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Delooping Symmetric Monoidal Categories

Dedicated to Professor A. Komatu on his 70th birthday

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Introduction

Recently various infinite loop space machines have been studied ([1], [3], [7], [11]). Among the interesting applications are the iterated deloopings of the classifying spaces of symmetric monoidal categories. Thus we can associate functorially an Ω-spectrum to any symmetric monoidal category.

However, as we start from a category, it seems natural and helpful to have a categorical construction corresponding to the space level delooping.

The main purpose of this paper is to show that there is such a "categorical delooping". Precisely, to each symmetric monoidal category C we shall associate functorially a new symmetric monoidal category BC such that its realization $|BC|$ becomes the delooping of $|C|$ in the sense of Segal [11].

Since BC is again a symmetric monoidal category, we can repeat this process so as to get a sequence of symmetric monoidal categories

$$C,\ BC,\ B^2C,\dots,\ B^nC,\dots.$$

Then the classifying spaces $|B^nC|$ form a connective spectrum $E(C)$, and hence we get a functor $C\mapsto E(C)$ from symmetric monoidal categories to spectra.

One advantage of our construction is that it gives a rather simple description of the multiplicative structure of the spectrum $E(C)$ associated with a symmetric bimonoidal category C. That is to say, starting from a pairing (cf. Definition 3.1)

$$\square_\times : C \times C \longrightarrow C,$$

we can canonically construct a coherent system of pairings

$$P_{m,n}\colon B^mC \times B^nC \longrightarrow B^{m+n}C$$

which induces a system of maps

$$\mu_{m,n}\colon |B^mC| \wedge |B^nC| \longrightarrow |B^{m+n}C|$$

making $E(C)$ a ring spectrum.

The paper is organized as follows. The categories B^nC are defined in § 1 and

§2. We construct the pairings $P_{m,n}$: $B^mC \times B^nC \to B^{m+n}C$ for any symmetric bimonoidal category C in §3. §4 shows that BC becomes an exact category if so is C and that QBC is isomorphic to BQC for the Quillen construction Q [10]. Finally §5 shows that almost all the above constructions and propositions can be extended to the equivariant cases. In particular we shall associate a G-spectrum to any symmetric monoidal G-category when G is a finite group.

§1. Γ-spaces and Γ-categories

In this paper all the spaces that we consider are compactly generated weak Hausdorff spaces [14] and $\mathscr{U}$ will denote the category of such spaces*).

Let Γ be the category whose objects are finite sets $\mathbf{n}=\{1,2,\dots,n\}$ and morphisms from $\mathbf{m}$ to $\mathbf{n}$ are the maps θ: $2^{\mathbf{m}} \to 2^{\mathbf{n}}$ which preserve unions and set differences. We use "Γ-space" to mean any contravariant functor from Γ to $\mathscr{U}$, and an original Γ-space of Segal [11] will be called a "special Γ-space". Thus a special Γ-space is a Γ-space A such that

(i) $A(\mathbf{0})$ is contractible, and

(ii) for any n the map P_n: $A(\mathbf{n}) \to A(\mathbf{1}) \times \cdots \times A(\mathbf{1})$ (n-times) induced by the maps i_k: $\mathbf{1} \to \mathbf{n}$ in Γ, where $i_k(1)=\{k\} \subset \mathbf{n}$, is a homotopy equivalence.

If A and B are Γ-spaces, then Γ-maps from A to B are natural transformations F: $A \xrightarrow{\cdot} B$.

For a (nondegenerately) based space X, define a covariant functor X: $\Gamma \to \mathscr{U}$ as follows. For any integer n we put $X(\mathbf{n})=X^n$ and for any map θ: $\mathbf{m} \to \mathbf{n}$ in Γ we put $\theta_*(x_1,\dots,x_m)=(x'_1,\dots,x'_n)$ where $x'_j=x_i$ for $j \in \theta(i)$ and $x'_j=*$ otherwise. Let A be a Γ-space.

DEFINITION 1.1. For any based space X, $X \otimes_\Gamma A$ denotes the quotient $\coprod_{n \geq 0} X^n \times A(\mathbf{n})/\sim$ in $\mathscr{U}$ where the equivalence relation $\sim$ is generated by $(\theta_*(x_1,\dots,x_m), a) \sim (x_1,\dots,x_m, \theta^* a)$ for all $x_j \in X$, $a \in A(\mathbf{n})$ and θ: $\mathbf{m} \to \mathbf{n}$. If F: $A \xrightarrow{\cdot} B$ is a Γ-map, the induced map from $X \otimes_\Gamma A$ to $X \otimes_\Gamma B$ is also denoted by $X \otimes_\Gamma F$.

REMARK. By Proposition 3.2 of [11], $S^1 \otimes_\Gamma A$ is homeomorphic to the geometric realization of A viewed as a simplicial space $\Delta^{\mathrm{op}} \longrightarrow \Gamma^{\mathrm{op}} \xrightarrow{A} \mathscr{U}$. Hence $S^1 \otimes_\Gamma A$ is homotopy equivalent to $BA(\mathbf{1})$ if A is a "good"special Γ-space. (See [11], §1 and Appendix A.)

Now let C be a small category and let OC (resp. MC) denote the object set (resp. morphism set) of C. Then C is called a topological category if

*) $X \in \mathscr{U}$ iff X is a compactly generated space and the image of X under the diagonal map is closed in $k(X \times X)$ (cf. [14]).

(i) both OC and MC are compactly generated weak Hausdorff spaces,

(ii) all the structure maps of C are continuous, and

(iii) the identity map $OC \to MC$ is a cofibration.

The category whose objects are topological categories and whose morphisms are continuous functors is denoted by $\mathscr{CAT}$.

Recall that a (special) Γ-category [11] is a contravariant functor $C: \Gamma \to \mathscr{CAT}$ such that

(i) $C(\mathbf{0})$ is equivalent to the category with a single morphism, and

(ii) for each n the functor $\prod_{k=1}^{n} i_k^*: C(\mathbf{n}) \to C(\mathbf{1}) \times \cdots \times C(\mathbf{1})$ (n-times) is an equivalence of categories.

A Γ-category C induces two Γ-spaces OC and MC such that $OC(\mathbf{n})$ (resp. $MC(\mathbf{n})$) is the space of objects (resp. morphisms) of $C(\mathbf{n})$. Moreover there are Γ-maps

$$MC \underset{\text{target}}{\overset{\text{source}}{\rightrightarrows}} OC,$$

$$(*) \qquad MC \times_{OC} MC \xrightarrow{\text{composition}} MC,$$

$$OC \xrightarrow{\text{identity}} MC,$$

given by the structure maps of each $C(\mathbf{n})$.

DEFINITION 1.2. For any based space X, $X \otimes_\Gamma C$ denotes the topological category whose object space and morphism space are $X \otimes_\Gamma OC$ and $X \otimes_\Gamma MC$ respectively, and whose structure maps are induced by the Γ-maps in $(*)$.

$X \otimes_\Gamma C$ is a coend (cf. [6]) of the functor

$$\begin{array}{ccc} \Gamma \times \Gamma^{\mathrm{op}} & \longrightarrow & \mathscr{CAT} \\ \Cup & & \Cup \\ (\mathbf{m}, \mathbf{n}) & \longleftrightarrow & X^m \times C(\mathbf{n}) \end{array}$$

in which X is considered as a category with identity morphisms.

