar{J}^{\mathrm{T}}R_5^{-1}\bar{J}\zeta(t). \tag{21}$$

As a tool of deriving a less conservative stability criterion, we add the following two zero equations with any matrices $\bar{N}$ and $\bar{M}$ to be chosen as

$$2\zeta^{\mathrm{T}}(t)\bar{N}\left[-\dot{x}(t) - Ax(t) + W_0g(x(t)) + W_1g(x(t-h(t))) + W_2\int_{t-\tau(t)}^{t} g(x(s))\,\mathrm{d}s + Dp(t)\right] = 0, \tag{22}$$

and

$$2\zeta^{\mathrm{T}}(t)\bar{M}\left[x(t) - x(t-h(t)) - \int_{t-h(t)}^{t}\dot{x}(s)\,\mathrm{d}s\right] = 0, \tag{23}$$

where $\bar{N}$ and $\bar{M}$ are defined in Appendix.

This can be represented as

$$\zeta^{\mathrm{T}}(t)(\Xi_1 + \Xi_2)\zeta(t) = 0, \tag{24}$$

where Ξ_1 and Ξ_2 are also defined in Appendix.

Here note that Eq. (7) means that

$$g_j(x_j(t))[g_j(x_j(t) - l_jx_j(t))] \leqslant 0 \quad (j = 1,\ldots,n), \tag{25}$$

and

$$g_j(x_j(t-h(t)))[g_j(x_j(t-h(t))) - l_jx_j(t-h(t))] \leqslant 0 \quad (j = 1,\ldots,n). \tag{26}$$

From the above two inequalities (25) and (26), for any diagonal positive matrices $K_1 = \mathrm{diag}\{k_{11},\ldots,k_{1n}\}$ and $K_2 = \mathrm{diag}\{k_{21},\ldots,k_{2n}\}$, the following inequalities hold:

$$\begin{aligned} 0 &\leqslant -2\sum_{j=1}^{n} k_{1j}g_j(x_j(t))[g_j(x_j(t)) - l_jx_j(t)] \\ &\quad -2\sum_{j=1}^{n} k_{2j}g_j(x_j(t-h(t)))[g_j(x_j(t-h(t))) - l_jx_j(t-h(t))] \\ &= 2[x^{\mathrm{T}}(t)LK_1g(x(t)) - g^{\mathrm{T}}(x(t))K_1g(x(t)) \\ &\quad + x^{\mathrm{T}}(t-h(t))LK_2g(x(t-h(t))) - g^{\mathrm{T}}(x(t-h(t)))K_2g(x(t-h(t)))]. \end{aligned} \tag{27}$$

Since the following inequality holds from (3) and (9):

$$p^{\mathrm{T}}(t)p(t) \leqslant q^{\mathrm{T}}(t)q(t) = \zeta^{\mathrm{T}}(t)\Psi^{\mathrm{T}}\Psi\zeta(t), \tag{28}$$

there exist a positive matrix H satisfying the following inequality:

$$\zeta^{\mathrm{T}}(t)\Psi^{\mathrm{T}}H\Psi\zeta(t) - p^{\mathrm{T}}(t)Hp(t) \geqslant 0, \tag{29}$$

where Ψ is defined in Appendix.

From Eqs. (16)–(29) and applying S-procedure [18], the time derivative of V has a new upper bound as

