Demand Management and Inventory Control for Substitutable Products

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Abstract

This paper studies dynamic inventory and pricing decisions for a set of substitutable products over a finite planning horizon. Demands are random and price sensitive. At the beginning of each period, based on the current inventory status, we decide the price and order quantity for each product. The replenishment arrives immediately, before demand realizations. The pricing decision influences the demand vector in the current period. Unsatisfied demands are backlogged subject to linear penalty costs. Inventories incur linear holding costs. The objective is to maximize the total expected discounted profit. We formulate the problem as a dynamic program. For tractability, we work with a transformed pricing vector, termed the market-share vector. We characterize the optimal policy and devise an algorithm to compute it. Both stationary and nonstationary data are considered. We also present a numerical study to illustrate the interplay of the pricing and inventory decisions.

1 Introduction

Imagine a retailer selling a set of products with similar functions and quality levels but various brands and styles, such as laptop computers or running shoes. Demand arises randomly and is price sensitive. Due to fast-changing technology and consumer tastes, the wholesale prices of the products may change over time. So, in addition to the regular inventory replenishment decisions, the retailer may set prices based on the current inventory status to influence future demands. This approach can help the sales of slow moving items and reduce potential losses due to obsolescence. By diverting some demands for products with low inventory to those with high inventory, it can also help reduce backlogs for some products and at the same time generate earlier realized revenues from other products. But, exactly how should one make such pricing decisions? When should one raise or lower the price of a product? And when should one order a product, and how large should the order be? These are the questions we aim to address in this paper.

Our model is a finite-horizon, periodic-review inventory system of multiple products with price-dependent random demand. At the beginning of each period, based on the current inventory status, we decide the price and order quantity for each product. The replenishment arrives immediately, before demand realizations. The pricing decision influences the demand vector in the current period. Unsatisfied demands are backlogged subject to linear penalty costs. Inventories incur linear holding costs. The objective is to maximize the total expected discounted profit.

In Section 3, we introduce a general nonlinear stochastic demand function. It encompasses both additive and multiplicative models. It also include the well-known logit and locational models as special cases.

In Section 4, we formulate the problem as a dynamic program. Both stationary and nonstationary data are considered. For tractability, we work with a transformed pricing vector, termed the market-share vector. We characterize the optimal policy and devise an algorithm to compute it. The optimal policy consists of three components: (1) the overstocking list records the products that need not to order, (2) the order-up-to levels, conditioned on the overstocking levels, and (3) the target market shares, also conditioned on the overstocking levels. The overstocking list determines what product the retailer does not need to order.

The order-up-to levels determine the ordering amounts. The target market shares provide the optimal market share assigned to each product, which, in turn, determines the optimal price for each product.

Section 5 studies the myopic policy and its optimality. There, we also report a numerical study to illustrate the interplay of the pricing and inventory decisions.

As discussed in the literature review in Section 2 below, our paper appears to be the first to study joint inventory and pricing decisions for system with multiple periods, multiple products, and nonlinear demand function. The closest prior work is a two-product model with a linear demand function. In Section 6 we summarize the theoretical results and discuss future research directions.

2 Literature Review

Many researchers have studied joint pricing and inventory decisions. However, most of these studies assume either a single product or multiple products but a single period. (See Chan et al. 2004 for a review.) Our multi-period and multi-product model extends both of these streams of research. Below we briefly review the main results of the most related literature.

Papers concerning single-product, periodic-review pricing and inventory control systems differ in their modeling assumptions on (1) the treatment of unsatisfied demand (backordering or lost-sales), (2) the cost structure (with or without a fixed ordering cost), and (3) the form of the demand function (additive or multiplicative). Assuming backordering and a linear ordering cost, Federgruen and Heching (1999) showed that a base-stock list-price policy is optimal. They also demonstrated that dynamic pricing can result in significant benefit compared to static pricing. Chen and Simchi-Levi (2004 a,b) considered both variable and fixed ordering costs. They proved that an (s, S, p) policy is optimal for additive demand, and an (s, S, A, p) policy is optimal for multiplicative demand, assuming the unsatisfied demand is fully backordered. With the same cost structure but assuming lost-sales and an additive demand function, Chen, et al. (2003) showed that an (s, S, p) policy is optimal. Song et al. (2005) extends this result to multiplicative demands. Finally, Huh and Janakiraman (2005) streamlined the above results by using a sample path argument. In our model, we consider multiple products, backordering, a linear ordering cost, and both additive and multiplicative demands.

There is a rich literature on inventory planning with substitutable products. But almost all those models assume predetermined prices and a single period. Some models allow hierarchy substitution, in which a product of higher quality can be used to fulfill the demand for products of lower quality. See, for example, Bassok, et al. (1999) and Chen, et al. (1999). Other authors focus on the stock-out based substitution, where consumers might switch to some other available products when their first choice is out of stock. See for example, Parlar and Goyal (1984), Ernst and Kouvelis (1999), Smith and Agrawal (2000), Rajaram and Tang (2001), and Honhon et al. (2006). Still other papers study assortment-based substitution, assuming a consumer may switch the most preferred product when the composition of a category changes. This is static substitution. That is, consumers do not look for a substitute in a given category, if their first choice is stocked out. Thus, in these papers the optimal size and composition of a category is of special interest. See, for example, van Ryzin and Mahajan (1999), Mahajan and van Ryzin (2001), Gaur and Honhon (2006). We refer the reader to KöK et al. (2006) for a complete review of inventory management of substitutable products and assortment management.

Our model differs from this literature in that we do not assume fixed prices and we focus on price-driven substitutions, i.e., a consumer's favorite within a category may change when prices change. We do not consider assortment-based substitution, since we study a certain category. We also ignore stock-out based substitution and assume that the consumers will delay their purchases when their favorite products are out of stock.

Aydin and Porteus (2005) studied joint pricing and inventory decisions for an assortment in a single-period setting. We are interested in the *dynamic* pricing and inventory decisions.

We are aware of only two previous studies that investigate dynamic pricing and inventory policies for multiple products within a category. Hall et al. (2003) study a setting with (1) non-stationary procurement costs, (2) joint setup costs of ordering, (3) both own-price and cross-price effects of all products within a category, and (4) deterministic demand. In our study, we assume demand is uncertain. Zhu and Thonemann (2005) investigated combined pricing and inventory decisions across *two* products, assuming (1) the cross-price effect is linear, and (2) the random demand has additive form. They segment the state space into four regions and characterize the optimal policy in each region. We consider an arbitrary number of products, linear and nonlinear cross-price substitutions, and random demands that are either additive or multiplicative.

3 Demand Model

We concern a retailer managing a category of J substitutable products, $\mathcal{J} = \{1, ..., J\}$. These products are differentiated in one of the following two ways.

1. *Vertical differentiation.* Products are vertically differentiated, if they differ in quality, where quality means the attributes having the "more is better" property. For example, cranberries are differentiated in terms of size and color, and digital cameras are differentiated in terms of Megapixal.

2. *Horizontal differentiation.* Products are horizontally differentiated, if they possess various styles, where style refers to the attributes not satisfying "more is better" property. For example, in the category of Classic-Fit Vintage Polo shirts, they differ only in color. In the category of Hershey's Kisses, chocolates are classified to almond chocolate, black chocolate, milk chocolate, and etc.

In this section, we present a general demand model to describe consumers' choices among different products in response to prices. We also make a transformation of variables and re-express the demand function as a function of the market-share vector. We then show that the logit and spatial models, which are widely used in the economic and marketing literature, are special cases of this general demand model.

3.1 The General Demand Model

Assume the planning horizon contains T periods. Consider any period $t \in \{1, ..., T\}$. Let p_{jt} be the price for product j at beginning of t. Denote $\mathbf{p}_t = (p_{1t}, ..., p_{Jt})$ be the price vector. Define $q_j(\mathbf{p}_t)$ be the probability that an arriving customer in period t selects product j, when observing $\mathbf{p}_t$. We also call $q_j(\mathbf{p}_t)$ the *market share* of product j in period t, a term coined in the literature of logit models (see Anderson et al., and etc.). Here, "q" is used to imply the *quota* of the market assigned to each variant in the category. Note that the total market share $\sum_{j \in \mathcal{J}} q_j(\mathbf{p}_t) \leq 1$. Denote $\mathbf{q}_t(\mathbf{p}_t) = (q_1 1(\mathbf{p}_t), ..., q_J(\mathbf{p}_t))$.

For any given price vector $\mathbf{p}$, we assume the demand in period t, $\mathbf{D}_t(\mathbf{p}) = (D_{1t}(\mathbf{p}), ..., D_{Jt}(\mathbf{p}))$, is of the following form:

$$\mathbf{D}_t(\mathbf{p}) = \mathbf{q}(\mathbf{p})\Lambda_t + \mathbf{L}(\mathbf{q}(\mathbf{p}))\varepsilon_t. \tag{1}$$

Here,

- $\mathbf{q}(\mathbf{p})$ is the market-share function defined above.
- Λ_t is a random variable with finite mean and variance, representing the volume of potential customers in period t. It characterizes the randomness of the market volume of the category. For example, consider a category of raincoats, the number of potential customers increases when they face freezing winter, and decreases when they meet warm winter.
- $\varepsilon_t = (\varepsilon_{1t}, ..., \varepsilon_{Jt})$ is the vector of random error terms, in which ε_{jt} has zero mean and finite variance. ε_{jt} indicates the uncertainty of the demand of product j. For example, if a movie star recently dressed a taupe color raincoat in a new released movie, the sales of taupe color raincoat should unpredictably increase. The random errors ε_{jt} can be correlated cross j, either positively or negatively. For simplicity, in the remaining analysis, we assume ε_{jt} is independent across j. However, the structure of the optimal policy holds for both dependent and independent error terms.
- $\mathbf{L}(\mathbf{q})$ is a $J \times J$-matrix.

Thus, the demand function consists of two parts – a controllable part $\mathbf{q}$, which is sensitive to prices, and an uncontrollable, stochastic part $(\Lambda_t, \varepsilon_t)$, which are independent across time periods. Note that the expected demand of product j is proportional to the market share $q_j(\mathbf{p})$.

When Λ_t is deterministic, the demand function is in additive form. When ε_t is deterministic, the demand function is in multiplicative form. When $\mathbf{L}(\mathbf{q}(\mathbf{p})) = \mathbf{I}$, a $J \times J$ identity matrix, the demands can be simplified to $\mathbf{q}(\mathbf{p})\Lambda_t + \varepsilon_t$. When $\mathbf{L}(\mathbf{q}(\mathbf{p})) = Diag(\mathbf{q}(\mathbf{p}))$, a diagonal matrix with the $(j, j)^{th}$ element $q_j(\mathbf{p})$, the demands can be expressed as $\mathbf{q}\Lambda_t + (q_1\varepsilon_{1t}, ..., q_J\varepsilon_{Jt})$.

To make the analysis tractable, we use the inverse of the market-share function to make a transformation of the decision variables. That is, we use $\mathbf{q}$ as the control variable instead of $\mathbf{p}$. Now, the demand function is of the following simpler expression:

$$\mathbf{D}_t(\mathbf{q}) = \mathbf{q}\Lambda_t + \mathbf{L}(\mathbf{q})\varepsilon_t. \tag{2}$$

Let

$$\begin{aligned} \mathbf{p}(\mathbf{q}_t) &= (p_1(\mathbf{q}_t), ..., p_J(\mathbf{q}_t)), \text{ the inverse market share function,} \\ &= \text{for } \mathbf{q}_t \in \mathcal{Q}_t = \{\mathbf{q}(\mathbf{p}_t) | \mathbf{p}_t \in \mathcal{P}_t\}; \\ r(\mathbf{q}) &= \mathbf{p}(\mathbf{q})\mathbf{q}, \text{ the revenue rate function.} \end{aligned}$$

Note that $r(\mathbf{q})$ is the ratio of the expected revenue to the expected volume of potential customers. We impose the following assumption about $r(\mathbf{q})$.

Assumption 1 $r(\mathbf{q})$ *is concave in* $\mathbf{q}$*, for* $\mathbf{q} \in \mathcal{Q}_t$*, where* $\mathcal{Q}_t$ *is a compact and convex set.*

In the following two subsections we present two special cases of the general demand function. In each case, we derive the market-share function and examine the property of the revenue rate function. We show that Assumption 1 is valid in both cases.

3.2 Special Case 1: Multinomial Logit Choice Model

A logit model can be used to study a set of products differentiated in both quality and style. A customer evaluates a product, based on the overall feeling of its attributes. A product can be described by the average perceptive value, which is the average value of the consumers' overall perception on a product. For example, a logit model can be applied to studying digital products. Concern a category of digital cameras, products are both vertically differentiated in quality and horizontally differentiated in brand, and they can be ranked in terms of the average evaluation by all consumers. Consider an MP3 category, in which some products are more popular than others. The popularity mainly comes from two sources, the high sound quality, or the appealing style. However, the logit model also assumes consumers are heterogeneous in their individual perception of each product. The heterogeneity is captured by independently and identically distributed random variables with double exponential distribution. To characterize the products, we define

Θ_j =the average perceptive value on product j.

