

Dendroidal Segal spaces and ∞ -operads

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ABSTRACT

We introduce the dendroidal analogues of the notions of complete Segal space and of Segal category, and construct two appropriate model categories for which each of these notions corresponds to the property of being fibrant. We prove that these two model categories are Quillen equivalent to each other, and to the monoidal model category for ∞ -operads which we constructed in an earlier paper. By slicing over the monoidal unit objects in these model categories, we derive as immediate corollaries the known comparison results between Joyal’s quasi-categories, Rezk’s complete Segal spaces, and Segal categories.

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Introduction

The category of dendroidal sets is an extension of that of simplicial sets, suitable for constructing nerves, not just of categories but also of (coloured) operads. It was introduced with this purpose, and with the aim of giving an inductive definition of weak higher categories, in [14, 15]. This category $dSet$ of dendroidal sets carries a symmetric monoidal closed structure which is closely related to the Boardman–Vogt tensor product of operads, and the inclusion of the category $sSet$ of simplicial sets into $dSet$ can in fact be identified with the forgetful functor, from the slice (or comma) category of dendroidal sets over the unit η of the monoidal structure back to dendroidal sets, via an explicit isomorphism of categories

$$dSet/\eta = sSet. \tag{I}$$

Dendroidal sets carry a very rich homotopical structure, which we began to explore in [7]. For example, there is a monoidal Quillen model structure on $dSet$, whose fibrant objects include all nerves of operads. In fact, these fibrant objects can be thought of as simple combinatorial models of the notion of operad-up-to-homotopy or ‘ ∞ -operad’. Like any Quillen model structure, this model structure on dendroidal sets induces another model structure on any slice category. Under the identification $dSet/\eta = sSet$, this induced model structure

can be shown to coincide with the Joyal model structure on simplicial sets, whose fibrant objects are most commonly known under the name ‘ ∞ -categories’ (and are also referred to as quasi-categories, weak Kan complexes, or inner Kan complexes [4, 11, 13]).

These ∞ -categories model a notion of category-up-to-homotopy. Other ways of modelling such a notion have occurred in the literature, including the theory of Segal categories [2, 17] and of complete Segal spaces [16]. The latter two concepts are both based on the much older observation that a simplicial set X is the nerve of a category if and only if the canonical map

$$X_n \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1 \quad (\text{II})$$

sending a simplex to its one-dimensional ribbons is an isomorphism. Indeed, Simpson and Rezk both base their theories on bisimplicial sets X for which the map (II) is a weak equivalence of simplicial sets (and replacing the fibred product on the right-hand side by its homotopy version). Building on the work of Simpson and Rezk, the relation between these different ways of modelling categories-up-to-homotopy was recently made precise through the works of Bergner, Joyal and Tierney, and Lurie. Indeed, Simpson’s Segal categories, Rezk’s complete Segal spaces, and Joyal’s ∞ -categories all arise as the fibrant objects in a specific Quillen model category structure, and these different model category structures have now been related to each other by explicit Quillen equivalences [2, 13]. Moreover, they are all Quillen equivalent to the model category of simplicial categories discovered by Bergner [2], thus providing a strictification or rigidification result for each of these notions of category-up-to-homotopy.

The goal of this paper and its sequel [8] is to develop analogous theories of Segal operads (rather than categories) and complete dendroidal (rather than simplicial) Segal spaces, to relate these to each other and to dendroidal sets via Quillen equivalences, and to prove a strictification result for each of them by relating them to simplicial operads. By a simple slicing procedure like in (I), the earlier results just mentioned for categories-up-to-homotopy can all be recovered from our results, which can in this sense be said to be more general.

