Demonstration of Negative Group Delays in a Simple Electronic Circuit

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Group Delays in Circuits

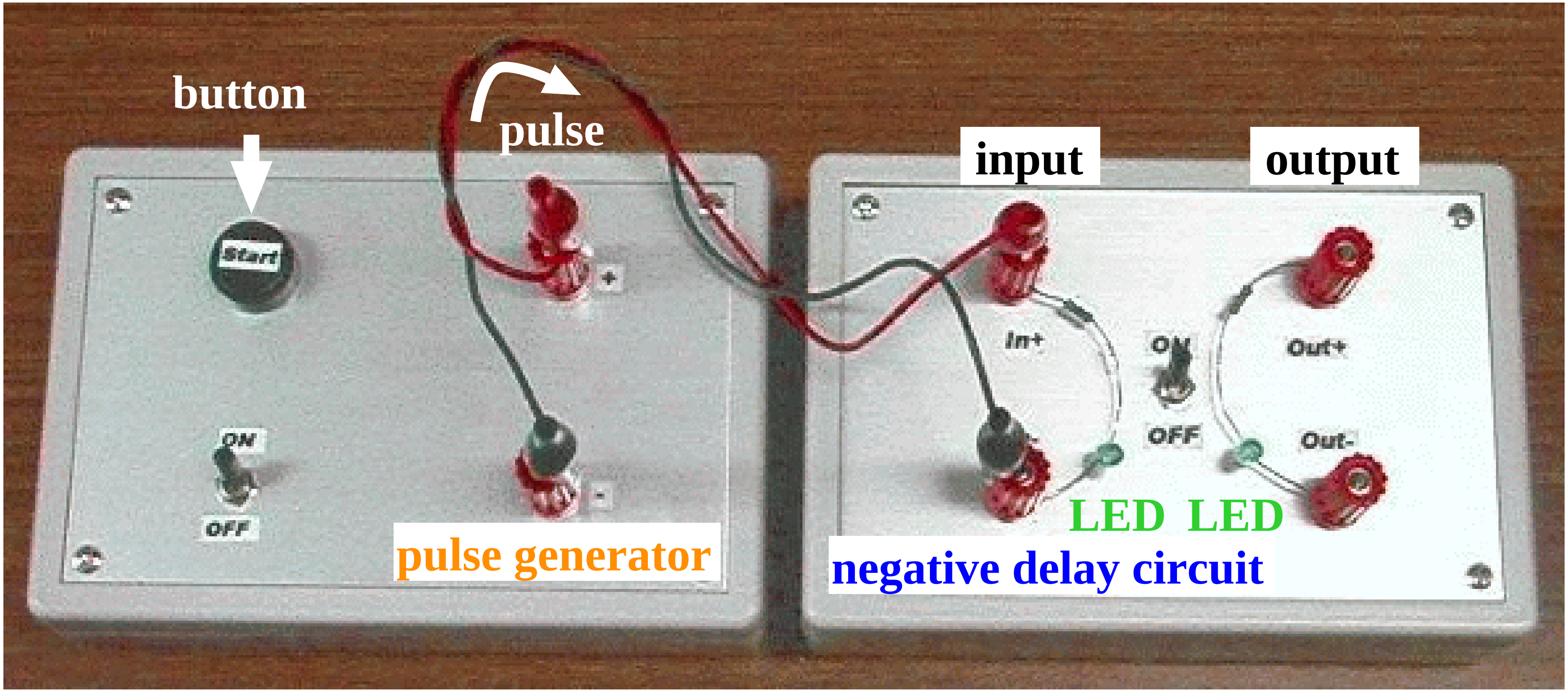
Group delays in lumped systems ($L = 0$ or $L \ll c/\omega$)

- Mitchell and Chiao: Am. J. Phys. **66**, 14 (1998)
  - Bandpass Amplifier (LC + opamp)
  - Arbitrary Waveform Generator
- Nakanishi, Sugiyama, and MK: quant-ph/0201001 (to appear in Am. J. Phys.)
  - Highpass Amplifier (RC + opamp)
  - Baseband pulse (No carriers)
  - Band limited signal from rectangular pulser + lowpass filter

Negative delay circuit

- The output LED is lit earlier than the input LED.
  — Negative delay
- Time constants could be order of seconds.
  — We can see it!

Experiment



1. rectangular pulse generator + low-pass filter
2. negative delay circuit

Group delays – ideal case

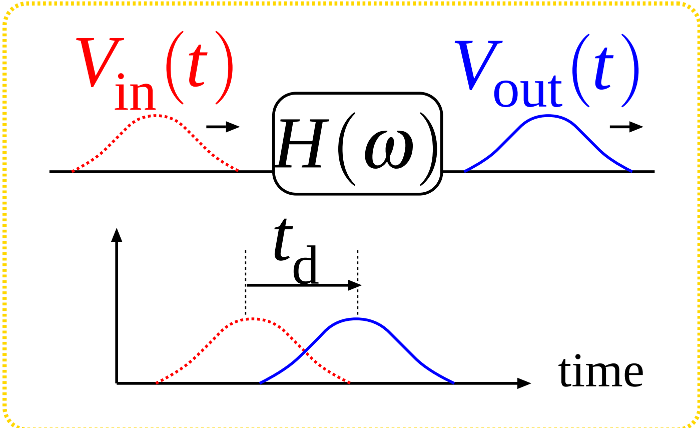
Group delay for base-band signals (delay time: t_d)

$$V_\mathrm{OUT}(t) = (h * V_\mathrm{IN})(t) = V_\mathrm{IN}(t - t_\mathrm{d})$$

$$h(t) = \delta(t - t_\mathrm{d})$$

Fourier Transformed: $\tilde{V}_\mathrm{OUT}(\omega) = H(\omega)\tilde{V}_\mathrm{IN}(\omega)$

$$H(\omega) = (\mathscr{F} h)(\omega) = \int \mathrm{d}t\, h(t) \mathrm{e}^{-\mathrm{i}\omega t} = \exp(-\mathrm{i}\omega t_\mathrm{d})$$



Positive and Negative delays

t_d	Causality	Physical realization
> 0	causal	distributed system
$= 0$	(locally, mutually) causal	lumped system
< 0	non causal	impossible

- Positive delays are easy, if you have an appropriate space.

  "Record and play" is also possible.
- No way to make ideal (unconditional) negative delays.
- No lumped systems ($L = 0$) can produce ideal positive or negative delays.

Approximate delay with lumped systems

- Ideal response function $H(\omega)$

$$\begin{aligned} A(\omega) &= |H(\omega)| = 1, \\ \phi(\omega) &= \arg H(\omega) = -t_{\mathrm{d}}\omega \end{aligned}$$

- Approximate realization #1 with lumped systems

$$\begin{aligned} H(\omega) &= \frac{1+\mathrm{i}\omega T}{1-\mathrm{i}\omega T} \quad \text{(1 pole, 1 zero)} \\ A(\omega) &= 1 \quad \text{(flat response)} \\ \phi(\omega) &= 2\tan^{-1}\omega T \sim 2T\omega \end{aligned}$$

Stability condition $\rightarrow T \leq 0$ (only positive delays)

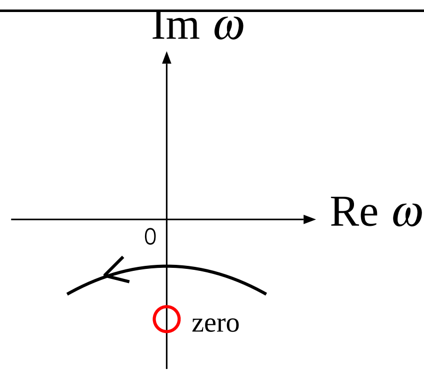
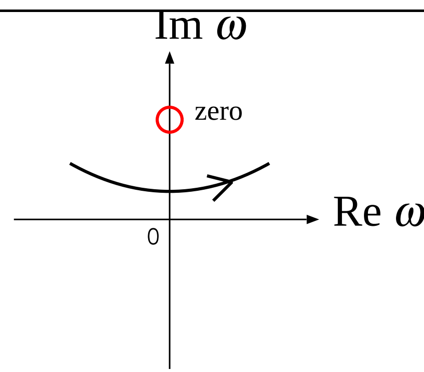
Approximate delay (2)

- Approximate realization #2 with lumped systems

$$\begin{aligned} H(\omega) &= 1 + \mathrm{i}\omega T \quad \text{(1 zero)} \\ A(\omega) &= \sqrt{1+(\omega T)^2} \sim 1 + \frac{(\omega T)^2}{2} \quad (\rightarrow \text{distortion}) \\ \phi(\omega) &= \tan^{-1}\omega T \sim T\omega \end{aligned}$$

No sign restriction on T (positive and negative delay)