ϑ_j =the random part of the individual perceptive value on product j

An individual's utility for product j is characterized by $U(\vartheta_j, j) = \Theta_j - p_j + \vartheta_j$, where Θ is determined in product design, and p_j is the pricing decision. The random part ϑ_j shows the heterogeneity of consumers' evaluation on product j. A consumer i's perceptive utility of these J products can be expressed as $(\Theta_1 - p_1 + \theta_{i1}, ..., \Theta_J - p_J + \theta_{iJ})$, where θ_{ij} is a realization of ϑ_{ij}, representing individual i's attitude to product j, for $j \in \mathcal{J}$. Consumer i selects product k, whenever $k \in \arg\max_{j\in\mathcal{J}}\{\Theta_j - p_j + \theta_{ij}\}$.

Guadagni and Little (1983) assumed a consumer will always purchase one product with the highest utility among these J products. We modify this assumption to count another *zero product* with $\Theta_0 = 0$ and $p_0 = 0$, which can be regarded as an option of non-purchase. Moreover, we can let $\Theta_0 > 0$ and $p_0 > 0$, representing the average value and price offered by the competitors. In the following analysis, we simply assume $\Theta_0 = 0$ and $p_0 = 0$. We make the assumption of utility.

Assumption 2 *(Logit Model)*

(1) A firm manages J products differentiated in their average perceptive value.

(2) A consumer's utility for product j contains two parts, where $\Theta_j - p_j$ shows consumers' homogeneous perceptive value on product j, and ϑ_j reflects the heterogeneity of consumers' evaluation.

$$U_j = \Theta_j - p_j + \vartheta_j, \quad \forall j \in \mathcal{J} \cup \{0\} \tag{3}$$

(3) The ϑ_j, for $j \in \mathcal{J} \cup \{0\}$, is i.i.d distributed with a double exponential distribution.

$$Pr(\vartheta_j \leq \theta) = e^{-e^{-\theta}}, \quad -\infty < \theta < \infty.$$

Given the above assumptions, it can be shown (Theil 1969, McFadden 1974, Anderson et al. 1992) that consumer i's choice probabilities have a simple form

$$q_k(\mathbf{p}) = \frac{\exp(\Theta_k - p_k)}{1 + \sum_{j\in\mathcal{J}} \exp(\Theta_j - p_j)}. \tag{4}$$

Then we specify the control space, and transform (4) to the inverse market share function.

Proposition 1 *Let $\mathcal{J}_s$ be any non empty subset of $\mathcal{J}$. For any $\mathbf{q} = (q_1, ..., q_J)$, satisfying*

$$0 \leq \sum_{j \in \mathcal{J}_s} q_j \leq \frac{\sum_{j \in \mathcal{J}_s} \exp(\Theta_j)}{1 + \sum_{j \in \mathcal{J}_s} \exp(\Theta_j)}, \quad \forall \mathcal{J}_s \subseteq \mathcal{J}, \tag{5}$$

which characterize $\mathcal{Q}$, we have

$$p_j = \Theta_j + \ln(1 - q_1 - ... - q_J) - \ln(q_j), \quad \forall j \in \mathcal{J}. \tag{6}$$

Proposition 2 *For J products, $r(\mathbf{q})$ is strictly concave in $(q_1, ..., q_J) \in \mathcal{Q}$, and $\dfrac{\partial^2 r(\mathbf{q})}{\partial q_j \partial q_k} < 0$, where*

$$r(\mathbf{q}) = \mathbf{p}(\mathbf{q})\mathbf{q} = \sum_{j \in \mathcal{J}} [\Theta_j q_j + q_j \ln(1 - q_1 - ... - q_J) - q_j \ln(q_j)]. \tag{7}$$

Thus the optimal policy can be applied to the logit model.

3.3 Special Case 2: Spatial Choice Model

A spatial model (locational model) concerns a set of products that provides similar quality but various styles. These products position in a space that describes consumers' taste. A consumer's taste can be represented as a point in the space, called *ideal point*. If one product doesn't happen to position at the ideal point of a consumer, the consumer feels a little unsatisfactory due to the taste mismatch. Locational model typically assumes the consumers are uniformly distributed on a horizontal line, which was introduced by Hotelling in 1929. A locational model can be adopted to analyze a shirt category consists of Polo T-shirts having various colors and patterns. Compared to the logit model, locational model provides a more tractable framework to analyze the horizontal differentiation, and avoid the independence of irrelevant alternatives (IIA) property, which is the shortcoming of logit model, see Guadagni and Little (1983), and Talluri and van Ryzin (2005).

In the model, each consumer makes purchase decision based on the price, quality, and style of products. We use following notations to characterize products and consumers.

Ψ_j =the average perceptive value of quality on product j.

τ_j =the position of product j on the taste line.

η_i =the ideal point of consumer i, which represents the taste of the consumer.

κ_i =the transportation cost of consumer i, measuring the strength of consumer's taste preference; $\kappa_i|\eta_i - \tau_j|$ measures the disutility of taste mismatch.

Consumers self-select a product in the category, which gives them the highest positive utility. The following assumption characterizes products and describe consumers' homogeneity and heterogeneity.

Assumption 3 *(Spatial Model)*

(1) The firm manages J products, having the same quality, Ψ, but different styles, positioned at $(\tau_1, ..., \tau_J)$ on the taste line.

(2) The consumers are homogeneous in their quality evaluation and their transportation cost κ, but heterogeneous in their ideal point η. Consumer i's utility function is

$$U(\eta, j) = \Psi - \kappa|\eta - \tau_j| - p_j, \quad for \quad j \in \{1, ..., J\}, \tag{8}$$

(3) Consumers' ideal point is uniformly distributed on the unit line, $\eta \sim \mathbf{U}(0, 1)$.

We fucus our attention on the simplest case of two products. Nevertheless, the results can be generalized. To clearly characterize the demand function, we use the following notations.

$\tau_j^l(p_j)$ =the most left point that product j can cover on the real axis, at p_j.
$\tau_j^r(p_j)$ =the most right point that product j can cover on the real axis, at p_j.
$\tau_{jk}^m(p_j, p_k)$ =the medium point of product j and k, at which customer feels indifferent between product j and k.

We briefly denote them as τ_j^l, τ_j^r, and τ_{jk}^m. According to (8), if a product is priced at Ψ, it does not provide positive utility to any buyer, and does not intervene the pricing of other products. So $p_{max} = \Psi$ can be regarded as the null price, which is introduced by Gallego and van Ryzin (1994) into dynamic pricing. Therefore, we can calculate τ_j^l and τ_j^r by setting the price of other products at Ψ, so that product j monopolizes the market.

$$\begin{cases} \tau_j^l = \tau_j - \frac{\Psi - p_j}{\kappa}, \\ \tau_j^r = \tau_j + \frac{\Psi - p_j}{\kappa}. \end{cases} \tag{9}$$

The medium point τ_{jk}^m can be expressed as in (10). However, if product j and k provides positive utility to the consumer at this point, it is an "actual" medium point, at which the price competition occurs, otherwise, it is a "virtual" medium point, in that the consumer at this point doesn't purchase.

$$\tau_{jk}^m = \frac{1}{2}\left(\tau_j^r + \tau_k^l\right), \quad for \quad j < k. \tag{10}$$

To establish a continuous market share function, we impose an assumption on the control space, which restricts the price gap.

Assumption 4 *The retailer sets the price* $\mathbf{p} \in \mathcal{P}$*, such that*

$$\mathcal{P} = [0, \Psi] \times [0, \Psi] \cup \{(p_1, p_2) \mid \kappa(\tau_2 - \tau_1) \geq |p_1 - p_2|\}. \tag{11}$$

We also illustrate a lemma to prove the concavity, for Proposition 3 and 4.

Lemma 3.1 *If* $g_i(\mathbf{z})$ *is concave in* $\mathbf{z}$*, for* $i = 1, ..., I$*, then* $\min_{i=1,...,I}\{g_i(\mathbf{z})\}$ *is concave in* $\mathbf{z}$*. If* $g_i(\mathbf{z})$ *is convex in* $\mathbf{p}$*, for* $i = 1, ..., I$*, then* $\max_{i=1,...,I}\{g_i(\mathbf{z})\}$ *is convex in* $\mathbf{z}$*.*

Proposition 3 *If the control space* $\mathcal{P}$ *satisfies Assumption 4, the market share function is given by the following equations.*

$$\begin{cases} q_1(p_1, p_2) = \min\{\tau_{12}^m, \tau_1^r\} - \max\{0, \tau_1^l\}, \\ q_2(p_1, p_2) = \min\{1, \tau_2^r\} - \max\{\tau_{12}^m, \tau_2^l\}, \end{cases} \tag{12}$$

The market share $q_j(\mathbf{p})$ *is concave in* $\mathbf{p}$*, for* $j \in \mathcal{J}$*.*

Next we establish the inverse market share $\mathbf{p}(\mathbf{q})$, and prove the concavity of $r(\mathbf{q})$.

Proposition 4 *(1) (Symmetric positioning) Assume* $\tau_1 = 0$ *and* $\tau_2 = 1$*, and* $\mathcal{Q} = \{(q_1, q_2)|\ 0 \leq q_1, q_2 \leq 1;\ q_1 + q_2 \leq 1\}$*. The inverse market share can be expressed as*

$$\begin{cases} p_1(q_1, q_2) = \Psi - \kappa q_1, \\ p_2(q_1, q_2) = \Psi - \kappa q_2, \end{cases}$$

which are concave in (q_1, q_2)*. The revenue function* $r(\mathbf{q})$ *is concave in* $\mathbf{q}$*.*

(2) (Asymmetric positioning) Assume $0 < \tau_1 < 1$ *and* $\tau_2 = 1$*, and* $\mathcal{Q} = \{(q_1, q_2)|\ 0 \leq q_1 \leq 1; 0 \leq q_2 \leq 1 - \tau_1;\ q_1 + q_2 \leq 1\}$*. The inverse demand function can be expressed as*

$$\begin{cases} p_1(q_1, q_2) = min\{\Psi - \frac{\kappa q_1}{2},\ \Psi - \kappa(q_1 - \tau_1),\ \Psi + \kappa(1 - \tau_1 - q_1 - q_2)\}, \\ p_2(q_1, q_2) = min\{\Psi - \kappa q_2,\ \Psi + \kappa(\frac{3}{2}(1 - \tau_1) - q_1 - 2q_2)\}, \end{cases}$$

which are concave in (q_1, q_2)*. The revenue function* $r(\mathbf{q})$ *can be expressed as*

$$r(q_1, q_2) = min \left\{ \begin{array}{l} q_1[\Psi - \frac{\kappa q_1}{2}] + q_2[\Psi - \kappa q_2], \\ q_1[\Psi - \kappa(q_1 - \tau_1)] + q_2[\Psi - \kappa q_2], \\ q_1[\Psi + \kappa(1 - \tau_1 - q_1 - q_2)] + q_2[\Psi + \kappa(\frac{3}{2}(1 - \tau_1) - q_1 - 2q_2)], \end{array} \right\}$$

which is concave in $\mathbf{q}$*.*

In the symmetric case, the market share of two products is still correlated, since $\mathcal{Q}$ requires $q_1+q_2 \leq 1$. The proposition shows the optimal policy can also be used in the locational model. The case of more than two products can be analyzed similarly, but much more complicated. We don't put effort on it in this paper.

4 Joint Inventory and Pricing Decisions

In this section, we analyze the dynamic program in (14), and characterize the structure of the optimal policy. We also develop an algorithm to compute the optimal policy. For expositional purposes, all proofs are provided in Appendix.

In general, we use F_X, f_X, and μ_X to denote the c.d.f., p.d.f. and mean of random variable X.

4.1 Dynamic Programming Formulation

We consider a T-period problem. At the beginning period t, $t \in \{1, ..., T\}$, we review the current inventory, set product prices and place replenishment orders. The replenishment orders arrive immediately, before the demand unfolds. The demand for the current period depends on the newly set product prices. Unsatisfied demands are backlogged. We use the following notations to describe the state variable, decision variables, and cost parameters of the joint pricing and inventory problem, where vectors are assumed to be *column* vectors, denoted by bold-faced letters:

x_{jt} =inventory level of product j before ordering at the beginning of period t, $\mathbf{x}_t = (x_{1t}, ..., x_{Jt})$.
y_{jt} =inventory level of product j after ordering at the beginning of period t, $\mathbf{y}_t = (y_{1t}, ..., y_{Jt})$.
q_{jt} =market share for product j in period t, $\mathbf{q}_t = (q_{1t}, ..., q_{Jt})$; $\mathbf{q}_t \in \mathcal{Q}_t$, the control space.
c_{jt} =unit procurement cost of product j, $\mathbf{c}_t = (c_{1t}, ..., c_{Jt})$.
h_{jt} =holding cost for unit excess of product j, $\mathbf{h}_t = (h_{1t}, ..., h_{Jt})$.
b_{jt} =penalty cost for unit shortage of product j, $\mathbf{b}_t = (b_{1t}, ..., b_{Jt})$.

Let $H_t(\mathbf{z}) = \mathbf{h}_t[\mathbf{z}]^+ + \mathbf{b}_t[\mathbf{z}]^-$, which is a strictly convex function of $(\mathbf{z})$. Then $EH_t(\mathbf{y} - \mathbf{D}_t(\mathbf{q}))$ is the expected inventory cost in period t.