If C and D are Γ-categories and $F: C \xrightarrow{\cdot} D$ is a natural transformation, then for any X we get a functor

$$F_* = X \otimes_\Gamma F: X \otimes_\Gamma C \longrightarrow X \otimes_\Gamma D.$$

Let C and C' be Γ-categories, and let $C \times C'$ denote the product $\mathbf{n} \mapsto C(\mathbf{n}) \times C'(\mathbf{n})$.

PROPOSITION 1.3. *$X \otimes_\Gamma (C \times C')$ is naturally isomorphic to $(X \otimes_\Gamma C) \times (X \otimes_\Gamma C')$.*

PROOF. Let $P: C\times C'\to C$ and $Q: C\times C'\to C'$ be the projections. Then we have a functor

$$P_* \times Q_*: X \otimes_\Gamma (C \times C') \longrightarrow (X \otimes_\Gamma C) \times (X \otimes_\Gamma C').$$

We will construct the inverse R of $P_* \times Q_*$ as follows. Let $u=[x_1,\dots,x_m,a]$ and $u'=[x'_1,\dots,x'_n,a']$ be objects of $X \otimes_\Gamma C$ and $X \otimes_\Gamma C'$ respectively, where $a\in C(\mathbf{m})$ and $a'\in C'(\mathbf{n})$. Then we define

$$R(u,u') = [x_1,\dots,x_m,x'_1,\dots,x'_n,(\theta^*a,\psi^*a')]$$

where θ: $\mathbf{m}+\mathbf{n}\to\mathbf{m}$ and ψ: $\mathbf{m}+\mathbf{n}\to\mathbf{n}$ are maps in Γ defined by

$$\theta(i) = \begin{cases} \{i\} & \text{for } i \leqq m \\ \emptyset & \text{for } i > m, \end{cases} \qquad \psi(i) = \begin{cases} \emptyset & \text{for } i \leqq m \\ \{i-m\} & \text{for } i > m \end{cases}$$

so that $(\theta^*a,\psi^*a')\in C(\mathbf{m}+\mathbf{n})\times C'(\mathbf{m}+\mathbf{n})$. For morphisms f and f' of $X\otimes_\Gamma C$ and $X\otimes_\Gamma C'$, $R(f,f')$ is defined by the same formula. Then we have $(P_*\times Q_*)R=\mathrm{Id}$, because

$$\begin{aligned} P_*R(u,u') &= [x_1,\dots,x_m,x'_1,\dots,x'_n,\theta^*a] \\ &= [\theta_*(x_1,\dots,x_m,x'_1,\dots,x'_n),a] \\ &= [x_1,\dots,x_m,a] = u, \\ Q_*R(u,u') &= [\psi_*(x_1,\dots,x_m,x'_1,\dots,x'_n),a'] \\ &= [x'_1,\dots,x'_n,a'] = u'. \end{aligned}$$

Similarly we can show that $R(P_*\times Q_*)=\mathrm{Id}$ and this completes the proof.

Now let C be a Γ-category and let $|C|'$ denote the Γ-space $\mathbf{n}\mapsto|C(\mathbf{n})|$ where $|C(\mathbf{n})|$ is the geometric realization of the nerve of $C(\mathbf{n})$ (i.e., the classifying space of $C(\mathbf{n})$). Obviously $|C|'$ becomes a special Γ-space.

PROPOSITION 1.4. *For any based space X, $|X\otimes_\Gamma C|$ is naturally homeomorphic to $X\otimes_\Gamma|C|'$. In particular $|S^1\otimes_\Gamma C|$ coincides with the geometric realization of the simplicial space $[n]\mapsto|C(\mathbf{n})|$.*

PROOF. We will use the notations of MacLane [6], e.g., $X\otimes_\Gamma C$ is denoted by $\int^n X^n\times C(\mathbf{n})$. Consider the functor S: $\Delta\times\Delta^{\mathrm{op}}\times\Gamma\times\Gamma^{\mathrm{op}}\to\mathscr{U}$ given by

$$S([p],[q],\mathbf{m},\mathbf{n}) = \Delta_p \times X^m \times N_qC(\mathbf{n})$$

where Δ_p is the standard p-simplex and $N_qC(\mathbf{n}) = MC(\mathbf{n})\times_{OC(\mathbf{n})}\cdots\times_{OC(\mathbf{n})}MC(\mathbf{n})$

(q-times). Since the geometric realization of the simplicial space $[q]\mapsto N_qC(\mathbf{n})$ is the classifying space $|C(\mathbf{n})|$, $X\otimes_\Gamma|C|'$ is identified with the iterated coend

$$\int^m\left[\int^p S([p],[p],\mathbf{m},\mathbf{m})\right].$$

On the other hand, $|X\otimes_\Gamma C|$ coincides with the iterated coend

$$\int^p\left[\int^m S([p],[p],\mathbf{m},\mathbf{m})\right],$$

because $\int^m X^m\times N_pC(\mathbf{m})$ is the p-th nerve of the category $X\otimes_\Gamma C$. Since these two iterated coends are naturally isomorphic (cf. [6]) we have the proposition.

§ 2. Symmetric Monoidal Categories

Let C be a topological category. C is called a symmetric monoidal category if there exists a functor

$$\square: C\times C\longrightarrow C$$

together with an object $e\in C$ and coherent natural isomorphisms

$$\begin{aligned}
a_{x,y,z}&: x\,\square\,(y\,\square\,z)\longrightarrow(x\,\square\,y)\,\square\,z,\\
c_{x,y}&: x\,\square\,y\longrightarrow y\,\square\,x,\\
l_y&: e\,\square\,y\longrightarrow y,\\
r_x&: x\,\square\,e\longrightarrow x.
\end{aligned}$$

Generalizing the method of Segal [11], we construct a Γ-category $\hat{C}$ (which is also denoted by $C^\wedge$) as follows.

DEFINITION 2.1. (i) For each n the objects of $\hat{C}(\mathbf{n})$ are the systems

$$\langle a_S;\alpha_{S,T}\rangle$$

where a_S is an object of C for each $S\subset\mathbf{n}$, and $\alpha_{S,T}$: $a_{S\amalg T}\to a_S\square a_T$ is an isomorphism for each pair of disjoint subsets S, $T\subset\mathbf{n}$, such that (a) $a_\emptyset=e$ and (b) the diagrams

$$\begin{tikzcd}
e\square a_S \arrow[r, dash, "\alpha_{\emptyset,S}"] \arrow[dr, "l"'] & a_S \arrow[r, "\alpha_{S,\emptyset}"] \arrow[d, equal] & a_S\square e, \arrow[dl, "r"] \\
 & a_S &
\end{tikzcd}$$

$$\begin{array}{ccc} a_{S\amalg T\amalg U} & \xrightarrow{\alpha_{S,T\amalg U}} a_S\Box a_{T\amalg U} \xrightarrow{\mathrm{id}\times\alpha_{T,U}} & a_S\Box(a_T\Box a_U) \\ \downarrow{\scriptstyle \alpha_{S\amalg T,U}} & & \downarrow{\scriptstyle a} \\ a_{S\amalg T}\Box a_U & \xrightarrow{\alpha_{S,T}\times\mathrm{id}} & (a_S\Box a_T)\Box a_U, \end{array}$$

$$\begin{array}{ccc} a_{S\amalg T} & \xrightarrow{\alpha_{S,T}} & a_S\Box a_T \\ \| & & \downarrow{\scriptstyle c} \\ a_{T\amalg S} & \xrightarrow{\alpha_{T,S}} & a_T\Box a_S \end{array}$$

commute for any disjoint subsets S, T and U.

