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Delay dynamic double integral inequalities on time scales with applications

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Abstract

In the article, we present the explicit bounds for three generalized delay dynamic Gronwall–Bellman type integral inequalities on time scales, which are the unification of continuous and discrete results. As applications, the boundedness for the solutions of delay dynamic integro-differential equations with initial conditions is discussed.

MSC: 26D10; 34C11; 39A12

Keywords: Delay integral inequality; Time scale; Dynamic equation; Discrete inequality; Boundedness

1 Introduction

Theory of time scales is the unification of both continuous and discrete analysis due to Stephen Hilger [1] in his PhD thesis, and it has wide applications in quantum calculus and difference and differential calculus. Due to its vast contributions in different branches of mathematics, it attracts the researchers and mathematicians to work on it. The role of inequalities cannot be forgot because they have huge contributions in the theory of differential equations [2–22], bivariate means [23–31], calculation and optimization [32–49], special functions [50–69], probability and statistics [70–75], and so on.

It is well known that the explicit bounds for integral inequalities on unknown functions are very useful in the qualitative and quantitative analysis for the solutions of differential, integral, and integro-differential equations [76–100]. One of the most important inequalities in mathematics is the Gronwall inequality, it is an indispensable tool to obtain various estimates in the theory of ordinary and stochastic differential equations. The differential form of this inequality was proved by Gronwall [101] and its integral form was proved by Bellman [102]. Gronwall–Bellman integral inequalities have a lot of contributions to analyzing the behavior of solutions of the differential and integral equations. Recently, to dealt with the difficulties encountered in the solutions of differential and difference equations, the Gronwall–Bellman integral inequality on time scales has become one of the most important topics in mathematics research. A lot of work has been done in this area [103–106]. To discuss the abstract analysis of the solutions of certain types of dynamic equations, Gronwall–Bellman delay type inequality on time scales can play a significant role, but it still cannot deal with the abstract analysis for the solutions of some more general differential and difference equations. This gives us the motivation to discuss some generalized delay integral inequalities on time scales. The main purpose of the article is to provide the explicit bounds for some delay double integral inequalities on time scales, present the characteristics of the solutions of certain integro-differential equations, and establish their discrete inequalities with initial conditions by use of the analytical and numerical methods in our obtained results.

2 Preliminaries

The time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers set $\mathbb{R}$, its forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ for $t \in \mathbb{T}$ is defined by $\sigma(t) := \inf\{r \in \mathbb{T} : r > t\}$ and its backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ for $t \in \mathbb{T}$ is defined by $\rho(t) := \sup\{r \in \mathbb{T} : r < t\}$. The derived set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

Lemma 2.1 (see [107]) *Let $a, x \in \mathbb{T}^k$ with $x > a$, $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ be continuous at (x, x), $f^\Delta(x, \cdot)$ be the derivative of f with respect to its first variable such that it is right-dense continuous on $[a, \sigma(x)]$, and $g(x) = \int_a^x f(x, \tau)\Delta\tau$. If, for each $\epsilon > 0$, there exists a neighborhood U of x independent of $\tau \in [a, \sigma(x)]$ such that*

$$\left|f\bigl(\sigma(x), \tau\bigr) - f(y, \tau) - f^\Delta(x, \tau)\bigl(\sigma(x) - y\bigr)\right| \le \epsilon\left|\sigma(x) - y\right|$$

for all $y \in U$, then

$$g^\Delta(x) = \int_a^x f^\Delta(x, \tau)\Delta\tau + f\bigl(\sigma(x), x\bigr).$$

Theorem 2.2 (see [108]) *Let $f_1 : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable and $f_2 : \mathbb{T} \to \mathbb{R}$ be delta differentiable. Then $f_1^\circ f_2 : \mathbb{T} \to \mathbb{R}$ is delta differentiable and*

$$\bigl(f_1^\circ f_2\bigr)^\Delta(x) = \left\{\int_0^1 f_1'\bigl(f_2(x) + h\mu(x)f_2^\Delta(x)\bigr)\,dh\right\} f_2^\Delta(x).$$

Theorem 2.3 (see [108]) *Let $h_1 : \mathbb{T} \to \mathbb{R}$ be strictly increasing such that $\overline{\mathbb{T}} = h_1(\mathbb{T})$ is a time scale, and $h_2 : \overline{\mathbb{T}} \to \mathbb{R}$. Then*

$$\bigl(h_2^\circ h_1\bigr)^\Delta = \bigl(h_2^{\overline{\Delta}} \circ h_1\bigr)h_1^\Delta$$

if $h_1^\Delta(x)$ and $h_2^{\overline{\Delta}}(h_1(x))$ exist for $x \in \mathbb{T}^k$.

3 Main results

Throughout the section, $\mathbb{R}$ represents the set of real numbers, $\mathbb{R}_0^+ = [0, \infty)$, $\mathbb{R}_1^+ = [1, \infty)$, $\mathbb{Z}$ represents the set of integers, $\mathbb{N}_0$ represents the set of nonnegative integers, $\mathbb{A}_j \subseteq \mathbb{N}_0$ $(1 \le j \le 2)$, $x_{0j} \in \mathbb{T}$, $\mathbb{T}_j = [x_{0j}, \infty)_{\mathbb{T}} \subseteq \mathbb{T}^k$ is the time scale, ρ_{ji} is the backward jump operator, $X^{\Delta y_i}(y_1, y_2, \dots, y_n)$ $(1 \le i \le n)$ is the partial delta-derivative of X with respect to its ith variable and $\Delta y_i X(y_1, y_2, \dots, y_n)$ is the forward difference of X with respect to its ith variable.

Theorem 3.1 *Let $u, r_i, a_j : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}_0^+$ and $f_i, g_i, f_i^{\Delta x_1} : \mathbb{T}_1^2 \times \mathbb{T}_2^2 \to \mathbb{R}_0^+$ be nonnegative and right-dense continuous such that a_j is nondecreasing with respect to its each* *variable, let* $\gamma_{ji} : \mathbb{T}_j \to \mathbb{R}_0^+$ *be nonnegative, nondecreasing, and right-dense continuous such that* $\gamma_{ji}(x_j) \leq x_j$ *and* $\gamma_{ji}^{\Delta}(x_j) > 0$, $\mu_{ji} : \mathbb{T}_j \to \mathbb{T}$ *such that* $\mu_{ji}(x_j) \leq x_j$ *and* $-\infty < \mathfrak{p}_j = \inf\{\min(\mu_{ji}(x_j)), x_j \in \mathbb{T}_j\} \leq x_{0j}$, $\mathfrak{a} : ([\mathfrak{p}_1, x_{01}] \times [\mathfrak{p}_2, x_{02}])_{\mathbb{T}^2} \to \mathbb{R}_0^+$ *be nonnegative and right-dense continuous,* $w, w_j : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be nonnegative, nondecreasing, and continuous such that* $w(p) > 0$, $w_j(p) > 0$ *for* $p > 0$ *and*

$$\begin{aligned} & w\big(u(x_1,x_2)\big) \\ & \leq a_1(x_1,x_2) + a_2(x_1,x_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\big(u\big(\mu_{1i}(t_1), \mu_{2i}(t_2)\big)\big) \\ & \quad \times \Bigg[f_i(x_1,t_1,x_2,t_2) \Bigg\{ w_2\big(u\big(\mu_{1i}(t_1), \mu_{2i}(t_2)\big)\big) + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\ & \quad \times w_2\big(u\big(\mu_{1i}(m_1), \mu_{2i}(m_2)\big)\big) \Delta m_2 \Delta m_1 \Bigg\} + r_i(t_1,t_2) \Bigg] \Delta t_2 \Delta t_1 \end{aligned} \tag{3.1}$$

for $(x_1,x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ *with the initial condition*

$$\begin{cases} w(u(x_1,x_2)) = \mathfrak{a}(x_1,x_2), & x_1 \in [\mathfrak{p}_1, x_{01}]_{\mathbb{T}} \text{ or } x_2 \in [\mathfrak{p}_2, x_{02}]_{\mathbb{T}}, \\ \mathfrak{a}(\mu_{1i}(x_1), \mu_{2i}(x_2)) \leq a_1(x_1,x_2), & \mu_{1i}(x_1) \leq x_{01} \text{ or } \mu_{2i}(x_2) \leq x_{02}. \end{cases} \tag{3.2}$$

Then

$$u(x_1,x_2) \leq w^{-1}\big(G_1^{-1}\big(G_2^{-1}\big(G_2\big(b_1(x_1,x_2)\big) + a_2(x_1,x_2)c(x_1,x_2)\big)\big)\big) \tag{3.3}$$

for all $x_{01} \leq x_1 \leq \tilde{x}_1$ *and* $x_{02} \leq x_2 \leq \tilde{x}_2$ *if*

$$\begin{aligned} b_1(x_1,x_2) &= G_1\big(a_1(x_1,x_2)\big) + a_2(x_1,x_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} r_i(t_1,t_2) \Delta t_2 \Delta t_1, \\ c(x_1,x_2) &= \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2) \\ & \quad \times \left(1 + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \Delta m_2 \Delta m_1 \right) \Delta t_2 \Delta t_1, \\ G_1(s) &= \int_{s_1}^{s} \frac{\Delta p}{w_1(w^{-1}(p))}, \quad s > s_1 > 0 \text{ with } G_1(\infty) = \infty, \\ G_2(s) &= \int_{s_2}^{s} \frac{\Delta p}{w_2(w^{-1}(G_1^{-1}(p)))}, \quad s > s_2 > 0 \text{ with } G_2(\infty) = \infty, \end{aligned}$$

where G_1^{-1} *and* G_2^{-1} *are respectively the inverse functions of* G_1 *and* G_2, *and* $\tilde{x}_1, \tilde{x}_2$ *are chosen such that*

$$\begin{aligned} & G_2\big(b_1(x_1,x_2)\big) + a_2(x_1,x_2)c(x_1,x_2) \in \operatorname{Dom}\big(G_2^{-1}\big), \\ & G_2^{-1}\big(G_2\big(b_1(x_1,x_2)\big) + a_2(x_1,x_2)c(x_1,x_2)\big) \in \operatorname{Dom}\big(G_1^{-1}\big), \\ & G_1^{-1}\big(G_2^{-1}\big(G_2\big(b_1(x_1,x_2)\big) + a_2(x_1,x_2)c(x_1,x_2)\big)\big) \in \operatorname{Dom}\big(w^{-1}\big). \end{aligned}$$

Proof For fixed numbers $\bar{x}_1 \in \mathbb{T}_1$, $\bar{x}_2 \in \mathbb{T}_2$ with $x_{01} \leq \bar{x}_1 \leq \tilde{x}_1$ and $x_{02} \leq \bar{x}_2 \leq \tilde{x}_2$, inequality (3.1) can be rewritten as follows:

$$\begin{aligned} & w\bigl(u(x_1,x_2)\bigr) \\ & \quad \leq a_1(\bar{x}_1,\bar{x}_2) + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\bigl(u\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr)\bigr) \\ & \qquad \times \biggl[f_i(x_1,t_1,x_2,t_2)\biggl\{w_2\bigl(u\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr)\bigr) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\ & \qquad \times w_2\bigl(u\bigl(\mu_{1i}(m_1),\mu_{2i}(m_2)\bigr)\bigr)\Delta m_2 \Delta m_1\biggr\} + r_i(t_1,t_2)\biggr]\Delta t_2 \Delta t_1. \end{aligned}$$

Let

$$\begin{aligned} & \xi_1(x_1,x_2) \\ & \quad = a_1(\bar{x}_1,\bar{x}_2) + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\bigl(u\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr)\bigr) \\ & \qquad \times \biggl[f_i(x_1,t_1,x_2,t_2)\biggl\{w_2\bigl(u\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr)\bigr) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\ & \qquad \times w_2\bigl(u\bigl(\mu_{1i}(m_1),\mu_{2i}(m_2)\bigr)\bigr)\Delta m_2 \Delta m_1\biggr\} + r_i(t_1,t_2)\biggr]\Delta t_2 \Delta t_1 \end{aligned} \tag{3.4}$$

for $x_1 \in [x_{01},\bar{x}_1]_{\mathbb{T}}$ and $x_2 \in [x_{02},\bar{x}_2]_{\mathbb{T}}$. Then

$$\xi_1(x_{01},x_2) = \xi_1(x_1,x_{02}) = a_1(\bar{x}_1,\bar{x}_2) \tag{3.5}$$

and

$$w\bigl(u(x_1,x_2)\bigr) \leq \xi_1(x_1,x_2), \qquad u(x_1,x_2) \leq w^{-1}\bigl(\xi_1(x_1,x_2)\bigr). \tag{3.6}$$

If $\mu_{1i}(x_1) \geq x_{01}$ and $\mu_{2i}(x_2) \geq x_{02}$ for $x_1 \in [x_{01},\bar{x}_1]_{\mathbb{T}}$ and $x_2 \in [x_{02},\bar{x}_2]_{\mathbb{T}}$, then $\mu_{1i}(x_1) \in [x_{01},\bar{x}_1]_{\mathbb{T}}$, $\mu_{2i}(x_1) \in [x_{02},\bar{x}_2]_{\mathbb{T}}$, and

$$u\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr) \leq w^{-1}\bigl(\xi_1\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr)\bigr) \leq w^{-1}\bigl(\xi_1(x_1,x_2)\bigr). \tag{3.7}$$

On the other hand, if $\mu_{1i}(x_1) \leq x_{01}$ or $\mu_{2i}(x_2) \leq x_{02}$, then from (3.2) we have

$$\begin{aligned} u\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr) &= w^{-1}\bigl(\mathfrak{a}\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr)\bigr) \\ &\leq w^{-1}\bigl(a_1(x_1,x_2)\bigr) \leq w^{-1}\bigl(\xi_1(x_1,x_2)\bigr). \end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$u\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr) \leq w^{-1}\bigl(\xi_1(x_1,x_2)\bigr) \tag{3.9}$$

for $x_1 \in [x_{01},\bar{x}_1]_{\mathbb{T}}$ and $x_2 \in [x_{02},\bar{x}_2]_{\mathbb{T}}$.