The expected revenue in period t is given by $R_t(\mathbf{q}) = E\{\mathbf{p}(\mathbf{q})(\mathbf{q}\Lambda_t + \mathbf{L}(\mathbf{q})\varepsilon_t)\}$. Note that $R_t(\mathbf{q}) = \mu_{\Lambda_t}\mathbf{p}(\mathbf{q})\mathbf{q}$, because $E\{\varepsilon_t\} = \mathbf{0}$. Let $r(\mathbf{q}) = \mathbf{p}(\mathbf{q})\mathbf{q}$, the revenue rate function, which is the ratio of the expected revenue to the expected volume of potential customers. We impose Assumption 1 about $r(\mathbf{q})$.

Let $v_t(\mathbf{x})$ be the maximum total expected discounted profit from period t to T, with a discount rate β. We have

$$\begin{aligned} & v_t(\mathbf{x}) = \max_{\mathbf{y}\geq\mathbf{x},\ \mathbf{q}\in\mathcal{Q}_t} \{\beta R_t(\mathbf{q}) - \mathbf{c}_t(\mathbf{y}-\mathbf{x}) - EH_t(\mathbf{y}-\mathbf{D}_t(\mathbf{q})) + \beta E v_{t+1}(\mathbf{y}-\mathbf{D}_t(\mathbf{q}))\}, \\ & v_{T+1}(\mathbf{x}) = 0. \end{aligned} \tag{13}$$

Replace $v_t(\mathbf{x})$ by $V_t(\mathbf{x})$, where $V_t(\mathbf{x}) = v_t(\mathbf{x}) - \mathbf{c}_t\mathbf{x}$. Then,

$$V_t(\mathbf{x}) = \max_{\mathbf{y}\geq\mathbf{x},\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q}), \tag{14}$$

where

$$\begin{aligned} & G_t(\mathbf{y},\mathbf{q}) \\ = \ & \beta R_t(\mathbf{q}) - \mathbf{c}_t\mathbf{y} - E\{H_t(\mathbf{y}-\mathbf{D}_t(\mathbf{q}))\} + \beta\mathbf{c}_{t+1}E\{\mathbf{y}-\mathbf{D}_t(\mathbf{q})\} + \beta E\{V_{t+1}(\mathbf{y}-\mathbf{D}_t(\mathbf{q}))\} \\ = \ & \beta\mu_{\Lambda_t}\ r(\mathbf{q}) - \beta\mu_{\Lambda_t}\ \mathbf{c}_{t+1}\mathbf{q} + (\beta\mathbf{c}_{t+1} - \mathbf{c}_t)\mathbf{y} - EH_t(\mathbf{y}-\mathbf{D}_t(\mathbf{q})) + \beta E V_{t+1}(\mathbf{y}-\mathbf{D}_t(\mathbf{q})). \end{aligned} \tag{15}$$

We shall use the following notation to describe the optimal policy.

$$\begin{aligned} \mathcal{J}^+ \ & = \text{overstocking list} = \Big\{ j : \text{not to order product} j,\ j \in \mathcal{J} = \{1,..,J\} \Big\}, \\ \mathcal{J}^- \ & = = \mathcal{J} \setminus \mathcal{J}^+, \\ \mathbf{x}_{\mathcal{J}^+} \ & = \text{vector of overstocking levels} = (x_j)_{j\in\mathcal{J}^+} = , \\ \mathbf{y}_{\mathcal{J}^+} \ & = \text{vector order-up-to levels for products in } \mathcal{J}^+ = (y_j)_{j\in\mathcal{J}^+}. \end{aligned}$$

Define $\mathbf{x}_{\mathcal{J}^-}$ and $\mathbf{y}_{\mathcal{J}^-}$ similarly. To ease the notation, let

$$\begin{aligned} \max\left\{G_t(\mathbf{y},\mathbf{q}) \mid y_j = x_j, j\in\mathcal{J}^+;\ y_j \geq x_j, j\in\mathcal{J}^-;\ \mathbf{q}\in\mathcal{Q}_t\right\} & = \max_{\mathbf{y}\geq\mathbf{x};\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q} \mid \mathbf{y}_{\mathcal{J}^+} = \mathbf{x}_{\mathcal{J}^+}), \\ \max\left\{G_t(\mathbf{y},\mathbf{q}) \mid y_j = x_j, j\in\mathcal{J}^+;\ y_j \in\mathcal{R}, j\in\mathcal{J}^-;\ \mathbf{q}\in\mathcal{Q}_t\right\} & = \max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q} \mid \mathbf{y}_{\mathcal{J}^+} = \mathbf{x}_{\mathcal{J}^+}). \end{aligned}$$

Similar notation is applied to arg max.

4.2 Optimal Policy and Algorithm

Our first result is:

Theorem 1 *Suppose Assumption 1 holds. Consider any time period t with initial inventory $\mathbf{x}$. We have*

(a) $G_t(\mathbf{y}, \mathbf{q})$ is jointly concave in $\mathbf{y}$ and $\mathbf{q}$, and $V_t(\mathbf{x})$ is concave and non-increasing in $\mathbf{x}$.

(b) $G_t(\mathbf{y}, \mathbf{q})$ has at least one maximizer,

$$(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) \in \arg \max_{\mathbf{y} \in \mathcal{R}^J,\ \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}),$$

where $\emptyset$ means no constraint is put on the ordering decision $\mathbf{y}$, while maximizing $G_t(\mathbf{y}, \mathbf{q})$.

(c) If $\mathbf{x} \le \mathbf{y}_t^*(\emptyset)$, the retailer will order up to $\mathbf{y}_t^*(\emptyset)$, thus retailer should order some non-negative amounts for all products.

The theorem indicates that there is a set of threshold inventory levels that achieves the global maximum of $G_t(\mathbf{y}, \mathbf{q})$, therefore these thresholds are the desired (optimal) inventory levels for period t. Consequently, if the initial inventory levels are all below the thresholds, it is optimal to order these products up to the thresholds. In other words, the theorem tells us what to do when the initial inventory levels of all products are lower than the thresholds.

However, what should one do if the initial inventory levels of some products are *higher* than the thresholds? If there is only a single product, then it is optimal not to order the production if its initial inventory is higher than the threshold. However, the case for multiple products is much more complicated. For one thing, the comparison between two vectors are much more complex than the comparison between two scalars. For example, suppose there are three products, and $\mathbf{y}_t^*(\emptyset) = (25, 27, 28)$. What should one do if the initial inventory level is $\mathbf{x}_t = (38, 24, 27)$? What if $\mathbf{x}_t = (38, 28, 27)$? The remaining of this section will be devoted to these issues.

To ease understanding, we first study the optimal policy for two products. Given any initial inventory levels, there are three possible scenarios: (i) the inventory of both products are lower than the thresholds; (ii) the inventory of one product is over the threshold, but the inventory of another one is below; (iii) both inventory levels are over the thresholds. The problem in the first scenario has been solved. Now we proceed to the second scenario.

Suppose $\mathbf{y}_t^*(\emptyset) = (25, 27)$, but the current inventory level happens to be $\mathbf{x} = (38, 24)$. The retailer faces the following questions: (1) should she stop ordering product 1? (2) should she still order product 2? (3) what price should she charge, or what levels of market share should she allocate to these products? The following lemma characterizes the structure of optimal action for these two products.

Proposition 5 *Consider two products,* 1 *and* 2*. Suppose Assumption 1 holds. Consider any time period t with initial inventory* $\mathbf{x}$*. If $x_1 > y_{1t}^*(\emptyset)$, and $x_2 \le y_{2t}^*(\emptyset)$, then:*

(a) It is optimal to stop ordering product 1, and the optimal solution can be found in the following constrained maximization problem:

$$V_t(\mathbf{x}) = \max_{y_1=x_1,\ y_2\ge x_2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q}) = \max_{\mathbf{y}\ge\mathbf{x}\ ;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q} \mid y_1 = x_1).$$

(b) We solve another less constrained maximization problem

$$\max_{y_1=x_1,\ y_2\in\mathcal{R};\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q}) = \max_{\mathbf{y}\in\mathcal{R}^2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q} \mid y_1 = x_1),$$

and obtain another set of thresholds,

$$(\mathbf{y}_t^*(x_1), \mathbf{q}_t^*(x_1)) \in \arg\max_{\mathbf{y}\in\mathcal{R}^2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q} \mid y_1 = x_1), \ \text{where } y_{1t}^*(x_1) = x_1.$$

If $x_2 \le y_{2t}^*(x_1)$, then the retailer orders product 2 up to $y_{2t}^*(x_1)$ and sets market share at $\mathbf{q}_t^*(x_1) = (q_{1t}^*(x_1), q_{2t}^*(x_1))$ for these two products. Otherwise, the retailer places no order and set the market share at $\mathbf{q}_t^*(\mathbf{x}) = (q_{1t}^*(x_1, x_2), q_{2t}^*(x_1, x_2))$, where

$$(\mathbf{y}_t^*(\mathbf{x}), \mathbf{q}_t^*(\mathbf{x})) \in \arg\max_{\mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q} \mid \mathbf{y} = \mathbf{x}).$$

Proposition 5 shows that in scenario 2 the overstocking vector, which records the inventory level of products that retailer does not need to order, can be computed recursively, and the thresholds are obtained conditioning on the overstocking vector.

Next, consider the third scenario. Suppose the first set of thresholds is $\mathbf{y}_t^*(\emptyset) = (25, 27)$, but the current inventory level happens to be $\mathbf{x} = (38, 28)$. Should the retailer stop ordering both products? The following lemma attempts to answer this question.

Proposition 6 *Suppose Assumption 1 holds. Consider any time period t with initial inventory* $\mathbf{x}$*. If $x_1 > y_{1t}^*(\emptyset)$ and $x_2 > y_{2t}^*(\emptyset)$, then*

(a) it is optimal to stop ordering at least one product, say product $k \in \{1, 2\}$. The optimal solution can be given by the following constrained maximization problem

$$V_t(\mathbf{x}) = \max_{\mathbf{y}\in\mathcal{R}^2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y},\mathbf{q} \mid y_k = x_k);$$

(b) we solve another constrained maximization problem, and obtain a new set of thresholds

$$(\mathbf{y}_t^*(x_k), \mathbf{q}_t^*(x_k)) \in \arg \max_{\mathbf{y}\in\mathcal{R}^2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_k = x_k),\ \textit{where } y_{kt}^*(x_k) = x_k.$$

Let $l = \{1,2\} \setminus \{k\}$. If $x_l \le y_{lt}^*(x_k)$. Then the retailer orders product l up to $y_{lt}^*(x_k)$, and sets market share at $\mathbf{q}_t^*(x_k)$ for these two products. Otherwise, the retailer places no order and sets the market share at $\mathbf{q}_t^*(\mathbf{x})$, where

$$(\mathbf{y}_t^*(\mathbf{x}), \mathbf{q}_t^*(\mathbf{x})) \in \arg \max_{\mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y} = \mathbf{x}).$$

Propositions 5 and 6 show the structure of the optimal policy when the initial inventory levels are not both below the original threshold levels. However, proposition 6 only proves the existence of the optimal policy, it is by no means an efficient algorithm to compute the policy. When both inventory levels are over the thresholds, the computational complexity increases, because we cannot immediately ascertain which one is overstocked. To obtain the optimal policy, we need to branch the program into two scenarios – product 1 is overstocked, or product 2 is overstocked – and then compared the outcome of the two scenarios. For more than two products, the computational complexity is beyond practicality. So, our next question is, under what condition can we confirm both products are overstocked? The following proposition provides a sufficient condition.

Proposition 7 *Consider two products. Take any time period t with initial inventory $\mathbf{x}$, where $x_1 > y_{1t}^*(\emptyset)$ and $x_2 > y_{2t}^*(\emptyset)$. Suppose Assumption 1 holds, and the thresholds possess the following decreasing property with respect to initial inventory:*

$$\begin{cases} y_{1t}^*(x_2) \ge y_{1t}^*(x_2'), & \textit{for } x_2 < x_2'; \\ y_{2t}^*(x_1) \ge y_{2t}^*(x_1'), & \textit{for } x_1 < x_1'; \end{cases}$$

$$\textit{where} \begin{cases} (\mathbf{y}_t^*(x_1), \mathbf{q}_t^*(x_1)) \in \arg \max_{\mathbf{y}\in\mathcal{R}^2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_1 = x_1); \\ (\mathbf{y}_t^*(x_2), \mathbf{q}_t^*(x_2)) \in \arg \max_{\mathbf{y}\in\mathcal{R}^2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_2 = x_2). \end{cases}$$

Then it is optimal to order neither *products in period t, and to set the market share at $\mathbf{q}_t^*(\mathbf{x})$, where*

$$(\mathbf{y}_t^*(\mathbf{x}), \mathbf{q}_t^*(\mathbf{x})) \in \arg \max_{\mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid \mathbf{y} = \mathbf{x}).$$

Thus, under the decreasing property of the thresholds, we can be ascertain that both products are overstocked, so no branching is needed in the algorithm.