In more detail, then, we will consider the category $sdSet$ of simplicial objects in dendroidal sets, or what is the same, dendroidal spaces. We will define a Segal-type condition on the objects of this category, based on an extension to trees of ‘the union of one-dimensional ribbons in an n -simplex’ to which we will refer as the Segal core of a tree. In Section 5, we will establish a closed model category structure on $sdSet$ whose fibrant objects satisfy a tree-like Segal condition involving these Segal cores, and a completeness condition like the one of Rezk, and prove (Corollary 6.7) that this model category is Quillen equivalent to our earlier model category structure on dendroidal sets [7]. The definitions and proofs of these results are based on some elementary observations about these Segal cores presented in Section 2, and on a characterization of weak equivalences between ∞ -operads as maps which are ‘essentially surjective and fully faithful’ in a suitable sense (Theorem 3.5). The proof also exploits the hybrid nature of the objects of $sdSet$, which can be viewed alternatively as simplicial objects in one category or as dendroidal objects in another. In fact, the first view point is taken in Section 4, while the second viewpoint underlies the notion of complete dendroidal Segal space and the formulation of the main equivalence (Corollary 6.7). The relation between these two view points is most clearly expressed by Theorem 6.6 which equates two seemingly different model category structures.

Again using the Segal cores, we define the notion of a Segal operad in Section 8. These Segal operads will then be shown to be the fibrant objects for a model category structure on a full subcategory of the category $sdSet$ of dendroidal spaces, the category of so-called Segal pre-operads (Theorems 8.13 and 8.17). Using most if not all of the earlier results, we will then be able to show that this model category with Segal operads as fibrant objects is Quillen equivalent to the model category having complete dendroidal Segal spaces as fibrant objects (Theorem 8.15), and hence also Quillen equivalent to the original model category of dendroidal sets.

We believe these results are of interest in themselves, and because they generalize important classical results from the simplicial-categorical context to the dendroidal-operadic one. In addition, they will all be used in our proof of the strictification theorem for ∞ -operads presented in [8].

1. Preliminaries

We begin by recalling the basic definitions related to dendroidal sets; see [7, 14, 15]. The starting point is a category Ω of trees. Its objects are finite (non-planar) trees. These trees have internal edges (between vertices) and external ones (attached to just one vertex); the root is one such external edge, the others are called ‘leaves’, or ‘input edges’. Each such tree freely generates a coloured operad, and the arrows in Ω are the maps between these operads. Thus, by definition, Ω is a full subcategory of the category of (symmetric coloured) operads.

Each natural number $n \geq 0$ defines a linear tree with n vertices and $n + 1$ edges, the input edge being labelled 0, and the output or root edge labelled n . The corresponding coloured operad is the category defined by the linear order $0 \leq \dots \leq n$. Thus, the simplicial category Δ is a full subcategory of Ω , and we denote the inclusion by

$$i : \Delta \longrightarrow \Omega, \quad [n] \longmapsto n = i[n].$$

The category $dSet$ of dendroidal sets is by definition the category of presheaves (that is, contravariant Set -valued functors) on Ω , just like the category of simplicial sets is that of presheaves on Δ . The inclusion functor i induces a pair of adjoint functors

$$i_! : sSet \rightleftarrows dSet : i^*,$$

where i^* is the restriction along i and $i_!$ is its fully faithful left adjoint (i^* also has a fully faithful right adjoint i_*).

We will write $\Omega[T]$ for the dendroidal set represented by a tree T . With the similar notation $\Delta[n]$ for representable simplicial sets, we thus have

$$i_! \Delta[n] = \Omega[n],$$

and this identification determines $i_!$ uniquely up to unique isomorphism (as colimit-preserving functor).

There is a natural identification of Δ with the slice category $\Omega/i[0]$, and this leads to an identification

$$sSet = dSet/\eta,$$

where $\eta = \Omega[0]$. Under this identification, the functor $i_!$ corresponds to the forgetful functor $dSet/\eta \rightarrow dSet$.

The full embedding of Ω into (coloured) operads gives an adjoint pair

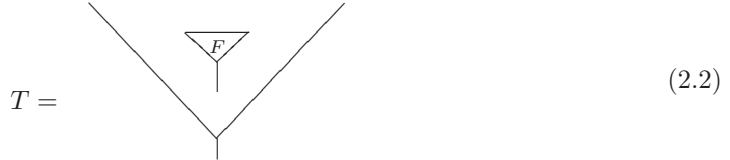
$$\tau_d : dSet \rightleftarrows Operad : N_d,$$

where the right adjoint N_d is called the dendroidal nerve. These functors restrict to the usual nerve of a small category and its left adjoint.