From Lemma 2.1, (3.4), and (3.9), one has

$$\begin{aligned}
&{\xi_1}^{\Delta x_1}(x_1,x_2)\\
&= a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\big(u\big(\mu_{1i}\big(\gamma_{1i}(x_1)\big),\mu_{2i}(t_2)\big)\big)\\
&\quad\times\bigg[f_i\big(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\big)\bigg\{w_2\big(u\big(\mu_{1i}\big(\gamma_{1i}(x_1)\big),\mu_{2i}(t_2)\big)\big)\\
&\quad+\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i\big(\gamma_{1i}(x_1),m_1,t_2,m_2\big)\\
&\quad\times w_2\big(u\big(\mu_{1i}(m_1),\mu_{2i}(m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}+r_i\big(\gamma_{1i}(x_1),t_2\big)\bigg]\Delta t_2\\
&\quad+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\\
&\quad\times\bigg[f_i(x_1,t_1,x_2,t_2)\bigg\{w_2\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\\
&\quad+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)w_2\big(u\big(\mu_{1i}(m_1),\mu_{2i}(m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}\\
&\quad+r_i(t_1,t_2)\bigg]\Delta t_2\bigg]^{\Delta x_1}\Delta t_1
\end{aligned}$$

and

$$\begin{aligned}
&{\xi_1}^{\Delta x_1}(x_1,x_2)\\
&\le a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\big(w^{-1}\big(\xi_1\big(\gamma_{1i}(x_1),t_2\big)\big)\big)\\
&\quad\times\bigg[f_i\big(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\big)\bigg\{w_2\big(w^{-1}\big(\xi_1\big(\gamma_{1i}(x_1),t_2\big)\big)\big)\\
&\quad+\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i\big(\gamma_{1i}(x_1),m_1,t_2,m_2\big)\\
&\quad\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}+r_i\big(\gamma_{1i}(x_1),t_2\big)\bigg]\Delta t_2\\
&\quad+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big)\\
&\quad\times\bigg[f_i(x_1,t_1,x_2,t_2)\bigg\{w_2\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big)\\
&\quad+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}\\
&\quad+r_i(t_1,t_2)\bigg]\Delta t_2\bigg]^{\Delta x_1}\Delta t_1.
\end{aligned}$$

Using the fact that w_1, w^{-1}, and ξ_1 are nondecreasing, we have

$${\xi_1}^{\Delta x_1}(x_1,x_2)$$

$$\begin{aligned}
&\le a_2(\bar{x}_1,\bar{x}_2)w_1\big(w^{-1}\big(\xi_1(x_1,x_2)\big)\big)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Big[f_i\big(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\big)\\
&\quad\times\bigg\{w_2\big(w^{-1}\big(\xi_1\big(\gamma_{1i}(x_1),t_2\big)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i\big(\gamma_{1i}(x_1),m_1,t_2,m_2\big)\\
&\quad\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}+r_i\big(\gamma_{1i}(x_1),t_2\big)\Big]\Delta t_2\\
&\quad+a_2(\bar{x}_1,\bar{x}_2)w_1\big(w^{-1}\big(\xi_1(x_1,x_2)\big)\big)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Big[f_i(x_1,t_1,x_2,t_2)\\
&\quad\times\bigg\{w_2\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2)\\
&\quad\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}+r_i(t_1,t_2)\Big]\Delta t_2\bigg]^{\Delta x_1}\Delta t_1.
\end{aligned}$$

Dividing both sides by $w_1(w^{-1}(\xi_1(x_1,x_2)))$, we obtain

$$\begin{aligned}
&\frac{\xi_1{}^{\Delta x_1}(x_1,x_2)}{w_1(w^{-1}(\xi_1(x_1,x_2)))}\\
&\le a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Big[f_i\big(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\big)\\
&\quad\times\bigg\{w_2\big(w^{-1}\big(\xi_1\big(\gamma_{1i}(x_1),t_2\big)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i\big(\gamma_{1i}(x_1),m_1,t_2,m_2\big)\\
&\quad\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}+r_i\big(\gamma_{1i}(x_1),t_2\big)\Big]\Delta t_2\\
&\quad+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Big[f_i(x_1,t_1,x_2,t_2)\\
&\quad\times\bigg\{w_2\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2)\\
&\quad\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}+r_i(t_1,t_2)\Big]\Delta t_2\bigg]^{\Delta x_1}\Delta t_1\\
&=a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\bigg[\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Big[f_i(x_1,t_1,x_2,t_2)\\
&\quad\times\bigg\{w_2\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2)\\
&\quad\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}+r_i(t_1,t_2)\Big]\Delta t_2\Delta t_1\bigg]^{\Delta x_1}.
\end{aligned}$$

Integrating over $[x_{01},x_1]$, then using the definition of G_1 and (3.5), we get

$$\begin{aligned}
&G_1\big(\xi_1(x_1,x_2)\big)\\
&\le G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}r_i(t_1,t_2)\Delta t_2\Delta t_1
\end{aligned}$$

$$
\begin{aligned}
&+ a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2)\\
&\times\left\{w_2\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\right.\\
&\left.\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\right\}\Delta t_2\Delta t_1\\
\le{}& G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(\bar{x}_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(\bar{x}_2)} r_i(t_1,t_2)\Delta t_2\Delta t_1\\
&+ a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2)\\
&\times\left\{w_2\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\right.\\
&\left.\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\right\}\Delta t_2\Delta t_1\\
={}& b_1(\bar{x}_1,\bar{x}_2)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2)\\
&\times\left\{w_2\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\right.\\
&\left.\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\right\}\Delta t_2\Delta t_1.
\end{aligned}
\tag{3.10}
$$

Let

$$
\begin{aligned}
\zeta_1(x_1,x_2) ={}& b_1(\bar{x}_1,\bar{x}_2)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2)\\
&\times\left\{w_2\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\right.\\
&\left.\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\right\}\Delta t_2\Delta t_1.
\end{aligned}
\tag{3.11}
$$

Then we have

$$
\zeta_1(x_{01},x_2)=\zeta_1(x_1,x_{02})=b_1(\bar{x}_1,\bar{x}_2)
\tag{3.12}
$$

and

$$
\xi_1(x_1,x_2)\le G_1^{-1}\big(\zeta_1(x_1,x_2)\big).
\tag{3.13}
$$

It follows from Lemma 2.1 and (3.11) that

$$
\begin{aligned}
{\zeta_1}^{\Delta x_1}(x_1,x_2) ={}& a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i\big(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\big)\\
&\times\left\{w_2\big(w^{-1}\big(\xi_1\big(\gamma_{1i}(x_1),t_2\big)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i\big(\gamma_{1i}(x_1),m_1,t_2,m_2\big)\right.
\end{aligned}
$$

$$\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}\Delta t_2$$

$$+ a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2)$$

$$\times\bigg\{w_2\big(w^{-1}\big(\xi_1(t_1,t_2)\big)\big) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)$$

$$\times w_2\big(w^{-1}\big(\xi_1(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}\Delta t_2\bigg]^{\Delta x_1}\Delta t_1.$$

Making use of (3.13) and the fact that w_2, w^{-1}, G_1^{-1}, and ζ_1 are nondecreasing, we get

$$\frac{\zeta_1{}^{\Delta x_1}(x_1,x_2)}{w_2(w^{-1}(G_1^{-1}(\zeta_1(x_1,x_2))))}$$

$$\le a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\bigg[\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i\big(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\big)$$

$$\times\bigg\{1+\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i\big(\gamma_{1i}(x_1),m_1,t_2,m_2\big)\Delta m_2\Delta m_1\bigg\}\Delta t_2$$

$$+\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2)$$

$$\times\bigg\{1+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\bigg\}\Delta t_2\bigg]^{\Delta x_1}\Delta t_1\bigg]$$

$$= a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\bigg[\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2)$$

$$\times\bigg\{1+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\bigg\}\Delta t_2\Delta t_1\bigg]^{\Delta x_1}.$$

Integrating over $[x_{01},x_1]$, then using the definition of G_2 and (3.12), we get

$$G_2\big(\zeta_1(x_1,x_2)\big) \le G_2\big(b_1(\bar{x}_1,\bar{x}_2)\big) + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2)$$

$$\times\bigg\{1+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\bigg\}\Delta t_2\Delta t_1 \qquad (3.14)$$

$$= G_2\big(b_1(\bar{x}_1,\bar{x}_2)\big) + a_2(\bar{x}_1,\bar{x}_2)c(x_1,x_2). \qquad (3.15)$$

A combination of (3.6), (3.13), and (3.15) gives the desired result (3.3). □

Remark 3.2 Let $\mathbb{T}=\mathbb{Z}$, $a_1(x_1,x_2)=c$, $a_2(x_1,x_2)=w_2=1$, $\gamma_{ji}=\mu_{ji}=I$, $n=1$, $f_i(x_1,t_1,x_2,t_2)=f(t_1,t_2)$, $g_i=r_i=0$. Then Theorem 3.1 coincides with Theorem 2.1 in [109]. Moreover, if $w(u)=u^p$, then it coincides with Theorem 2.1 in [110].

Remark 3.3 Let $\mathbb{T}=\mathbb{R}$, $a_1(x_1,x_2)=c$, $a_2(x_1,x_2)=1$, $w_1=\mu_{ji}=I$, $f_i(x_1,t_1,x_2,t_2)=f_i(t_1,t_2)$, $g_i=0$ $(1\le i\le n)$. Then Theorem 3.1 leads to Theorem 2.3 in [111]. Moreover, if $r_i=0$, then it reduces to Theorem 2.2 in [111].

Remark 3.4 Let $\mathbb{T} = \mathbb{R}$, $w_1(u) = u^q$, $\mu_{ji} = I$, $f_i(x_1, t_1, x_2, t_2) = f_i(t_1, t_2)$, $g_i = 0$ $(1 \le i \le n)$. Then Theorem 3.1 becomes Theorem 2.1 in [112].

Remark 3.5 Let $n = 1$, $\gamma_{ji} = I$, $w_2 = 1$, $f_i(x_1, t_1, x_2, t_2) = f(t_1, t_2)$, and $g_i = r_i = 0$. Theorem 3.1 coincides with Theorem 1 in [113].

Remark 3.6 Let $\mathbb{T} = \mathbb{Z}$, $w_2 = 1$, $\gamma_{ji} = \mu_{ji} = I$, $n = 1$, $f_i(x_1, t_1, x_2, t_2) = f(t_1, t_2)$, and $g_i = r_i = 0$. Then Theorem 3.1 leads to Theorem 1 in [114].

Remark 3.7 Let $\mathbb{T} = \mathbb{R}$, $\mu_{ji} = I$, $f_i(x_1, t_1, x_2, t_2) = f_i(t_1, t_2)$, and $g_i = 0$. Then Theorem 3.1 reduces to Theorem 1 in [115].