The decreasing property above describes the situation in which the threshold of one product goes down as the initial inventory level of another product increases. For example, suppose the retailer confirms that one product, say product 1, is overstocked, she probably will decrease the price of product 1 and increase (or hold) the price of product 2. Then the market share for product 1 increases while that for product 2 decreases. As the degree of overstocking of product 1 increases, she is most likely to set a lower market share, therefore a lower threshold for product 2. The inventory threshold of product 2 conditioning on the inventory of product 1 is decreasing as the inventory of product 1 increases, until it goes below the initial inventory level of product 2. We shall demonstrate this effect numerically in Section 3.

The above propositions can be extended to the multiple product setting. Proposition 5 and 6 can be extended to J products. The optimal policy consists of three components: (1) the overstocking list records the products that need not to order, (2) the order-up-to levels, conditioned on the overstocking levels, and (3) the target market shares, also conditioned on the overstocking levels. The overstocking list determines what product the retailer does not need to order. The order-up-to levels determine the ordering amounts. The target market shares provide the optimal market share assigned to each product, which, in turn, determines the optimal price for each product. The following theorem rigorously describes the optimal policy.

Theorem 2 *(Structure of Optimal Policy) Suppose Assumption 1 holds. Consider any time period t with initial inventory $\mathbf{x}$. We can always recursively make a complete overstocking list, $\mathcal{J}_M^+$, which contains the products for not ordering. M counts the number of total iterations to reach the final list; $M \leq J$ (see Algorithm 1). The optimal policy for time t consists of $\mathcal{J}_M^+$ and the following inventory thresholds (order-up-to levels) and the market share values:*

$$\left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_M^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{J}_M^+})\right) \in \arg \max_{\mathbf{y} \in \mathcal{R}^J;\ \mathbf{q} \in \mathcal{Q}_t} G_t \left(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{J}_M^+} = \mathbf{x}_{\mathcal{J}_M^+}\right),$$
$$\textit{with} \quad y_{jt}^*(\mathbf{x}_{\mathcal{J}_M^+}) = x_j, \forall j \in \mathcal{J}_M^+.$$

We use an iterative algorithm to prove the existence of the optimal policy. In the algorithm, we construct two sequences of lists, $\mathcal{J}_m^+$ and $\mathcal{J}_m^-$. The former contains the products that are confirmed to be overstocked, and the latter contains the remaining products, which can be either overstocked or under-stocked. Hence, $\mathcal{J}_m^+$ is an updated list of overstocked products.

Algorithm 1 *(Existence of Optimal Policy) Suppose Assumption 1 holds. We design two sequences of lists, $\mathcal{J}_m^+$ and $\mathcal{J}_m^-$. The former updates the list of products that do not need to be ordered at the m^{th} step, and $\mathcal{J}_m^- = \mathcal{J} \setminus \mathcal{J}_m^+$. The algorithm can be terminated in M steps, where $M \leq J$.*

***Step 0**: Let $\mathcal{J}_0^+ = \emptyset$. Set $m = 1$ and go to Step m.*

***Step m**: Solve $\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y},\mathbf{q} \mid \mathbf{y}_{\mathcal{J}_{m-1}^+} = \mathbf{x}_{\mathcal{J}_{m-1}^+}\right)$ to obtain the thresholds*

$$\left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_{m-1}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{J}_{m-1}^+})\right) \in \arg\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y},\mathbf{q} \mid \mathbf{y}_{\mathcal{J}_{m-1}^+} = \mathbf{x}_{\mathcal{J}_{m-1}^+}\right),$$
$$\text{where} \quad y_{jt}^*(\mathbf{x}_{\mathcal{J}_{m-1}^+}) = x_j,\ \forall j \in \mathcal{J}_{m-1}^+.$$

If $\mathbf{x} \leq \mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_{m-1}^+})$, let $\mathcal{J}_m^+ = \mathcal{J}_{m-1}^+$, set $M = m$, and stop the iteration.

Otherwise, Define $\mathcal{A}_m = \left\{j \mid x_j > y_{jt}^*(\mathbf{x}_{\mathcal{J}_{m-1}^+})\right\}$, a non-empty set. There must exist some $k \in \mathcal{A}_m$ such that it is optimal not to order product k. Let $\mathcal{J}_m^+ = \mathcal{J}_{m-1}^+ \cup \{k\}$, and $\mathcal{J}_m^- = \mathcal{J} \subset \mathcal{J}_m^+$.

If $\mathcal{J}_m^- = \emptyset$, let $M = m$, stop ordering and set market share at $\mathbf{q}_t^*(\mathbf{x})$. Otherwise, set $m = m + 1$ and go to step m.

The above algorithm can be very time consuming when $\mathcal{A}_m$ contains several elements. Below we present a condition (Assumption 5), which is the extension of the decreasing property of the thresholds in the two-product case. Under this condition, we can update $\mathcal{J}_m^+ = \mathcal{J}_{m-1}^+ \cup \mathcal{A}_m$. In other words, $\mathcal{A}_m$ is exactly the set of products adding on the overstocking list $\mathcal{J}_{m-1}^+$ in the m^{th} iteration. This leads to an efficient algorithm, which goes from step 1 to M, where $1 \leq M \leq J$. At the beginning of step m, we have two lists, where $\mathcal{J}_{m-1}^+$ contains the products already confirmed to be overstocked in the previous steps, and $\mathcal{J}_{m-1}^-$ contains the remaining products. It the m^{th} step, we recalculate a set of thresholds, $\left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_{m-1}^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{J}_{m-1}^+})\right)$, conditioning on the inventory levels of the products in $\mathcal{J}_{m-1}^+$. Then we compare the thresholds to the inventory levels of the products in $\mathcal{J}_{m-1}^-$. If we find any product in $\mathcal{J}_{m-1}^-$ such that its inventory level is higher

than the threshold, then the product is confirmed to overstocked. The overstocked products are then moved into $\mathcal{J}_m^+$. The algorism continues, unless we cannot find any product that is overstocked in the m^{th} step.

Assumption 5 *Let $\mathcal{J}^+$, $\mathcal{J}^-$ be the sets defined at the beginning of Section 4. The inventory thresholds derived from $G_t(\mathbf{y},\mathbf{q})$ possesses the following property.*

If $\mathbf{x}_{\mathcal{J}^+} \leq \mathbf{x}'_{\mathcal{J}^+}$, then $y^*_{jt}(\mathbf{x}_{\mathcal{J}^+}) \geq y^*_{jt}(\mathbf{x}'_{\mathcal{J}^+})$, for $j \in \mathcal{J}^-$,

$$\text{where} \begin{cases} (\mathbf{y}^*_t(\mathbf{x}_{\mathcal{J}^+}), \mathbf{q}^*_t(\mathbf{x}_{\mathcal{J}^+})) \in \arg\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y},\mathbf{q} \mid \mathbf{y}_{\mathcal{J}^+} = \mathbf{x}_{\mathcal{J}^+}\right); \\ \left(\mathbf{y}^*_t(\mathbf{x}'_{\mathcal{J}^+}), \mathbf{q}^*_t(\mathbf{x}'_{\mathcal{J}^+})\right) \in \arg\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y},\mathbf{q} \mid \mathbf{y}_{\mathcal{J}^+} = \mathbf{x}'_{\mathcal{J}^+}\right). \end{cases}$$

Algorithm 2 *(Efficient Computation of the Optimal Policy) Suppose Assumptions 1 and 5 hold. We design two sequences of lists $\mathcal{J}^+_m$ and $\mathcal{J}^-_m$, where $\mathcal{J}^-_m = \mathcal{J}/\mathcal{J}^+_m$. The complete overstocking list, $\mathcal{J}^+_M$, can always be made recursively in M steps, where $M \leq J$.*

***Step 0**: Let $\mathcal{J}^+_0 = \emptyset$. Set $m = 1$ and go to Step m.*

***Step m**: Solve $\max\limits_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y},\mathbf{q} \mid \mathbf{y}_{\mathcal{J}^+_{m-1}} = \mathbf{x}_{\mathcal{J}^+_{m-1}}\right)$ to obtain the thresholds*

$$\left(\mathbf{y}^*_t(\mathbf{x}_{\mathcal{J}^+_{m-1}}), \mathbf{q}^*_t(\mathbf{x}_{\mathcal{J}^+_{m-1}})\right) \in \arg\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y},\mathbf{q} \mid \mathbf{y}_{\mathcal{J}^+_{m-1}} = \mathbf{x}_{\mathcal{J}^+_{m-1}}\right).$$

If $\mathbf{x} \leq \mathbf{y}^*_t(\mathbf{x}_{\mathcal{J}^+_{m-1}})$, let $\mathcal{J}^+_m = \mathcal{J}^+_{m-1}$, set $M = m$, and stop.

Otherwise, define $\mathcal{A}_m = \left\{j \mid x_j > y^*_{jt}(\mathbf{x}_{\mathcal{J}^+_{m-1}})\right\}$, a non-empty set. Let $\mathcal{J}^+_m = \mathcal{J}^+_{m-1} \cup \mathcal{A}_m$, and $\mathcal{J}^-_m = \mathcal{J} \setminus \mathcal{J}^+_m$.

If $\mathcal{J}^-_m = \emptyset$, let $M = m$, stop ordering and set market share at $\mathbf{q}^*_t(\mathbf{x})$. Otherwise, set $m = m + 1$ and go to Step m.

We are still working on finding sufficient condition to guarantee the parametric monotonicity. We believe the condition $\frac{\partial^2 r(\mathbf{q})}{\partial q_j \partial q_k} \leq 0$ plays an important role in ascertaining the property. The condition states that if the retailer has to increase q_1 from the global optima, it is better to lower q_2 rather than increase q_2 to maximize the expected single-period revenue.

The above theorems and algorithms show the way to calculate at least one optimal solution. It does not theoretically count all optimal solutions. When we have $r(\mathbf{q})$ strictly concave in $\mathbf{q}$, the optimal solution is guaranteed to be unique, which is stated in the following theorem.

Theorem 3 *Assume $r(\mathbf{q})$ is strictly concave in $\mathbf{q}$, for $\mathbf{q} \in \mathcal{Q}_t$. Then (1) $G_t(\mathbf{y},\mathbf{q})$ is strictly concave in $(\mathbf{y},\mathbf{q})$; (2) the optimal solution generated by Algorithm 1 or Algorithm 2 is unique.*

5 Stationary Analysis

In this section, we assume the demand process is stationary, and the cost parameters are stable in time. We therefore suppress the index t from these parameters. In particular, the demand function now is of the following form:

$$\mathbf{D}_t(\mathbf{q}) = \mathbf{D}(\mathbf{q}) = \Lambda\mathbf{q} + \mathbf{L}(\mathbf{q})\varepsilon. \tag{16}$$

5.1 Myopic Policy

The following analysis of myopic policy is suggested by Heyman and Sobel (1984). The total expected discount profit during T periods can be expressed as

$$\Pi^T = \sum_{t=1}^{T} \beta^{t-1}[\beta\ \mathbf{p}(\mathbf{q}_t)\mathbf{D}(\mathbf{q}_t) - \mathbf{c}(\mathbf{y}_t - \mathbf{x}_t) - H(\mathbf{y}_t - \mathbf{D}(\mathbf{q}_t))] + \beta^T \mathbf{c}(\mathbf{y}_T - \mathbf{D}(\mathbf{q}_T)), \tag{17}$$

where the last term outside the bracket is the salvage value, based on the assumption that the supplier is willing to buy back the remaining products at the wholesales price at the end of the planning horizon (e.g., a business season), or a discount store is willing to buy the remains at the price equal to the procurement cost. Substitute $\mathbf{y}_{t-1} - \mathbf{D}(\mathbf{q}_{t-1})$ for $\mathbf{x}_t$, if $t > 1$, and rewrite Π^T:

$$\begin{aligned} \Pi^T &= \beta\ \mathbf{p}(\mathbf{q}_1)\mathbf{D}(\mathbf{q}_1) - \mathbf{c}(\mathbf{y}_1 - \mathbf{x}_1) - H(\mathbf{y}_1 - \mathbf{D}(\mathbf{q}_1)) \\ &\quad + \sum_{t=2}^{T} \beta^{t-1}[\beta\ \mathbf{p}(\mathbf{q}_t)\mathbf{D}(\mathbf{q}_t) - \mathbf{c}(\mathbf{y}_t - (\mathbf{y}_{t-1} - \mathbf{D}(\mathbf{q}_{t-1}))) - H(\mathbf{y}_t - \mathbf{D}(\mathbf{q}_t))] \\ &\quad + \beta^T \mathbf{c}(\mathbf{y}_T - \mathbf{D}(\mathbf{q}_T)) \\ &= \mathbf{c}\mathbf{x}_1 + \sum_{t=1}^{T} \beta^{t-1}[\beta\ \mathbf{p}(\mathbf{q}_t)\mathbf{D}(\mathbf{q}_t) - \mathbf{c}\mathbf{y}_t + \beta\mathbf{c}(\mathbf{y}_t - \mathbf{D}(\mathbf{q}_t)) - H(\mathbf{y}_t - \mathbf{D}(\mathbf{q}_t))]. \end{aligned}$$

We have $E\Pi^T = \mathbf{c}\mathbf{x}_1 + \sum_{t=1}^{T} \beta^{t-1} E\pi(\mathbf{y}_t,\ \mathbf{q}_t)$, where

$$\pi(\mathbf{y}_t,\ \mathbf{q}_t) = \beta\ \mathbf{p}(\mathbf{q}_t)\mathbf{D}(\mathbf{q}_t) - \mathbf{c}\mathbf{y}_t + \beta\mathbf{c}(\mathbf{y}_t - \mathbf{D}(\mathbf{q}_t)) - H(\mathbf{y}_t - \mathbf{D}(\mathbf{q}_t)). \tag{18}$$

$$E\pi(\mathbf{y}_t,\ \mathbf{q}_t) = \beta\ \mu_{\Lambda}\ r(\mathbf{q}_t) - (1-\beta)\mathbf{c}\mathbf{y}_t - \beta\ \mu_{\Lambda}\ \mathbf{c}\mathbf{q}_t - EH(\mathbf{y}_t - \mathbf{D}(\mathbf{q}_t)).$$

The following theorem states the existence of the myopic policy, which is optimal for a single period. It shows that the myopic policy is optimal for multiple periods if the initial inventory level is under the myopic threshold.