Asymmetry — positive / negative

<table>
<tr><th>Transfer function</th><th>positive delay $T < 0$</th><th>negative delay $T > 0$</th></tr>
<tr><td>$H(\omega) = 1 + \mathrm{i}\omega T$</td><td>
</td><td>
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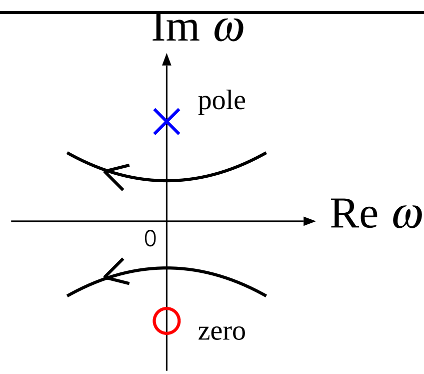
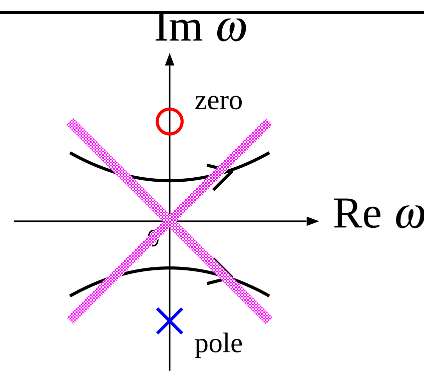
<tr><td>$H(\omega) = \dfrac{1 + \mathrm{i}\omega T}{1 - \mathrm{i}\omega T}$</td><td>
</td><td>
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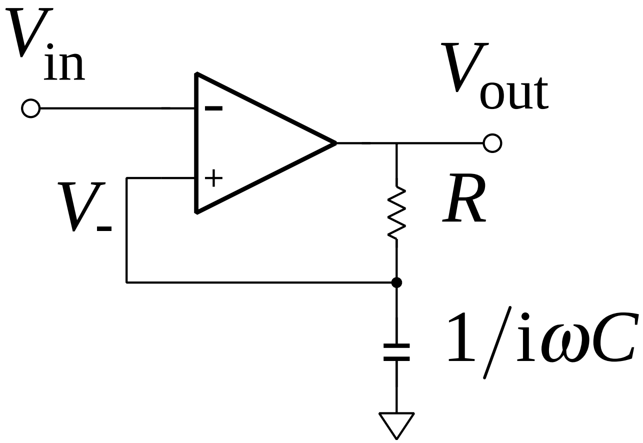
The arrows shows the direction of the phase increase of $H(\omega)$.

Stability condition requires that poles can be only in the upper-half plane, but zero can be anywhere.

Negative Group Delay Circuit



$$V_- = \frac{(i\omega C)^{-1}}{R + (i\omega C)^{-1}} V_{out} = \frac{1}{1 + i\omega CR} V_{out}$$

$V_{in} \sim V_-$ for large gain of operational amplifier

$$V_{out} = (1 + i\omega CR) V_{in}$$

A pole is converted into a zero by the feedback circuit.

Finite bandwidth

- Transfer Function

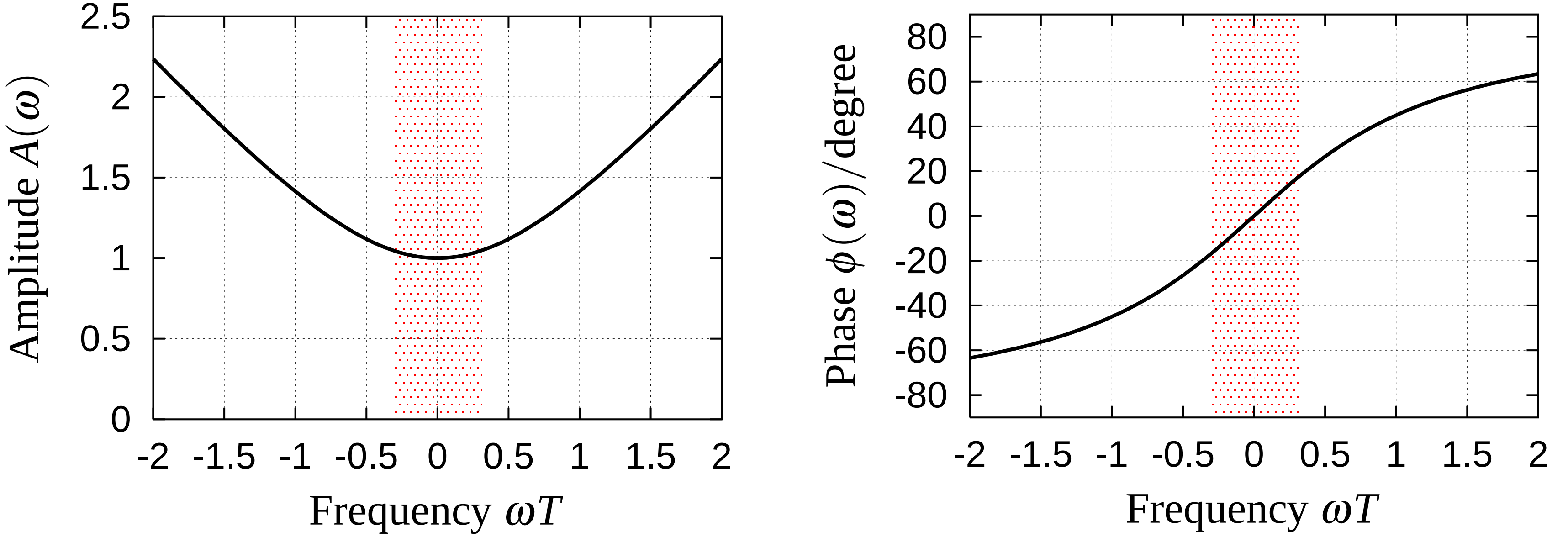
$$
\begin{aligned}
H(\omega) &= 1 + \mathrm{i}\omega T \ \ (T = CR > 0) \\
A(\omega) &= 1 + O(\omega^2 T^2), \\
\phi(\omega) &= \omega T + O(\omega^3 T^3)
\end{aligned}
$$

- Spectral condition for input signals

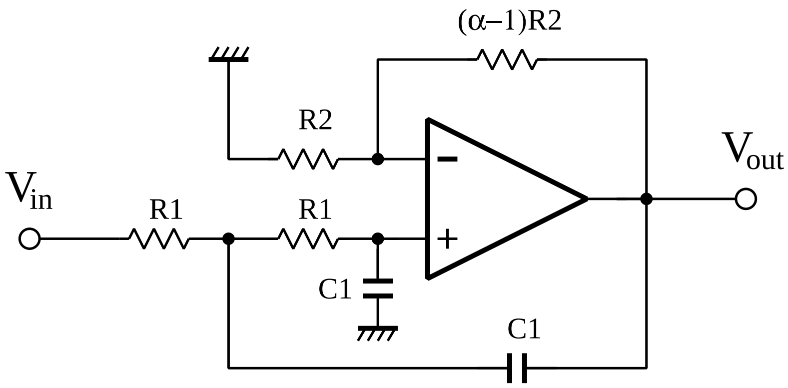
$$
|\omega| < \frac{1}{T}
$$

Bandwidth

Works only for band-limited signals
— otherwise outputs are distorted.



Band-Limit Circuit (Low-pass filter)



- Bessel filter (2nd order; $m = 2$)

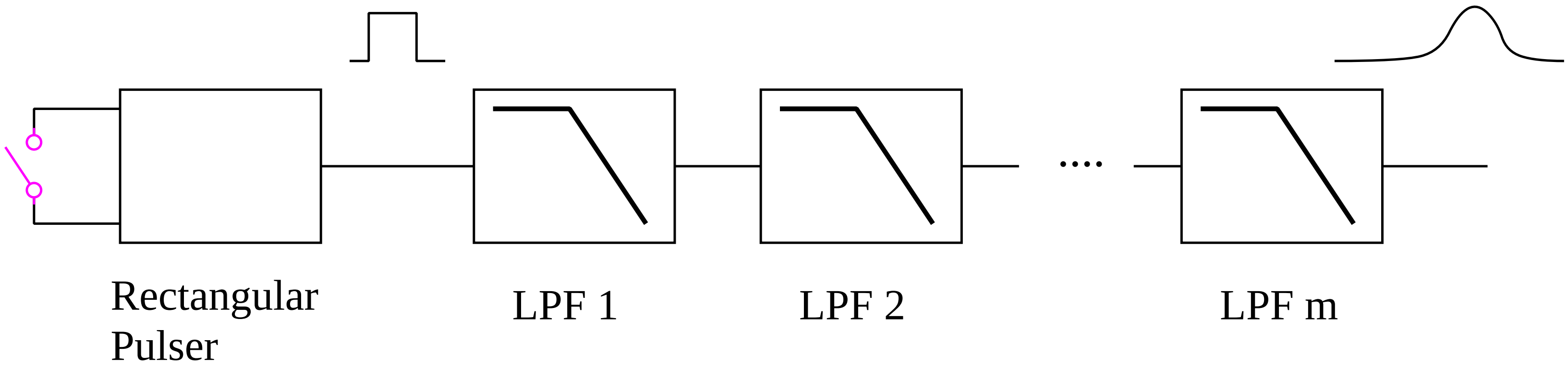
$$H_{\mathrm{LP}}(\omega) = \frac{\alpha}{1 + i\omega T_{\mathrm{LP}}(3-\alpha) + (i\omega T_{\mathrm{LP}})^2}$$