$$
\begin{aligned}
\dot{V}(t) \leqslant\ & x^{\mathrm{T}}(t)(-PA - A^{\mathrm{T}}P + PG + G^{\mathrm{T}}P)x(t) - 2x^{\mathrm{T}}(t)PGx(t-h(t)) \\
& - 2x^{\mathrm{T}}(t)PG\int_{t-h(t)}^{t}\dot{x}(s)\,\mathrm{d}s + 2x^{\mathrm{T}}(t)PW_0 g(x(t)) + 2x^{\mathrm{T}}(t)PW_1 g(x(t-h(t))) \\
& + 2x^{\mathrm{T}}(t)PW_2\int_{t-\tau(t)}^{t} g(x(s))\,\mathrm{d}s + 2x^{\mathrm{T}}(t)PDp(t) + 2g^{\mathrm{T}}(x(t))R_1\dot{x}(t) \\
& + g^{\mathrm{T}}(x(t))R_2 g(x(t)) - (1-\mu)g^{\mathrm{T}}(x(t-h(t)))R_2 g(x(t-h(t))) \\
& + x^{\mathrm{T}}(t)R_3 x(t) - (1-\mu)x^{\mathrm{T}}(t-h(t))R_3 x(t-h(t)) \\
& + \bar{h}\dot{x}^{\mathrm{T}}(t)R_4\dot{x}(t) + \zeta^{\mathrm{T}}(t)\widetilde{F}\zeta(t) + \bar{h}\zeta^{\mathrm{T}}(t)\bar{F}^{\mathrm{T}}R_4^{-1}\bar{F}\zeta(t) \\
& + \bar{\tau}g^{\mathrm{T}}(x(t))R_5 g(x(t)) + \zeta^{\mathrm{T}}(t)\widetilde{J}\zeta(t) + \bar{\tau}\zeta^{\mathrm{T}}(t)\bar{J}^{\mathrm{T}}R_5^{-1}\bar{J}\zeta(t) + \zeta^{\mathrm{T}}(t)(\Xi_1 + \Xi_2)\zeta(t) \\
& + 2[x^{\mathrm{T}}(t)LK_1 g(x(t)) - g^{\mathrm{T}}(x(t))K_1 g(x(t)) \\
& + x^{\mathrm{T}}(t-h(t))LK_2 g(x(t-h(t))) - g^{\mathrm{T}}(x(t-h(t)))K_2 g(x(t-h(t)))] \\
& + \zeta^{\mathrm{T}}(t)\Psi^{\mathrm{T}}H\Psi\zeta(t) - p^{\mathrm{T}}(t)Hp(t) \\
=\ & \zeta^{\mathrm{T}}(t)[\Omega + \bar{h}\bar{F}^{\mathrm{T}}R_4^{-1}\bar{F} + \bar{\tau}\bar{J}^{\mathrm{T}}R_5^{-1}\bar{J} + \Psi^{\mathrm{T}}H\Psi]\zeta(t),
\end{aligned} \tag{30}
$$

where

$$
\Omega = \begin{bmatrix}
(1,1) & (1,2) & (1,3) & \Sigma_{14} & \Sigma_{15} & \Sigma_{16} & \Sigma_{17} & \Sigma_{18} \\
\star & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} & \Sigma_{26} & \Sigma_{27} & \Sigma_{28} \\
\star & \star & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} & \Sigma_{36} & \Sigma_{37} & \Sigma_{38} \\
\star & \star & \star & \Sigma_{44} & \Sigma_{45} & \Sigma_{46} & \Sigma_{47} & \Sigma_{48} \\
\star & \star & \star & \star & \Sigma_{55} & \Sigma_{56} & \Sigma_{57} & \Sigma_{58} \\
\star & \star & \star & \star & \star & \Sigma_{66} & \Sigma_{67} & \Sigma_{68} \\
\star & \star & \star & \star & \star & \star & \Sigma_{77} & \Sigma_{78} \\
\star & \star & \star & \star & \star & \star & \star & \Sigma_{88}
\end{bmatrix},
$$

$$
\begin{aligned}
(1,1) &= P(-A+G) + (-A+G)^{\mathrm{T}}P + R_3 - N_1 A - A^{\mathrm{T}}N_1^{\mathrm{T}} + M_1 + M_1^{\mathrm{T}}, \\
(1,2) &= -PG - A^{\mathrm{T}}N_2^{\mathrm{T}} - M_1 + M_2^{\mathrm{T}}, \\
(1,3) &= -PG + F_1^{\mathrm{T}} - A^{\mathrm{T}}N_3^{\mathrm{T}} - M_1 + M_3^{\mathrm{T}},
\end{aligned}
$$

and other Σ_{ij} are defined in Appendix.

By defining $Y_1 = PG$ and using Fact 1, the inequality $\Omega + \bar{h}\bar{F}^{\mathrm{T}}R_4^{-1}\bar{F} + \bar{\tau}\bar{J}^{\mathrm{T}}R_5^{-1}\bar{J} + \Psi^{\mathrm{T}}H\Psi < 0$, which guarantees the stability of system (1) by the Lyapunov stability theory, is equivalent to the LMI (13). This completes our proof. □

Remark 1. Since the LMIs (13) in Theorem 1 can be easily solved by various efficient convex algorithms. In this paper, we utilize Matlab's LMI Control Toolbox [19] which implements the interior-point algorithm. This algorithm is significantly faster than classical convex optimization algorithms [18].

Remark 2. By iteratively solving the LMIs given in Theorem 1 with respect to $\bar{h}$ for fixed $\bar{\tau}$, one can find the maximum upper bound of time delay $\bar{h}$ for guaranteeing asymptotic stability of system (1).