Theorem 4 *(1)* $\pi(\mathbf{y},\ \mathbf{q})$ *is jointly concave in* $(\mathbf{y},\ \mathbf{q})$*, and admits an optimum* $(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*)$*.*

$$(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*) \in \arg\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} E\pi(\mathbf{y},\ \mathbf{q}) \tag{19}$$

(2) If $r(\mathbf{q})$ *is strictly concave in* $\mathbf{q}$*, then* $\pi(\mathbf{y},\ \mathbf{q})$ *is strictly joint concave in* $(\mathbf{y},\ \mathbf{q})$*, and admits a unique optimum.*

(3) If $\mathbf{x}_1 \leq \mathbf{y}_\pi^*$*, then* $(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*)$ *is the optimal policy.*

(4) If there is any $j \in \mathcal{J}$*, s.t.* $x_{j1} > y_{\pi j}^*$*, it is optimal not to order product* j *in period* t*.*

(5) Due to the concavity of $\pi(\mathbf{y},\ \mathbf{q})$*, we can recursively compute a myopic policy with the overstocking list.*

The proof of the first two parts is similar to that of Theorem 1 and 3. The proof of the third and fourth part abides by the analysis suggested by Heyman and Sobel (1984). The computation of the fifth part is the same as Algorithm 1 and 2.

5.2 Interplay of Marketing and Operations Decisions

The myopic policy depends on the specific form of randomness of demand. We study three kinds of demand function, specify the myopic policy, and show the degree of interaction between pricing and replenishment. These can be described as marketing and operations decision. The proof of the following propositions is provided in Appendix.

*Additive-***I** *Demand*

We first analyze the additive demand function with $\mathbf{L}(\mathbf{q}) = \mathbf{I}$. The demand function can be simply expressed as $\Lambda\mathbf{q} + \varepsilon$. Assume $\Lambda \equiv \mu_\Lambda$, and $\varepsilon = (\varepsilon_1, ..., \varepsilon_J)$, where ε_j is i.i.d., for $j \in \{1, ..., J\}$. The inventory related cost can be specified as

$$EH(\mathbf{y} - \mathbf{D}(\mathbf{q})) = \sum_{j=1}^{J} \left\{ h_j \int_{-\infty}^{y_j - \mu_\Lambda q_j} (y_j - \mu_\Lambda q_j - \epsilon_j) f_{\varepsilon_j}(\epsilon_j) d\epsilon_j + b_j \int_{y_j - \mu_\Lambda q_j}^{\infty} (-y_j + \mu_\Lambda q_j + \epsilon_j) f_{\varepsilon_j}(\epsilon_j) d\epsilon_j \right\}.$$

Define

$$\begin{aligned} \xi_j &= \frac{b_j - c_j(1-\beta)}{b_j + h_j}, \\ \omega_j &= F_{\varepsilon_j}^{-1}[\xi_j]. \end{aligned}$$

We have

Proposition 8 *For the additive demand function with the linear transformation* $\mathbf{L}(\mathbf{q}) = \mathbf{I}$, *we can obtain the unconstrained optima* $(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*)$ *by solving the following equations.*

$$\begin{cases} y_j = \omega_j + \mu_\Lambda q_j \ ; \\ \beta \frac{\partial r(\mathbf{q})}{\partial q_j} = c_j \ . \end{cases}$$

If $\mathbf{q}_\pi^* \in \mathcal{Q}_t$, *and* $\mathbf{y}_\pi^* \geq \mathbf{x}_1$, *then* $(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*)$ *is the optimal policy.*

Under additive-I demand, the retailer can make marketing and operations decision separately, when having none product overstocked. The retailer can set market share to make the marginal revenue equal to the marginal procurement cost, regardless of the inventory decision. However, when having product overstocked, the retailer must integrally use the marketing and operations lever to diminish the negative effect of overstock.

Additive-Diag($\mathbf{q}$) *Demand*

Second, we analyze the additive demand function with $\mathbf{L}(\mathbf{q}) = Diag(\mathbf{q})$. The demands can be expressed as $\mu_\Lambda \mathbf{q} + (q_1\varepsilon_1, ..., q_J\varepsilon_J)$. The inventory-related cost can be expressed as

$$\begin{aligned} EH(\mathbf{y} - \mathbf{D}(\mathbf{q})) &= h_j \sum_{j=1}^{J} \int_{-\infty}^{\frac{y_j - \mu_\Lambda q_j}{q_j}} (y_j - \mu_\Lambda q_j - \epsilon_j q_j) f_{\varepsilon_j}(\epsilon_j) d\epsilon_j \\ &+ b_j \sum_{j=1}^{J} \int_{\frac{y_j - \mu_\Lambda q_j}{q_j}}^{\infty} (-y_j + \mu_\Lambda q_j + \epsilon_j q_j) f_{\varepsilon_j}(\epsilon_j) d\epsilon_j. \end{aligned}$$

Proposition 9 *For the additive demand function with the linear transformation* $\mathbf{L}(\mathbf{q}) = Diag(\mathbf{q})$, *we can obtain the unconstrained optima* $(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*)$ *by solving the following equations.*

$$\begin{cases} y_j = (\omega_j + \mu_\Lambda) q_j \\ \beta \frac{\partial r(\mathbf{q})}{\partial q_j} = c_j + (b_j + h_j) \frac{1}{\mu_\Lambda} \int_{\omega_j}^{\infty} \epsilon_j f_{\varepsilon_j}(\epsilon_j) d\epsilon_j. \end{cases}$$

If $\mathbf{q}_\pi^* \in \mathcal{Q}_t$, *and* $\mathbf{y}_\pi^* \geq \mathbf{x}_1$, *then* $(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*)$ *is the optimal policy.*

Under additive-$Diag(\mathbf{q})$ demand, even though the retailer does not incur overstocking, she still needs to take into account some operations cost when doing pricing. The operations cost is measured as $(b_j + h_j)\dfrac{1}{\mu_\Lambda}\displaystyle\int_{\omega_j}^{\infty} \epsilon_j f_{\varepsilon_j}(\epsilon_j) d\epsilon_j$. Her marketing decision is not only based on the controllable part $r(\mathbf{q})$, but also on the random part ε. Moreover, her decision should concern not only the procurement cost, but also the inventory cost. Such interaction is static, since it doesn't depend on the realtime operations information (e.g. inventory level). So we call it *static* interaction between operations and marketing management. However, when she has some products overstocked, she has to price based on the overstocking list. We call it *dynamic* interaction.

Multiplicative Demand

Finally, we analyze the multiplicative demand function, which can be simplified as $\Lambda\mathbf{q}$. The corresponding inventory related cost can be expressed as

$$EH(\mathbf{y}_t - \mathbf{D}(\mathbf{q}_t)) = h_j \sum_{j=1}^{J} \int_0^{y_j/q_j} (y_j - \lambda q_j) f_\Lambda(\lambda) d\lambda + b_j \sum_{j=1}^{J} \int_{y_j/q_j}^{\infty} (-y_j + \lambda q_j) f_\Lambda(\lambda) d\lambda.$$

Proposition 10 *For the multiplicative demand function, we can obtain the unconstrained optima* $(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*)$ *by solving the following equations.*

$$\begin{cases} y_j = q_j \omega_j \ ; \\ \beta \dfrac{\partial r(\mathbf{q})}{\partial q_j} = c_j + (b_j + h_j) \dfrac{1}{\mu_\Lambda} \displaystyle\int_0^{\omega_j} (\mu_\Lambda - \lambda) f_\Lambda(\lambda) d\lambda. \end{cases}$$

If $\mathbf{q}_\pi^* \in \mathcal{Q}_t$*, and* $\mathbf{y}_\pi^* \geq \mathbf{x}_1$*, then* $(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*)$ *is the optimal policy.*

Under the multiplicative demand, when the retailer doesn't incur overstocking, she should consider an additional operations cost, $(b_j + h_j)\frac{1}{\mu_\Lambda}\int_0^{\omega_j}(\mu_\Lambda - \lambda) f_\Lambda(\lambda) d\lambda$, when doing pricing. If overstocking takes place, she has to decide market share conditioning on the overstocking list, thereafter, the dynamic interaction takes place.

5.3 Numerical Study of Myopic Policy

We now numerically study the dynamic interaction, showing the dynamics of myopic policy conditioning on the inventory level. To clearly illustrate the main results, we study the myopic policy, $(\mathbf{y}_\pi^*,\ \mathbf{q}_\pi^*) \in \arg\max_{\mathbf{y} \geq \mathbf{x};\ \mathbf{q} \in \mathcal{Q}_t} E\pi(\mathbf{y},\ \mathbf{q})$, for two substitutable products in the numerical research.

We have following assumptions about demands. First, $\mathbf{q}$ and $r(\mathbf{q})$ are established in the logit model with parameter $(\Theta_1, \Theta_2) = (13.2, 13)$, which means product 1 has higher average perceptive value than product 2. Second, the demands are in additive form with error terms proportional to the market share, $\mu_\Lambda \mathbf{q} + (q_1\varepsilon_1, ..., q_J\varepsilon_J)$, where $\mu_\Lambda = 100$, ε_j is i.i.d., and $\varepsilon_j \sim \mathbf{U}(-50, 50)$, for $j = 1, 2$. We have the symmetric cost structure, such that $\mathbf{h} = (0.5, 0.5)$, $\mathbf{b} = (4.5, 4.5)$, and $\mathbf{c} = (10, 10)$. We assume the discount rate $\beta = 0.95$.

We run simulation in two scenarios. (a) We fixed the initial inventory of product 2 at 30, and control the initial inventory level of product 1 from 30 to 69. We tend to see the dynamics of myopic policy conditioning on the initial inventory level of the relatively popular product. (b) We fixed the initial inventory level of product 1 at 30, and manipulate the initial inventory level of product 2 from 30 to 69, to detect the dynamics of myopic policy conditioning on the initial inventory level of less popular product. In the following pictures, the left figure describes the myopic policy in scenario a, and the right figure describes the myopic policy in scenario b.

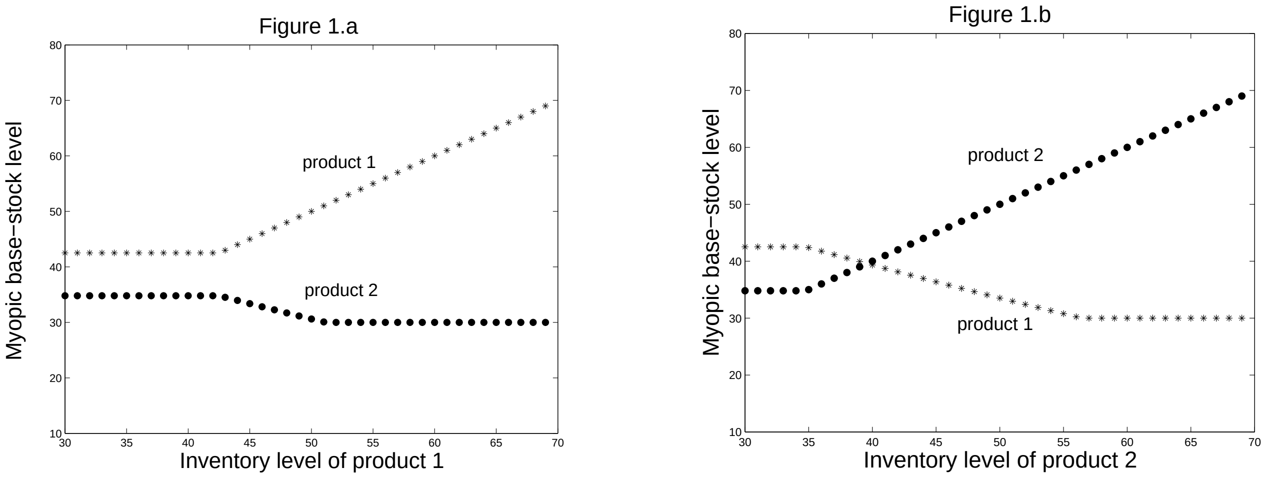

Figure 1: Optimal myopic base-stock level conditioning on the initial inventory of products.