$$T_{\mathrm{LP}} = R_1 C_1, \quad \alpha = (1 + R_3/R_2) = 1.268$$

- Cutoff frequency: $\omega_{\mathrm{C}} = 0.7861/T_{\mathrm{LP}}$

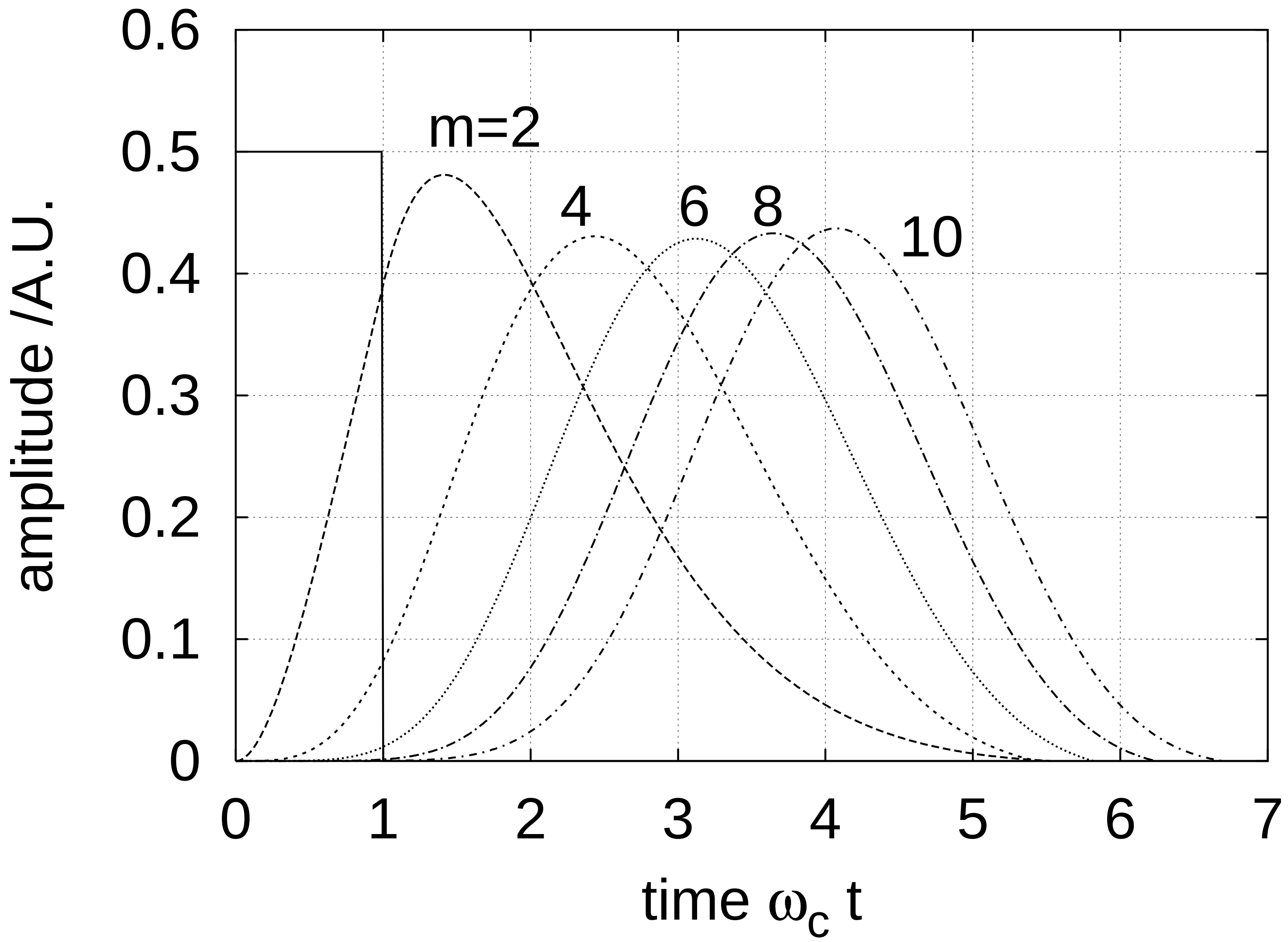
Low-pass filters

A rectangular pulser and a series of lowpass filters (m stages) are used to generate band-limited pulses.

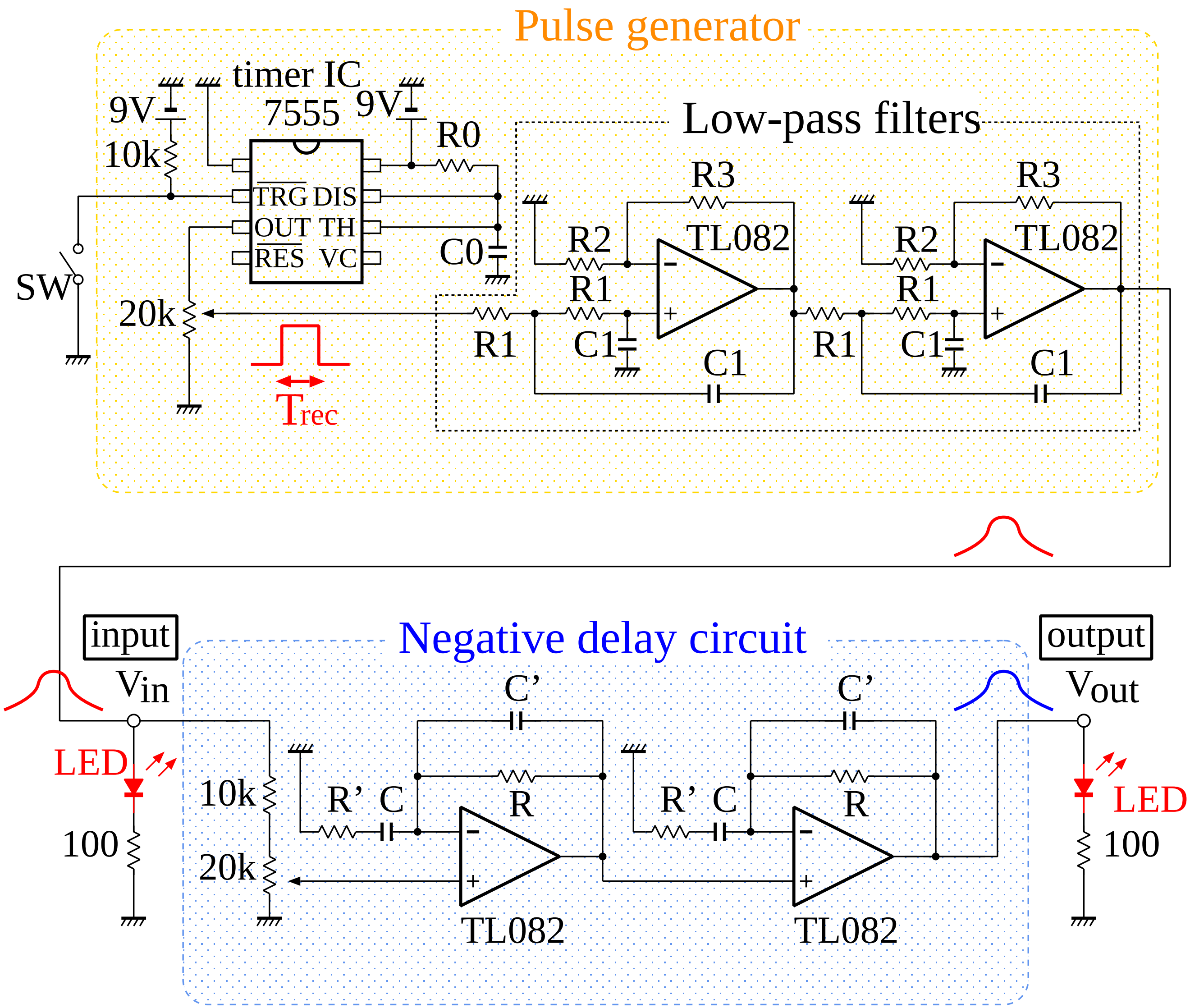


- Pushing the button (at $t = 0$) starts the event.
- The band-limited output has a smooth leading edge.
- A delay comparable to the pulse width is unavoidable.

