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Delayed positive feedback can stabilize oscillatory systems

Chaouki T. Abdallah

Peter Dorato

J. Benites-Read

R. Byrne

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Delayed Positive Feedback Can Stabilize Oscillatory Systems

C. Abdallah, P. Dorato, J. Benites-Read
Department of Electrical and Computer Engineering
University of New Mexico
Albuquerque, NM 87131, USA

R. Byrne
Sandia National Laboratories, Department 9616
Albuquerque, NM 87185

Abstract

This paper expands on a method proposed in [1] for stabilising oscillatory systems with positive, delayed feedback. The closed-loop system obtained is shown (using the Nyquist criterion) to be stable for a range of delays.

1 Introduction

The stabilization of oscillatory systems finds applications in robotics [2] and flexible structures [1]. A simple example of an oscillatory system is given by the second-order system

$$\ddot{y} + w_0^2 y = u \quad (1)$$

This class of systems can be stabilized with negative derivative feedback, i.e.

$$u(t) = -k\dot{y}(t)\ ; \quad k > 0 \quad (2)$$

The closed-loop system then becomes

$$\ddot{y} + k\dot{y} + w_0^2 y = 0 \quad (3)$$

which is obviously stable for $k > 0$. This feedback will require the differentiation of the output, or the use of an observer to estimate $\dot{y}$ from the measurement of y. This paper will present an *exact analysis* of a method given in [1] to stabilise this system using instead *positive delayed output feedback* only, i.e.

$$u(t) = ky(t - \tau) \quad (4)$$

In [1], the analysis of the closed-loop system was done using a first-order Padé approximation of the pure delay. In addition, no attempt was made to determine the range of allowable delays in order to guarantee stability. A root locus approach was presented for such systems in [3], [4] and more recently in [5]. Note that in general, a double-integrator system described by

$$\ddot{y}(t) = u(t) \quad (5)$$

can be reduced to the oscillatory problem above by use of output-plus-delayed-output feedback of the form

$$u(t) = -w_0^2 y(t) + ky(t - \tau) \quad (6)$$

A double-integrator system will result, for example, from applying feedback-linearisation to many nonlinear systems [6]. By stabilizing these systems using output feedback only, savings in sensors (tachometers) or observers are achieved. This paper will analyze the closed-loop stability of this type of system

The remaining of the paper is organized as follows. Section 2 contains the analysis of the delayed, positive-feedback control as applied to an oscillatory system and Section 3 contains our conclusions.

2 Analysis

Consider the plant given by

$$G(s) = \frac{1}{s^2 + w_0^2} \quad (7)$$

and the positive-feedback, time-delay compensator

$$C(s) = ke^{-s\tau} \quad (8)$$

where $k > 0$ in a simple unity-feedback loop shown in Figure 1, such that the closed-loop system is given by

$$\begin{aligned} T(s) &= \frac{G(s)C(s)}{1 - G(s)C(s)} \\ &= \frac{ke^{-s\tau}}{s^2 + w_0^2 - ke^{-s\tau}} \end{aligned} \quad (9)$$

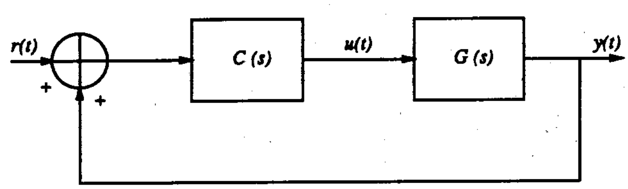

Figure 1: Block Diagram of Oscillatory System with Positive, Delay Feedback

We will study the stability of the closed-loop system by exploring the Nyquist plot of

$$-G(s)C(s) = \frac{-ke^{-s\tau}}{s^2 + w_0^2} \quad (10)$$

The Nyquist contour is assumed to be indented at the open-loop poles $\pm jw_0$ so that no poles exist in the RHP. Thus for closed-loop stability there should be no clockwise encirclements of the $(-1, 0)$ point. First, note that with $\tau = 0$, the closed-loop system is unstable because the Nyquist plot will always encircle the (-1,0) point. Consider then the case where $\tau > 0$, and note that a necessary condition for stability is that

$$k < w_0^2 \quad (11)$$

If (11) does not hold there will always be at least one clockwise encirclement. Assuming that this condition holds, let us consider the instability mechanisms by counting the number of encirclements of -1 by the polar plot of

$$-G(jw)C(jw) = \frac{-ke^{-jw\tau}}{w_0^2 - w^2} \quad (12)$$

Note that we have 3 important regions: 1) $w < w_0$, 2) $w = w_0$, and 3) $w > w_0$. At $w = w_0$, the magnitude of the polar plot goes to infinity. This point will be studied later. Let us consider what happens to both magnitude and phase as w goes from 0 to $w_0 - \epsilon$, and then from $w_0 + \epsilon$ to ∞. The phase is given by

$$\begin{aligned} \theta(w) &= -\pi - w\tau; \;\; 0 \le w < w_0 \\ &= -2\pi - w\tau; \;\; w > w_0 \end{aligned} \tag{13}$$

and the magnitude by

$$\begin{aligned} \mid G(jw)C(jw) \mid &= \frac{k}{w_0^2 - w^2}; \;\; 0 \le w < w_0 \\ &= \frac{k}{w^2 - w_0^2}; \;\; w > w_0 \end{aligned} \tag{14}$$