Figure 1 illustrates the dynamics of myopic base-stock level. It is optimal to set a higher base-stock level for the popular product, when no product is overstocked. It is optimal to stop ordering the product with initial inventory level over the threshold. When one product is overstocked, the myopic threshold of another product decreases dynamically, conditioning on the degree of overstock. When the threshold finally goes below the initial inventory level of another product, both products are overstocked. We can explicitly see the parametric monotonicity, that is, the inventory threshold of the under-stocked product is conditioning on the initial inventory of the overstocked product, and the threshold decreases as the degree of overstock increases. The parametric monotonicity of inventory thresholds is consistent with the monotone property of market share conditioning on the degree of overstock.

Figure 2 reveals the dynamics of myopic market share. It is optimal to set higher market share for the popular product, when no overstocking occurs. It is good to increase the market

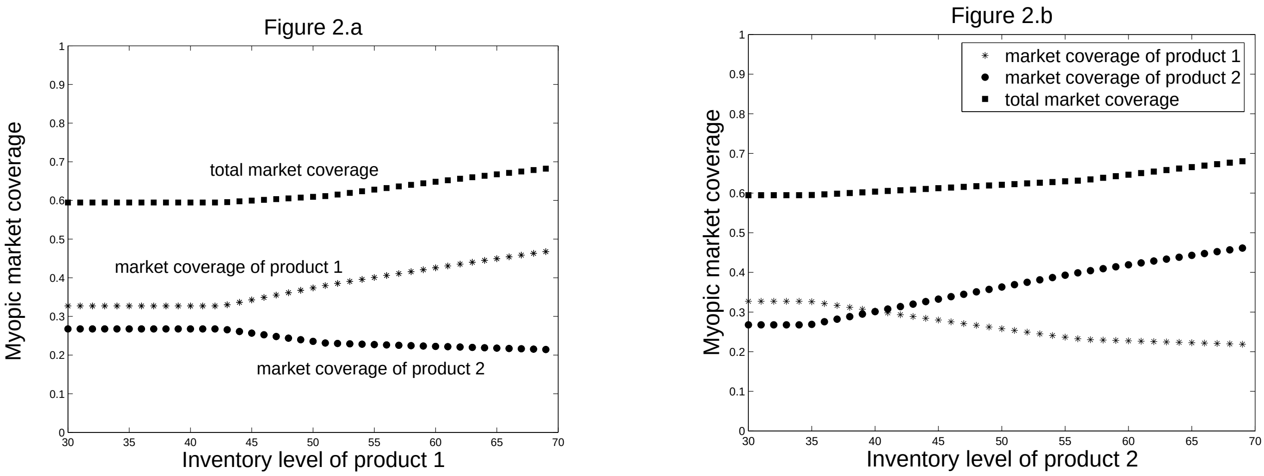

Figure 2: Optimal myopic market share conditioning on the initial inventory of products.

share of overstocked product, and decrease the market share of under-stocked product. For the understocked product, the base-stock level is proportional to the market share, therefore, we have the monotone property of inventory threshold conditioning on the overstocking level.

Figure 3 shows the dynamics of myopic pricing scheme. There are two findings of interest. First, the same price is charged to both products, although one of them is generally perceived more valuable by customers. Therefore, it is reasonable to charge a single category price for all the products having the same cost structure, whenever none of them is overstocked. However it is not true when the cost structure is different. Second, when one product is overstocked, it is good to decrease the price of the overstocked product and simultaneously increase the price of the under-stocked, which results the monotonicity of market share. When both products are overstocked, price goes down for both of them.

Further, Figure 3 tells a consumer it is not likely to see the markdown of his/her favorite product, when he/she observes high inventory of some other products in the same category. However, if the retailer always charges a single price for the whole category, the situation might be different (see Song and Xue 2007 b).

6 Concluding Remarks

This paper studied demand management and inventory control for substitutable products. In our model, a retailer makes joint price and inventory decision at the beginning of each period, based on the inventory level. We consider the price driven substitution, so the random

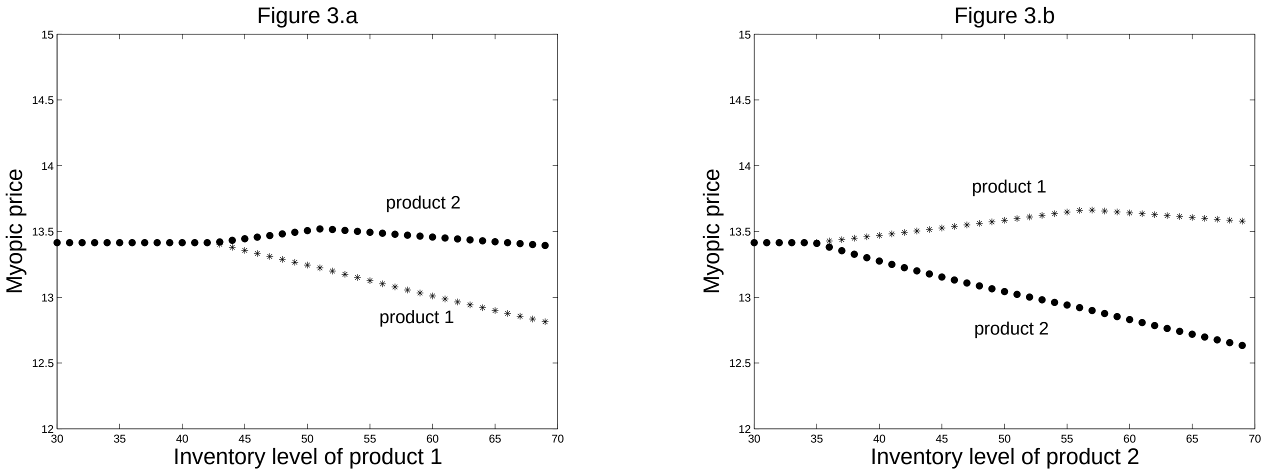

Figure 3: Optimal myopic price conditioning on the initial inventory of products.

demands are affected by the price vector.

We showed that the optimal policy consists of three components, the overstocking list, the inventory thresholds, and the target value of market share. It is optimal for the decision maker to stop ordering the products on the list, and set market share and inventory thresholds conditioning on the overstocking level. The overstocking list and inventory thresholds can be recursively calculated by some algorithm. The optimal policy and algorithm are established under certain mild condition, which holds in the logit and location model. The structure of optimal policy can also be extended to the lost sales case.

We further studied the interaction between marketing and operations decisions. The optimal policy shows the demand management depends on the inventory levels, when some products are overstocked. The myopic policy reveals the sales department should count operations cost, when setting marginal revenue equal to marginal cost, even if the demand is stationary and no product is overstocked.

There are still many topics for future research. First, the effective heuristics and simple form solution are critically important to the quick response in the management of multiple products. The computation of the optimal policy is demanding and time consuming, due to the large state space. The myopic policy provides one acceptable heuristic in the stationary system, but it is not expected to perform well under the non-stationary demand. Second, we can extend the study to include the fixed ordering cost and lead time. The (s, S) type policy might be optimal to the system with fixed ordering cost, however the parameter s may be very difficult to characterize. Third, we may consider stock-out based substitutions.

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Appendix

Proposition 1

Proof : The qualification condition of market share comes from the fact that price should be non-negative. Consider any set of products, $\mathcal{J}_s \subseteq \mathcal{J}$. When retailer sets price at zero for the products within the set and sets price to infinity (or the highest level allowed) for the other products, this set occupies the highest possible market share, $\frac{\sum_{j\in\mathcal{J}_s}\exp(\Theta_j)}{1+\sum_{j\in\mathcal{J}_s}\exp(\Theta_j)}$.

We derive the inverse market share function by direct transformation. The equations of (4) can be transformed to

$$\exp(\Theta_j - p_j) = \frac{q_j}{1 - q_1 - ... - q_J}, \quad \forall j \in \mathcal{J}.$$

When $\mathbf{q}$ satisfies the qualification conditions, we obtain the inverse market share function.

$$p_j(\mathbf{q}) = \Theta_j + \ln(1 - q_1 - ... - q_J) - \ln(q_j), \quad \forall j \in \mathcal{J}.$$

■

Proposition 3

Proof : When Assumption 4 holds, no product will reach the position of other products. We first consider the left edge of the share of product 1. Product 1 cannot cover the zero point, if p_1 is high enough, otherwise it covers the left end of line, so the left edge of product 1 should be $\max\{0, \tau_1^l\}$. Secondly, we consider the right edge of product 1. Product 1 act as monopolist, if p_2 is high enough, otherwise it compete with product 2, thus the right edge of product 1 should be $\min\{\tau_{12}^m, \tau_1^r\}$. Therefore the share range $q_1(p_1, p_2)$ is equal to the distance between the right edge and left edge. Similarly the share of product 2 is given after the same argument. ■

Theorem 1

Proof : (a) Similar to Federgruen and Heching (1999), we prove the joint concavity of $G_t(\mathbf{y}, \mathbf{p})$ by induction. The first term of (15) is concave in $\mathbf{q}$, according to Assumption 1. The second term is linear in $\mathbf{q}$, and the third term is linear in $\mathbf{y}$. The fourth term is jointly concave, in that the inventory related cost is jointly convex in $(\mathbf{y}, \mathbf{q})$. As we set $V_{T+1}(\mathbf{x}) = 0$, $G_T(\mathbf{y}, \mathbf{q})$

is jointly concave in $(\mathbf{y}, \mathbf{q})$. Thus $V_T(\mathbf{x})$ is concave in $\mathbf{x}$ (see Sundaram 1996 Theorem 9.17), and it is also non-increasing, since $\mathbf{y}$ is controlled in the region such that $\mathbf{y} \geq \mathbf{x}$.

Now we argue by induction. Assuming $G_{t+1}(\mathbf{y}, \mathbf{q})$ is jointly concave for some $t \in \{T-1, ..., 1\}$, and then $V_{t+1}(\mathbf{x})$ is non-increasingly concave in $\mathbf{x}$, we can show $G_t(\mathbf{y}, \mathbf{q})$ is jointly concave also. The joint concavity of the first four terms of $G_{t+1}(\mathbf{y}, \mathbf{q})$ has been proven. To prove the last term $EV_{t+1}(\mathbf{y} - \mathbf{D}_t(\mathbf{q}))$ is jointly concave, fix the random part $\mathbf{D}_t$ to its realization $\mathbf{d}_t$, consider any pair $(\mathbf{y}, \mathbf{q})$ and $(\mathbf{y}', \mathbf{q}')$, we have

$$\mathbf{d}_t(\alpha\mathbf{q} + (1-\alpha)\mathbf{q}') = \alpha\mathbf{d}_t(\mathbf{q}) + (1-\alpha)\mathbf{d}_t(\mathbf{q}'),$$

since $\mathbf{d}_t(\mathbf{q}) = \lambda_t\mathbf{q} + \mathbf{L}(\mathbf{q})\epsilon_t$, and $\mathbf{L}$ is linear transformation, satisfying $\mathbf{L}(\alpha\mathbf{q} + (1-\alpha)\mathbf{q}') = \alpha\mathbf{L}(\mathbf{q}) + (1-\alpha)\mathbf{L}(\mathbf{q}')$, for $0 \leq \alpha \leq 1$.

Because $V_{t+1}(\mathbf{x})$ is non-increasingly concave, we obtain

$$\begin{aligned}
& V_{t+1}\left(\alpha\mathbf{y} + (1-\alpha)\mathbf{y}' - \mathbf{d}_t(\alpha\mathbf{q} + (1-\alpha)\mathbf{q}')\right) \\
= \; & V_{t+1}\left(\alpha\mathbf{y} + (1-\alpha)\mathbf{y}' - \alpha\mathbf{d}_t(\mathbf{q}) - (1-\alpha)\mathbf{d}_t(\mathbf{q}')\right) \\
= \; & V_{t+1}\left(\alpha(\mathbf{y} - \mathbf{d}_t(\mathbf{q})) + (1-\alpha)(\mathbf{y}' - \mathbf{d}_t(\mathbf{q}'))\right) \\
\geq \; & \alpha V_{t+1}\left(\mathbf{y} - \mathbf{d}_t(\mathbf{q})\right) + (1-\alpha)V_{t+1}\left(\mathbf{y}' - \mathbf{d}_t(\mathbf{q}')\right).
\end{aligned}$$

(b) The proof needs to discuss the limit of $G_t(\mathbf{y}, \mathbf{q})$ as y_j goes to positive and negative infinity, which is similar to that of Federgruen and Heching (1999).

(c) Part (c) immediately results from (a) and (b). ■

Proposition 5

Proof : (a) Assume $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) \in \arg\max_{\mathbf{y}\in\mathcal{R}^2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q})$ is one optimal solution of the less constrained maximization problem, and $(\mathbf{y}_t^*, \mathbf{q}_t^*) \in \arg\max_{\mathbf{y}\geq\mathbf{x}\ ;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q})$ be any optimal solution under the full constraints.