Theorem 3.8 *Let a_j, r_i, w, w_j, f_i, g_i, γ_{ji}, μ_{ji}, and $\mathfrak{a}$ be defined as in Theorem 3.1 such that w_{j+2} has the same conditions of w_j, and let $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}_1^+$ be a right-dense continuous function such that*

$$\begin{aligned} & w\bigl(u(x_1, x_2)\bigr) \\ & \quad \le a_1(x_1, x_2) + a_2(x_1, x_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\bigl(u\bigl(\mu_{1i}(t_1), \mu_{2i}(t_2)\bigr)\bigr) \\ & \qquad \times \Bigl[f_i(x_1, t_1, x_2, t_2) w_2\bigl(u\bigl(\mu_{1i}(t_1), \mu_{2i}(t_2)\bigr)\bigr) \Bigl\{ w_3\bigl(u\bigl(\mu_{1i}(t_1), \mu_{2i}(t_2)\bigr)\bigr) \\ & \qquad + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1, m_1, t_2, m_2) \times w_3\bigl(u\bigl(\mu_{1i}(m_1), \mu_{2i}(m_2)\bigr)\bigr) \Delta m_2 \Delta m_1 \Bigr\} \\ & \qquad + r_i(t_1, t_2) w_4\bigl(\log\bigl(u\bigl(\mu_{1i}(t_1), \mu_{2i}(t_2)\bigr)\bigr)\bigr) \Bigr] \Delta t_2 \Delta t_1 \end{aligned} \tag{3.16}$$

for $(x_1, x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with initial condition (3.2). Then the following statements are true:

(1) *If $w_2(u) \ge w_4(\log(u))$, then*

$$u(x_1, x_2) \le w^{-1}\Bigl(G_1^{-1}\Bigl(G_2^{-1}\Bigl(Q_1^{-1}\bigl(Q_1\bigl(d_1(x_1, x_2)\bigr) + a_2(x_1, x_2) c(x_1, x_2)\bigr)\Bigr)\Bigr)\Bigr) \tag{3.17}$$

for all $x_{01} \le x_1 \le \tilde{x}_1$ and $x_{02} \le x_2 \le \tilde{x}_2$;

(2) *If $w_2(u) < w_4(\log(u))$, then*

$$u(x_1, x_2) \le w^{-1}\Bigl(G_1^{-1}\Bigl(G_3^{-1}\Bigl(Q_2^{-1}\bigl(Q_2\bigl(d_2(x_1, x_2)\bigr) + a_2(x_1, x_2) c(x_1, x_2)\bigr)\Bigr)\Bigr)\Bigr) \tag{3.18}$$

for all $x_{01} \le x_1 \le \tilde{x}_3$ and $x_{02} \le x_2 \le \tilde{x}_4$, where

$$\begin{aligned} & d_j(x_1, x_2) = G_{j+1}\bigl(G_1\bigl(a_1(x_1, x_2)\bigr)\bigr) + a_2(x_1, x_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} r_i(t_1, t_2) \Delta t_2 \Delta t_1, \\ & G_3(s) = \int_{s_3}^{s} \frac{\Delta p}{w_4(w^{-1}(G_1^{-1}(p)))}, \quad s > s_3 > 0 \text{ with } G_3(\infty) = \infty, \\ & Q_j(s) = \int_{s_{3+j}}^{s} \frac{\Delta p}{w_3(w^{-1}(G_1^{-1}(G_{j+1}^{-1}(p))))}, \quad s > s_{3+j} > 0 \text{ with } Q_j(\infty) = \infty, \end{aligned}$$

c, G_1, and G_2 are defined as in Theorem 3.1, G_1^{-1}, G_2^{-1}, G_3^{-1}, and G_4^{-1} are respectively the inverse functions of G_1, G_2, G_3, and G_4, and $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$, and $\tilde{x}_4$ are chosen such that

$$\begin{aligned}
&Q_j\big(d_j(x_1,x_2)\big) + a_2(x_1,x_2)c(x_1,x_2) \in \operatorname{Dom}\big(Q_j^{-1}\big),\\
&Q_j^{-1}\big(Q_j\big(d_j(x_1,x_2)\big) + a_2(x_1,x_2)c(x_1,x_2)\big) \in \operatorname{Dom}\big(G_{j+1}^{-1}\big),\\
&G_{j+1}^{-1}\big(Q_j^{-1}\big(Q_j\big(d_j(x_1,x_2)\big) + a_2(x_1,x_2)c(x_1,x_2)\big)\big) \in \operatorname{Dom}\big(G_1^{-1}\big),\\
&G_1^{-1}\big(G_{j+1}^{-1}\big(Q_j^{-1}\big(Q_j\big(d_j(x_1,x_2)\big) + a_2(x_1,x_2)c(x_1,x_2)\big)\big)\big) \in \operatorname{Dom}\big(w^{-1}\big).
\end{aligned}$$

Proof Let $\bar{x}_1 \in \mathbb{T}_1$ and $\bar{x}_2 \in \mathbb{T}_2$ with $x_{01} \le \bar{x}_1 \le \tilde{x}_1$ and $x_{02} \le \bar{x}_2 \le \tilde{x}_2$. Then inequality (3.16) can be rewritten as

$$\begin{aligned}
&w\big(u(x_1,x_2)\big)\\
&\quad\le a_1(\bar{x}_1,\bar{x}_2) + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\\
&\qquad\times\Bigg[f_i(x_1,t_1,x_2,t_2)w_2\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\Bigg\{w_3\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\\
&\qquad+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\times w_3\big(u\big(\mu_{1i}(m_1),\mu_{2i}(m_2)\big)\big)\Delta m_2\Delta m_1\Bigg\}\\
&\qquad+r_i(t_1,t_2)w_4\big(\log\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\big)\Bigg]\Delta t_2\Delta t_1
\end{aligned}$$

for $x_{01} \le x_1 \le \bar{x}_1$ and $x_{02} \le x_2 \le \bar{x}_2$.

Let

$$\begin{aligned}
&\xi_2(x_1,x_2)\\
&\quad= a_1(\bar{x}_1,\bar{x}_2) + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\\
&\qquad\times\Bigg[f_i(x_1,t_1,x_2,t_2)w_2\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\Bigg\{w_3\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\\
&\qquad+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\times w_3\big(u\big(\mu_{1i}(m_1),\mu_{2i}(m_2)\big)\big)\Delta m_2\Delta m_1\Bigg\}\\
&\qquad+r_i(t_1,t_2)w_4\big(\log\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\big)\Bigg]\Delta t_2\Delta t_1.
\end{aligned}$$

Then one has

$$\xi_2(x_{01},x_2) = \xi_2(x_1,x_{02}) = a_1(\bar{x}_1,\bar{x}_2)$$

and

$$u(x_1,x_2) \le w^{-1}\big(\xi_2(x_1,x_2)\big). \tag{3.19}$$

On taking the identical steps as in (3.7)–(3.10), we get

$$\begin{aligned} &G_1\big(\xi_2(x_1,x_2)\big) \\ &\quad\leq G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Big[f_i(x_1,t_1,x_2,t_2) \\ &\qquad\times w_2\big(w^{-1}\big(\xi_2(t_1,t_2)\big)\big)\Big\{w_3\big(w^{-1}\big(\xi_2(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2) \\ &\qquad\times w_3\big(w^{-1}\big(\xi_2(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\Big\} \\ &\qquad+r_i(t_1,t_2)w_4\big(\log\big(w^{-1}\big(\xi_2(t_1,t_2)\big)\big)\big)\Big]\Delta t_2\Delta t_1. \end{aligned}\tag{3.20}$$

For $w_2(\mathrm{u})\geq w_4(\log(\mathrm{u}))$, inequality (3.20) can be rewritten as follows:

$$\begin{aligned} &G_1\big(\xi_2(x_1,x_2)\big) \\ &\quad\leq G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}w_2\big(w^{-1}\big(\xi_2(t_1,t_2)\big)\big) \\ &\qquad\times\Big[f_i(x_1,t_1,x_2,t_2)\Big\{w_3\big(w^{-1}\big(\xi_2(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2) \\ &\qquad\times w_3\big(w^{-1}\big(\xi_2(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\Big\}+r_i(t_1,t_2)\Big]\Delta t_2\Delta t_1. \end{aligned}\tag{3.21}$$

Let

$$\begin{aligned} \zeta_2(x_1,x_2)={}&G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}w_2\big(w^{-1}\big(\xi_2(t_1,t_2)\big)\big) \\ &\times\Big[f_i(x_1,t_1,x_2,t_2)\Big\{w_3\big(w^{-1}\big(\xi_2(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2) \\ &\times w_3\big(w^{-1}\big(\xi_2(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\Big\}+r_i(t_1,t_2)\Big]\Delta t_2\Delta t_1. \end{aligned}$$

Then

$$\zeta_2(x_{01},x_2)=\zeta_2(x_1,x_{02})=G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big)$$

and

$$\xi_2(x_1,x_2)\leq G_1^{-1}\big(\zeta_2(x_1,x_2)\big).\tag{3.22}$$

On taking the identical steps as those in (3.13)–(3.14), one has

$$\begin{aligned} &G_2\big(\zeta_2(x_1,x_2)\big) \\ &\quad\leq G_2\big(G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big)\big)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Big[f_i(x_1,t_1,x_2,t_2) \\ &\qquad\times w_3\big(w^{-1}\big(G_1^{-1}\big(\zeta_2(t_1,t_2)\big)\big)\big) \end{aligned}$$

$$\times \left\{1+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\right\}+r_i(t_1,t_2)\Bigg]\Delta t_2\Delta t_1$$

$$\le d_1(\bar{x}_1,\bar{x}_2)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_1)$$

$$\times w_3\bigl(w^{-1}\bigl(G_1^{-1}\bigl(\zeta_2(t_1,t_2)\bigr)\bigr)\bigr)$$

$$\times \left\{1+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\right\}\Delta t_2\Delta t_1.$$

Let

$$\vartheta(x_1,x_2)=d_1(\bar{x}_1,\bar{x}_2)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_1)$$
$$\times w_3\bigl(w^{-1}\bigl(G_1^{-1}\bigl(\zeta_2(t_1,t_2)\bigr)\bigr)\bigr)$$
$$\times \left\{1+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\right\}\Delta t_2\Delta t_1. \tag{3.23}$$

Then

$$\vartheta(x_{01},x_2)=\vartheta(x_1,x_{02})=d_1(\bar{x}_1,\bar{x}_2) \tag{3.24}$$

and

$$\zeta_2(x_1,x_2)\le G_2^{-1}\bigl(\vartheta(x_1,x_2)\bigr). \tag{3.25}$$

It follows from Lemma 2.1 and (3.23) that

$$\vartheta^{\Delta x_1}(x_1,x_2)$$
$$=a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\Bigg[\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i\bigl(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\bigr)$$
$$\times w_3\bigl(w^{-1}\bigl(G_1^{-1}\bigl(\zeta_2\bigl(\gamma_{1i}(x_1),t_2\bigr)\bigr)\bigr)\bigr)$$
$$\times \left\{1+\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i\bigl(\gamma_{1i}(x_1),m_1,t_2,m_2\bigr)\Delta m_2\Delta m_1\right\}\Delta t_2$$
$$+\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\Bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_1)w_3\bigl(w^{-1}\bigl(G_1^{-1}\bigl(\zeta_2(t_1,t_2)\bigr)\bigr)\bigr)$$
$$\times \left\{1+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\right\}\Delta t_2\Bigg]^{\Delta x_1}\Delta t_1\Bigg].$$

From (3.25) and the fact that w_3, w^{-1}, G_1^{-1}, G_2^{-1}, and ϑ are nondecreasing, we have

$$\frac{\vartheta^{\Delta x_1}(x_1,x_2)}{w_3(w^{-1}(G_1^{-1}(G_2^{-1}(\vartheta(x_1,x_2)))))}$$
$$\le a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\Bigg[\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i\bigl(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\bigr)$$

$$\times \left\{1 + \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i\big(\gamma_{1i}(x_1), m_1, t_2, m_2\big) \Delta m_2 \Delta m_1\right\} \Delta t_2$$

$$+ \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \left[\left[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1, t_1, x_2, t_1)\right.\right.$$

$$\left.\left.\times \left\{1 + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1, m_1, t_2, m_2) \Delta m_2 \Delta m_1\right\} \Delta t_2\right]^{\Delta x_1} \Delta t_1\right]$$

$$= a_2(\bar{x}_1, \bar{x}_2) \sum_{i=1}^{n} \left[\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1, t_1, x_2, t_1)\right.$$

$$\left.\times \left\{1 + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1, m_1, t_2, m_2) \Delta m_2 \Delta m_1\right\} \Delta t_2 \Delta t_1\right]^{\Delta x_1}.$$

Using the definition of Q_1 and (3.24), integrating over $[x_{01}, x_1]$, we get

$$Q_1\big(\vartheta(x_1, x_2)\big) \leq Q_1\big(d_1(\bar{x}_1, \bar{x}_2)\big) + a_2(\bar{x}_1, \bar{x}_2) c(x_1, x_2). \tag{3.26}$$

It follows from (3.19), (3.22), (3.25), and (3.26) that

$$u(x_1, x_2) \leq w^{-1}\Big(G_1^{-1}\Big(G_2^{-1}\Big(Q_1^{-1}\Big(Q_1\big(d_1(\bar{x}_1, \bar{x}_2)\big) + a_2(\bar{x}_1, \bar{x}_2) c(x_1, x_2)\Big)\Big)\Big)\Big).$$

Let $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$. Then

$$u(\bar{x}_1, \bar{x}_2) \leq w^{-1}\Big(G_1^{-1}\Big(G_2^{-1}\Big(Q_1^{-1}\Big(Q_1\big(d_1(\bar{x}_1, \bar{x}_2)\big) + a_2(\bar{x}_1, \bar{x}_2) c(\bar{x}_1, \bar{x}_2)\Big)\Big)\Big)\Big). \tag{3.27}$$

Since $x_{01} \leq \bar{x}_1 \leq \tilde{x}_1$ and $x_{02} \leq \bar{x}_2 \leq \tilde{x}_2$ are chosen arbitrarily, so after substituting $\bar{x}_1$ and respectively $\bar{x}_2$ with x_1 and x_2, we obtain the desired result (3.17).