Remark 3. In [11,12], the additional condition $\dot{h}(t) \leqslant \mu < 1$ is required to guarantee the stability of DCNNs with time-varying delays. However, our criterion do not need this condition, which is more general cases than the previous in other literature.

Remark 4. When the bound of time-delay derivative μ is unknown, we can obtain a delay-dependent stability criterion using similar method in Theorem 1. With the Lyapunov functional candidate,

$$V = x^{\mathrm{T}}(t)Px(t) + 2\sum_{i=1}^{n} r_i \int_0^{x_i(t)} g_i(s)\,\mathrm{d}s + \int_{t-\bar{h}}^{t}\int_s^t \dot{x}^{\mathrm{T}}(u)R_4\dot{x}(u)\,\mathrm{d}u\,\mathrm{d}s + \int_{t-\bar{\tau}}^{t}\int_s^t g^{\mathrm{T}}(x(u))R_5 g(x(u))\,\mathrm{d}u\,\mathrm{d}s, \quad (31)$$

the delay-dependent stability criterion can be obtained by letting $R_2 = R_3 = 0$ in Theorem 1.

4. Numerical example

Example 1. Consider the uncertain DCNNs of form (1) in [15], where

$$A = \begin{bmatrix} 2.3 & 0 & 0 \\ 0 & 3.4 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.9 & -1.5 & 0.1 \\ -1.2 & 1 & 0.2 \\ 0.2 & 0.3 & 0.8 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ 0.5 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.5 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}, \quad L = \mathrm{diag}\{0.20.20.2\}, \quad D = I, \quad E_a = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.1 \end{bmatrix},$$

$$E_0 = \begin{bmatrix} 0.2 & 0 & 0.3 \\ 0.2 & 0 & 0.3 \\ 0.2 & 0 & 0.3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}, \quad E_2 = [0 \;\; 0.1 \;\; 0.1].$$

Table 1
Comparison of delay bounds with $\bar{h} = \bar{\tau}$ (Example 1)

	$\mu = 0$	$\mu = 0.8$	$\mu = 1$ or unknown
[15]	19.5	16.3	2.861
Ours	∞	∞	3.40

By applying Theorem 1 and Remark 4 to the above system, we can obtain the maximum allowable delay bounds as shown in Table 1 which compares our results with ones in [15] for the case $\bar{h} = \bar{\tau}$. From Table 1, one can see that our results guarantee much larger delay bounds.

5. Conclusion

In this paper, the problems of delay-dependent stability criterion for cellular neural networks with discrete and distributed time-varying delays have been studied. By constructing a suitable Lyapunov–Krasovskii functionals, the stability criterion are derived in terms of LMIs which can be easily solved via convex optimization algorithm. A numerical example is included to show that our proposed method provides a larger time-delay bound than other results.

Appendix

$$\begin{aligned}
&\Sigma = (\Sigma_{ij}) \quad (i = 1,\ldots,8,\ j = 1,\ldots,8),\\
&\Sigma_{11} = -PA - A^{\mathrm{T}}P + Y_1 + Y_1^{\mathrm{T}} + R_3 - N_1A - A^{\mathrm{T}}N_1^{\mathrm{T}} + M_1 + M_1^{\mathrm{T}},\\
&\Sigma_{12} = -Y_1 - A^{\mathrm{T}}N_2^{\mathrm{T}} - M_1 + M_2^{\mathrm{T}},\\
&\Sigma_{13} = -Y_1 + F_1^{\mathrm{T}} - A^{\mathrm{T}}N_3^{\mathrm{T}} - M_1 + M_3^{\mathrm{T}},\\
&\Sigma_{14} = -N_1 - A^{\mathrm{T}}N_4^{\mathrm{T}} + M_4^{\mathrm{T}},\\
&\Sigma_{15} = PW_0 + LK_1 + N_1W_0 - A^{\mathrm{T}}N_5^{\mathrm{T}} + M_5^{\mathrm{T}},\\
&\Sigma_{16} = PW_1 + N_1W_1 - A^{\mathrm{T}}N_6^{\mathrm{T}} + M_6^{\mathrm{T}},\\
&\Sigma_{17} = PW_2 + N_1W_2 - A^{\mathrm{T}}N_7^{\mathrm{T}} + M_7^{\mathrm{T}} + J_1^{\mathrm{T}},\\
&\Sigma_{18} = PD + N_1D - A^{\mathrm{T}}N_8^{\mathrm{T}} + M_8^{\mathrm{T}},\\
&\Sigma_{22} = -(1-\mu)R_3 - M_2 - M_2^{\mathrm{T}},\\
&\Sigma_{23} = F_2^{\mathrm{T}} - M_2 - M_3^{\mathrm{T}},\\
&\Sigma_{24} = -N_2 - M_4^{\mathrm{T}},\\
&\Sigma_{25} = N_2W_0 - M_5^{\mathrm{T}},\\
&\Sigma_{26} = LK_2 + N_2W_1 - M_6^{\mathrm{T}},\\
&\Sigma_{27} = N_2W_2 - M_7^{\mathrm{T}} + J_2^{\mathrm{T}},\\
&\Sigma_{28} = N_2D - M_8^{\mathrm{T}},\\
&\Sigma_{33} = F_3 + F_3^{\mathrm{T}} - M_3 - M_3^{\mathrm{T}},\\
&\Sigma_{34} = F_4 - N_3 - M_4^{\mathrm{T}},\\
&\Sigma_{35} = F_5 + N_3W_0 - M_5^{\mathrm{T}},\\
&\Sigma_{36} = F_6 + N_3W_1 - M_6^{\mathrm{T}},\\
&\Sigma_{37} = F_7 + N_3W_2 - M_7^{\mathrm{T}} + J_3^{\mathrm{T}},
\end{aligned}$$