When $x_1 > y_{1t}^*(\emptyset)$ and $x_2 \leq y_{2t}^*(\emptyset)$, we want to prove it is unnecessary to order any positive amount of product 1. We assume conversely that any optimal solution must have $y_{1t}^* > x_1$, that is, it is optimal only if retailer orders some positive amount of item 1. Thus, under the converse assumption, there is no optimal solution such that $y_{1t}^* = x_1$. Now we have $y_{1t}^* > x_1 > y_{1t}^*(\emptyset)$, and obtain a α, where $\alpha = \dfrac{x_1 - y_{1t}^*(\emptyset)}{y_{1t}^* - y_{1t}^*(\emptyset)}$, satisfying $0 < \alpha < 1$. Let

$$(\mathbf{y}_t', \mathbf{q}_t') = \alpha(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1-\alpha)(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)).$$

We can prove $(\mathbf{y}_t', \mathbf{q}_t')$ is feasible and suboptimal. First, we have $\mathbf{y}_t' \geq \mathbf{x}$, since $y_{1t}' = \alpha y_{1t}^* + (1-\alpha) y_{1t}^*(\emptyset) = x_1$, and $y_{2t}' = \alpha y_{2t}^* + (1-\alpha) y_{2t}^*(\emptyset) \geq \alpha x_2 + (1-\alpha) x_2 = x_2$. Second, we have $\mathbf{q}_t' \in \mathcal{Q}_t$, since $\mathbf{q}_t' = \alpha \mathbf{q}_t^* + (1-\alpha)\mathbf{q}_t^*(\emptyset)$, and the control space $\mathcal{Q}_t$ is convex. Last, it is suboptimal, since $y_{1t}' = x_1$.

As $(\mathbf{y}_t^*, \mathbf{q}_t^*)$ is an optimal solution, and $(\mathbf{y}_t', \mathbf{q}_t')$ is feasible and suboptimal, we have $G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) > G_t(\mathbf{y}_t', \mathbf{q}_t')$. As $(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset))$ is the maximum solution of a less constrained maximization problem, we have $G_t(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) > G_t(\mathbf{y}_t', \mathbf{q}_t')$. Thus, we obtain

$$\alpha G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1-\alpha) G_t(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) > G_t(\mathbf{y}_t', \mathbf{q}_t').$$

It contradicts the concavity of G_t, hence we must have some optimal solution such that $y_{1t}^* = x_1$, i.e., it is optimal for the retailer to stop ordering item 1.

(b) We add product 1 on the overstocking list, fix $y_{1t} = x_1$, solve

$$\max_{\mathbf{y} \in \mathcal{R}^2;\ \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_1 = x_1),$$

and obtain some maximum solution $(\mathbf{y}_t^*(x_1), \mathbf{q}_t^*(x_1))$, where $y_{1t}^*(x_1) = x_1$.

If $y_{2t}^*(x_1) \geq x_2$, then it is optimal to order product 2 up to $y_{2t}^*(x_1)$, and set market share at $(q_{1t}^*(x_1), q_{2t}^*(x_1))$. Otherwise, by the similar argument of part (a), it is optimal for retailer to stop ordering product 2. Further, we solve the following constrained maximum problem,

$$\max_{\mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} \mid y_1 = x_1, y_2 = x_2),$$

and obtain the market share, $\mathbf{q}_t^*(\mathbf{x}) = (q_{1t}^*(x_1, x_2), q_{2t}^*(x_1, x_2))$. ■

Proposition 6

Proof : (a) When $x_1 > y_{1t}^*(\emptyset)$ and $x_2 > y_{2t}^*(\emptyset)$, we want to prove it is optimal to stop ordering at least one product, say k, $k \in \{1, 2\}$. We assume conversely that any optimal solution must have $\mathbf{y}^*(\emptyset) > \mathbf{x}$, that is, it is optimal only if retailer orders some positive amount of both products. Then $\mathbf{y}^* > \mathbf{x} > \mathbf{y}^*(\emptyset)$. We can locate a k, where $k \in \arg\max_{j \in \{1,2\}} \left\{ \dfrac{x_j - y_{jt}^*(\emptyset)}{y_{jt}^* - y_{jt}^*(\emptyset)} \right\}$, and construct an $\alpha = \dfrac{x_k - y_{kt}^*(\emptyset)}{y_{kt}^* - y_{kt}^*(\emptyset)}$. Then we have $x_k = \alpha y_{kt}^* + (1-\alpha) y_{kt}^*(\emptyset)$. Let

$$(\mathbf{y}_t', \mathbf{q}_t') = \alpha(\mathbf{y}_t^*, \mathbf{q}_t^*) + (1-\alpha)(\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)).$$

Similar to the proof of proposition 5, we can prove $(\mathbf{y}_t', \mathbf{q}_t')$ is feasible and suboptimal. As $(\mathbf{y}_t^*, \mathbf{q}_t^*)$ is an optimal solution, and $(\mathbf{y}_t', \mathbf{q}_t')$ is feasible and suboptimal, we have $G_t(\mathbf{y}_t^*, \mathbf{q}_t^*) >$

$G_t(\mathbf{y}'_t, \mathbf{q}'_t)$. As $(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset))$ is an optima of the less constrained maximization problem, we have $G_t(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset)) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$. Thus, we obtain

$$\alpha G_t(\mathbf{y}^*_t, \mathbf{q}^*_t) + (1-\alpha) G_t(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset)) > G_t(\mathbf{y}'_t, \mathbf{q}'_t).$$

It destroys the concavity of G_t. Therefore, we have $y^*_{kt} = x_k$, i.e., it can be optimal for the retailer to stop ordering item k.

(b) The proof is similar to that of proposition 5. ∎

Proposition 7

Proof : Proposition 6 shows it is optimal to stop ordering at least one product. Without loss of generality, assume product 1 is overstocked. We tend to prove product 2 is also overstocked, by using parametric monotonicity. Now we have $y^*_{1t} = x_1 > y^*_{1t}(\emptyset)$, and $x_2 > y^*_{2t}(\emptyset)$. First, we are going to show $y^*_{2t}(\emptyset) \geq y^*_{2t}(x_1)$, where

$$\begin{cases} (\mathbf{y}^*_t(x_1), \mathbf{q}^*_t(x_1)) \in \arg \max\limits_{\mathbf{y}\in\mathcal{R}^2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} |\ y_1 = x_1); \\ (\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset)) \in \arg \max\limits_{\mathbf{y}\in\mathcal{R}^2;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q} |\ y_1 = y^*_{1t}(\emptyset)). \end{cases}$$

The inequality holds due to the assumption of parametric monotonicity, since $x_1 > y^*_{1t}(\emptyset)$. It means if we put product 1 on the overstocking list, and compute an inventory threshold of product 2, conditioning on x_1, the new threshold is no higher than the previous threshold. Therefore, we have $x_2 > y^*_{2t}(x_1)$. Next we show $y^*_{2t} = x_2$ by converse argument.

We conversely assume that any optimal solution must have $y^*_{1t} = x_1$, and $y^*_{2t} > x_2$ that is, it is optimal only if retailer orders some positive amount product 2. Then $y^*_{1t} = x_1 > y^*_{1t}(\emptyset)$, and $y^*_{2t} > x_2 > y^*_{2t}(\emptyset)$. Let

$$(\mathbf{y}'_t, \mathbf{q}'_t) = \alpha(\mathbf{y}^*_t, \mathbf{q}^*_t) + (1-\alpha)(\mathbf{y}^*_t(x_1), \mathbf{q}^*_t(x_1)),$$

where $\alpha = \dfrac{x_2 - y^*_{2t}(x_1)}{y^*_{2t} - y^*_{2t}(x_1)}$. We have $x_2 = \alpha y^*_{2t} + (1-\alpha) y^*_{2t}(x_1)$.

It is not difficult to show $(\mathbf{y}'_t, \mathbf{q}'_t)$ is feasible and suboptimal, since $\mathbf{y}'_t = \mathbf{x}$. We have $G_t(\mathbf{y}^*_t, \mathbf{q}^*_t) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$, and $G_t(\mathbf{y}^*_t(x_1), \mathbf{q}^*_t(x_1)) \geq G_t(\mathbf{y}'_t, \mathbf{q}'_t)$. The second inequality holds, in that $\mathbf{y}'_t = \mathbf{x}$, and then $G_t(\mathbf{y}^*_t(x_1), \mathbf{q}^*_t(x_1)) \geq G_t(\mathbf{y}^*_t(\mathbf{x}), \mathbf{q}^*_t(\mathbf{x})) \geq G_t(\mathbf{y}'_t, \mathbf{q}'_t)$. Then, we obtain

$$\alpha G_t(\mathbf{y}^*_t, \mathbf{q}^*_t) + (1-\alpha) G_t(\mathbf{y}^*_t(x_1), \mathbf{q}^*_t(x_1)) > G_t(\mathbf{y}'_t, \mathbf{q}'_t),$$

which contradicts the concavity of G_t. Hence it is optimal for the retailer to stop ordering item 2 also. ∎

Theorem 2 & Algorithm 1

Proof : We prove it step by step. In each step, the key is to prove whenever $\mathcal{A}_m = \left\{ j \mid y^*_{jt}(\mathbf{x}_{\mathcal{J}^+_{m-1}}) < x_j \right\}$ is a non-empty set, we can always find a $k \in \mathcal{A}_m$, for which it is optimal to stop ordering.

In step 1, let

$$(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset)) \in \arg \max_{\mathbf{y} \in \mathcal{R}^J;\ \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}),$$

which is the maximum solution without the constraints $\mathbf{y} \geq \mathbf{x}$, and

$$(\mathbf{y}^*_t, \mathbf{q}^*_t) \in \arg \max_{\mathbf{y} \geq \mathbf{x},\ \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}).$$

which is the optimal solution under the constraints such that $\mathbf{y} \geq \mathbf{x}$.

If $\mathbf{y}^*_t(\emptyset) \geq \mathbf{x}$, then the optimal policy is given by $(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset))$. Otherwise, there is some $j \in \mathcal{J}^-_0 = \mathcal{J}$, such that $y^*_{jt} < x_j$, i.e., $\mathcal{A}_1 = \left\{ j \mid y^*_{jt}(\emptyset) < x_j \right\}$ is a non-empty set.

We tend to prove there always exist a $k \in \mathcal{A}_1$, for which it is optimal to stop ordering. Now we assume conversely that it is optimal only if retailer order some positive amount for all the products in $\mathcal{A}_1$, i.e. $y^*_{jt} > x_j$, for any $j \in \mathcal{A}_1$.

We can always locate a $k \in \mathcal{A}_1$, where $k \in \arg \max_{\{j \in \mathcal{A}_1\}} \{ \frac{x_j - y^*_{jt}(\emptyset)}{y^*_{jt} - y^*_{jt}(\emptyset)} \}$, and construct an $\alpha \in (0,1)$, where $\alpha = \frac{x_k - y^*_{kt}(\emptyset)}{y^*_{kt} - y^*_{kt}(\emptyset)}$. Then we have $x_j \leq \alpha y^*_{jt} + (1-\alpha) y^*_{jt}(\emptyset)$, for all $j \in \mathcal{A}_1$, and $x_k = \alpha y^*_{kt} + (1-\alpha) y^*_{kt}(\emptyset)$. Let

$$(\mathbf{y}'_t, \mathbf{q}'_t) = \alpha(\mathbf{y}^*_t, \mathbf{q}^*_t) + (1-\alpha)(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset)).$$

We can show $(\mathbf{y}'_t, \mathbf{q}'_t)$ is a feasible and suboptimal solution, by the similar argument in the proof of Proposition 6.

As $(\mathbf{y}^*_t, \mathbf{q}^*_t)$ is assumed to be the optimal solution of the problem, and $(\mathbf{y}'_t, \mathbf{q}'_t)$ is a feasible and suboptimal solution, we have $G_t(\mathbf{y}^*_t, \mathbf{q}^*_t) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$. As $(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset))$ is a maximum solution of less constrained problem, we have $G_t(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset)) > G_t(\mathbf{y}'_t, \mathbf{q}'_t)$. Thus, we obtain

$$\alpha G_t(\mathbf{y}^*_t, \mathbf{q}^*_t) + (1-\alpha) G_t(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset)) > G_t(\mathbf{y}'_t, \mathbf{q}'_t).$$

It ruins the concavity of G_t. Hence we must have $y^*_{kt} = x_k$, i.e., it is optimal for the retailer to stop ordering product k. Then we update $\mathcal{J}^+_1 = \mathcal{J}^+_0 \cup \{k\}$, and continue searching the optimal solution in a maximization problem with more equality constraints.

$$\max_{\mathbf{y} \geq \mathbf{x};\ \mathbf{q} \in \mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}) = \max_{\mathbf{y} \geq \mathbf{x};\ \mathbf{q} \in \mathcal{Q}_t} G_t \left(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{J}^+_1} = \mathbf{x}_{\mathcal{J}^+_1} \right).$$

If $\mathcal{J}_1^+ \subset \mathcal{J}$, we go to the next step.