For $w_2(u) < w_4(\log(u))$, inequality (3.20) can be rewritten as follows:

$$G_1\big(\xi_2(x_1, x_2)\big)$$

$$\leq G_1\big(a_1(\bar{x}_1, \bar{x}_2)\big) + a_2(\bar{x}_1, \bar{x}_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_4\Big(\log\big(w^{-1}\big(\xi_2(t_1, t_2)\big)\big)\Big)$$

$$\times \left[f_i(x_1, t_1, x_2, t_2) \left\{w_3\big(w^{-1}\big(\xi_2(t_1, t_2)\big)\big) + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1, m_1, t_2, m_2)\right.\right.$$

$$\left.\left.\times w_3\big(w^{-1}\big(\xi_2(m_1, m_2)\big)\big) \Delta m_2 \Delta m_1\right\} + r_i(t_1, t_2)\right] \Delta t_2 \Delta t_1$$

$$\leq G_1\big(a_1(\bar{x}_1, \bar{x}_2)\big) + a_2(\bar{x}_1, \bar{x}_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_4\big(w^{-1}\big(\xi_2(t_1, t_2)\big)\big)$$

$$\times \left[f_i(x_1, t_1, x_2, t_2) \left\{w_3\big(w^{-1}\big(\xi_2(t_1, t_2)\big)\big) + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1, m_1, t_2, m_2)\right.\right.$$

$$\left.\left.\times w_3\big(w^{-1}\big(\xi_2(m_1, m_2)\big)\big) \Delta m_2 \Delta m_1\right\} + r_i(t_1, t_2)\right] \Delta t_2 \Delta t_1.$$

On taking the identical steps as those (3.21)–(3.27), then after substituting $\bar{x}_1$ and $\bar{x}_2$ with x_1 and x_2, we obtain the desired result (3.18). □

Remark 3.9 Let $\mathbb{T} = \mathrm{R}$, $w_3 = 1$, $\mu_{ji} = I$, $f_i(x_1,t_1,x_2,t_2) = f_i(t_1,t_2)$, and $g_i = 0$. Then Theorem 3.8 leads to Theorem 2 in [115].

Theorem 3.10 *Let a_j, r_i, w, w_j, f_i, g_i, γ_{ji}, μ_{ji}, and $\mathfrak{a}$ be defined as in Theorem 3.1, and $\mathrm{L},\mathrm{M} : \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be right-dense continuous on $\mathbb{T}_1 \times \mathbb{T}_2$ and continuous on $\mathbb{R}_0^+$ such that*

$$0 \le \mathrm{L}(x_1,x_2,\mathfrak{u}) - \mathrm{L}(x_1,x_2,\mathfrak{v}) \le \mathrm{M}(x_1,x_2,\mathfrak{v})(\mathfrak{u}-\mathfrak{v})$$

for $\mathfrak{u} > \mathfrak{v} \ge 0$ and $(x_1,x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, and

$$\begin{aligned} &w\big(u(x_1,x_2)\big) \\ &\quad\le a_1(x_1,x_2) + a_2(x_1,x_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_1\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big) \\ &\qquad\times\Big[f_i(x_1,t_1,x_2,t_2)\Big\{w_2\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\ &\qquad\times w_2\big(u\big(\mu_{1i}(m_1),\mu_{2i}(m_2)\big)\big)\Delta m_2\Delta m_1\Big\} \\ &\qquad+ r_i(t_1,t_2)\mathrm{L}\big(t_1,t_2,w_2\big(u(t_1,t_2)\big)\big)\Big]\Delta t_2\Delta t_1 \end{aligned} \tag{3.28}$$

for $(x_1,x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with initial condition (3.2). Then

$$\begin{aligned} u(x_1,x_2) \le w^{-1}\Bigg(G_1^{-1}\Bigg(G_2^{-1}\Bigg(&G_2\big(b_2(x_1,x_2)\big) + a_2(x_1,x_2)\Big\{c(x_1,x_2) \\ &+ \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\Delta t_2\Delta t_1\Big\}\Bigg)\Bigg)\Bigg) \end{aligned} \tag{3.29}$$

for all $x_{01} \le x_1 \le \tilde{x}_1$ and $x_{02} \le x_2 \le \tilde{x}_2$, where

$$\begin{aligned} b_2(x_1,x_2) = {}& G_1\big(a_1(x_1,x_2)\big) + a_2(x_1,x_2) \\ &\times \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} r_i(t_1,t_2)\mathrm{L}(t_1,t_2,0)\Delta t_2\Delta t_1, \end{aligned}$$

c, G_1, and G_2 are as defined in Theorem 3.1, and $\tilde{x}_1$ and $\tilde{x}_2$ are chosen such that

$$\begin{aligned} &G_2\big(b_2(x_1,x_2)\big) + a_2(x_1,x_2)\Big\{c(x_1,x_2) \\ &\quad+ \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\Delta t_2\Delta t_1\Big\} \in \mathrm{Dom}\big(G_2^{-1}\big), \\ &G_2^{-1}\Bigg(G_2\big(b_2(x_1,x_2)\big) + a_2(x_1,x_2)\Big\{c(x_1,x_2) \\ &\quad+ \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\Delta t_2\Delta t_1\Big\}\Bigg) \in \mathrm{Dom}\big(G_1^{-1}\big), \end{aligned}$$

$$G_1^{-1}\left(G_2^{-1}\left(G_2\big(b_2(x_1,x_2)\big)+a_2(x_1,x_2)\left\{c(x_1,x_2)\right.\right.\right.$$
$$\left.\left.\left.+\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\Delta t_2\Delta t_1\right\}\right)\right)\in \mathrm{Dom}\big(w^{-1}\big).$$

Proof Let $\bar{x}_1\in\mathbb{T}_1$ and $\bar{x}_2\in\mathbb{T}_2$ with $x_{01}\le\bar{x}_1\le\tilde{x}_1$ and $x_{02}\le\bar{x}_2\le\tilde{x}_2$. Then inequality (3.28) can be rewritten as follows:

$$\begin{aligned}
&w\big(u(x_1,x_2)\big)\\
&\quad\le a_1(\bar{x}_1,\bar{x}_2)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}w_1\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\\
&\qquad\times\Big[f_i(x_1,t_1,x_2,t_2)\Big\{w_2\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2)\\
&\qquad\times w_2\big(u\big(\mu_{1i}(m_1),\mu_{2i}(m_2)\big)\big)\Delta m_2\Delta m_1\Big\}+r_i(t_1,t_2)\mathrm{L}\big(t_1,t_2,w_2\big(u(t_1,t_2)\big)\big)\Big]\Delta t_2\Delta t_1
\end{aligned}$$

for $x_{01}\le x_1\le\bar{x}_1$ and $x_{02}\le x_2\le\bar{x}_2$. Let

$$\begin{aligned}
&\xi_3(x_1,x_2)\\
&\quad= a_1(\bar{x}_1,\bar{x}_2)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}w_1\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)\\
&\qquad\times\Big[f_i(x_1,t_1,x_2,t_2)\Big\{w_2\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2)\\
&\qquad\times w_2\big(u\big(\mu_{1i}(m_1),\mu_{2i}(m_2)\big)\big)\Delta m_2\Delta m_1\Big\}+r_i(t_1,t_2)\mathrm{L}\big(t_1,t_2,w_2\big(u(t_1,t_2)\big)\big)\Big]\Delta t_2\Delta t_1.
\end{aligned}$$

Then

$$\xi_3(x_{01},x_2)=\xi_3(x_1,x_{02})=a_1(\bar{x}_1,\bar{x}_2)$$

and

$$u(x_1,x_2)\le w^{-1}\big(\xi_3(x_1,x_2)\big). \tag{3.30}$$

On taking the identical steps as in (3.17)–(3.10), one has

$$\begin{aligned}
&G_1\big(\xi_3(x_1,x_2)\big)\\
&\quad\le G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Big[f_i(x_1,t_1,x_2,t_2)\\
&\qquad\times\Big\{w_2\big(w^{-1}\big(\xi_3(t_1,t_2)\big)\big)+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2)\\
&\qquad\times w_2\big(w^{-1}\big(\xi_3(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\Big\}
\end{aligned}$$

$$+ r_i(t_1,t_2)\mathrm{L}\big(t_1,t_2,w_2\big(w^{-1}\big(\xi_3(t_1,t_2)\big)\big)\big)\Big]\Delta t_2\Delta t_1$$

$$\begin{aligned}
&\le G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big) + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Big[f_i(x_1,t_1,x_2,t_2)\\
&\quad\times\bigg\{w_2\big(w^{-1}\big(\xi_3(t_1,t_2)\big)\big) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\\
&\quad\times w_2\big(w^{-1}\big(\xi_3(m_1,m_2)\big)\big)\Delta m_2\Delta m_1\bigg\}\\
&\quad + r_i(t_1,t_2)\big\{\mathrm{L}(t_1,t_2,0) + \mathrm{M}(t_1,t_2,0)w_2\big(w^{-1}\big(\xi_3(t_1,t_2)\big)\big)\big\}\Big]\Delta t_2\Delta t_1\\
&\le G_1\big(a_1(\bar{x}_1,\bar{x}_2)\big) + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}((\bar{x}_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(\bar{x}_2)} r_i(t_1,t_2)\mathrm{L}(t_1,t_2,0)\Delta t_2\Delta t_1\\
&\quad + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_2\big(w^{-1}\big(\xi_3(t_1,t_2)\big)\big)\\
&\quad\times\bigg[f_i(x_1,t_1,x_2,t_2)\bigg\{1 + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\bigg\}\\
&\quad + r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\bigg]\Delta t_2\Delta t_1\\
&= b_2(\bar{x}_1,\bar{x}_2) + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_2\big(w^{-1}\big(\xi_3(t_1,t_2)\big)\big)\\
&\quad\times\bigg[f_i(x_1,t_1,x_2,t_2)\bigg\{1 + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\bigg\}\\
&\quad + r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\bigg]\Delta t_2\Delta t_1.
\end{aligned}$$

Let

$$\begin{aligned}
\zeta_3(x_1,x_2) &= b_2(\bar{x}_1,\bar{x}_2) + a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} w_2\big(w^{-1}\big(\xi_3(t_1,t_2)\big)\big)\\
&\quad\times\bigg[f_i(x_1,t_1,x_2,t_2)\bigg\{1 + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\bigg\}\\
&\quad + r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\bigg]\Delta t_2\Delta t_1.
\end{aligned}$$

Then

$$\zeta_3(x_{01},x_2) = \zeta_3(x_1,x_{02}) = b_2(\bar{x}_1,\bar{x}_2)$$

and

$$\xi_3(x_1,x_2) \le G_1^{-1}\big(\zeta_3(x_1,x_2)\big). \tag{3.31}$$

On taking the identical steps as in (3.13) to (3.14), one has

$$\begin{aligned}
&G_2\big(\zeta_3(x_1,x_2)\big)\\
&\quad\le G_2\big(b_2(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\bigg\{f_i(x_1,t_1,x_2,t_2)\\
&\qquad\times\bigg(1+\int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2}g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\bigg)\\
&\qquad+r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\bigg\}\Delta t_2\Delta t_1\\
&\quad=G_2\big(b_2(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\Bigg\{c(x_1,x_2)\\
&\qquad+\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\Delta t_2\Delta t_1\Bigg\}.
\end{aligned}\tag{3.32}$$