$$\Sigma_{38} = F_8 + N_3 D - M_8^{\mathrm{T}},$$
$$\Sigma_{44} = \bar{h} R_4 - N_4 - N_4^{\mathrm{T}},$$
$$\Sigma_{45} = R_1 + N_4 W_0 - N_5^{\mathrm{T}},$$
$$\Sigma_{46} = N_4 W_1 - N_6^{\mathrm{T}},$$
$$\Sigma_{47} = N_4 W_2 - N_7^{\mathrm{T}} + J_4^{\mathrm{T}},$$
$$\Sigma_{48} = N_4 D - N_8^{\mathrm{T}},$$
$$\Sigma_{55} = R_2 + \tau R_5 - 2K_1 + N_5 W_0 + W_0^{\mathrm{T}} N_5^{\mathrm{T}},$$
$$\Sigma_{56} = N_5 W_1 + W_0^{\mathrm{T}} N_6^{\mathrm{T}},$$

$$\Sigma_{57} = N_5 W_2 + W_0^{\mathrm{T}} N_7^{\mathrm{T}} + J_5^{\mathrm{T}},$$
$$\Sigma_{58} = N_5 D + W_0^{\mathrm{T}} N_8^{\mathrm{T}},$$
$$\Sigma_{66} = -(1-\mu) R_2 - 2K_2 + N_6 W_1 + W_1^{\mathrm{T}} N_6^{\mathrm{T}},$$
$$\Sigma_{67} = N_6 W_2 + W_1^{\mathrm{T}} N_7^{\mathrm{T}} + J_6^{\mathrm{T}},$$
$$\Sigma_{68} = N_6 D + W_1^{\mathrm{T}} N_8^{\mathrm{T}},$$
$$\Sigma_{77} = N_7 W_2 + W_2^{\mathrm{T}} N_7^{\mathrm{T}} + J_7^{\mathrm{T}} + J_7,$$
$$\Sigma_{78} = N_7 D + W_2^{\mathrm{T}} N_8^{\mathrm{T}} + J_8,$$
$$\Sigma_{88} = -H + N_8 D + D^{\mathrm{T}} N_8^{\mathrm{T}},$$
$$\bar{F} = [F_1 \;\; F_2 \;\; F_3 \;\; F_4 \;\; F_5 \;\; F_6 \;\; F_7 \;\; F_8],$$
$$\bar{J} = [J_1 \;\; J_2 \;\; J_3 \;\; J_4 \;\; J_5 \;\; J_6 \;\; J_7 \;\; J_8],$$

$$\widetilde{F} = \begin{bmatrix} 0 & 0 & F_1^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\ \star & 0 & F_2^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & F_3^{\mathrm{T}} + F_3 & F_4 & F_5 & F_6 & F_7 & F_8 \\ \star & \star & \star & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & 0 \end{bmatrix},$$