The category of dendroidal sets carries a symmetric monoidal closed structure, denoted by \otimes and Hom . Its unit object is the representable dendroidal set $\eta = \Omega[0]$. This structure is compatible with the product of simplicial sets as well as with the Boardman–Vogt tensor product of operads, in the sense that, for any simplicial sets M and N , and for any dendroidal sets X and Y , we have natural identifications

$$i_!(M \times N) = i_!(M) \otimes i_!(N) \quad \text{and} \quad \tau_d(X \otimes Y) = \tau_d(X) \otimes_{BV} \tau_d(Y).$$

In general, we say that a face map $F \rightarrow T$ is a *subtree* if $F \rightarrow T$ is a composition of external faces. In other words, a face map $F \rightarrow T$ is a subtree if F is obtained by successively pruning away top vertices, or pruning away root vertices which have only one internal edge attached to them.



DEFINITION 2.2. Given a tree T with at least one vertex, we define its *Segal core* $\text{Sc}[T]$ as the subobject of $\Omega[T]$ defined as the union of all the images of those maps $\Omega[C_n] \rightarrow \Omega[T]$ corresponding to subtrees of shape $C_n \rightarrow T$. Remark that, up to isomorphism, such a map $C_n \rightarrow T$ is completely determined by the vertex of T in its image, so we can write

$$\text{Sc}[T] = \bigcup_v \Omega[C_{n(v)}],$$

where the union is over all the vertices of T , and $n(v)$ is the number of input edges at v .

If $T = [0]$ is the tree with no vertices (so that $\Omega[T] = \Omega[0] = \eta$ is the unit object of the Boardman–Vogt tensor product), then it will be convenient to define $\text{Sc}[T] = \eta$.

Recall from [7, Paragraph 1.2] that an (elementary) face $S \rightarrow T$ of a tree T is called *outer* or *external* if S is obtained from T by pruning away an external vertex, that is, a vertex with exactly one inner edge attached to it.

DEFINITION 2.3. Let T be a tree. The *external boundary* of $\Omega[T]$ is the subobject $\partial^{\text{ext}}\Omega[T]$ of $\Omega[T]$ obtained as the union of all the external faces of T .

PROPOSITION 2.4. For any tree T , the inclusion $\text{Sc}[T] \rightarrow \Omega[T]$ is inner anodyne.

Proof. Note that, if T has at most one vertex, then this inclusion is an isomorphism, while, if T has exactly two vertices, then this is an inner horn. So we may assume that T has at least three vertices.

If T has N vertices, then $\Omega[T]$ has a natural filtration by subobjects

$$\Omega[T]_1 \subset \Omega[T]_2 \subset \cdots \subset \Omega[T]_{N-1} \subset \Omega[T]_N = \Omega[T],$$

where, for $1 \leq n \leq N$,

$$\Omega[T]_n = \bigcup_{\xi} \Omega[F_{\xi}]$$

is the union over all subtrees F_{ξ} of T with at most n vertices. Note that, by definition, we have

$$\Omega[T]_1 = \text{Sc}[T] \quad \text{and} \quad \Omega[T]_{N-1} = \partial^{\text{ext}}\Omega[T].$$

By virtue of [15, Lemma 5.1], the inclusion $\partial^{\text{ext}}\Omega[S] \rightarrow \Omega[S]$ is inner anodyne for any tree S with at least two vertices. We shall use this to prove that the inclusion $\Omega[T]_{n-1} \rightarrow \Omega[T]_n$ is inner anodyne for $2 \leq n \leq N$, which will prove the proposition.