It follows from (3.30)–(3.32) that

$$\begin{aligned}
u(x_1,x_2)\le w^{-1}\Bigg(G_1^{-1}\Bigg(G_2^{-1}\Bigg(&G_2\big(b_2(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\Bigg\{c(x_1,x_2)\\
&+\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\Delta t_2\Delta t_1\Bigg\}\Bigg)\Bigg)\Bigg).
\end{aligned}$$

Let $x_1=\bar{x}_1$ and $x_2=\bar{x}_2$. Then we have

$$\begin{aligned}
u(\bar{x}_1,\bar{x}_2)\le w^{-1}\Bigg(G_1^{-1}\Bigg(G_2^{-1}\Bigg(&G_2\big(b_2(\bar{x}_1,\bar{x}_2)\big)+a_2(\bar{x}_1,\bar{x}_2)\Bigg\{c(\bar{x}_1,\bar{x}_2)\\
&+\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(\bar{x}_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(\bar{x}_2)}r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\Delta t_2\Delta t_1\Bigg\}\Bigg)\Bigg)\Bigg).
\end{aligned}$$

Since $x_{01}\le\bar{x}_1\le\tilde{x}_1$ and $x_{02}\le\bar{x}_2\le\tilde{x}_2$ are chosen arbitrarily, we obtain

$$\begin{aligned}
u(x_1,x_2)\le w^{-1}\Bigg(G_1^{-1}\Bigg(G_2^{-1}\Bigg(&G_2\big(b_2(x_1,x_2)\big)+a_2(x_1,x_2)\Bigg\{c(x_1,x_2)\\
&+\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}r_i(t_1,t_2)\mathrm{M}(t_1,t_2,0)\Delta t_2\Delta t_1\Bigg\}\Bigg)\Bigg)\Bigg).
\end{aligned}$$

□

Remark 3.11 Let $\mathbb{T}=\mathbb{R}$, $w_2=\mu_{ji}=I$, $f_i(x_1,t_1,x_2,t_2)=f_i(t_1,t_2)$, and $g_i=0$. Then Theorem 3.10 coincides with Theorem 3 in [115].

Corollary 3.12 *Let u, r_i, w_2, f_i, g_i, γ_{ji}, and μ_{ji} be defined as in Theorem 3.1, $\bar{\mathfrak{a}}(x_1,x_2)$ be a nonnegative and right-dense continuous function defined on $([\bar{\mathfrak{p}}_1,x_{01}]\times[\bar{\mathfrak{p}}_2,x_{02}])_{\mathbb{T}^2}$, and*

$q_1 > q_2 > 0$ and $\mathfrak{C} \geq 0$ be constants such that

$$\begin{aligned} u^{q_1}(x_1,x_2) \leq \mathfrak{C} + \frac{q_1}{q_1-q_2} \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} u^{q_2}\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big) \\ \times \Bigg[f_i(x_1,t_1,x_2,t_2) \bigg\{ w_2\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big) \\ + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\ \times w_2\big(u\big(\mu_{1i}(m_1),\mu_{2i}(m_2)\big)\big) \Delta m_2 \Delta m_1 \bigg\} + r_i(t_1,t_2) \Bigg] \Delta t_2 \Delta t_1 \end{aligned} \tag{3.33}$$

for $(x_1,x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with the initial condition

$$\begin{cases} u(x_1,x_2) = \overline{\mathfrak{a}}(x_1,x_2), & x_1 \in [\overline{\mathfrak{p}}_1, x_{01}]_{\mathbb{T}} \text{ or } x_2 \in [\overline{\mathfrak{p}}_2, x_{02}]_{\mathbb{T}}, \\ \overline{\mathfrak{a}}(\mu_{1i}(x_1),\mu_{2i}(x_2)) \leq \sqrt[q_1]{\mathfrak{C}}, & \mu_{1i}(x_1) \leq x_{01} \text{ or } \mu_{2i}(x_2) \leq x_{02}. \end{cases} \tag{3.34}$$

Then

$$u(x_1,x_2) \leq \sqrt[q_1-q_2]{H_1{}^{-1}\big(H_1\big(\overline{b}_1(x_1,x_2)\big) + c(x_1,x_2)\big)} \tag{3.35}$$

for all $x_{01} \leq x_1 \leq \tilde{x}_1$ and $x_{02} \leq x_2 \leq \tilde{x}_2$, where

$$\begin{aligned} \overline{b}_1(x_1,x_2) &:= \mathfrak{C}^{\frac{q_1-q_2}{q_1}} + \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} r_i(t_1,t_2) \Delta t_2 \Delta t_1, \\ H_1(s) &= \int_{s_6}^{s} \frac{\Delta p}{w_2(\sqrt[q_1-q_2]{p})}, \quad s > s_6 > 0 \text{ with } H_1(\infty) = \infty, \end{aligned}$$

c is defined as in Theorem 3.1, H_1^{-1} is the inverse function of H_1, and $\tilde{x}_1$ and $\tilde{x}_2$ are chosen such that

$$H_1\big(\overline{b}_1(x_1,x_2)\big) + c(x_1,x_2) \in \operatorname{Dom}\big(H_1{}^{-1}\big).$$

Proof Let

$$\begin{aligned} \overline{\xi}_1(x_1,x_2) = \mathfrak{C} + \frac{q_1}{q_1-q_2} \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} u^{q_2}\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big) \\ \times \Bigg[f_i(x_1,t_1,x_2,t_2) \bigg\{ w_2\big(u\big(\mu_{1i}(t_1),\mu_{2i}(t_2)\big)\big) \\ + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\ \times w_2\big(u\big(\mu_{1i}(m_1),\mu_{2i}(m_2)\big)\big) \Delta m_2 \Delta m_1 \bigg\} + r_i(t_1,t_2) \Bigg] \Delta t_2 \Delta t_1. \end{aligned} \tag{3.36}$$

Then

$$\bar{\xi}_1(x_{01},x_2) = \bar{\xi}_1(x_1,x_{02}) = \mathfrak{C} \tag{3.37}$$

and

$$u(x_1,x_2) \le \sqrt[q_1]{\overline{\xi}_1(x_1,x_2)} \quad \text{for } x_1 \in [x_{01},\bar{x}_1]_{\mathbb{T}} \text{ and } x_2 \in [x_{02},\bar{x}_2]_{\mathbb{T}}, \tag{3.38}$$

where $\bar{x}_1 \in \mathbb{T}_1$ and $\bar{x}_2 \in \mathbb{T}_2$ are fixed numbers with $x_{01} \le \bar{x}_1 \le \tilde{x}_1$ and $x_{02} \le \bar{x}_2 \le \tilde{x}_2$.

For $x_1 \in [x_{01},\bar{x}_1]_{\mathbb{T}}$ and $x_2 \in [x_{02},\bar{x}_2]_{\mathbb{T}}$, if $\mu_{1i}(x_1) \ge x_{01}$ and $\mu_{2i}(x_2) \ge x_{02}$, then $\mu_{1i}(x_1) \in [x_{01},\bar{x}_1]_{\mathbb{T}}$, $\mu_{2i}(x_1) \in [x_{02},\bar{x}_2]_{\mathbb{T}}$, and

$$u\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr) \le \sqrt[q_1]{\overline{\xi}_1\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr)} \le \sqrt[q_1]{\overline{\xi}_1(x_1,x_2)}. \tag{3.39}$$

If $\mu_{1i}(x_1) \le x_{01}$ or $\mu_{2i}(x_2) \le x_{02}$, then from (3.34) we obtain

$$u\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr) = \overline{\mathfrak{a}}\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr) \le \sqrt[q_1]{\mathfrak{C}} \le \sqrt[q_1]{\overline{\xi}_1(x_1,x_2)}. \tag{3.40}$$

It follows from (3.39) and (3.40) that

$$u\bigl(\mu_{1i}(x_1),\mu_{2i}(x_2)\bigr) \le \sqrt[q_1]{\overline{\xi}_1(x_1,x_2)} \tag{3.41}$$

for $x_1 \in [x_{01},\bar{x}_1]_{\mathbb{T}}$ and $x_2 \in [x_{02},\bar{x}_2]_{\mathbb{T}}$.

From Lemma 2.1 and (3.36) we know that

$$\begin{aligned}
&\overline{\xi}_1^{\Delta x_1}(x_1,x_2) \\
&\quad = \frac{q_1}{q_1-q_2}\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} u^{q_2}\bigl(\mu_{1i}\bigl(\gamma_{1i}(x_1)\bigr),\mu_{2i}(t_2)\bigr) \\
&\qquad \times \Bigl[f_i\bigl(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\bigr)\Bigl\{w_2\bigl(u\bigl(\mu_{1i}\bigl(\gamma_{1i}(x_1)\bigr),\mu_{2i}(t_2)\bigr)\bigr) \\
&\qquad + \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i\bigl(\gamma_{1i}(x_1),m_1,t_2,m_2\bigr) \\
&\qquad \times w_2\bigl(u\bigl(\mu_{1i}(m_1),\mu_{2i}(m_2)\bigr)\bigr)\Delta m_2\Delta m_1\Bigr\} + r_i\bigl(\gamma_{1i}(x_1),t_2\bigr)\Bigr]\Delta t_2 \\
&\qquad + \frac{q_1}{q_1-q_2}\sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\Biggl[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} u^{q_2}\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr) \\
&\qquad \times \Bigl[f_i(x_1,t_1,x_2,t_2)\Bigl\{w_2\bigl(u\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr)\bigr) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\
&\qquad \times w_2\bigl(u\bigl(\mu_{1i}(m_1),\mu_{2i}(m_2)\bigr)\bigr)\Delta m_2\Delta m_1\Bigr\} + r_i(t_1,t_2)\Bigr]\Delta t_2\Biggr]^{\Delta x_1}\Delta t_1.
\end{aligned}$$

Making use of (3.41) and the fact that $\overline{\xi}_1$ is nondecreasing, we have

$$\begin{aligned}
&\frac{\overline{\xi}_1^{\Delta x_1}(x_1,x_2)}{\overline{\xi}_1^{\frac{q_2}{q_1}}(x_1,x_2)} \\
&\quad \le \frac{q_1}{q_1-q_2}\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(x_1)\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)}\Bigl[f_i\bigl(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\bigr)
\end{aligned}$$

$$\times \left\{ w_2\left(\sqrt[q_1]{\overline{\xi}_1\left(\gamma_{1i}(x_1), t_2\right)}\right) + \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i\left(\gamma_{1i}(x_1), m_1, t_2, m_2\right) \times w_2\left(\sqrt[q_1]{\overline{\xi}_1(m_1, m_2)}\right) \Delta m_2 \Delta m_1 \right\} + r_i\left(\gamma_{1i}(x_1), t_2\right) \Bigg] \Delta t_2$$

$$+ \frac{q_1}{q_1 - q_2} \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \Bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} \Bigg[f_i(x_1, t_1, x_2, t_2) \left\{ w_2\left(\sqrt[q_1]{\overline{\xi}_1(t_1, t_2)}\right) + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1, m_1, t_2, m_2) w_2\left(\sqrt[q_1]{\overline{\xi}_1(m_1, m_2)}\right) \Delta m_2 \Delta m_1 \right\} + r_i(t_1, t_2) \Bigg] \Delta t_2 \Bigg]^{\Delta x_1} \Delta t_1$$

$$= \frac{q_1}{q_1 - q_2} \sum_{i=1}^{n} \Bigg[\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} \Bigg[f_i(x_1, t_1, x_2, t_2) \left\{ w_2\left(\sqrt[q_1]{\overline{\xi}_1(t_1, t_2)}\right) + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1, m_1, t_2, m_2) w_2\left(\sqrt[q_1]{\overline{\xi}_1(m_1, m_2)}\right) \Delta m_2 \Delta m_1 \right\} + r_i(t_1, t_2) \Bigg] \Delta t_2 \Delta t_1 \Bigg]^{\Delta x_1}. \tag{3.42}$$