$$\widetilde{J} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & J_1^{\mathrm{T}} & 0 \\ \star & 0 & 0 & 0 & 0 & 0 & J_2^{\mathrm{T}} & 0 \\ \star & \star & 0 & 0 & 0 & 0 & J_3^{\mathrm{T}} & 0 \\ \star & \star & \star & 0 & 0 & 0 & J_4^{\mathrm{T}} & 0 \\ \star & \star & \star & \star & 0 & 0 & J_5^{\mathrm{T}} & 0 \\ \star & \star & \star & \star & \star & 0 & J_6^{\mathrm{T}} & 0 \\ \star & \star & \star & \star & \star & \star & J_7^{\mathrm{T}} + J_7 & J_8^{\mathrm{T}} \\ \star & \star & \star & \star & \star & \star & \star & 0 \end{bmatrix},$$

$$\bar{N} = [N_1 \;\; N_2 \;\; N_3 \;\; N_4 \;\; N_5 \;\; N_6 \;\; N_7 \;\; N_8],$$

$$\bar{M} = [M_1 \;\; M_2 \;\; M_3 \;\; M_4 \;\; M_5 \;\; M_6 \;\; M_7 \;\; M_8],$$

$$\Xi_1 = \begin{bmatrix} -N_1A - AN_1^{\mathrm{T}} & -A^{\mathrm{T}}N_2^{\mathrm{T}} & -A^{\mathrm{T}}N_3^{\mathrm{T}} & -N_1 - A^{\mathrm{T}}N_4^{\mathrm{T}} & N_1W_0 - A^{\mathrm{T}}N_5^{\mathrm{T}} & N_1W_1 - A^{\mathrm{T}}N_6^{\mathrm{T}} & N_1W_2 - A^{\mathrm{T}}N_7^{\mathrm{T}} & N_1D - A^{\mathrm{T}}N_8^{\mathrm{T}} \\ \star & 0 & 0 & -N_2 & N_2W_0 & N_2W_1 & N_2W_2 & N_2D \\ \star & \star & 0 & -N_3 & N_3W_0 & N_3W_1 & N_3W_2 & N_3D \\ \star & \star & \star & -N_4 - N_4^{\mathrm{T}} & N_4W_0 - N_5^{\mathrm{T}} & N_4W_1 - N_6^{\mathrm{T}} & N_4W_2 - N_7^{\mathrm{T}} & N_4D - N_8^{\mathrm{T}} \\ \star & \star & \star & \star & N_5W_0 + W_0^{\mathrm{T}}N_5^{\mathrm{T}} & N_5W_1 + W_0N_6^{\mathrm{T}} & N_5W_2 + W_0^{\mathrm{T}}N_7^{\mathrm{T}} & N_5D + W_0^{\mathrm{T}}N_8^{\mathrm{T}} \\ \star & \star & \star & \star & \star & N_6W_1 + W_1^{\mathrm{T}}N_6^{\mathrm{T}} & N_6W_2 + W_1^{\mathrm{T}}N_7^{\mathrm{T}} & N_6D + W_1^{\mathrm{T}}N_8^{\mathrm{T}} \\ \star & \star & \star & \star & \star & \star & N_7W_2 + W_2^{\mathrm{T}}N_7^{\mathrm{T}} & N_7D + W_2^{\mathrm{T}}N_8^{\mathrm{T}} \\ \star & \star & \star & \star & \star & \star & \star & N_8D + D^{\mathrm{T}}N_8^{\mathrm{T}} \end{bmatrix},$$

$$\Xi_2 = \begin{bmatrix} M_1 + M_1^{\mathrm{T}} & -M_1 + M_2^{\mathrm{T}} & -M_1 + M_3^{\mathrm{T}} & M_4^{\mathrm{T}} & M_5^{\mathrm{T}} & M_6^{\mathrm{T}} & M_7^{\mathrm{T}} & M_8^{\mathrm{T}} \\ \star & -M_2 - M_2^{\mathrm{T}} & -M_2 - M_3^{\mathrm{T}} & -M_4^{\mathrm{T}} & -M_5^{\mathrm{T}} & M_6^{\mathrm{T}} & M_7^{\mathrm{T}} & M_8^{\mathrm{T}} \\ \star & \star & -M_3 - M_3^{\mathrm{T}} & -M_4^{\mathrm{T}} & -M_5^{\mathrm{T}} & M_6^{\mathrm{T}} & M_7^{\mathrm{T}} & M_8^{\mathrm{T}} \\ \star & \star & \star & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & 0 \end{bmatrix},$$

$$\Psi = [E_a \;\; 0 \;\; 0 \;\; 0 \;\; E_0 \;\; E_1 \;\; E_2 \;\; 0].$$

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