Let F_0, \dots, F_k be all subtrees of T having n vertices. For $0 \leq j \leq k$, we put

$$S_j = \bigcup_{0 \leq i \leq j} \Omega[F_i] \subset \Omega[T].$$

We shall prove by induction on j that the map

$$S_j \cap \Omega[T]_{n-1} \longrightarrow S_j$$

is inner anodyne. The case $j = 0$ follows from the identification $\Omega[F_i] \cap \Omega[T]_{n-1} = \partial^{\text{ext}} \Omega[F_i]$, $0 \leq i \leq k$. Assume that $j > 0$. Note that, since

$$S_{j-1} \cap S_j \cap \Omega[T]_{n-1} = S_{j-1} \cap \Omega[T]_{n-1},$$

the following diagram is a pushout:

$$\begin{array}{ccc} S_{j-1} \cap \Omega[T]_{n-1} & \longrightarrow & S_{j-1} \\ \downarrow & & \downarrow \\ S_j \cap \Omega[T]_{n-1} & \longrightarrow & S_{j-1} \cup (\Omega[F_j] \cap \Omega[T]_{n-1}) \end{array}$$

Moreover, since $\Omega[F_p] \cap \Omega[F_q] \subset \Omega[T]_{n-1}$ for $p \neq q$, we have

$$\Omega[F_j] \cap (S_{j-1} \cup (\Omega[F_j] \cap \Omega[T]_{n-1})) = \Omega[F_j] \cap \Omega[T]_{n-1},$$

which gives the following pushout square:

$$\begin{array}{ccc} \Omega[F_j] \cap \Omega[T]_{n-1} & \longrightarrow & \Omega[F_j] \\ \downarrow & & \downarrow \\ S_{j-1} \cup (\Omega[F_j] \cap \Omega[T]_{n-1}) & \longrightarrow & S_j \end{array}$$

Since the top arrows in these two squares are inner anodyne, so are the lower ones, and we obtain that the composite

$$S_j \cap \Omega[T]_{n-1} \longrightarrow S_{j-1} \cup (\Omega[F_j] \cap \Omega[T]_{n-1}) \longrightarrow S_j$$

is inner anodyne as well. For $j = k$, we conclude that the map

$$\Omega[T]_{n-1} \longrightarrow S_k \cup \Omega[T]_{n-1} = \Omega[T]_n$$

is the pushout of an inner anodyne extension. □

PROPOSITION 2.5. *Let W be a class satisfying the following three conditions.*

- (i) *The class W is closed under transfinite compositions, pushouts, and retracts.*
- (ii) *Any Segal core inclusion belongs to W .*
- (iii) *For any normal monomorphisms between normal dendroidal sets $u : X \rightarrow Y$ and $v : Y \rightarrow Z$, if u and vu are in W , so is v .*

Then any inner anodyne extension belongs to W .

Proof. As, by definition, the class of inner anodyne extensions is the smallest class of maps which satisfies condition (i) and contains the inner horn inclusions, it is sufficient to prove that any inner horn inclusion belongs to W . For a tree T with at least two vertices, and any internal edge e in T , we have the following natural inclusions:

$$\text{Sc}[T] \longrightarrow \partial^{\text{ext}} \Omega[T] \longrightarrow \Lambda^e[T] \longrightarrow \Omega[T]. \tag{2.3}$$

We shall prove by induction on the number $|T|$ of vertices of T that all these inclusions belong to W (note that this is not so if T has only one vertex). In fact, if A is a union of at least two external faces of T , then $\text{Sc}[T] \subset A$, while, if B is the union of $\partial^{\text{ext}}\Omega[T]$ and of a collection of internal faces not including the one contracting e , then there are interpolating inclusions

$$\text{Sc}[T] \longrightarrow A \longrightarrow \partial^{\text{ext}}\Omega[T] \longrightarrow B \longrightarrow \Lambda^e[T] \longrightarrow \Omega[T]. \tag{2.4}$$

Our induction on $|T|$ will proceed by showing that all these inclusions belong to W .