Low-pass filters(2)



Circuit Diagram

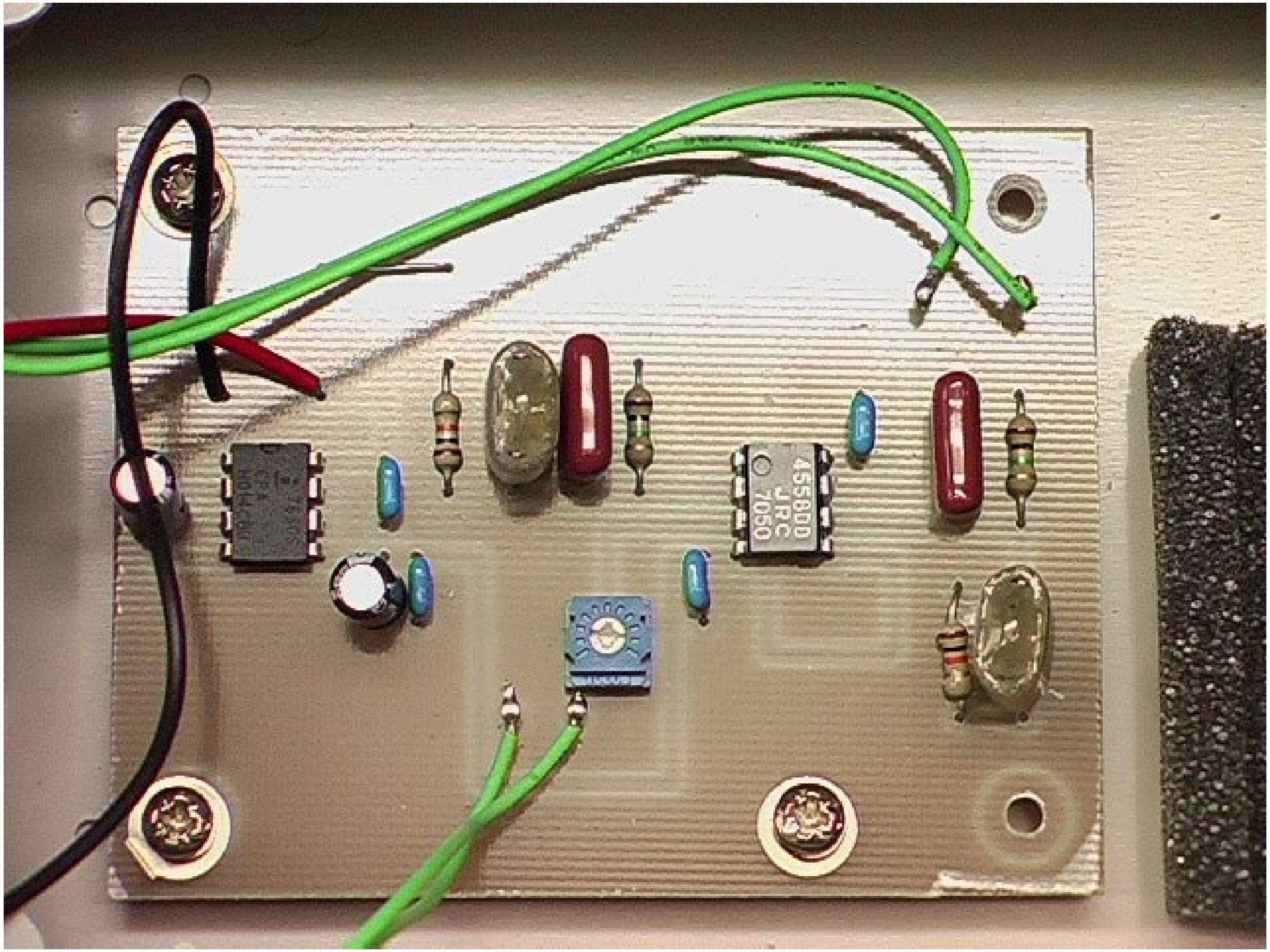


Circuit parameters

Pulse generator	
R_0	$6.8\,\mathrm{M\Omega}$
C_0	$0.22\,\mu\mathrm{F}$
T_{rec}	$1.5\,\mathrm{s}$
R_1	$2.2\,\mathrm{M\Omega}$
C_1	$0.22\,\mu\mathrm{F}$
R_2	$10\,\mathrm{k\Omega}$
R_3	$2.2\,\mathrm{k\Omega}$
ω_{c}	$1.6\,\mathrm{Hz}$

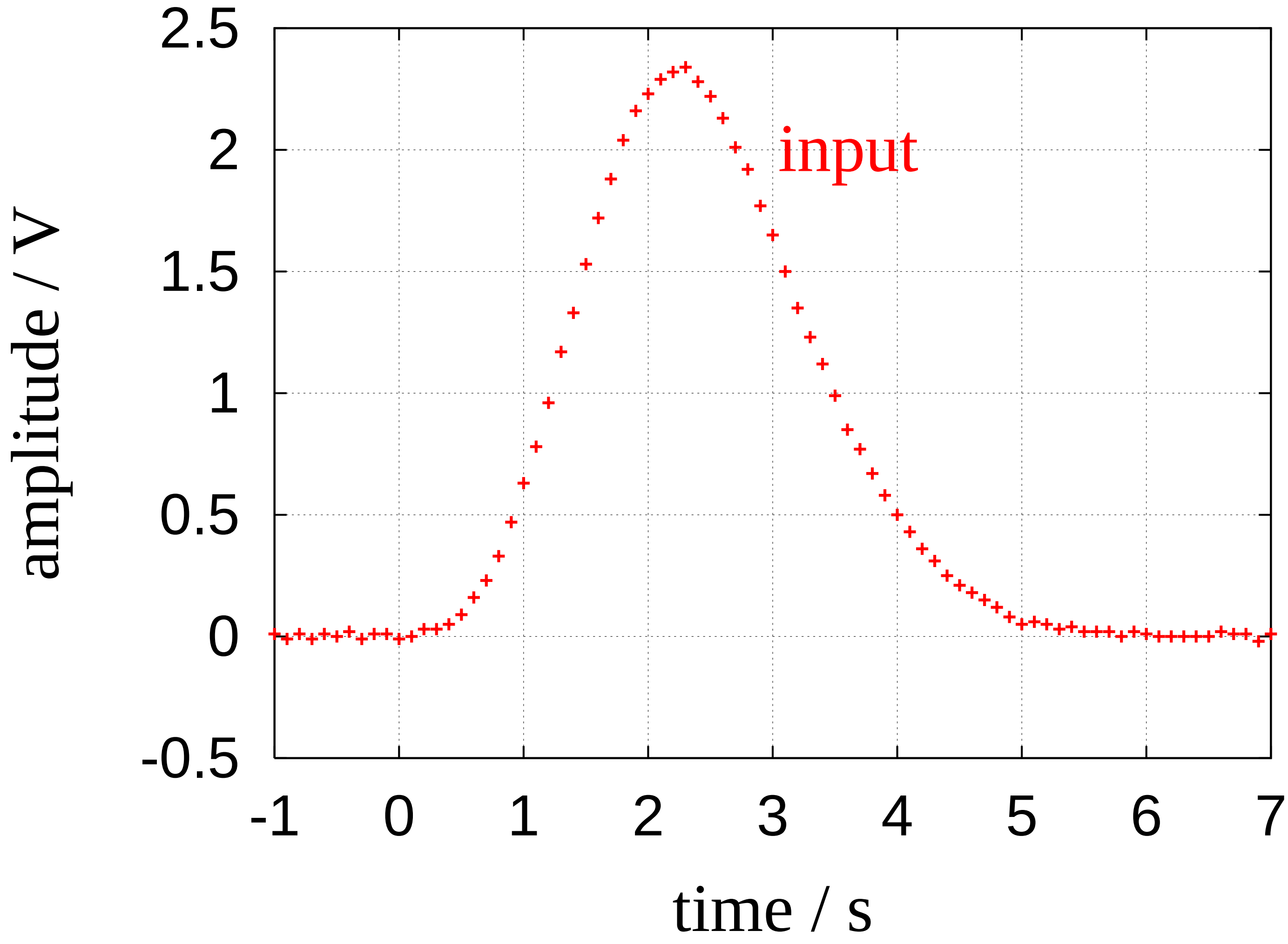
Negative delay circuit	
R	$1\,\mathrm{M\Omega}$
C	$0.22\,\mu\mathrm{F}$
R'	$10\,\mathrm{k\Omega}$
C'	$22\,\mathrm{nF}$
$T\ (= RC)$	$0.22\,\mathrm{s}$