Theorem 2.2 and $\overline{\xi}_1^{\Delta x_1}(x_1, x_2) \geq 0$ lead to the conclusion that

$$\begin{aligned}
&\left(\frac{q_1}{q_1 - q_2} \overline{\xi}_1^{\frac{q_1 - q_2}{q_1}}(x_1, x_2) \right)^{\Delta x_1} \\
&= \overline{\xi}_1^{\Delta x_1}(x_1, x_2) \int_0^1 \left\{ \overline{\xi}_1(x_1, x_2) + h\mu(x_1, x_2) \overline{\xi}_1^{\Delta x_1}(x_1, x_2) \right\}^{-\frac{q_2}{q_1}} dh \\
&= \frac{\overline{\xi}_1^{\Delta x_1}(x_1, x_2)}{\overline{\xi}_1^{\frac{q_2}{q_1}}(x_1, x_2)} \int_0^1 \left\{ 1 + h\mu(x_1, x_2) \frac{\overline{\xi}_1^{\Delta x_1}(x_1, x_2)}{\overline{\xi}_1(x_1, x_2)} \right\}^{-\frac{q_2}{q_1}} dh \\
&= \frac{\overline{\xi}_1^{\Delta x_1}(x_1, x_2)}{\overline{\xi}_1^{\frac{q_2}{q_1}}(x_1, x_2)} \frac{\{1 + \mu(x_1, x_2) \frac{\overline{\xi}_1^{\Delta x_1}(x_1, x_2)}{\overline{\xi}_1(x_1, x_2)}\}^{-\frac{q_2}{q_1}+1} - 1}{\mu(x_1, x_2) \frac{\overline{\xi}_1^{\Delta x_1}(x_1, x_2)}{\overline{\xi}_1(x_1, x_2)} (1 - \frac{q_2}{q_1})} \leq \frac{\overline{\xi}_1^{\Delta x_1}(x_1, x_2)}{\overline{\xi}_1^{\frac{q_2}{q_1}}(x_1, x_2)}.
\end{aligned} \tag{3.43}$$

From (3.42) and (3.43) we have

$$\begin{aligned}
&\left(\frac{q_1}{q_1 - q_2} \overline{\xi}_1^{\frac{q_1 - q_2}{q_1}}(x_1, x_2) \right)^{\Delta x_1} \\
&\leq \frac{q_1}{q_1 - q_2} \sum_{i=1}^{n} \Bigg[\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} \Bigg[f_i(x_1, t_1, x_2, t_2) \\
&\quad \times \left\{ w_2\left(\sqrt[q_1]{\overline{\xi}_1(t_1, t_2)}\right) + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1, m_1, t_2, m_2) \right. \\
&\quad \left. \times w_2\left(\sqrt[q_1]{\overline{\xi}_1(m_1, m_2)}\right) \Delta m_2 \Delta m_1 \right\} + r_i(t_1, t_2) \Bigg] \Delta t_2 \Delta t_1 \Bigg]^{\Delta x_1}.
\end{aligned}$$

Making use of (3.37) and integrating over $[x_{01}, x_1]$, we get

$$\begin{aligned}
\overline{\xi}_1^{\frac{q_1-q_2}{q_1}}(x_1,x_2) \le{}& \mathfrak{C}^{\frac{q_1-q_2}{q_1}} + \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} r_i(t_1,t_2)\Delta t_2\Delta t_1 \\
&+ \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2) \\
&\times \biggl\{ w_2\bigl(\sqrt[q_1]{\overline{\xi}_1(t_1,t_2)}\bigr) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\
&\times w_2\bigl(\sqrt[q_1]{\overline{\xi}_1(m_1,m_2)}\bigr)\Delta m_2\Delta m_1 \biggr\}\Delta t_2\Delta t_1 \\
\le{}& \mathfrak{C}^{\frac{q_1-q_2}{q_1}} + \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(\bar{x}_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(\bar{x}_2)} r_i(t_1,t_2)\Delta t_2\Delta t_1 \\
&+ \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2) \\
&\times \biggl\{ w_2\bigl(\sqrt[q_1]{\overline{\xi}_1(t_1,t_2)}\bigr) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\
&\times w_2\bigl(\sqrt[q_1]{\overline{\xi}_1(m_1,m_2)}\bigr)\Delta m_2\Delta m_1 \biggr\}\Delta t_2\Delta t_1 \\
={}& \overline{b}_1(\bar{x}_1,\bar{x}_2) + \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2) \\
&\times \biggl\{ w_2\bigl(\sqrt[q_1]{\overline{\xi}_1(t_1,t_2)}\bigr) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\
&\times w_2\bigl(\sqrt[q_1]{\overline{\xi}_1(m_1,m_2)}\bigr)\Delta m_2\Delta m_1 \biggr\}\Delta t_2\Delta t_1.
\end{aligned}$$

Let

$$\begin{aligned}
\overline{\zeta}_1(x_1,x_2) ={}& \overline{b}_1(\bar{x}_1,\bar{x}_2) + \sum_{i=1}^{n}\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)}\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2) \\
&\times \biggl\{ w_2\bigl(\sqrt[q_1]{\overline{\xi}_1(t_1,t_2)}\bigr) + \int_{\gamma_{1i}(x_{01})}^{t_1}\int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) \\
&\times w_2\bigl(\sqrt[q_1]{\overline{\xi}_1(m_1,m_2)}\bigr)\Delta m_2\Delta m_1 \biggr\}\Delta t_2\Delta t_1. \qquad (3.44)
\end{aligned}$$

Then we have

$$\overline{\zeta}_1(x_{01},x_2) = \overline{\zeta}_1(x_1,x_{02}) = \overline{b}_1(\bar{x}_1,\bar{x}_2) \qquad (3.45)$$

and

$$\overline{\xi}_1(x_1,x_2) \le \overline{\zeta}_1^{\frac{q_1}{q_1-q_2}}(x_1,x_2). \qquad (3.46)$$

It follows from Lemma 2.1 and (3.44) that

$$\begin{aligned}
&\overline{\zeta}_1^{\Delta x_1}(x_1,x_2) \\
&= \sum_{i=1}^{n} \gamma_{1i}^{\Delta}(x_1) \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i\big(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\big)\bigg\{ w_2\big(\sqrt[q_1]{\overline{\xi}_1\big(\gamma_{1i}(x_1),t_2\big)}\big) \\
&\quad + \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i\big(\gamma_{1i}(x_1),m_1,t_2,m_2\big) \times w_2\big(\sqrt[q_1]{\overline{\xi}_1(m_1,m_2)}\big)\Delta m_2 \Delta m_1\bigg\}\Delta t_2 \\
&\quad + \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2) \times \bigg\{ w_2\big(\sqrt[q_1]{\overline{\xi}_1(t_1,t_2)}\big) \\
&\quad + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2) w_2\big(\sqrt[q_1]{\overline{\xi}_1(m_1,m_2)}\big)\Delta m_2 \Delta m_1\bigg\}\Delta t_2\bigg]^{\Delta x_1} \Delta t_1.
\end{aligned}$$

It follows from (3.46) and the fact that w_2 and $\overline{\zeta}_1$ are nondecreasing that

$$\begin{aligned}
&\frac{\overline{\zeta}_1^{\Delta x_1}(x_1,x_2)}{w_2\big(\sqrt[q_1-q_2]{\overline{\zeta}_1(x_1,x_2)}\big)} \\
&\leq \sum_{i=1}^{n} [\gamma_{1i}^{\Delta}(x_1) \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i\big(\sigma(x_1),\gamma_{1i}(x_1),x_2,t_2\big) \\
&\quad \times \bigg\{1 + \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i\big(\gamma_{1i}(x_1),m_1,t_2,m_2\big)\Delta m_2 \Delta m_1\bigg\}\Delta t_2 \\
&\quad + \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \bigg[\int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2) \\
&\quad \times \bigg\{1 + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2 \Delta m_1\bigg\}\Delta t_2\bigg]^{\Delta x_1} \Delta t_1 \\
&= \sum_{i=1}^{n} \bigg[\int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2) \\
&\quad \times \bigg\{1 + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2 \Delta m_1\bigg\}\Delta t_2 \Delta t_1\bigg]^{\Delta x_1}.
\end{aligned}$$

Making use of (3.45) and the definition of H_1, integrating over $[x_{01},x_1]$ gives

$$\begin{aligned}
H_1\big(\overline{\zeta}_1(x_1,x_2)\big) &\leq H_1\big(\overline{b}_1(\bar{x}_1,x_2)\big) + \sum_{i=1}^{n} \int_{\gamma_{1i}(x_{01})}^{\gamma_{1i}(x_1)} \int_{\gamma_{2i}(x_{02})}^{\gamma_{2i}(x_2)} f_i(x_1,t_1,x_2,t_2) \\
&\quad \times \bigg\{1 + \int_{\gamma_{1i}(x_{01})}^{t_1} \int_{\gamma_{2i}(x_{02})}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2 \Delta m_1\bigg\}\Delta t_2 \Delta t_1 \\
&= H_1\big(\overline{b}_1(\bar{x}_1,x_2)\big) + c(x_1,x_2). \qquad (3.47)
\end{aligned}$$

Combining (3.38), (3.46), and (3.47), we get

$$u(x_1,x_2) \leq \sqrt[q_1-q_2]{H_1^{-1}\big(H_1\big(\overline{b}_1(\bar{x}_1,\bar{x}_2)\big) + c(x_1,x_2)\big)}.$$

Letting $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$ in (3.3), and considering $\bar{x}_1 \in \mathbb{T}_1$ and $\bar{x}_2 \in \mathbb{T}_2$ are arbitrary, after substituting $\bar{x}_1$ and $\bar{x}_2$ with x_1 and x_2, we obtain the desired result (3.35). □

Let $\mathbb{T} = Z$. Then Theorem 3.1 leads to Corollary 3.13 immediately.

Corollary 3.13 *Let $u, r_i, a_j : A_1 \times A_2 \to \mathbb{R}_0^+$ and $f_i, g_i, f_i^{\Delta x_1} : A_1^2 \times A_2^2 \to \mathbb{R}_0^+$ be nonnegative real-valued functions such that a_j is nondecreasing with respect to its each variable, let $\gamma_{ji} : A_j \to \mathbb{R}_0^+$ be a nonnegative and nondecreasing function, $\tilde{\mathfrak{a}} : \{-\rho_{1i}, \dots, -1, 0\} \times \{-\rho_{2i}, \dots, -1, 0\} \to \mathbb{R}_0^+$ be a nonnegative function, $-\infty < \tilde{\mathfrak{p}}_j = \inf\{\min(x_j - \rho_{ji}), x_j \in A_j\} \le 0$, w and w_j be as defined in Theorem 3.1. If $u(x_1, x_2)$ satisfies the following discrete inequality*

$$\begin{aligned} &w\big(u(x_1,x_2)\big) \\ &\quad\le a_1(x_1,x_2) + a_2(x_1,x_2)\sum_{i=1}^{n}\sum_{t_1=\gamma_{1i}(0)}^{\gamma_{1i}(x_1)-1}\sum_{t_2=\gamma_{2i}(0)}^{\gamma_{2i}(x_2)-1} w_1\big(u(t_1-\rho_{1i},t_2-\rho_{2i})\big) \\ &\qquad\times\Bigg[f_i(x_1,t_1,x_2,t_2)\Bigg\{w_2\big(u(t_1-\rho_{1i},t_2-\rho_{2i})\big) + \prod_{l=1}^{2}\sum_{m_l=\gamma_{li}(0)}^{t_l-1} g_i(t_1,m_1,t_2,m_2) \\ &\qquad\times w_2\big(u(m_1-\rho_{1i},m_2-\rho_{2i})\big)\Bigg\} + r_i(t_1,t_2)\Bigg] \end{aligned} \tag{3.48}$$

with the following initial condition

$$\begin{cases} w(u(x_1,x_2)) = \tilde{\mathfrak{a}}(x_1,x_2), & x_1 \in [\tilde{\mathfrak{p}}_1, 0] \text{ or } x_2 \in [\tilde{\mathfrak{p}}_2, 0], \\ \tilde{\mathfrak{a}}(x_1-\rho_{1i}, x_2-\rho_{2i}) \le a_1(x_1,x_2), & x_1 \le \rho_{1i}, \text{ or } x_2 \le \rho_{2i}, \end{cases} \tag{3.49}$$

then

$$u(x_1,x_2) \le w^{-1}\big(H_2^{-1}\big(H_3^{-1}\big(H_3\big(\tilde{b}_1(x_1,x_2)\big) + a_2(x_1,x_2)\tilde{c}(x_1,x_2)\big)\big)\big) \tag{3.50}$$

for all $0 \le x_1 \le \tilde{x}_1$ and $0 \le x_2 \le \tilde{x}_2$, where

$$\begin{aligned} \tilde{c}(x_1,x_2) &= \sum_{i=1}^{n}\sum_{t_1=\gamma_{1i}(0)}^{\gamma_{1i}(x_1)-1}\sum_{t_2=\gamma_{2i}(0)}^{\gamma_{2i}(x_2)-1} f_i(x_1,t_1,x_2,t_2) \\ &\quad\times\Bigg(1 + \sum_{m_1=\gamma_{1i}(0)}^{t_1-1}\sum_{m_2=\gamma_{2i}(0)}^{t_2-1} g_i(t_1,m_1,t_2,m_2)\Bigg), \\ \tilde{b}_1(x_1,x_2) &= H_2\big(a_1(x_1,x_2)\big) + a_2(x_1,x_2)\sum_{i=1}^{n}\sum_{t_1=\gamma_{1i}(0)}^{\gamma_{1i}(x_1)-1}\sum_{t_2=\gamma_{2i}(0)}^{\gamma_{2i}(x_2)-1} r_i(t_1,t_2), \\ H_2(s) &= \int_{s_7}^{s}\frac{dp}{w_1(w^{-1}(p))}, \quad s > s_7 > 0 \text{ with } H_2(\infty) = \infty, \\ H_3(s) &= \int_{s_8}^{s}\frac{dp}{w_2(w^{-1}(H_2^{-1}(p)))}, \quad s > s_8 > 0 \text{ with } H_3(\infty) = \infty, \end{aligned}$$