In step m, assume the above conclusions are true, by a similar argument, we can show the results also holds in the $m+1^{th}$ step. The proof is conducted to (1) make some converse assumption about the optimal solution, (2) find an $\alpha \in (0,1)$, (3) locate a feasible and suboptimal solution $(\mathbf{y}'_t, \mathbf{q}'_t)$, where $(\mathbf{y}'_t, \mathbf{q}'_t) = \alpha(\mathbf{y}^*_t, \mathbf{q}^*_t) + (1-\alpha)\left(\mathbf{y}^*_t(\mathbf{x}_{\mathcal{J}^+_{m-1}}), \mathbf{q}^*_t(\mathbf{x}_{\mathcal{J}^+_{m-1}})\right)$, and (4) show the concavity of G_t is destroyed on the line from $\left(\mathbf{y}^*_t(\mathbf{x}_{\mathcal{J}^+_{m-1}}), \mathbf{q}^*_t(\mathbf{x}_{\mathcal{J}^+_{m-1}})\right)$ to $(\mathbf{y}'_t, \mathbf{q}'_t)$ to $(\mathbf{y}^*_t, \mathbf{q}^*_t)$.

Keep doing the iteration, the structure of optimal policy can be obtained in less than J steps in each period t. ∎

Algorithm 2

Proof : We also prove it step by step. In each step, the key is to prove whenever $\mathcal{A}_m = \left\{j \mid y^*_{jt}(\mathbf{x}_{\mathcal{J}^+_{m-1}}) < x_j\right\}$ is a non-empty set, it is optimal to stop ordering any product $k \in \mathcal{A}_m$.

In step 1, let

$$(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset)) \in \arg \max_{\mathbf{y}\in\mathbf{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}),$$

which is the maximum solution without the constraints $\mathbf{y} \geq \mathbf{x}$, and

$$(\mathbf{y}^*_t, \mathbf{q}^*_t) \in \arg \max_{\mathbf{y}\geq\mathbf{x},\ \mathbf{q}\in\mathcal{Q}_t} G_t(\mathbf{y}, \mathbf{q}).$$

which is the optimal solution under the constraints such that $\mathbf{y} \geq \mathbf{x}$.

If $\mathbf{y}^*_t(\emptyset) \geq \mathbf{x}$, then the optimal policy is given by $(\mathbf{y}^*_t(\emptyset), \mathbf{q}^*_t(\emptyset))$. Otherwise, there is some $j \in \mathcal{J}_0^- = \mathcal{J}$, such that $y^*_{jt} < x_j$, i.e., $\mathcal{A}_1 = \left\{j \mid y^*_{jt}(\emptyset) < x_j\right\}$ is a non-empty set.

Algorithm 1 states there always exist at least one $k_1 \in \mathcal{A}_1$, for which it can be optimal to stop ordering. Here we tend to prove if $\mathcal{A}_1$ contains more than one products, it is optimal to stop ordering all of them under Assumption 5. First, we make converse assumption that it is optimal only if retailer order some positive amount for some products in $\mathcal{A}_1$, when $\mathcal{A}_1$ contains more than one elements, that is, there exists some $l \in \mathcal{A}_1$, which is understocked. Hence, $\mathcal{A}_1$ can be divided into two mutual exclusive non-empty subsets, $\mathcal{A}_1^+$ and $\mathcal{A}_1^-$. The former contains products overstocked, and the latter contains products understocked. So $k_1 \in \mathcal{A}_1^+$, and $l \in \mathcal{A}_1^-$.

We can show

$$\left(\mathbf{y}^*_t(\mathbf{x}_{\mathcal{A}_1^+})\right)_{\mathcal{A}_1^-} \leq (\mathbf{y}^*_t(\emptyset))_{\mathcal{A}_1^-} < \mathbf{x}_{\mathcal{A}_1^-}, \tag{20}$$

$$\text{where} \quad \begin{cases} \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{A}_1^+})\right) \in \arg\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{A}_1^+} = \mathbf{x}_{\mathcal{A}_1^+}\right); \\ (\mathbf{y}_t^*(\emptyset), \mathbf{q}_t^*(\emptyset)) \in \arg\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{A}_1^+} = (\mathbf{y}_t^*(\emptyset))_{\mathcal{A}_1^+}\right). \end{cases}$$

The first inequality results from the parametric monotonicity, since $(\mathbf{y}_t^*(\emptyset))_{\mathcal{A}_1^+} \leq \mathbf{x}_{\mathcal{A}_1^+}$. The second inequality holds, since $(\mathbf{y}_t^*(\emptyset))_{\mathcal{A}_1} < \mathbf{x}_{\mathcal{A}_1}$, and $\mathcal{A}_1^- \subseteq \mathcal{A}_1$.

Let $\mathcal{J}_1^+ = \mathcal{J}_0^+ \cup \mathcal{A}_1^+ = \mathcal{A}_1^+$, updating the overstocking list, and go to the next step.

In step 2, there always exists a non-empty set, $\mathcal{A}_2 = \left\{j \mid y_{jt}^*(\mathbf{x}_{\mathcal{J}_1^+}) < x_j\right\}$, where $\mathcal{A}_1^- \subseteq \mathcal{A}_2$. It is non-empty, since at least product l is still in $\mathcal{A}_2$.

According to Algorithm 1, there always exists another element $k_2 \in \mathcal{A}_2$, which is overstocked. If k_2 comes from $\mathcal{A}_1^-$, then it contradicts the assumption that any element in $\mathcal{A}_1^-$ is understocked. Therefore, k_2 must be some product other than l. Then we can divide $\mathcal{A}_2$ into two mutual exclusive non-empty subsets, $\mathcal{A}_2^+$ and $\mathcal{A}_2^-$, where we have $\mathcal{A}_1^- \subseteq \mathcal{A}_2^- \subset \mathcal{A}_2$, $\mathcal{A}_1^- \cap \mathcal{A}_2^+ = \emptyset$, $\mathcal{A}_1^+ \cap \mathcal{A}_2^+ = \emptyset$, and $\mathcal{A}_2^+ \subseteq \mathcal{J}_1^- / \mathcal{A}_1^-$.

We can show

$$\left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+})\right)_{\mathcal{A}_1^-} \leq \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+})\right)_{\mathcal{A}_1^-} < \mathbf{x}_{\mathcal{A}_1^-}, \tag{21}$$

$$\text{where} \quad \begin{cases} \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+})\right) \in \arg\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+} = \mathbf{x}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+}\right); \\ \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+}), \mathbf{q}_t^*(\mathbf{x}_{\mathcal{A}_1^+})\right) \in \arg\max_{\mathbf{y}\in\mathcal{R}^J;\ \mathbf{q}\in\mathcal{Q}_t} G_t\left(\mathbf{y}, \mathbf{q} \mid \mathbf{y}_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+} = \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+})\right)_{\mathcal{A}_1^+ \cup \mathcal{A}_2^+}\right). \end{cases}$$

The first inequality comes from the parametric monotonicity, since $\left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+})\right)_{\mathcal{A}_1^+} = \mathbf{x}_{\mathcal{A}_1^+}$, and $\left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{A}_1^+})\right)_{\mathcal{A}_2^+} < \mathbf{x}_{\mathcal{A}_2^+}$, where $\mathcal{A}_2^+ \subset \mathcal{A}_2 = \left\{j \mid y_{jt}^*(\mathbf{x}_{\mathcal{J}_1^+}) < x_j\right\}$, and $\mathcal{J}_1^+ = \mathcal{A}_1^+$.

The second inequality directly comes from the results in the first step.

Let $\mathcal{J}_2^+ = \mathcal{J}_1^+ \cup \mathcal{A}_2^+ = \mathcal{A}_1^+ \cup \mathcal{A}_2^+$, updating the overstocking list.

By induction, if we insist any product l in $\mathcal{A}_1^-$ is understocked, we have to assume $\mathcal{A}_1^- \cap \mathcal{A}_m^+ = \emptyset$, for $m = 1, 2, ...$ However, under Assumption 5, we always have

$$\mathbf{x}_{\mathcal{A}_1^-} > \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_0^+})\right)_{\mathcal{A}_1^-} \geq \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_1^+})\right)_{\mathcal{A}_1^-} \geq \left(\mathbf{y}_t^*(\mathbf{x}_{\mathcal{J}_2^+})\right)_{\mathcal{A}_1^-} \geq \ldots \quad , \tag{22}$$

where $\mathcal{J}_m^+ = \mathcal{A}_1^+ \cup \ldots \cup \mathcal{A}_m^+$. Then Algorithm 1 continues forever. However, $\mathcal{J}$ is a finite set, the algorithm cannot infinitely continue adding $\mathcal{A}_m^+$ on $\mathcal{J}_{m-1}^+$. Therefore, sooner or later, we have to put l on the overstocking list.

■

Theorem 3

Proof : Obviously, $EH_t(\mathbf{z}) = E\{\mathbf{h}_t[\mathbf{z}]^+ + \mathbf{b}_t[\mathbf{z}]^-\}$ is strictly convex in $\mathbf{z}$.

Consider two vectors of decision variables, $(\mathbf{y}, \mathbf{q})$ and $(\mathbf{y}', \mathbf{q}')$, and $(\mathbf{y}, \mathbf{q}) \neq (\mathbf{y}', \mathbf{q}')$. We consider two cases.

Case 1: If $\mathbf{y} - \mathbf{d}_t(\mathbf{q}) \neq \mathbf{y}' - \mathbf{d}_t(\mathbf{q}')$, then we have

$$\begin{aligned} & H_t\left(\alpha\mathbf{y} + (1-\alpha)\mathbf{y}' - \mathbf{d}_t(\alpha\mathbf{q} + (1-\alpha)\mathbf{q}')\right) \\ = \; & H_t\left(\alpha(\mathbf{y} - \mathbf{d}_t(\mathbf{q})) + (1-\alpha)(\mathbf{y}' - \mathbf{d}_t(\mathbf{q}'))\right) \\ < \; & \alpha H_t\left(\mathbf{y} - \mathbf{d}_t(\mathbf{q})\right) + (1-\alpha) H_t\left(\mathbf{y}' - \mathbf{d}_t(\mathbf{q}')\right). \end{aligned}$$

Case 2: If $\mathbf{y} - \mathbf{d}_t(\mathbf{q}) = \mathbf{y}' - \mathbf{d}_t(\mathbf{q}')$, then we have

$$\begin{aligned} & H_t\left(\alpha\mathbf{y} + (1-\alpha)\mathbf{y}' - \mathbf{d}_t(\alpha\mathbf{q} + (1-\alpha)\mathbf{q}')\right) \\ = \; & \alpha H_t\left(\mathbf{y} - \mathbf{d}_t(\mathbf{q})\right) + (1-\alpha) H_t\left(\mathbf{y}' - \mathbf{d}_t(\mathbf{q}')\right). \end{aligned}$$

However, we cannot have $\mathbf{q} = \mathbf{q}'$, otherwise we must have $\mathbf{y} = \mathbf{y}'$, contradicting the assumption that $(\mathbf{y}, \mathbf{q}) \neq (\mathbf{y}', \mathbf{q}')$. Then

$$r(\alpha\mathbf{q} + (1-\alpha)\mathbf{q}') > \alpha r(\mathbf{q}) + (1-\alpha) r(\mathbf{q}').$$

Therefore, $G_t(\mathbf{y}, \mathbf{q})$ is always strictly concave in $(\mathbf{y}, \mathbf{q})$, and the algorism can guarantee a unique optimal solution. ∎

Proposition 8, Proposition 9, & Proposition 10

Proof : The solution results from the first order condition, by solving $\bigtriangledown_{(\mathbf{y},\ \mathbf{q})} E\pi(\mathbf{y},\ \mathbf{q}) = 0$.

In proposition 8, we have

$$\begin{cases} F_{\varepsilon_j}(y_j - \mu_\Lambda q_j) = \frac{b_j - c_j(1-\beta)}{b_j + h_j}; \\ F_{\varepsilon_j}(y_j - \mu_\Lambda q_j) = \frac{b_j - \beta\frac{\partial r(\mathbf{q})}{\partial q_j} + \beta c_j}{b_j + h_j}. \end{cases}$$

In proposition 9, we have

$$\begin{cases} \displaystyle\int_{-\infty}^{\frac{y_j - \mu_\Lambda q_j}{q_j}} f_{\varepsilon_j}(\epsilon_j) d\epsilon_j = \frac{b_j - c_j(1-\beta)}{b_j + h_j}; \\ \displaystyle\int_{-\infty}^{\frac{y_j - \mu_\Lambda q_j}{q_j}} \epsilon_j f_{\varepsilon_j}(\epsilon_j) d\epsilon_j = \mu_\Lambda \left\{ \frac{b_j - \beta\frac{\partial r(\mathbf{q})}{\partial q_j} + \beta c_j}{b_j + h_j} - F_{\varepsilon_j}\left(\frac{y_j - \mu_\Lambda q_j}{q_j}\right) \right\}. \end{cases}$$

In proposition 10, we have

$$\begin{cases} \displaystyle\int_0^{y_j/q_j} f_\Lambda(\lambda)d\lambda = \frac{b_j - c_j(1-\beta)}{b_j + h_j}; \\ \displaystyle\int_0^{y_j/q_j} \lambda f_\Lambda(\lambda)d\lambda = \mu_{\Lambda}\frac{b_j - \beta\frac{\partial r(\mathbf{q})}{\partial q_j} + \beta c_j}{b_j + h_j}. \end{cases}$$

By doing transformation and simplification, we can gain the result. ■