Circuit Board



Experimental result

$m = 4, n = 2$



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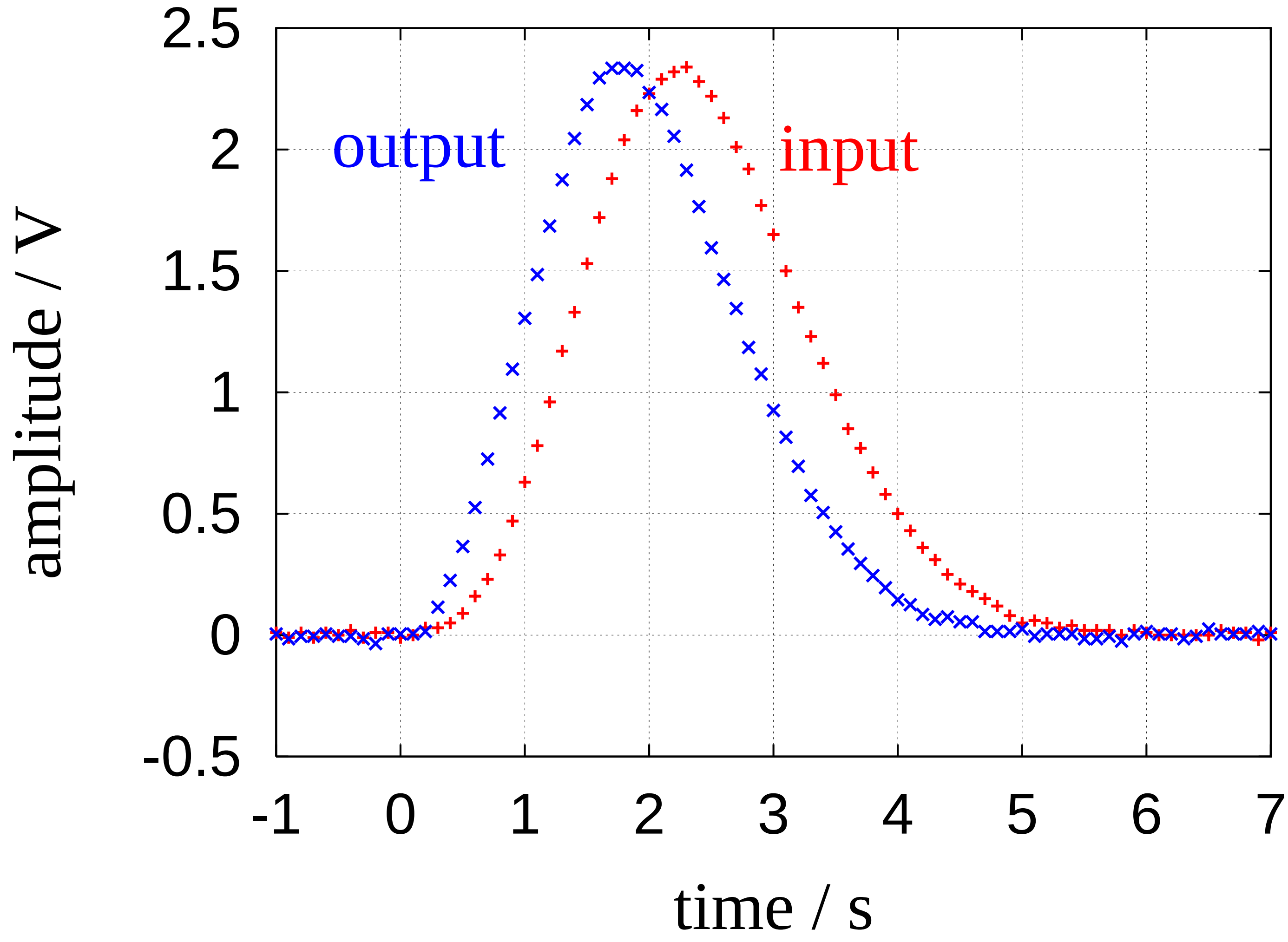
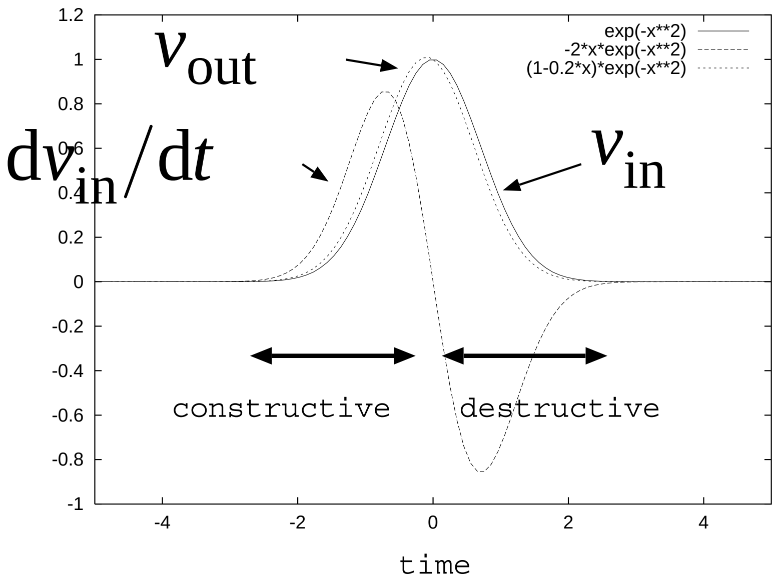


Interference in time domain

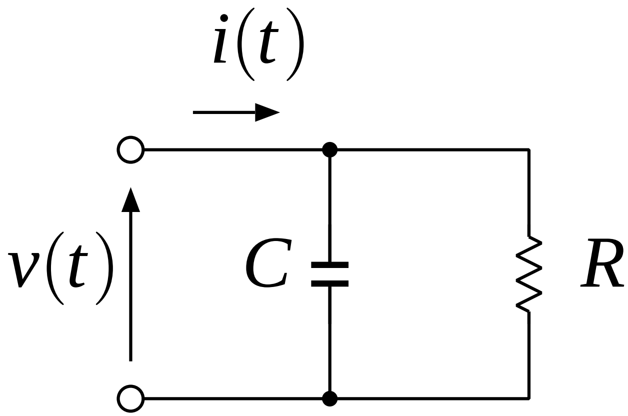
In time domain (cf. $H(\omega) = 1 + i\omega T$)

$$v_{\text{out}}(t) = \left(1 + T\frac{d}{dt}\right) v_{\text{in}}(t) = v_{\text{in}}(t) + T\frac{dv_{\text{in}}}{dt}(t)$$

Two terms interfere — constructively at the leading edge and destructively at the trailing edge.



All-passive circuit



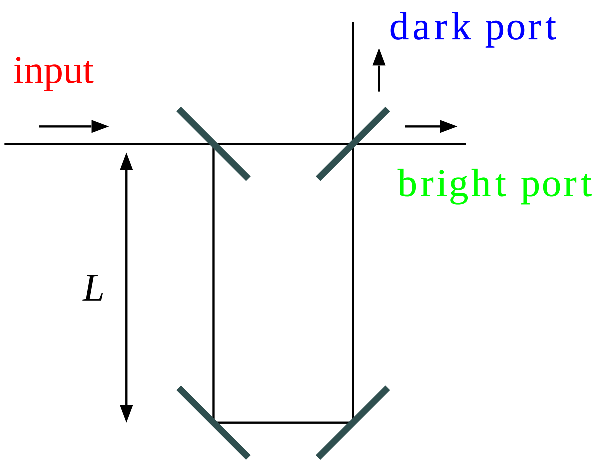
$$i(t) = \frac{1}{R}v(t) + C\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \frac{1}{R}\left(1 + RC\frac{\mathrm{d}}{\mathrm{d}t}\right)v(t)$$

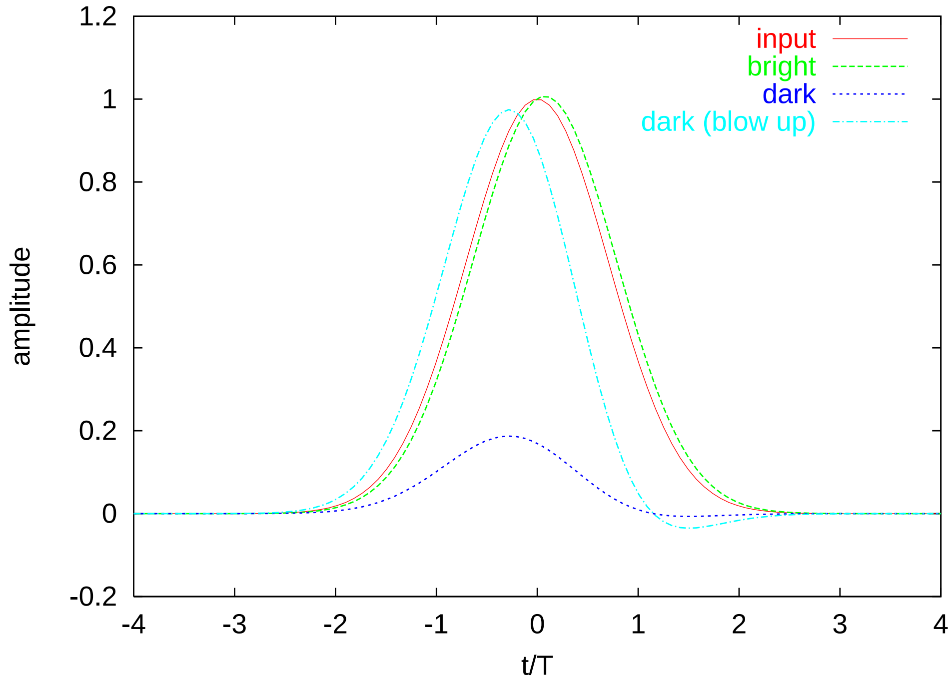
This circuit gives negative group delays, if we consider $v(t)$ as the input and $i(t)$ as the output.