Let us then find all intersections of the polar plot with the negative real axis. The intersections will take place whenever the phase is $-(2n+1)\pi$, $n = 0, 1, \cdots$. Therefore, they will take place at the frequencies w_c

$$\begin{aligned} -\pi - w_c\tau &= -(2n+1)\pi; \;\; 0 \le w_c < w_0 \\ -2\pi - w_c\tau &= -(2n+1)\pi; \;\; w_c > w_0 \end{aligned} \tag{15}$$

$$\begin{aligned} w_c\tau &= 2n\pi; \;\; 0 \le w_c < w_0 \\ w_c\tau &= (2n+1)\pi; \;\; w_c > w_0 \end{aligned} \tag{16}$$

In order to make sure that no encirclements of the -1 point take place, we must guarantee that the magnitude $\mid G(jw)C(jw) \mid$ evaluated at w_c is less than 1, i.e.

$$\begin{aligned} \frac{k}{w_0^2 - (4n^2\pi^2)/\tau^2} &< 1; \;\; 0 \le 2n\pi/\tau < w_0 \\ \frac{k}{(2n+1)^2\pi^2/\tau^2 - w_0^2} &< 1; \;\; (2n+1)\pi/\tau > w_0 \end{aligned} \tag{17}$$

Combining both conditions we get, given that $k < w_0^2$,

$$\frac{2n\pi}{\sqrt{w_0^2 - k}} < \tau < \frac{(2n+1)\pi}{\sqrt{w_0^2 + k}} \tag{18}$$

Now, let us consider what happens at $w = w_0$. Since the magnitude is infinite at $w = w_0$, we should make sure that the phase can never be $-(2n+1)\pi$ at that frequency. In other words, we need to make sure that

$$\frac{2n\pi}{w_0} < \tau < \frac{(2n+1)\pi}{w_0} \tag{19}$$

Therefore, combining all conditions, we have the following 2 conditions

$$k < w_0^2 \tag{20}$$

$$\frac{2n\pi}{w_0} < \frac{2n\pi}{\sqrt{w_0^2 - k}} < \tau < \frac{(2n+1)\pi}{\sqrt{w_0^2 + k}} < \frac{(2n+1)\pi}{w_0} \tag{21}$$

For all $n = 0, 1, \cdots$. Note that w_0^2 can be modified if necessary by proportional feedback $-fy(t)$ in (4), i.e.

$$u(t) = -fy(t) + ky(t - \tau) \tag{22}$$

so that w_0^2 becomes

$$W_n^2 = w_0^2 + f \tag{23}$$

Also note that we can solve for the allowable region of k explicitly by finding the point of intersection of the lower and upper bounds in (21) to obtain

$$\begin{aligned} 0 \; &< \; k \le \frac{1+4n}{1+4n+8n^2} w_0^2 \\ \frac{2n\pi}{\sqrt{w_0^2 - k}} \; &< \; \tau < \frac{(2n+1)\pi}{\sqrt{w_0^2 + k}} \end{aligned} \tag{24}$$

See the plots in Figure 2, for $w_0^2 = 1$. In particular, note that the region of stabilizing k shrinks as the delay τ gets larger.

3 Conclusions

One normally thinks of positive feedback and pure delays as destabilizing effects in a feedback system. However for purely oscillatory systems as illustrated by the second-order system in this paper, this type of feedback is actually stabilizing ; and indeed since it involves only output feedback, it can result in a simpler controller.

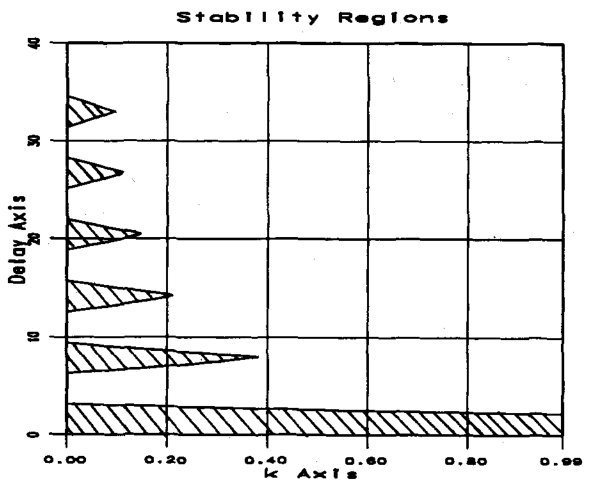

Figure 2: Stability Regions (shaded) for $w_0^2 = 1$

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