H_2^{-1} and H_3^{-1} are respectively the inverse functions of H_2 and H_3, and $\tilde{x}_1$ and $\tilde{x}_2$ are chosen such that

$$\begin{aligned}
&H_3\big(\tilde{b}_1(x_1,x_2)\big) + a_2(x_1,x_2)\tilde{c}(x_1,x_2) \in \mathrm{Dom}\big(H_3^{-1}\big),\\
&H_3^{-1}\big(H_3\big(\tilde{b}_1(x_1,x_2)\big) + a_2(x_1,x_2)\tilde{c}(x_1,x_2)\big) \in \mathrm{Dom}\big(H_2^{-1}\big),\\
&H_2^{-1}\big(H_3^{-1}\big(H_3\big(\tilde{b}_1(x_1,x_2)\big) + a_2(x_1,x_2)\tilde{c}(x_1,x_2)\big)\big) \in \mathrm{Dom}\big(w^{-1}\big).
\end{aligned}$$

4 Applications

Example 4.1 Consider the following integro-differential equation with several arguments:

$$\begin{aligned}
\big[u(x_1,x_2)\big]^{\Delta x_1\Delta x_2} = \mathrm{F}\Big[&x_1,t_1,x_2,t_2,u\big(\mu_{11}(t_1),\mu_{21}(t_2)\big),\ldots,u\big(\mu_{1n}(t_1),\mu_{2n}(t_2)\big),\\
&\int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2}\mathbf{Q}\big(t_1,m_1,t_2,m_2,u\big(\mu_{11}(m_1),\mu_{21}(m_2)\big),\\
&\ldots,u\big(\mu_{1n}(m_1),\mu_{2n}(m_2)\big)\big)\Delta m_2\Delta m_1\Big]
\end{aligned}\tag{4.1}$$

with the initial condition

$$\begin{cases}
[u(x_1,x_{02})]^{\Delta x_2} = \mathfrak{a}_1^{\Delta}(x_1), \qquad u(x_{01},x_2) = \mathfrak{a}_2(x_2),\\
u(x_1,x_2) = \mathfrak{a}(x_1,x_2), \quad x_1 \in [\mathfrak{p}_1,x_{01}]_{\mathbb{T}} \text{ or } x_2 \in [\mathfrak{p}_2,x_{02}]_{\mathbb{T}},\\
|\mathfrak{a}(\mu_{1i}(x_1),\mu_{2i}(x_2))| \le |a_1(x_1,x_2)|, \quad \mu_{1i}(x_1) \le x_{01} \text{ or } \mu_{2i}(x_2) \le x_{02}
\end{cases}\tag{4.2}$$

for $\mathrm{F}:\mathbb{T}_1^2\times\mathbb{T}_2^2\times \mathrm{R}^{n+1}\to\mathbb{R}$ is right-dense continuous on $\mathbb{T}_1^2\times\mathbb{T}_2^2$ and continuous on $\mathbb{R}^{n+1}$, $\mathbf{Q}:\mathbb{T}_1^2\times\mathbb{T}_2^2\times\mathbb{R}^n\to\mathbb{R}$ is right-dense continuous on $\mathbb{T}_1^2\times\mathbb{T}_2^2$ and continuous on $\mathbb{R}^n$, $u:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}/\{0\}$, $\mathfrak{a}_j:\mathbb{T}_j\to\mathbb{R}$, $\mathfrak{a}:([\mathfrak{p}_1,x_{01}]\times[\mathfrak{p}_2,x_{02}])_{\mathbb{T}^2}\to\mathbb{R}$, $a_1:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ are right-dense continuous functions and μ_{ji} is as defined in Theorem 3.1.

Theorem 4.2 *Assume that*

$$\left.\begin{aligned}
&|\mathrm{F}(x_1,t_1,x_2,t_2,\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_n,k)| \le a_2(x_1,x_2)\textstyle\sum_{i=1}^n w_1(|\mathfrak{k}_i|)\\
&\qquad\qquad\times[f_i(x_1,t_1,x_2,t_2)\{w_2(|\mathfrak{k}_i|)+|k|\}+r_i(t_1,t_2)],\\
&|\mathbf{Q}(x_1,t_1,x_2,t_2,\mathfrak{k}_1,\ldots,\mathfrak{k}_n)| \le g_i(x_1,t_1,x_2,t_2)w_2(|\mathfrak{k}_i|),
\end{aligned}\right\}\tag{4.3}$$

where f_i, g_i, r_i, a_2 are as defined in Theorem 3.1, $a_1(x_1,x_2)=\sum_{j=1}^2\mathfrak{a}_j(x_j)$, $w_1(\eta)=\sqrt[3]{\sigma^2(\eta)}+\sqrt[3]{\sigma(\eta)\eta}+\sqrt[3]{\eta^2}$, $w_2(\eta)=\sqrt{\sigma(\sqrt[3]{\eta})}+\sqrt[6]{\eta}$ for $\eta\in\mathbb{R}_0^+$, and $u(x_1,x_2)$ is a solution of equation (4.1) with initial condition (4.2), then

$$\big|u(x_1,x_2)\big| \le \Big(\sqrt{\mathfrak{b}_3(x_1,x_2)}+a_2(x_1,x_2)\mathfrak{b}_4(x_1,x_2)\Big)^6,$$

where

$$\mathfrak{b}_3(x_1,x_2) = \sqrt[3]{\big|a_1(x_1,x_2)\big|} + a_2(x_1,x_2)\sum_{i=1}^n\int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2} r_i(t_1,t_2)\Delta t_2\Delta t_1,$$

$$\mathfrak{b}_4(x_1,x_2) = \sum_{i=1}^{n}\int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2} f_i(x_1,t_1,x_2,t_2) \times \left(1+\int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2} g_i(t_1,m_1,t_2,m_2)\Delta m_2\Delta m_1\right)\Delta t_2\Delta t_1. \tag{4.4}$$

Proof Let $\tilde{\mathbb{T}} = \varrho(\mathbb{T}_1,x_2)$ and $\tilde{G}_j(\eta) = \sqrt[4-j]{\eta}$, where $\varrho(x_1,x_2)$ is strictly increasing with respect to $x_1 \in \mathbb{T}_1$. Then it follows from Theorem 2.3 that

$$\begin{aligned}
\left[\tilde{G}_1\bigl(\varrho(x_1,x_2)\bigr)\right]^{\Delta x_1} &= \tilde{G}_1^{\tilde{\Delta}}(\varrho)\varrho^{\Delta x_1}(x_1,x_2) \\
&= \frac{\varrho^{\Delta x_1}(x_1,x_2)}{\sqrt[3]{\sigma^2(\varrho(x_1,x_2))} + \sqrt[3]{\sigma(\varrho(x_1,x_2))\varrho(x_1,x_2)} + \sqrt[3]{\varrho^2(x_1,x_2)}} \\
&= \frac{\varrho^{\Delta x_1}(x_1,x_2)}{w_1(\varrho(x_1,x_2))} = \frac{\varrho^{\Delta x_1}(x_1,x_2)}{w_1(w^{-1}(\varrho(x_1,x_2)))}, \\
\left[\tilde{G}_2\bigl(\varrho(x_1,x_2)\bigr)\right]^{\Delta x_1} &= \tilde{G}_2^{\tilde{\Delta}}(\varrho)\varrho^{\Delta x_1}(x_1,x_2) \\
&= \frac{\varrho^{\Delta x_1}(x_1,x_2)}{\sqrt{\sigma(\varrho(x_1,x_2))} + \sqrt{\varrho(x_1,x_2)}} \\
&= \frac{\varrho^{\Delta x_1}(x_1,x_2)}{w_2(\varrho^3(x_1,x_2))} = \frac{\varrho^{\Delta x_1}(x_1,x_2)}{w_2(w^{-1}(\tilde{G}_1^{-1}(\varrho(x_1,x_2))))}.
\end{aligned}$$

From (4.1) and (4.2) we get

$$\begin{aligned}
u(x_1,x_2) = a_1(x_1,x_2) + \int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2} \mathrm{F}\Bigl[&x_1,t_1,x_2,t_2,u\bigl(\mu_{11}(t_1),\mu_{21}(t_2)\bigr), \\
&\dots,u\bigl(\mu_{1n}(t_1),\mu_{2n}(t_2)\bigr),\int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2}\mathbf{Q}\bigl(t_1,m_1,t_2,m_2,u\bigl(\mu_{11}(m_1),\mu_{21}(m_2)\bigr), \\
&\dots,u\bigl(\mu_{1n}(m_1),\mu_{2n}(m_2)\bigr)\bigr)\Delta m_2\Delta m_1\Bigr]\Delta t_2\Delta t_1.
\end{aligned} \tag{4.5}$$

From (4.3) we know that (4.5) has the form

$$\begin{aligned}
\bigl|u(x_1,x_2)\bigr| \le \bigl|a_1(x_1,x_2)\bigr| + \int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2}\Bigl|\mathrm{F}\Bigl[&x_1,t_1,x_2,t_2,u\bigl(\mu_{11}(t_1),\mu_{21}(t_2)\bigr), \\
&\dots,u\bigl(\mu_{1n}(t_1),\mu_{2n}(t_2)\bigr),\int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2}\mathbf{Q}\bigl(t_1,m_1,t_2,m_2,u\bigl(\mu_{11}(m_1),\mu_{21}(m_2)\bigr), \\
&\dots,u\bigl(\mu_{1n}(m_1),\mu_{2n}(m_2)\bigr)\bigr)\Delta m_2\Delta m_1\Bigr]\Bigr|\Delta t_2\Delta t_1 \\
\le \bigl|a_1(x_1,x_2)\bigr| + a_2(x_1,x_2)&\sum_{i=1}^{n}\int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2} w_1\bigl(\bigl|u\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr)\bigr|\bigr) \\
&\times\Bigl[f_i(x_1,t_1,x_2,t_2)\Bigl\{w_2\bigl(\bigl|u\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr)\bigr|\bigr) + \int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2} g_i(t_1,m_1,t_2,m_2) \\
&\times w_2\bigl(\bigl|u\bigl(\mu_{1i}(m_1),\mu_{2i}(m_2)\bigr)\bigr|\bigr)\Delta m_2\Delta m_1\Bigr\} + r_i(t_1,t_2)\Bigr]\Delta t_2\Delta t_1.
\end{aligned} \tag{4.6}$$

Therefore, the desired result follows easily from (3.3) and (4.6). □

Example 4.3 Consider another integro-differential equation with several arguments:

$$\begin{aligned}&\bigl[u^{\frac{2}{5}}(x_1,x_2)u^{\Delta x_1}(x_1,x_2)\bigr]^{\Delta x_2}\mathrm{F}\biggl[x_1,t_1,x_2,t_2,u\bigl(\mu_{11}(t_1),\mu_{21}(t_2)\bigr),\\&\quad\dots,u\bigl(\mu_{1n}(t_1),\mu_{2n}(t_2)\bigr),\int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2}\mathbf{Q}\bigl(t_1,m_1,t_2,m_2,u\bigl(\mu_{11}(m_1),\mu_{21}(m_2)\bigr),\\&\quad\dots,u\bigl(\mu_{1n}(m_1),\mu_{2n}(m_2)\bigr)\bigr)\Delta m_2\Delta m_1\biggr]\end{aligned}\tag{4.7}$$

with the initial condition

$$\begin{cases}u^{\frac{2}{5}}(x_1,x_{02})u^{\Delta x_1}(x_1,x_{02})=\frac{5\mathfrak{f}_1^{\Delta}(x_1)}{3}, & u^{\frac{3}{5}}(x_{01},x_2)=\mathfrak{f}_2(x_2),\\ u(x_1,x_2)=\mathfrak{a}(x_1,x_2), & x_1\in[\mathfrak{p}_1,x_{01}]_{\mathbb{T}}\text{ or }x_2\in[\mathfrak{p}_2,x_{02}]_{\mathbb{T}},\\ |\mathfrak{a}(\mu_{1i}(x_1),\mu_{2i}(x_2))|\le\mathfrak{C}^{\frac{1}{5}}, & \mu_{1i}(x_1)\le x_{01}\text{ or }\mu_{2i}(x_2)\le x_{02}\end{cases}\tag{4.8}$$

for $\mathrm{F}:\mathbb{T}_1^2\times\mathbb{T}_2^2\times\mathbb{R}^{n+1}\to\mathbb{R}$ is right-dense continuous on $\mathbb{T}_1^2\times\mathbb{T}_2^2$ and continuous on $\mathbb{R}^{n+1}$, $\mathbf{Q}:\mathbb{T}_1^2\times\mathbb{T}_2^2\times\mathbb{R}^n\to\mathbb{R}$ is right-dense continuous on $\mathbb{T}_1^2\times\mathbb{T}_2^2$ and continuous on $\mathbb{R}^n$, $u:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}/\{0\}$, $\mathfrak{f}_j:\mathbb{T}_j\to\mathbb{R}$, $\mathfrak{a}:([\mathfrak{p}_1,x_{01}]\times[\mathfrak{p}_2,x_{02}])_{\mathbb{T}^2}\to\mathbb{R}$ are right-dense continuous functions, $\mathfrak{C}$ is a nonzero constant such that $\mathfrak{C}\ge\sum_{j=1}^{2}|\mathfrak{f}_j(x_j)|$ and μ_{ji} is as defined in Theorem 3.1.