Unbalanced interferometer

Reflectivity of beam splitters: $R = 1/2 - \rho$, Delay: $\tau = 2L/c$

$$E_{\text{dark}}(t) = (1-R)E_{\text{in}}(t) - RE_{\text{in}}(t-\tau) \sim 2\rho \left(1 + \frac{\tau}{4\rho}\frac{\mathrm{d}}{\mathrm{d}t}\right) E_{\text{in}}(t)$$



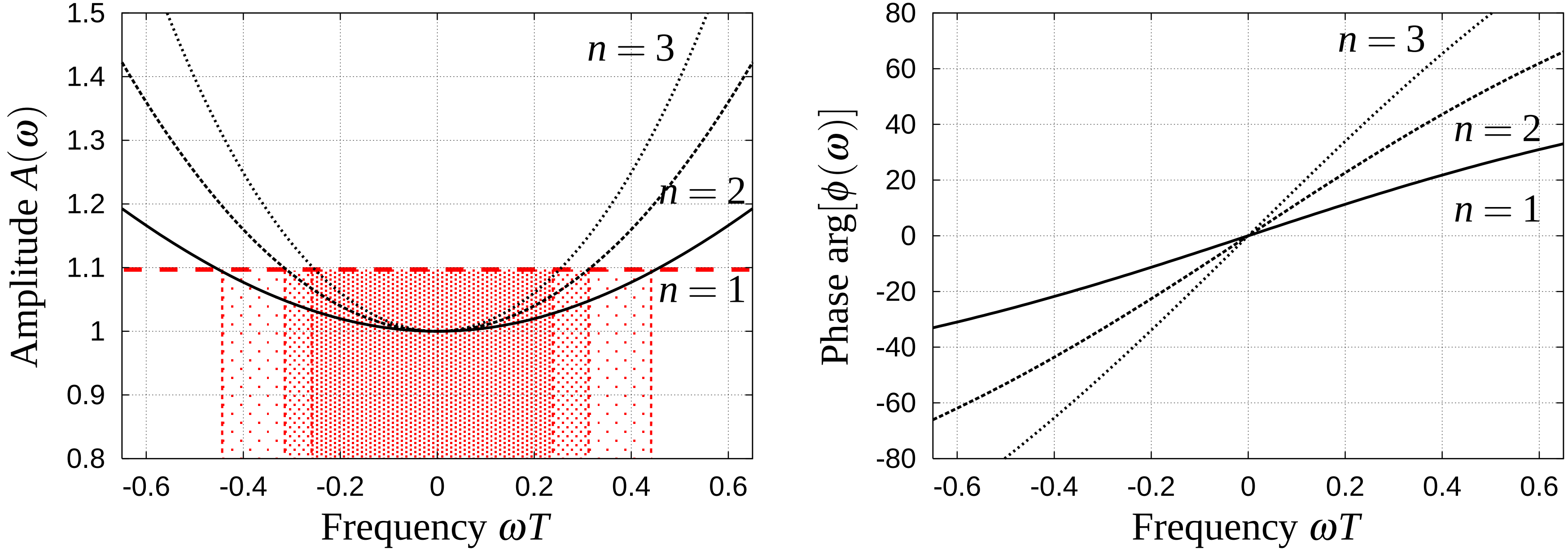


$\rho = 0.08$, $\tau = 0.15T$, T: gaussian pulse width

Cascading — for larger advancement

Transfer function for n stages: $H^n(\omega) = (1 + \mathrm{i}\omega T)^n$

$$A^n(\omega) \sim 1 + \frac{n(\omega T)^2}{2} = 1 + \frac{(\sqrt{n}\omega T)^2}{2}, \quad n\phi(\omega) \sim n\omega T$$



Bandwidth is reduced as n increased.

Cascading (2)

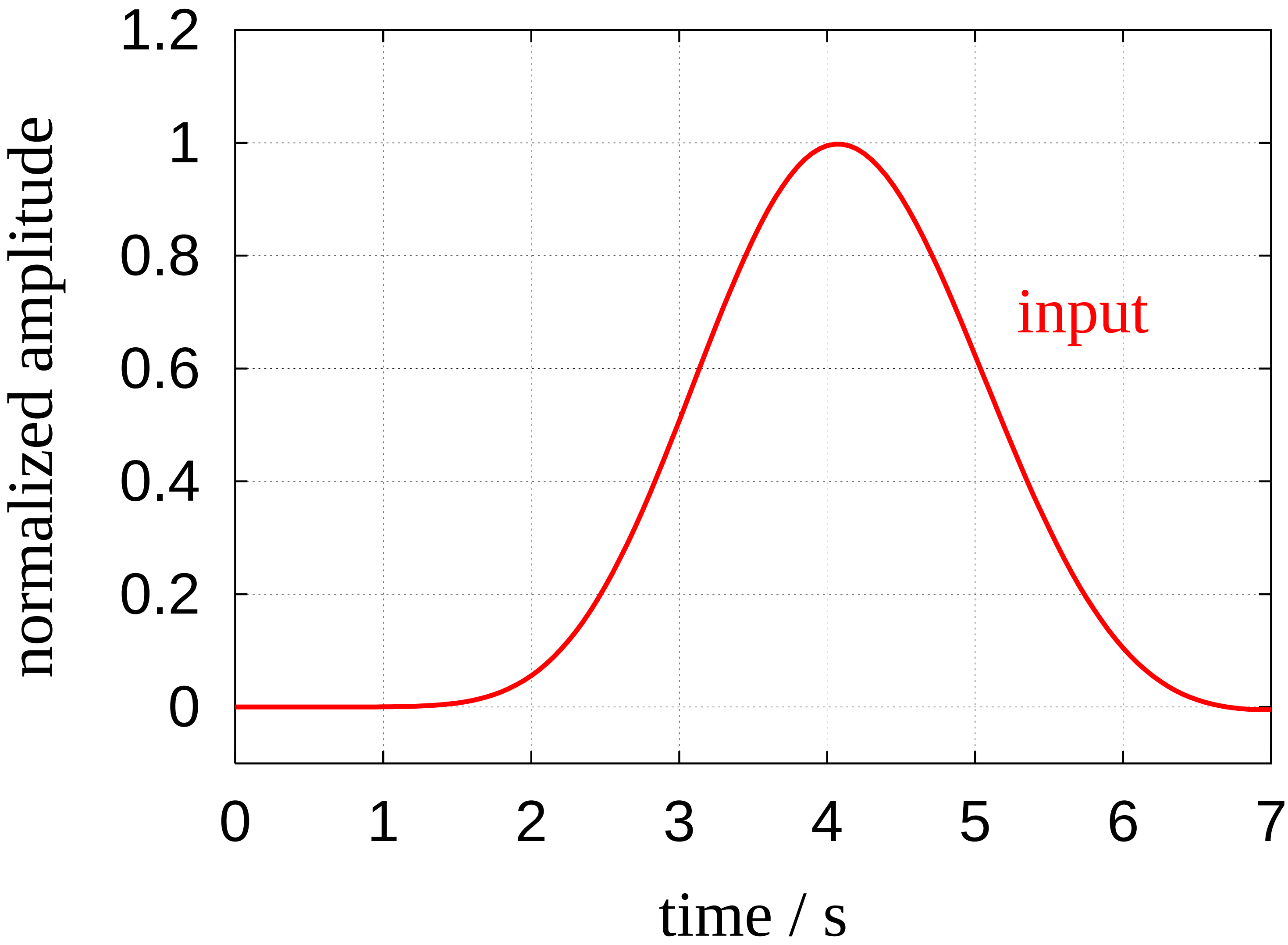
- Usable bandwidth decreases as $1/\sqrt{n}$, if T is given.
- For a given input pulse (T_w), $T = CR$ must be reduced as $T_\mathrm{w}/\sqrt{n}$.
- Total advancement scales $1/\sqrt{n}$;

$$-t_\mathrm{d} \propto n \times (T_\mathrm{w}/\sqrt{n}) = \sqrt{n}T_\mathrm{w}$$