A morphism from $\langle a_S;\alpha_{S,T}\rangle$ to $\langle b_S;\beta_{S,T}\rangle$ in $\hat{C}(\mathbf{n})$ is a system $\langle f_S\rangle$ of morphisms $f_S\colon a_S\to b_S$ ($f_\emptyset=\mathrm{id}_e$) such that the diagram

$$\begin{array}{ccc} a_{S\amalg T} & \xrightarrow{f_{S\amalg T}} & b_{S\amalg T} \\ \downarrow{\scriptstyle \alpha_{S,T}} & & \downarrow{\scriptstyle \beta_{S,T}} \\ a_S\Box a_T & \xrightarrow{f_S\Box f_T} & b_S\Box b_T \end{array}$$

commutes for any S, $T\subset\mathbf{n}$ with $S\cap T=\emptyset$.

(ii) For any $\theta\colon \mathbf{m}\to\mathbf{n}$, $\theta^*\colon \hat{C}(\mathbf{n})\to\hat{C}(\mathbf{m})$ is given by

$$\theta^*\langle a_S;\alpha_{S,T}\rangle = \langle a_{\theta U};\alpha_{\theta U,\theta V}\rangle \qquad (U,\ V\subset\mathbf{m},\quad U\cap V=\emptyset),$$

$$\theta^*\langle f_S\rangle = \langle f_{\theta U}\rangle \qquad (U\subset\mathbf{m}).$$

We consider $O\hat{C}(\mathbf{n})$ as a subspace of

$$\textstyle\prod_{S\subset\mathbf{n}}(OC)_S\times\prod_{S,T\subset\mathbf{n},S\cap T=\emptyset}(MC)_{S,T}$$

where $(OC)_S$ and $(MC)_{S,T}$ are copies of OC and MC respectively. $M\hat{C}(\mathbf{n})$ is regarded as a subspace of

$$O\hat{C}(\mathbf{n})\times\textstyle\prod_{S\subset\mathbf{n}}(MC)_S\times O\hat{C}(\mathbf{n})$$

by the assignment $f=\langle f_S\rangle\mapsto(\mathrm{source}(f),\ (f_S),\ \mathrm{target}(f))$.

Note that $\hat{C}(\mathbf{0})$ is the category with a single morphism, and $\hat{C}(\mathbf{1})$ is isomorphic to C. In order to see that $\hat{C}$ is a (special) Γ-category, it suffices to prove

LEMMA 2.2. *The functor $P_n=\prod_{k=1}^n i_k^*\colon \hat{C}(\mathbf{n})\to\hat{C}(\mathbf{1})^n=C^n$ induced by the maps $i_k\colon \mathbf{1}\to\mathbf{n}$, $i_k(1)=\{k\}\subset\mathbf{n}$, is an equivalence of categories.*

PROOF. Define a functor $I_n\colon C^n\to\hat{C}(\mathbf{n})$ as follows. For any object $(a_1,\dots,a_n)$ of C^n, we take

$$I_n(a_1,\dots,a_n)=\langle a_S;\alpha_{S,T}\rangle$$

where $a_S = a_{i_1}\Box(\cdots\Box(a_{i_{k-1}}\Box a_{i_k})\cdots)$ for each $S=\{i_1,\dots,i_{k-1},i_k\}\subset\mathbf{n}$ with $i_1<\cdots<i_{k-1}<i_k$ and $\alpha_{S,T}$ is the uniquely determined isomorphism $a_{S\amalg T}\cong a_S\Box a_T$. For a morphism $(f_1,\dots,f_n)$ of C^n, $I_n(f_1,\dots,f_n)=\langle f_S\rangle$ is given by $f_S=f_{i_1}\Box(\cdots\Box(f_{i_{k-1}}\Box f_{i_k})\cdots)$ for $S=\{i_1,\dots,i_{k-1},i_k\}$. Obviously we have $P_nI_n=\mathrm{Id}$. On the other hand, there is a natural isomorphism $\lambda_n\colon \mathrm{Id}\xrightarrow{\cdot} I_nP_n$ given by the composition

$$\begin{aligned} a_S &\longrightarrow a_{\{i_1\}}\Box a_{S-\{i_1\}} \longrightarrow \cdots \\ &\longrightarrow a_{\{i_1\}}\Box(\cdots\Box(a_{\{i_{k-2}\}}\Box a_{\{i_{k-1},i_k\}})\cdots) \\ &\longrightarrow a_{\{i_1\}}\Box(\cdots\Box(a_{\{i_{k-2}\}}\Box(a_{\{i_{k-1}\}}\Box a_{\{i_k\}}))\cdots) \end{aligned}$$

of $\mathrm{id}\,\Box(\cdots\Box(\mathrm{id}\,\Box\alpha_{\{i_j\},\{i_{j+1},\dots i_k\}})\cdots)$ for each object $\langle a_S;\alpha_{S,T}\rangle$ of $\hat{C}(\mathbf{n})$. This proves that P_n is an equivalence.

REMARK 2.3. In fact $\hat{C}$ is a functor from Γ to the category of symmetric monoidal categories. For objects $a=\langle a_S;\alpha_{S,T}\rangle$ and $b=\langle b_S;\beta_{S,T}\rangle$ of $\hat{C}(\mathbf{n})$ we take $a\hat{\Box}b=\langle a_S\Box b_S;(\mathrm{id}\Box c\Box\mathrm{id})(\alpha_{S,T}\Box\beta_{S,T})\rangle$.

DEFINITION 2.4. For any based space X, $X\otimes C$ denotes the category $X\otimes_\Gamma\hat{C}$.

By definition, there is a commutative diagram

$$\begin{tikzcd} & X\times\hat{C}(\mathbf{1}) \arrow[d] & \\ X\times\hat{C}(\mathbf{0}) \arrow[ur, "\mathrm{id}\times\rho^*"] \arrow[dr, "\rho_*\times\mathrm{id}"'] & X\otimes C & *\times\hat{C}(\mathbf{1}) \arrow[ul, "\varepsilon_*\times\mathrm{id}"'] \arrow[dl, "\mathrm{id}\times\varepsilon^*"] \\ & *\times\hat{C}(\mathbf{0}) \arrow[u] & \end{tikzcd}$$

where X is regarded as a category having only identity morphisms, and ε (resp. ρ) is a unique morphism from $\mathbf{0}$ (resp. $\mathbf{1}$) to $\mathbf{1}$ (resp. $\mathbf{0}$) in Γ. Since $|\hat{C}(\mathbf{0})|=*$ the functor $X\times\hat{C}(\mathbf{1})\to X\otimes C$ induces a continuous map

$$X\wedge|C|\longrightarrow|X\otimes C|.$$

Now let $C=\langle C,\Box_C,e_C\rangle$ and $D=\langle D,\Box_D,e_D\rangle$ be symmetric monoidal categories. A monoidal functor from C to D is a functor $F\colon C\to D$ together with an isomorphism $Fe_C\to e_D$ and coherent natural isomorphisms $\psi\colon F(a\,\Box_C\,b)\cong Fa\,\Box_D\,Fb$. It is checked that F induces a natural transformation