Theorem 4.4 *Assume that*

$$\left.\begin{aligned}&|\mathrm{F}(x_1,t_1,x_2,t_2,\mathfrak{k}_1,\mathfrak{k}_2,\dots,\mathfrak{k}_n,k)|\\&\quad\le\sum_{i=1}^{n}\frac{|\mathfrak{k}_i|^2}{3}[f_i(x_1,t_1,x_2,t_2)\{w_2(|\mathfrak{k}_i|)+|k|\}+r_i(t_1,t_2)],\\&|\mathrm{Q}(x_1,t_1,x_2,t_2,\mathfrak{k}_1,\dots,\mathfrak{k}_n)|\le g_i(x_1,t_1,x_2,t_2)w_2(|\mathfrak{k}_i|),\end{aligned}\right\}\tag{4.9}$$

where f_i, g_i, and r_i are as defined in Theorem 3.1, $w_2(\eta)=\sqrt[3]{\sigma^2(\eta^3)}+\sqrt[3]{\sigma(\eta^3)\eta^3}+\eta^2$ for $\eta\in\mathbb{R}_0^+$. If $u(x_1,x_2)$ is a solution of equation (4.7) satisfying initial condition (4.8), then

$$\bigl|u(x_1,x_2)\bigr|\le\sqrt[3]{\overline{\mathfrak{b}}_3(x_1,x_2)}+\mathfrak{b}_4(x_1,x_2),$$

where $\mathfrak{b}_4(x_1,x_2)$ is defined by (4.4) and

$$\overline{\mathfrak{b}}_3(x_1,x_2):=\mathfrak{C}^{\frac{3}{5}}+\sum_{i=1}^{n}\int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2}r_i(t_1,t_2)\Delta t_2\Delta t_1.$$

Proof Let $\overline{\mathbb{T}}=\varpi(\mathbb{T}_1,x_2)$ and $\overline{G}_2(\eta)=\sqrt[3]{\eta}$, where ϖ is strictly increasing with respect to $x_1\in\mathbb{T}_1$. Then by Theorem 2.3 we have

$$\begin{aligned}\bigl[\overline{G}_2\bigl(\varpi(x_1,x_2)\bigr)\bigr]^{\Delta x_1}&=\overline{G}_2^{\overline{\Delta}}(\varpi)\varpi^{\Delta x_1}(x_1,x_2)\\&=\frac{\varpi^{\Delta x_1}(x_1,x_2)}{\sqrt[3]{\sigma^2(\varpi(x_1,x_2))}+\sqrt[3]{\sigma(\varpi(x_1,x_2))\varpi(x_1,x_2)}+\sqrt[3]{\varpi^2(x_1,x_2)}}\\&=\frac{\varpi^{\Delta x_1}(x_1,x_2)}{w_2(\sqrt[3]{\varpi(x_1,x_2)})}.\end{aligned}$$

Integrating equation (4.7) over $[x_{02}, x_2]$ gives

$$\begin{aligned} & u^{\frac{2}{5}}(x_1,x_2)u^{\Delta x_1}(x_1,x_2) \\ &\quad = u^{\frac{2}{5}}(x_1,x_{02})u^{\Delta x_1}(x_1,x_{02}) + \int_{x_{02}}^{x_2} \mathrm{F}\Big[x_1,t_1,x_2,t_2,u\big(\mu_{11}(t_1),\mu_{21}(t_2)\big),\ldots, \\ &\qquad u\big(\mu_{1n}(t_1),\mu_{2n}(t_2)\big), \int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2} \mathbf{Q}\big(t_1,m_1,t_2,m_2,u\big(\mu_{11}(m_1),\mu_{21}(m_2)\big),\ldots, \\ &\qquad u\big(\mu_{1n}(m_1),\mu_{2n}(m_2)\big)\big)\Delta m_2\Delta m_1\Big]\Delta t_2. \end{aligned} \tag{4.10}$$

It follows from Theorem 2.2 and $\frac{u^{\Delta x_1}(x_1,x_2)}{u(x_1,x_2)} \geq 0$ that

$$\begin{aligned} \left(\frac{5}{3}u^{\frac{3}{5}}(x_1,x_2)\right)^{\Delta x_1} &= u^{\Delta x_1}(x_1,x_2)\int_0^1 \big\{u(x_1,x_2)+h\mu(x_1,x_2)u^{\Delta x_1}(x_1,x_2)\big\}^{-\frac{2}{5}}\,dh \\ &= \frac{u^{\Delta x_1}(x_1,x_2)}{u^{\frac{2}{5}}(x_1,x_2)}\int_0^1 \left\{1+h\mu(x_1,x_2)\frac{u^{\Delta x_1}(x_1,x_2)}{u(x_1,x_2)}\right\}^{-\frac{2}{5}}\,dh \\ &= \frac{u^{\Delta x_1}(x_1,x_2)}{u^{\frac{2}{5}}(x_1,x_2)} \times \frac{5\{1+\mu(x_1,x_2)\frac{u^{\Delta x_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u(x_1,x_2)}\}^{\frac{3}{5}}-5}{3\mu(x_1,x_2)\frac{u^{\Delta x_1}(x_1,\mathfrak{x}_2)}{u(x_1,x_2)}} \\ &\leq \frac{u^{\Delta x_1}(x_1,x_2)}{u^{\frac{2}{5}}(x_1,x_2)}. \end{aligned} \tag{4.11}$$

From (4.10) and (4.11) one has

$$\begin{aligned} & \left(\frac{5}{3}u^{\frac{3}{5}}(x_1,x_2)\right)^{\Delta x_1} \\ &\quad \leq \frac{5\mathfrak{f}_1^{\Delta}(x_1)}{3} + \int_{x_{02}}^{x_2} \mathrm{F}\Big[x_1,t_1,x_2,t_2,u\big(\mu_{11}(t_1),\mu_{21}(t_2)\big), \\ &\qquad \ldots,u\big(\mu_{1n}(t_1),\mu_{2n}(t_2)\big), \int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2} \mathbf{Q}\big(t_1,m_1,t_2,m_2,u\big(\mu_{11}(m_1),\mu_{21}(m_2)\big),\ldots, \\ &\qquad u\big(\mu_{1n}(m_1),\mu_{2n}(m_2)\big)\big)\Delta m_2\Delta m_1\Big]\Delta t_2. \end{aligned}$$

Integrating over $[\mathfrak{x}_{01}, \mathfrak{x}_1]$ leads to

$$\begin{aligned} u^{\frac{3}{5}}(x_1,x_2) \leq \mathfrak{f}_1(x_1)+\mathfrak{f}_2(x_2)+\frac{3}{5}\int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2} &\mathrm{F}\Big[x_1,t_1,x_2,t_2,u\big(\mu_{11}(t_1),\mu_{21}(t_2)\big), \\ &\ldots,u\big(\mu_{1n}(t_1),\mu_{2n}(t_2)\big), \int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2} \mathbf{Q}\big(t_1,m_1,t_2,m_2,u\big(\mu_{11}(m_1),\mu_{21}(m_2)\big), \\ &\ldots,u\big(\mu_{1n}(m_1),\mu_{2n}(m_2)\big)\big)\Delta m_2\Delta m_1\Big]\Delta t_2\Delta t_1. \end{aligned} \tag{4.12}$$

Table 1 The value of $u(x;y)$ from (4.8) and (4.10)

(x,y)	(4.8)	(4.10)
(2, 2)	3.0692e+11	5.7300e+11
(2, 5)	3.1140e+11	1.9384e+12
(2, 9)	3.1481e+11	9.2602e+12
(3, 4)	3.1193e+11	2.8268e+12
(3, 8)	3.1605e+11	3.9592e+13
(7, 3)	3.1497e+11	1.7927e+13
(11, 5)	3.2673e+26	6.3683e+26
(15, 2)	3.1803e+11	1.2336e+14
(40, 1)	3.2160e+11	1.9787e+14

From (4.9) we know that inequality (4.12) has the form

$$\begin{aligned} \left|u^{\frac{3}{5}}(x_1,x_2)\right| \le{}& \mathfrak{C} + \frac{3}{5}\int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2}\Bigl|\mathrm{F}\Bigl[x_1,t_1,x_2,t_2,u\bigl(\mu_{11}(t_1),\mu_{21}(t_2)\bigr),\\ &\ldots,u\bigl(\mu_{1n}(t_1),\mu_{2n}(t_2)\bigr),\int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2}\mathbf{Q}\bigl(t_1,m_1,t_2,m_2,u\bigl(\mu_{11}(m_1),\mu_{21}(m_2)\bigr),\\ &\ldots,u\bigl(\mu_{1n}(m_1),\mu_{2n}(m_2)\bigr)\bigr)\Delta m_2\Delta m_1\Bigr]\Bigr|\Delta t_2\Delta t_1\\ \le{}& \mathfrak{C} + \frac{1}{5}\sum_{i=1}^{n}\int_{x_{01}}^{x_1}\int_{x_{02}}^{x_2}\bigl|u\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr)\bigr|^2\Bigl[f_i(x_1,t_1,x_2,t_2)\\ &\times\Bigl\{w_2\bigl(\bigl|u\bigl(\mu_{1i}(t_1),\mu_{2i}(t_2)\bigr)\bigr|\bigr) + \int_{x_{01}}^{t_1}\int_{x_{02}}^{t_2} g_i(t_1,m_1,t_2,m_2)\\ &\times w_2\bigl(\bigl|u\bigl(\mu_{1i}(m_1),\mu_{2i}(m_2)\bigr)\bigr|\bigr)\Delta m_2\Delta m_1\Bigr\} + r_i(t_1,t_2)\Bigr]\Delta t_2\Delta t_1. \end{aligned} \tag{4.13}$$

Therefore, the desired result follows from (3.5) and (4.13). □

Example 4.5 Consider the delay discrete inequality (4.8) satisfying initial condition (4.9) with $u(x_1,x_2) = 27^{x_1x_2}$, $\rho_{ji} = ji$, $a_1(x_1,x_2) = 27^8$, $a_2(x_1,x_2) = \sqrt[10]{x_1x_2}$, $f_i(x_1,t_1,x_2,t_2) = \arctan(\sqrt[i+1]{x_1+t_1+x_2+t_2})$, $r_i(x_1,x_2) = \sqrt[i+1]{\exp(x_1x_2)}$, $\gamma_{ji} = w = I$, $g_i(x_1,t_1,x_2,t_2) = 10^{-i-1}\times\sqrt[i+1]{x_1+t_1+x_2+t_2}$ $(1 \le i \le 2)$, and w_j is defined as in Theorem 4.2.

We find that the numerical solution agrees with the analytical solution for some discrete inequalities by calculating the value of $u(x,y)$ from (4.8) and (4.10) (see Table 1).

5 Conclusion

In the article, we have presented several explicit bounds for the delay double integral inequalities on time scales and have given their applications to the solutions of certain integro-differential equations. Our results are the improvements and generalizations of some previously known results. Furthermore, we have found some new estimates for the integral inequalities in the form of exponential function.

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