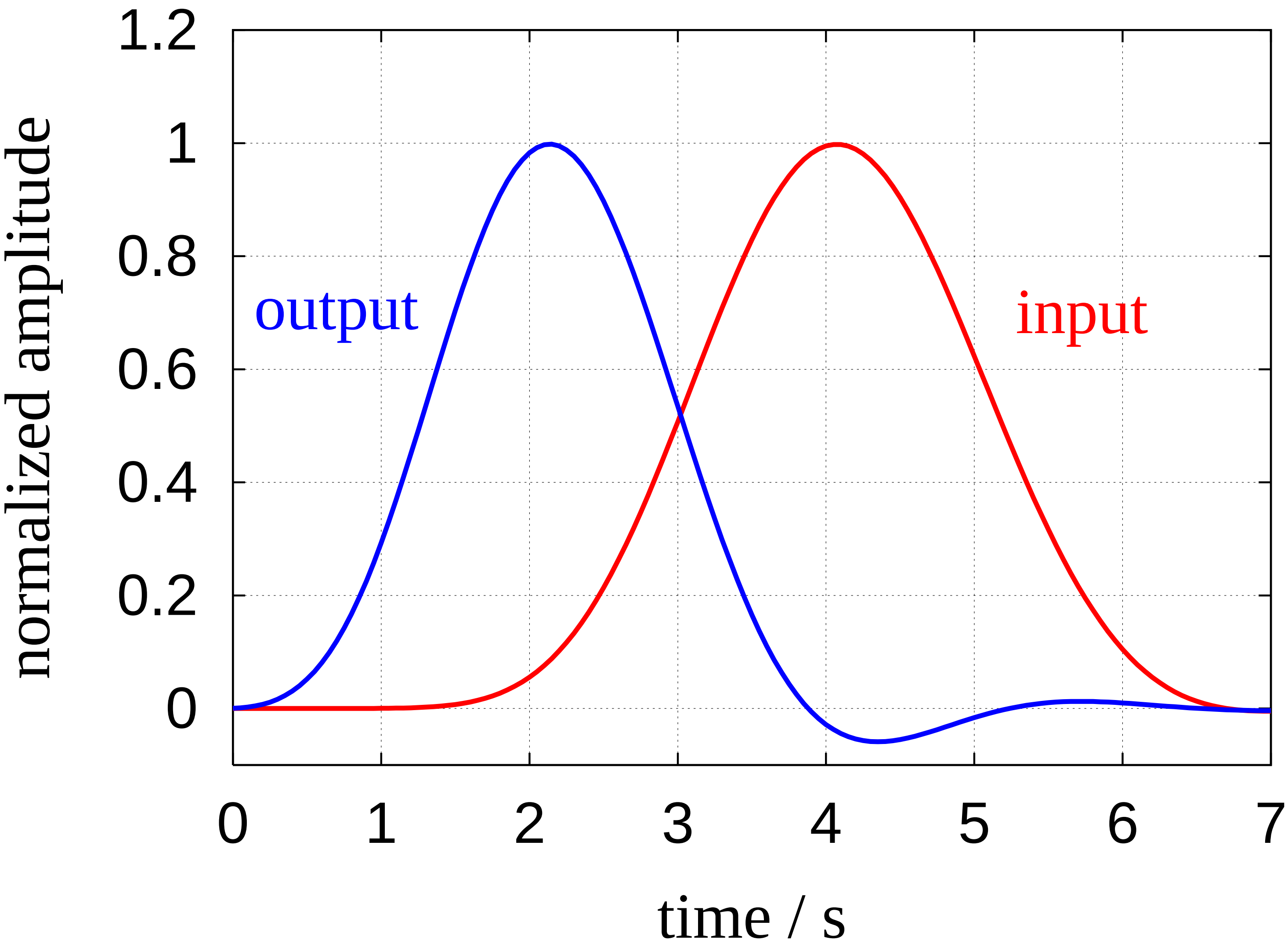
- Order m of lowpass filters must be increased accordingly;

$$m > n$$

Cascading, $n = 10$



Cascading, $n = 10$

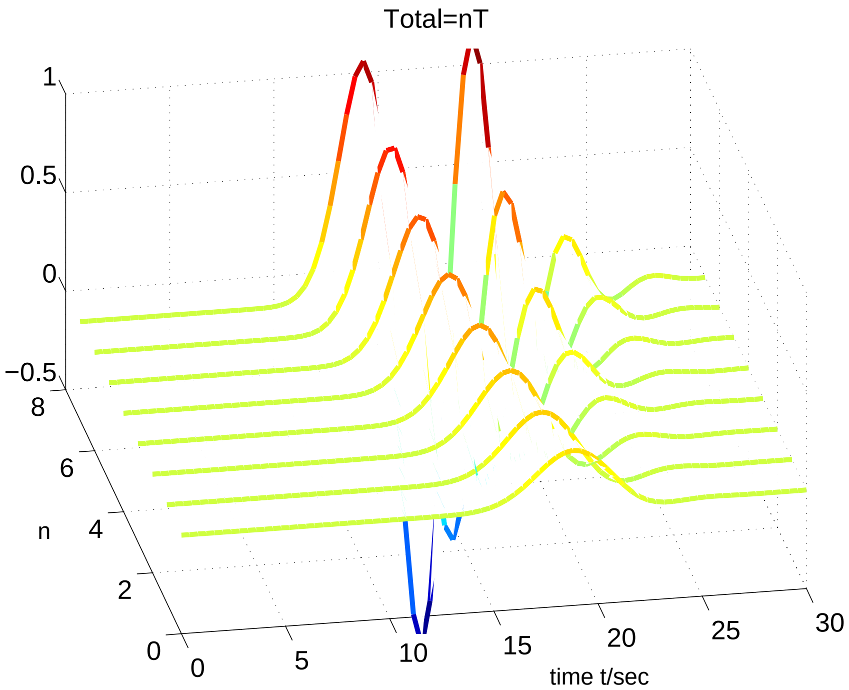


Problems in cascading

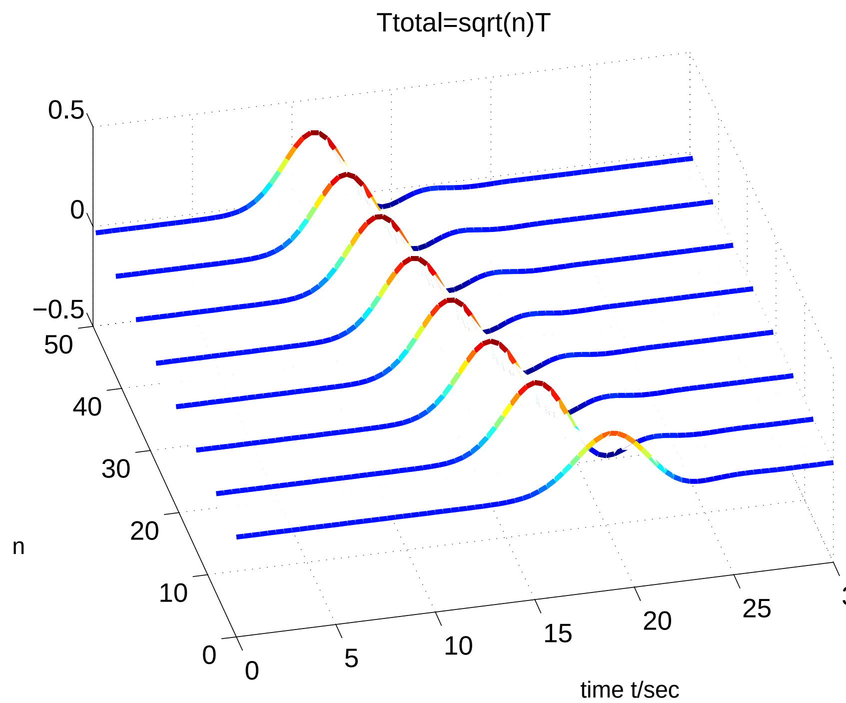
It is possible to increase the advancement as large as the pulse width or more, but

- The advancement increases slowly; $1/\sqrt{n}$.
- The order m of lowpass filters must be increased.
- The system becomes very sensitive to noises. (A huge gain outside of the band)

Pushing the limit

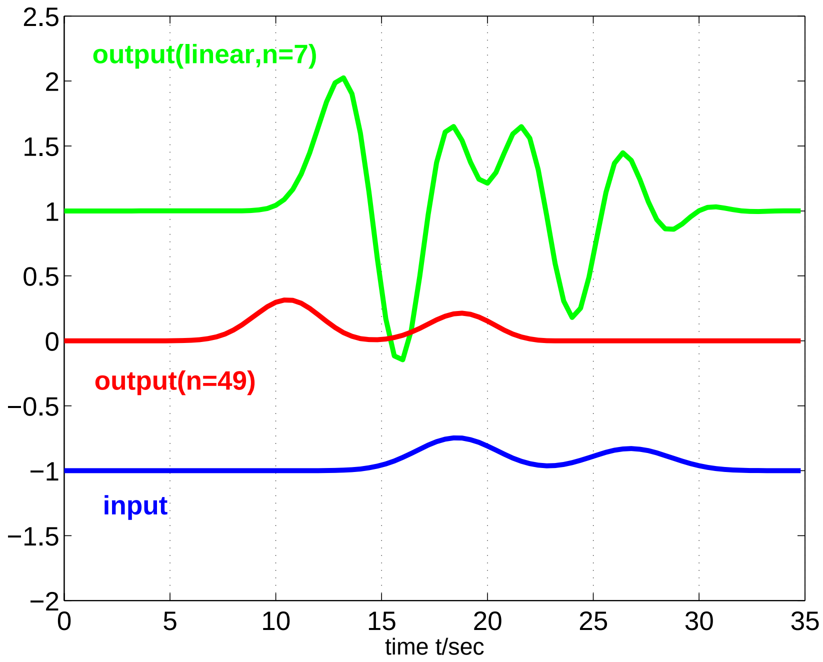


Simple-minded cascading



$1/\sqrt{n}$ cascading

Pushing the limit — two pulse case



Out-of-band gain

Outside gain — Significant obstacle toward large n.