$$\hat{F}\colon\hat{C}\xrightarrow{\cdot}\hat{D}$$

such that $\hat{F}\langle a_S;\alpha_{S,T}\rangle=\langle a'_S;\alpha'_{S,T}\rangle$ where $a'_{S,}=Fa_S$ (for $S\neq\emptyset$) and $\alpha'_{ST}=\psi F\alpha_{S,T}$: $Fa_{S\amalg T}\to F(a_S\,\Box_C\,a_T)\to Fa_S\,\Box_D\,Fa_T$. Consequently we get a functor

$$X \otimes F = X \otimes_\Gamma \hat{F}\colon X \otimes C \longrightarrow X \otimes D$$

for any based space X. Moreover if $\lambda\colon F \xrightarrow{\cdot} F'$ is a natural transformation of monoidal functors (i.e., the diagram

$$\begin{tikzcd} F(a\square_C b) \arrow[r, "\lambda"] \arrow[d, "\phi"] & F'(a\square_C b) \arrow[d, "\phi'"] \\ Fa\square_D Fb \arrow[r, "\lambda\square_D\lambda"] & F'a\square_D F'b \end{tikzcd}$$

commutes for any a, $b \in OC$), we can define a natural transformation $\bar{\lambda}\colon X\otimes F \xrightarrow{\cdot} X\otimes F'$ in the following way. For any object $u=[x_1,\ldots,x_n,\langle a_S;\alpha_{S,T}\rangle]$ of $X \otimes C$ the objects $X\otimes F(u)$ and $X\otimes F'(u)$ are represented by $(x_1,\ldots,x_n,\hat{F}\langle a_S;\alpha_{S,T}\rangle)$ and $(x_1,\ldots,x_n,\hat{F}'\langle a_S;\alpha_{S,T}\rangle)$ respectively. Since the diagram

$$\begin{tikzcd} Fa_{S\amalg T} \arrow[r, "F\alpha_{S,T}"] \arrow[d, "\lambda"] & F(a_S\square_C a_T) \arrow[r, "\phi"] \arrow[d, "\lambda"] & Fa_S\square_D Fa_T \arrow[d, "\lambda\square_D\lambda"] \\ F'a_{S\amalg T} \arrow[r, "F'\alpha_{S,T}"] & F'(a_S\square_C a_T) \arrow[r, "\phi'"] & F'a_S\square_D F'a_T \end{tikzcd}$$

commutes for any disjoint subsets S, $T\subset \mathbf{n}$, there is a natural isomorphism $\langle\lambda_S\rangle$: $\hat{F}\langle a_S;\alpha_{S,T}\rangle \cong \hat{F}'\langle a_S;\alpha_{S,T}\rangle$ with $\lambda_S=\lambda_{a_S}\colon Fa_S\to F'a_S$, which induces the desired isomorphism

$$\bar{\lambda}_u\colon X \otimes F(u) \cong X \otimes F'(u).$$

In particular, if $C=\langle C,\square,e,a,c,r,l\rangle$ is a symmetric monoidal category, the functor $\square\colon C\times C\to C$ becomes a monoidal functor. Since the Γ-category $(C\times C)^\wedge$ is canonically isomorphic to $\hat{C}\times\hat{C}$, we get a functor

$$\begin{aligned} \bar{\square}\colon (X \otimes C) \times (X \otimes C) &\cong X \otimes_\Gamma (\hat{C} \times \hat{C}) \\ &= X \otimes (C \times C) \\ &\xrightarrow{X\otimes\square} X \otimes C. \end{aligned}$$

Furthermore the natural isomorphisms a, c, r, l induce coherent natural isomorphisms

$$\bar{a}: u \,\bar{\square}\, (v \,\bar{\square}\, w) \cong (u \,\bar{\square}\, v) \,\bar{\square}\, w,$$

$$\bar{c}: u \,\bar{\square}\, v \cong v \,\bar{\square}\, u,$$

$$\bar{r}: u \,\bar{\square}\, \bar{e} \cong u, \qquad \bar{l}\colon \bar{e} \,\bar{\square}\, v \cong v$$

in which $\bar{e}\in X\otimes C$ is the class of $\langle e\rangle \in \hat{C}(\mathbf{0})$. Thus we have

Theorem 2.5. *$X\otimes C=\langle X\otimes C, \square, \bar{e}, \bar{a}, \bar{c}, \bar{r}, \bar{l}\rangle$ is a symmetric monoidal category.*

Note. If $u=[x_1,\dots,x_m,\langle a_S;\alpha_{S,T}\rangle]$ and $v=[y_1,\dots,y_n,\langle b_U;\beta_{U,V}\rangle]$ are objects of $X\otimes C$, then $u\square v$ is represented by the object

$$(x_1,\dots,x_m,y_1,\dots,y_n,\langle a_{\theta Q}\ \square\ b_{\varphi Q};\gamma_{Q,R}\rangle)$$

of $X^{m+n}\times\hat{C}(\mathbf{m}+\mathbf{n})$ where $\theta\colon \mathbf{m}+\mathbf{n}\to\mathbf{m}$ and $\varphi\colon \mathbf{m}+\mathbf{n}\to\mathbf{n}$ are the maps defined in the proof of Proposition 1.3 and $\gamma_{Q,R}$ is the composition: $a_{\theta(Q\amalg R)}\square b_{\varphi(Q\amalg R)}\to(a_{\theta Q}\square a_{\theta R})\square(b_{\varphi Q}\square b_{\varphi R})\to(a_{\theta Q}\square b_{\varphi Q})\square(a_{\theta R}\square b_{\varphi R})$. Note that if $Q=\{i_1,\dots,i_p,j_1,\dots,j_q\}\subset\mathbf{m}+\mathbf{n}$, $i_1<\cdots<i_p\leqq m<j_1<\cdots<j_q$, then $\theta Q=\{i_1,\dots,i_p\}\subset\mathbf{m}$ and $\varphi Q=\{j_1-m,\dots,j_q-m\}\subset\mathbf{n}$.

Since $X\otimes C$ is again a symmetric monoidal category, we can construct a new symmetric monoidal category $Y\otimes(X\otimes C)$ for a based space Y.

Lemma 2.6. *$Y\otimes(X\otimes C)$ is naturally isomorphic to $X\otimes(Y\otimes C)$.*

Proof. Let $\tilde{C}\colon \Gamma\times\Gamma\to\mathscr{C}\mathscr{A}\mathscr{T}$ be a contravariant functor such that

$$\tilde{C}(\mathbf{m},\mathbf{n})=(\hat{C}(\mathbf{m}))\hat{\ }(\mathbf{n})\qquad\text{(cf. Remark 2.3)}.$$

Then $Y\otimes(X\otimes C)$ is canonically isomorphic to the iterated coend

$$\int^n Y^n\times\left[\int^m X\times\tilde{C}(\mathbf{m},\mathbf{n})\right],$$

for there is an isomorphism of Γ-categories

$$\begin{array}{ccc}\displaystyle\int^m X^m\times(\hat{C}(\mathbf{m}))\hat{\ }(\mathbf{n}) & \longrightarrow & (X\otimes C)\hat{\ }(\mathbf{n})\,. \\ \Downarrow\!\!\in & & \Downarrow\!\!\in \\ [x_1,\dots,x_m,\langle a_S;\alpha_{S,T}\rangle] & \longmapsto & \langle[x_1,\dots,x_m,a_S];[x_1,\dots,x_m,\alpha_{S,T}]\rangle\end{array}$$

$$(a_S\in O\hat{C}(\mathbf{m}),\ \alpha_{S,T}\in M\hat{C}(\mathbf{m}))\,.$$

On the other hand, there is a natural isomorphism

$$\begin{array}{ccc}\tau\colon \tilde{C}(\mathbf{m},\mathbf{n}) & \cong & \tilde{C}(\mathbf{n},\mathbf{m})\,. \\ \Downarrow\!\!\in & & \Downarrow\!\!\in \\ \langle a_S;\alpha_{S,T}\rangle & \longmapsto & \langle b_U;\beta_{U,V}\rangle\end{array}$$

given as follows: For $a_S=\langle c_U^S;\gamma_{U,V}^S\rangle\in O\hat{C}(\mathbf{m})$ and $\alpha_{S,T}=\langle\varphi_U^{S,T}\colon c_U^{S\amalg T}\to c_U^S\square c_U^T\rangle\in M\hat{C}(\mathbf{m})$, $b_U=\langle c_U^S;\varphi_U^{S,T}\rangle\in O\hat{C}(\mathbf{n})$ and $\beta_{U,V}=\langle\gamma_{U,V}^S\colon c_{U\amalg V}^S\to c_U^S\square c_V^S\rangle\in M\hat{C}(\mathbf{n})$. Thus we have

$$\begin{aligned} Y \otimes (X \otimes C) &\cong \int^n Y^n \times \left[\int^m X^m \times \tilde{C}(\mathbf{m}, \mathbf{n})\right] \\ &\cong \int^m X^m \times \left[\int^n Y^n \times \tilde{C}(\mathbf{m}, \mathbf{n})\right] \\ &\xrightarrow[\tau_*]{\cong} \int^m X^m \times \left[\int^n Y^n \times \tilde{C}(\mathbf{n}, \mathbf{m})\right] \\ &\cong X \otimes (Y \otimes C). \end{aligned}$$

DEFINITION 2.7. For any symmetric monoidal category C, we put

$$BC = S^1 \otimes C$$

and inductively

$$B^{n+1}C = S^1 \otimes B^n C \qquad \text{for} \quad n \geqq 1.$$

We have already shown that the functor $S^1 \times B^n C \to S^1 \otimes B^n C = B^{n+1}C$ induces a continuous map

$$\varepsilon_n \colon S^1 \wedge |B^n C| \longrightarrow |B^{n+1}C|.$$

Consequently we have a spectrum

$$E(C) = \{|B^n C|, \varepsilon_n\}.$$

Since $|BC| = S^1 \otimes_\Gamma |\hat{C}|'$ (the geometric realization of the good simplical space $|\hat{C}|'$) is equivalent to the classifying space $B|\hat{C}|'(\mathbf{1})$ of Segal [11], the following proposition is a direct consequence of the main theorem of [11]. (See also [9].)

PROPOSITION 2.8. *The adjoint* $\bar{\varepsilon} \colon |C| \to \Omega|BC|$ *of* ε_0 *is a group completion.*

Thus $\bar{\varepsilon}$ is a homotopy equivalence if $\pi_0(|C|, *)$ is a group. In particular $|B^n C|$ is homotopically equivalent to the loop space of $|B^{n+1}C|$ for $n \geqq 1$.

§ 3. Symmetric Bimonoidal Categories and Ring Spectra

Let $A = \langle A, \square_A, e_A \rangle$, $B = \langle B, \square_B, e_B \rangle$ and $C = \langle C, \square_C, e_C \rangle$ be symmetric monoidal categories.

DEFINITION 3.1. A functor $P \colon A \times B \to C$ is called a *pairing* if there exist natural isomorphisms

$$\delta \ \colon P(a \,\square_A\, a', b) \longrightarrow P(a, b) \,\square_C\, P(a', b),$$

$$\delta' \colon P(a, b \,\square_B\, b') \longrightarrow P(a, b) \,\square_C\, P(a, b')$$

such that (i) for any object b of B the functor $P(\cdot, b)\colon A \to C$, given by $a \to P(a, b)$ for $a \in OA$ and by $f \mapsto P(f, \mathrm{id}_b)$ for $f \in MA$, is a monoidal functor, and similarly the functor $P(a, \cdot)\colon B \to C$ becomes a monoidal functor for any $a \in OA$, and (ii) the diagram

$$\begin{tikzcd}
P(a\square_A a', b\square_B b') \arrow[r, "\delta"] \arrow[d, "\delta'"] & P(a, b\square_B b')\square_C P(a', b\square_B b') \arrow[d, "\delta'\square_C\delta'"] \\
P(a\square_A a', b)\square_C P(a\square_A a', b') \arrow[d, "\delta\square_C\delta"] & (P(a, b)\square_C P(a, b'))\square_C(P(a', b)\square_C P(a', b')) \\
(P(a, b)\square_C P(a, b'))\square_C(P(a, b')\square_C P(a', b')) \arrow[ur, equal] &
\end{tikzcd}$$

commutes for any $a, a' \in OA$ and $b, b' \in OB$.

EXAMPLE 3.2. Let R be a commutative ring with unit and let $\mathscr{P}(R)$ denote the category of finitely generated projective R-modules. Then $\mathscr{P}(R)$ has a structure of symmetric monoidal category which comes from the direct sums, and also has a pairing

$$\otimes_R\colon \mathscr{P}(R) \times \mathscr{P}(R) \longrightarrow \mathscr{P}(R)$$

which assigns to a pair $(M, N) \in \mathscr{P}(R) \times \mathscr{P}(R)$ the tensor product $M \otimes_R N$.

In fact $\mathscr{P}(R)$ is an example of symmetric bimonoidal category. A topological category C is called a symmetric bimonoidal category if it has two distinct structures of symmetric monoidal category $\langle C, \square_+, 0\rangle$ and $\langle C, \square_\times, 1\rangle$ and if the functor $\square_\times\colon C \times C \to C$ is a pairing with respect to $\langle C, \square_+, 0\rangle$.

Now let A, B and C be symmetric monoidal categories and let $P\colon A \times B \to C$ be a pairing. Then we have

PROPOSITION 3.3. *For any based spaces X and Y, there are pairings*

$$P'_X\colon (X \otimes A) \times B \longrightarrow X \otimes C,$$

$$P''_Y\colon A \times (Y \otimes B) \longrightarrow Y \otimes C$$

such that the diagram

$$\begin{tikzcd}
 & X\otimes(Y\otimes C) \arrow[d, "\cong"] \\
(X\otimes A)\times(Y\otimes B) \arrow[ur, "(P''_Y)'_X"] \arrow[r, "(P'_X)''_Y"'] & Y\otimes(X\otimes C)
\end{tikzcd}$$

commutes.