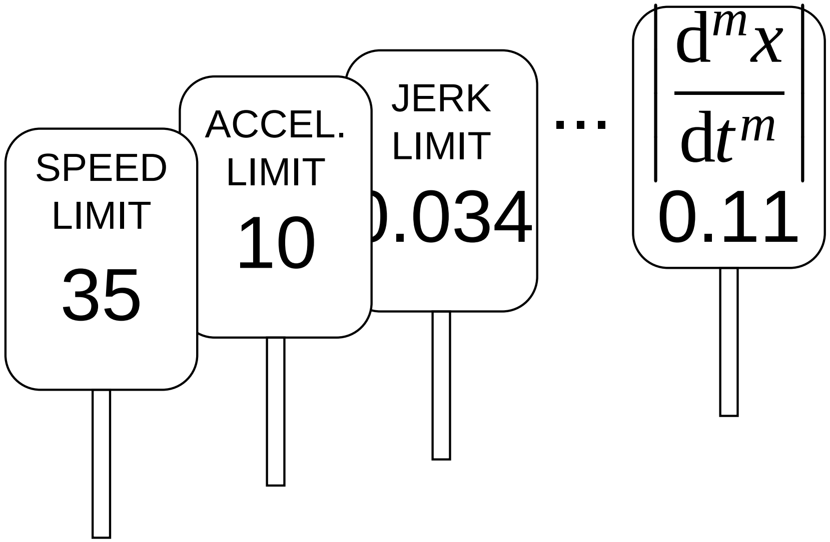
- Realistic transfer function (finite gain at $\omega = \infty$):

$$\begin{aligned} H(\omega) &= \frac{1 + i\omega T}{1 + i\omega T/\alpha}, \quad \alpha > 1 \\ \phi(\omega) &= \omega(1 - \alpha^{-1})T \\ A(\omega) &= \sqrt{\frac{1 + (\omega T)^2}{1 + (\omega T/\alpha)^2}} \sim \begin{cases} 1 & (\omega = 0) \\ \alpha & (\omega = \infty) \end{cases} \end{aligned}$$

- For $\alpha = 0.5$, $n = 50$; $A^n(\infty) = \alpha^n = 2^{50} = 10^{15}$!

Role of lowpass filters

- Predictability — Your move will be predicted if some restrictions are imposed.

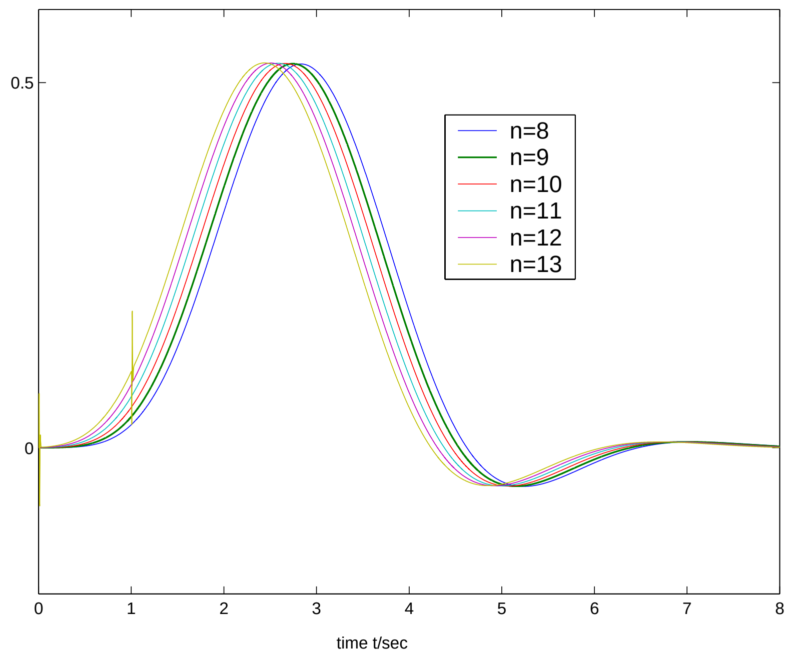


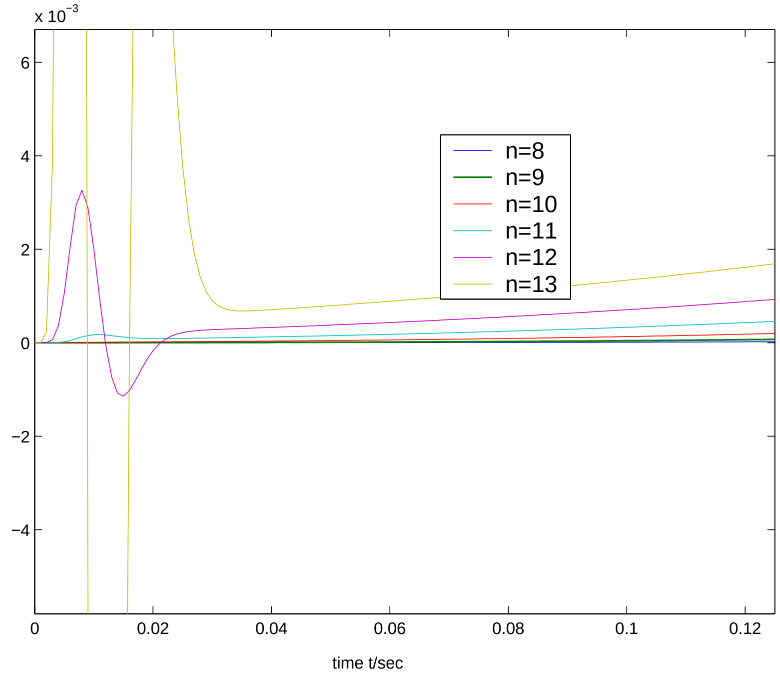
- Resemblance of functions — A negative delay circuit reveals the difference.

$$\mathrm{dist}(f,g) = \max_{0\le j\le m} \left\| \frac{d^j f}{dt^j} - \frac{d^j g}{dt^j} \right\|$$

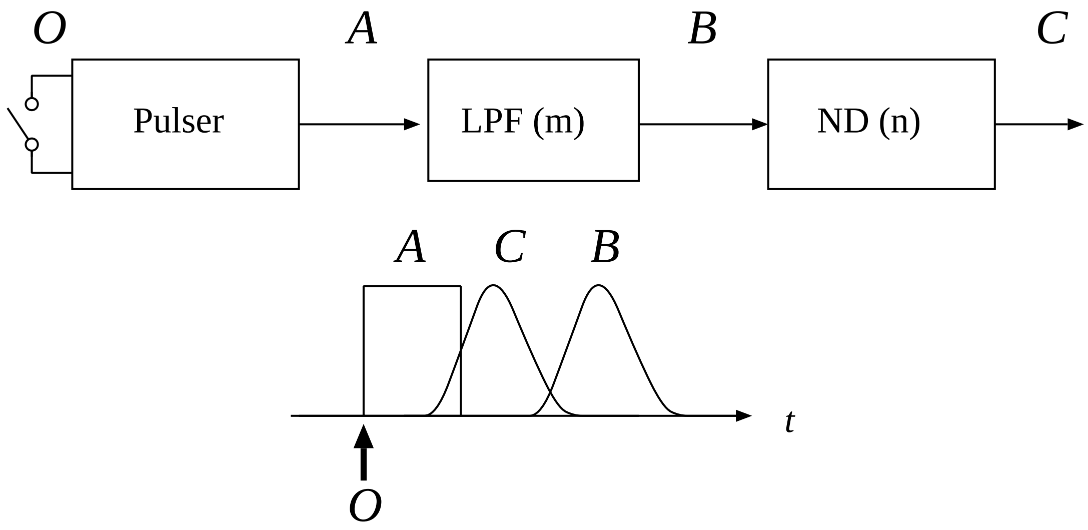
Role of filters (2)

On the condition $m > n$





Causality



- Causal relation in a casual sense

$$O \longrightarrow A \longrightarrow B \stackrel{?}{\longrightarrow} C$$

- Causal relation in a strict sense

$$A \Leftrightarrow B \Leftrightarrow C \quad (\Leftrightarrow A)$$

Conclusion

- A stand-alone (battery-operated) demonstration box
  - Battery operated
  - No extra equipment such as an oscilloscope or a function generator needed
- Large negative delays $\sim 0.5\,\mathrm{s}$ are achieved experimentally.

  25% of pulse width
- Larger delay with cascading is possible (in priciple).

  Advancement can be larger than the pulse width.
- Filter stage plays an essential role.

  The start of event (cause) is located before the filters.