PROOF. For each m, define a pairing

$$\hat{P}'_m : \hat{A}(\mathbf{m}) \times B \longrightarrow \hat{C}(\mathbf{m})$$

by $\hat{P}'_m(\langle a_S; \alpha_{S,T}\rangle, b) = \langle P(a_S, b); \delta P(\alpha_{S,T}, \mathrm{id}_b)\rangle$. Then for any map $\theta: \mathbf{k} \to \mathbf{m}$ in Γ we have the commutative diagram

$$\begin{array}{ccc} \hat{A}(\mathbf{m}) \times B & \xrightarrow{\hat{P}'_m} & \hat{C}(\mathbf{m}) \\ \downarrow{\scriptstyle \theta^* \times \mathrm{Id}} & & \downarrow{\scriptstyle \theta^*} \\ \hat{A}(\mathbf{k}) \times B & \xrightarrow{\hat{P}'_k} & \hat{C}(\mathbf{k}). \end{array}$$

Hence we get an induced pairing

$$P'_X = X \otimes_\Gamma \hat{P}' : (X \otimes A) \times B \longrightarrow X \otimes C.$$

Similarly we have a natural transformation (of Γ-categories)

$$\begin{array}{ccc} \hat{P}''_n : A \times B(\mathbf{n}) & \longrightarrow & \hat{C}(\mathbf{n}) \\ \Downarrow & & \Downarrow \\ (a, \langle b_U, \beta_{U,V}\rangle) & \longmapsto & \langle P(a, b_U), \delta' P(\mathrm{id}_a, \beta_{U,V})\rangle \end{array}$$

which induces a pairing

$$P''_Y = Y \otimes_\Gamma \hat{P}'' : A \times (Y \otimes B) \longrightarrow Y \otimes C.$$

Now it is easily proved that the diagram

$$\begin{array}{ccc} & \overset{(\hat{P}'_m)''_n}{\nearrow} & (\hat{C}(\mathbf{m}))\hat{\ }(\mathbf{n}) = \tilde{C}(\mathbf{m}, \mathbf{n}) \\ \hat{A}(\mathbf{m}) \times \hat{B}(\mathbf{n}) & & \downarrow{\scriptstyle \tau} \\ & \underset{(\hat{P}''_n)'_m}{\searrow} & (\hat{C}(\mathbf{n}))\hat{\ }(\mathbf{m}) = \tilde{C}(\mathbf{n}, \mathbf{m}) \end{array}$$

commutes for any m and n. Thus we have the commutative diagram

$$\begin{array}{ccc} & & X \otimes (Y \otimes C) \\ & & \| \\ & \overset{(P''_Y)'_X}{\nearrow} & \int^m X^m \times \left[\int^n Y^n \times \tilde{C}(\mathbf{n}, \mathbf{m})\right] \\ (X \otimes A) \times (Y \otimes B) = \left[\int^m X^m \times \hat{A}(\mathbf{m})\right] \times \left[\int^n Y^n \times \hat{B}(\mathbf{n})\right] & & \uparrow{\scriptstyle \tau} \\ & \underset{(P'_X)''_Y}{\searrow} & \int^n Y^n \times \left[\int^m X^m \times \tilde{C}(\mathbf{m}, \mathbf{n})\right] \\ & & \| \\ & & Y \otimes (X \otimes C) \end{array}$$

Now let $C=\langle C, \square_+, 0, \square_\times, 1\rangle$ be a symmetric bimonoidal category and let us take the categories $B^n C$ using the structure $\langle C, \square_+, 0\rangle$.

THEOREM 3.4. *For any pair of integers $m, n \geqq 0$ there is a pairing*

$$P_{m,n}\colon B^m C \times B^n C \longrightarrow B^{m+n} C$$

such that

(i) $P_{0,0}=\square_\times\colon C\times C\to C$,

(ii) *the diagrams*

$$\begin{tikzcd} B^l C\times B^m C\times B^n C \arrow[r, "\mathrm{Id}\times P_{m,n}"] \arrow[d, "P_{l,m}\times \mathrm{Id}"] & B^l C\times B^{m+n} C \arrow[d, "P_{l,m+n}"] \\ B^{l+m} C\times B^n C \arrow[r, "P_{l+m,n}"] & B^{l+m+n} C \end{tikzcd}$$

and

$$\begin{tikzcd} B^m C\times B^n C \arrow[rr, "T"] \arrow[dr, "P_{m,n}"'] & & B^n C\times B^m C \arrow[dl, "P_{n,m}"] \\ & B^{m+n} C & \end{tikzcd}$$

commute up to coherent natural isomorphisms $\lambda_{l,m,n}\colon P_{l,m+n}(a, P_{m,n}(b,c))\cong P_{l+m,n}(P_{l,m}(a,b),c)$ and $\tau_{m,n}\colon P_{m,n}(b,c)\cong P_{n,m}(c,b)$.

PROOF. Assume that a pairing $P_{m,n}\colon B^m C\times B^n C\to B^{m+n} C$ has been defined. Then from Proposition 3.3 we get pairings

$$P_{m+1,n} = (P_{m,n})'_{S^1}\colon (S^1\otimes B^m C)\times B^n C \longrightarrow S^1\otimes B^{m+n} C,$$

$$P_{m,n+1} = (P_{m,n})''_{S^1}\colon B^m C\times (S^1\otimes B^n C) \longrightarrow S^1\otimes B^{m+n} C$$

such that $(P_{m+1,n})''_{S^1}\colon B^{m+1}C\times B^{n+1}C\to B^{m+n+2}C$ coincides with $(P_{m,n+1})'_{S^1}$ $(=P_{m+1,n+1})$. Thus starting from $P_{0,0}=\square_\times\colon C\times C\to C$, we can inductively construct the pairings $P_{m,n}\colon B^m C\times B^n C\to B^{m+n} C$.

The natural isomorphisms $\lambda_{l,m,n}$ and $\tau_{m,n}$ are also defined inductively. The inductive step is as follows. Assume that there is a diagram of pairings

$$\begin{tikzcd} C_1\times C_2\times C_3 \arrow[r, "\mathrm{Id}\times E"] \arrow[d, "G\times \mathrm{Id}"] & C_1\times C_{2,3} \arrow[d, "F"] \\ C_{1,2}\times C_3 \arrow[r, "H"] & C_{1,2,3} \end{tikzcd}$$

together with coherent natural isomorphisms

$$\lambda\colon F(a, E(b,c)) \cong H(G(a,b),c).$$

Then we have a diagram

$$\begin{tikzcd}
(S^1\otimes C_1)\times C_2\times C_3 \arrow[r, "\mathrm{Id}\times E"] \arrow[d, "G'\times\mathrm{Id}"] & (S^1\otimes C_1)\times C_{2,3} \arrow[d, "F'"] \\
(S^1\otimes C_{1,2})\times C_3 \arrow[r, "H'"] & S^1\otimes C_{1,2,3}
\end{tikzcd}$$

Now define a natural isomorphism

$$\lambda'\colon F'(u,E(b,c)) \cong H'(G'(u,b),c)$$

as follows. For any objects $u=[x_1,\dots,x_k,\langle a_S;\alpha_{S,T}\rangle]\in S^1\otimes C_1$, $b\in C_2$ and $c\in C_3$, we have

$$F'(u,E(b,c)) = [x_1,\dots,x_k,\langle a'_S;\alpha'_{S,T}\rangle],$$

$$H'(G'(u,b),c) = [x_1,\dots,x_k,\langle a''_S;\alpha''_{S,T}\rangle]$$

in which $a'_S=F(a_S,E(b,c))$ and $a''_S=H(G(a_S,b),c)$ for any $S\subset\mathbf{k}$. Then the diagram

$$\begin{tikzcd}
F(a_{S\amalg T},E(b,c)) \arrow[r, "\lambda"] \arrow[d] & H(G(a_{S\amalg T},b),c) \arrow[d] \\
F(a_S\Box a_T,E(b,c)) \arrow[dd] & H(G(a_S\Box a_T,b),c) \arrow[d] \\
 & H(G(a_S,b)\Box G(a_T,b),c) \arrow[d] \\
F(a_S,E(b,c))\Box F(a_T,E(b,c)) \arrow[r, "\lambda\Box\lambda"] \arrow[uuu, bend left=60, "\alpha'_{S,T}"] & H(G(a_S,b),c)\Box H(G(a_T,b),c) \arrow[uuu, bend right=60, "\alpha''_{S,T}"']
\end{tikzcd}$$

Commutes for any $S,T\subset\mathbf{k}$ with $S\cap T=\emptyset$. Hence we get an isomorphism $\langle a'_S;\alpha'_{S,T}\rangle\cong\langle a''_S;\alpha''_{S,T}\rangle$ which induces $\lambda'\colon F'(u,E(b,c))\cong H'(G'(u,b),c)$. Similarly we can define natural isomorphisms

$$\lambda''\colon F''(a,E'(v,c))\cong H'(G''(a,v),c),$$

$$\lambda'''\colon F''(a,E''(a,E''(b,w))\cong H''(G(a,b),w)$$

where $E'\colon(S^1\otimes C_2)\times C_3\to S^1\otimes C_{2,3}$, $E''\colon C_2\times(S^1\otimes C_3)\to S^1\otimes C_{2,3}$, $F''\colon C_1\times(S^1\otimes C_{2,3})\to S^1\otimes C_{1,2,3}$ and $H''\colon C_{1,2}\times(S^1\otimes C_3)\to S^1\otimes C_{1,2,3}$. Moreover we can show that $(\lambda')''=(\lambda'')'$, $(\lambda'')'''=(\lambda''')''$, and $(\lambda''')'=(\lambda')'''$ (as natural transformations). Thus starting from the coherent natural isomorphisms

$$a\Box_\times(b\Box_\times c)\cong(a\Box_\times b)\Box_\times c,$$

we can define natural isomorphisms

$$\lambda_{l,m,n}\colon P_{l,m+n}(u,P_{m,n}(v,w))\cong P_{l+m,n}(P_{l,m}(u,v),w).$$

Similarly we can define coherent natural isomorphisms

$$\tau_{m,n}\colon P_{m,n}(v, w) \cong P_{n,m}(w, v).$$

This proves (ii).

Now let $E(C)=\{|B^nC|, \varepsilon_n\}$ be the spectrum associated with $C=\langle C, \Box_+, 0\rangle$. Then the pairing $P_{m,n}$ induces a map

$$\mu_{m,n}\colon |B^mC| \wedge |B^nC| \longrightarrow |B^{m+n}C|.$$

Furthermore there is a system of maps $\{\iota_n\colon S^n \to |B^nC|\}$ induced by a map ι_0: $S^0=\{0,1\}\to|C|$ where $\iota_0(1)$ is the class of $1 \in OC$. The system $\{\mu_{m,n}, \iota_n\}$ defines a multiplication on the spectrum $E(C)$, which is associative, unital and commutative up to homotopy (cf. [13], § 13). Thus we have

Corollary 3.5. *Let C be a symmetric bimonoidal category. Then the spectrum $E(C)$ becomes a commutative ring spectrum.*

§ 4. Exact Categories

Let $M=\langle M, E\rangle$ be an exact category where M is a small additive category and E a family of "exact sequences" in M (cf. [10]). One may regard M as a full subcategory of an abelian category A, closed under extensions and containing the zero object, and E a family of all sequences

$$0 \longrightarrow m' \overset{i}{\rightarrowtail} m \overset{j}{\twoheadrightarrow} m'' \longrightarrow 0$$

which are exact in A. (i is called an admissible monomorphism and j an admissible epimorphism.)

For an exact category M, Quillen [10] defined higher K-groups of M by

$$K_i(M) = \pi_{i+1}(|QM|, *) \qquad \text{for} \quad i \geqq 0$$

where QM is a category defined as follows.

Definition 4.1. QM has the same objects as M. A morphism in QM from m' to m is an isomorphism class of diagrams

$$m' \overset{p}{\twoheadleftarrow} n \overset{i}{\rightarrowtail} m$$

where i (resp. p) is an admissible monomorphism (resp. admissible epimorphism). (For details, see [10].)

Since M is an additive category, we have a Γ-category $\hat{M}$.

Lemma 4.2. *We can make each $\hat{M}(\mathbf{n})$ an exact category such that the functor $\theta^*\colon \hat{M}(\mathbf{m})\to\hat{M}(\mathbf{n})$, induced by a map $\theta\colon \mathbf{n}\to\mathbf{m}$ in Γ, is an exact functor.*

PROOF. Let E_n be the family of all sequences

$$\langle a_S;\alpha_{S,T}\rangle \xrightarrow{\langle i_S\rangle} \langle b_S;\beta_{S,T}\rangle \xrightarrow{\langle j_S\rangle} \langle c_S;\gamma_{S,T}\rangle$$

in $\hat{M}(\mathbf{n})$ such that the sequence $a_S \xrightarrow{i_S} b_S \xrightarrow{j_S} c_S$ belongs to E for each $S\subset\mathbf{n}$. Then it is easily proved that $\langle\hat{M}(\mathbf{n}), E_n\rangle$ satisfies the conditions to be an exact category. If $\theta\colon \mathbf{n}\to\mathbf{m}$ is a map in Γ, then θ^* takes the sequences in E_m into those in E_n and therefore θ^* becomes an exact functor.

Note that QM is also a symmetric monoidal category: Given two objects $m, m'\in OQM=OM$ we take their sum $m\oplus m'$. Consequently we have a Γ-category $(QM)^\wedge$. Since the isomorphisms in QM are in one-to-one correspondence with the isomorphisms in M, any object $\langle a_S;\alpha_{S,T}\rangle$ of $(QM)^\wedge(\mathbf{n})$ may be regarded as an object of $\hat{M}(\mathbf{n})$, and conversely. Thus $O(QM)^\wedge(\mathbf{n})$ coincides with $O\hat{M}(\mathbf{n})=OQ(\hat{M}(\mathbf{n}))$. Now obviously we have

LEMMA 4.3. *$Q\hat{M}\colon\mathbf{n}\mapsto Q(\hat{M}(\mathbf{n}))$ is naturally isomorphic to $(QM)^\wedge$ as a Γ-category.*

Now let X be a based space. Then we have

THEOREM 4.4. *The category $X\otimes M$ becomes an exact category. Furthermore there is a natural isomorphism*

$$X\otimes QM \cong Q(X\otimes M).$$

PROOF. Let $\bar{E}$ be the family of all sequences $u'\to u\to u''$ in $X\otimes M$, represented by

$$(\mathbf{x},\langle a_S;\alpha_{S,T}\rangle)\xrightarrow{(\mathrm{id}_\mathbf{x},\langle i_S\rangle)}(\mathbf{x},\langle b_S;\beta_{S,T}\rangle)\xrightarrow{(\mathrm{id}_\mathbf{x},\langle j_S\rangle)}(\mathbf{x},\langle c_S;\gamma_{S,T}\rangle)$$

in which $\langle a_S;\alpha_{S,T}\rangle \xrightarrow{\langle i_S\rangle} \langle b_S;\beta_{S,T}\rangle \xrightarrow{\langle j_S\rangle} \langle c_S;\gamma_{S,T}\rangle$ is an exact sequence in $\hat{M}(\mathbf{n})$. By Proposition 4.2 this definition does not depend on the choice of representatives. Therefore $\bar{E}$ is well-defined and enjoys all the requisite properties of exact sequences. This proves the first part of the theorem. Now we prove the second. From the above argument we have an isomorphism

$$Q(X\otimes M)=X\otimes_\Gamma Q\hat{M}$$

where $Q\hat{M}$ is the Γ-category $\mathbf{n}\mapsto Q(\hat{M}(\mathbf{n}))$. On the other hand, by Lemma 4.3, $Q\hat{M}$ is isomorphic to the Γ-category $\mathbf{n}\mapsto(QM)^\wedge(\mathbf{n})$, and hence we get an isomorphism

$$X\otimes Q\hat{M}\cong X\otimes_\Gamma (QM)^\wedge = X\otimes QM.$$

COROLLARY 4.5. *Let M be an exact category. Then, for any $n \geqq 1$ there is a natural isomorphism*

$$B^n(QM) \cong Q(B^n M).$$

§5. Equivariant Delooping

In this section we shall summarize (without proof) how our previous constructions and propositions extend to the equivariant cases. Let G be a finite group.

DEFINITION 5.1. A *symmetric monoidal G-category* is a G-category C with a structure of a symmetric monoidal category, i.e., $C = \langle C, \square, e, a, c, r, l \rangle$ having a G-action $G \times C \to C$ such that

(i) the functor $\square : C \times C \to C$ is equivariant, i.e., $g(a \square a') = ga \square ga'$ for any a, $a' \in OC$ and $g(f \square f') = gf \square gf'$ for any f, $f' \in MC$ and $g \in G$,

(ii) $e \in OC$ is a fixed point, and

(iii) the diagrams

$$\begin{array}{ccc} g(x\square(y\square z)) & \xrightarrow{ga_{x,y,z}} & g((x\square y)\square z) \\ \| & & \| \\ gx\square(gy\square gz) & \xrightarrow{a_{gx,gy,gz}} & (gx\square gy)\square gz, \end{array} \qquad \begin{array}{ccc} g(x\square y) & \xrightarrow{gc_{x,y}} & g(y\square x) \\ \| & & \| \\ gx\square gy & \xrightarrow{c_{gx,gy}} & gy\square gx, \end{array}$$

$$\begin{array}{ccccc} g(e\square x) & \xrightarrow{gl_x} & gx & \xrightarrow{gr_x} & g(x\square e) \\ \| & & \| & & \| \\ e\square gx & \xrightarrow{l_{gx}} & gx & \xrightarrow{r_{gx}} & gx\square e \end{array}$$

commute for any x, y, $z \in OC$ and $g \in G$.

EXAMPLE 5.2. Let $\mathscr{R}$ be a category whose objects are natural numbers and whose morphisms are invertible complex matrices. Thus

$$O\mathscr{R} = \boldsymbol{N}, \qquad M\mathscr{R} = \amalg_{n \geqq 0} GL(n, \boldsymbol{C})$$

with source(A) = target$(A) = n$ for $A \in GL(n, \boldsymbol{C})$. Define a structure of symmetric monoidal category as follows

$$\begin{cases} m \square n = m + n & \text{for} \quad m, n \in \boldsymbol{N} \\ A \square B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} & \text{for} \quad A, B \in \amalg_n GL(n, \boldsymbol{C}). \end{cases}$$

On the other hand, the complex conjugation defines an involution on $\mathscr{R}$, and hence $\mathscr{R}$ becomes a symmetric monoidal $\boldsymbol{Z}/2\boldsymbol{Z}$-category.

Let X be a (nice) based G-space, and let C be a symmetric monoidal G-category. Then we have

THEOREM 5.3. *The category* $X\otimes C=X\otimes_{\Gamma}\hat{C}$ *becomes a symmetric monoidal* G*-category.*

Let W be a G-module containing exactly one copy of each irreducible G-module as a direct summand and let S^W denote the one-point compactification of W. Put

$$BC = S^W \otimes C \quad \text{and} \quad B^{n+1}C = S^W \otimes B^nC \qquad \text{for} \quad n \geqq 1.$$

Then it follows that the functor

$$S^W \times B^nC \longrightarrow S^W \otimes B^nC = B^{n+1}C$$

is equivariant and hence induces a G-map

$$\varepsilon_n\colon S^W \wedge |B^nC| \longrightarrow |B^{n+1}C|.$$

THEOREM 5.4. $E(C)=\{|B^nC|, \varepsilon_n\}$ *is a* G*-spectrum in the sense of Araki-Murayama* [2].

REMARK. Segal [12] has announced that if A is a G-Γ-space (e.g., $A=|\hat{C}|'$) and if $A(\mathbf{1})$ is group-like, then there is a G-homotopy equivalence $A(\mathbf{1})\simeq\Omega^W(S^W\otimes A)$.

By the result of [2], $E(C)$ defines an $RO(G)$-graded G-cohomology theory. In particular the category $\mathscr{R}$ of Example 5.2 induces a (connective) KR-thoery.

Now assume that C has a pairing

$$\square_{\times}\colon C \times C \longrightarrow C$$

which commutes with the G-action on C. Then the induced pairings

$$P_{m,n}\colon B^mC \times B^nC \longrightarrow B^{m+n}C$$

are also equivariant. Thus we have

COROLLARY 5.5. *If* C *is a symmetric bimonoidal* G*-category, then* $E(C)$ *becomes a ring* G*-spectrum.*

References

[1] D. W. Anderson, Spectra and Γ-sets; Algebraic topology (Proc. Sympos. Pure Math. **XXII**), pp. 23–30, Amer. Math. Soc. 1971.

[2] S. Araki and M. Murayama, G-homotopy types of G-complexes and representation of G-cohomology theories, Publ. RIMS Kyoto Univ. **14** (1978), 203–222.

[3] J. M. Boardman and R. M. Vogt, Homotopy everything H-spaces, Bull. Amer. Math. Soc. **74** (1968), 1117–1122.

[4] ———, Homotopy invariant algebraic structures on topological spaces, Lecture Notes in Math. **347**, Springer, 1972.

[5] M. Laplaza, Coherence for distributivity; Coherence in categories, pp. 29–65, Lecture Notes in Math. **281**, Springer, 1972.

[6] S. MacLane, Categories for the working mathematician, Graduate Texts in Mathematics **5**, Springer, 1971.

[7] J. P. May, The geometry of iterated loop spaces, Lecture Notes in Math. **271**, Springer, 1972.

[8] ———, E_∞ ring spaces and E_∞ ring spectra, Lecture Notes in Math. **577**, Springer, 1977.

[9] ———, The uniqueness of infinite loop space machines, (preprint).

[10] D. Quillen, Higher algebraic K-theory I; Algebraic K-theory I, pp. 85–147, Lecture Notes in Math. **341**, Springer, 1973.

[11] G. Segal, Categories and cohomology theories, Topology **13** (1974), 293–312.

[12] ———, Some results in equivariant homotopy theory, (preprint).

[13] R. M. Switzer, Algebraic topology—homotopy and homology, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen **212**, Springer, 1975.

[14] F. R. Cohen, T. J. Lada and J. P. May, The homology of iterated loop spaces, Lecture Notes in Math. **533**, Springer, 1977.

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