To begin with, if $|T| = 2$, then $\Omega[T]$ has just two external faces, and one internal one (given by the edge e), so $\text{Sc}[T] = A = \partial^{\text{ext}}\Omega[T] = B = \Lambda^e[T]$, and hence, as W contains isomorphisms and Segal core inclusions, all the inclusions in (2.4) are in W .

Consider now a tree S with $|S| > 2$, and assume that, for any tree T such that $2 \leq |T| < |S|$, all the maps in (2.4) are in W . We shall first show that, for any set $\{R_i\}_{0 \leq i \leq j}$ of at least two external faces of S , the map

$$\text{Sc}[S] \longrightarrow A = \bigcup_{0 \leq i \leq j} \Omega[R_i] \tag{2.5}$$

is in W . For the set of all external faces of S , the map (2.5) is the inclusion $\text{Sc}[S] \rightarrow \partial^{\text{ext}}\Omega[S]$, so that we shall have shown that this map belongs to W as well. By condition (iii), it then follows that the map $A \rightarrow \partial^{\text{ext}}\Omega[S]$ also belongs to W .

To prove that (2.5) is in W , consider the case of just two distinct external faces R_0 and R_1 in S . Then the map

$$\text{Sc}[R_0 \cap R_1] \longrightarrow \Omega[R_0 \cap R_1]$$

belongs to W (also if $R_0 \cap R_1$ is a tree with just one vertex), as does the map $\text{Sc}[R_1] \rightarrow \Omega[R_1]$. Now, consider the commutative diagram below.

$$\begin{array}{ccccc} \text{Sc}[R_0 \cap R_1] & \longrightarrow & \text{Sc}[R_1] & & \\ \downarrow & & \downarrow & \searrow & \\ \Omega[R_0] \cap \Omega[R_1] & \longrightarrow & \text{Sc}[R_1] \cup (\Omega[R_0] \cap \Omega[R_1]) & \longrightarrow & \Omega[R_1] \end{array}$$

The square is a pushout, so the right-hand vertical map belongs to W . As the slanted map also does by assumption, we find that the right-hand horizontal map belongs to W . Next, the two pushout diagrams

$$\begin{array}{ccc} \text{Sc}[R_1] \cup (\Omega[R_0] \cap \Omega[R_1]) & \longrightarrow & \Omega[R_1] \\ \downarrow & & \downarrow \\ \text{Sc}[R_1] \cup \Omega[R_0] & \longrightarrow & \Omega[R_0] \cup \Omega[R_1] \end{array}$$

and

$$\begin{array}{ccc} \text{Sc}[R_0] & \longrightarrow & \Omega[R_0] \\ \downarrow & & \downarrow \\ \text{Sc}[R_0] \cup \text{Sc}[R_1] & \longrightarrow & \Omega[R_0] \cup \text{Sc}[R_1] \end{array}$$

show that

$$\text{Sc}[S] = \text{Sc}[R_0] \cup \text{Sc}[R_1] \longrightarrow \Omega[R_0] \cup \Omega[R_1]$$

belongs to W . This shows that (2.5) belongs to W for any collection of two distinct external faces. Consider now a collection $\{R_i\}_{0 \leq i \leq j}$ of distinct external faces of S , with $j \geq 1$. We shall

prove by induction on j that (2.5) belongs to W . The case $j = 1$ just having been dealt with, we may assume that $j \geq 2$. By induction, we have that the map

$$\text{Sc}[R_0] \longrightarrow \bigcup_{0 < i \leq j} \Omega[R_0 \cap R_i]$$

belongs to W , because $R_0 \cap R_i$ is an external face of R_0 for $0 < i \leq j$. Also, the map $\text{Sc}[R_0] \rightarrow \Omega[R_0]$ is in W , so that, by condition (iii), the map

$$\bigcup_{0 < i \leq j} \Omega[R_0 \cap R_i] \longrightarrow \Omega[R_0]$$

belongs to W . By induction, the map

$$\text{Sc}[S] \longrightarrow \bigcup_{0 < i \leq j} \Omega[R_i]$$

is in W , whence we deduce from the pushout

$$\begin{array}{ccc} \bigcup_{0 < i \leq j} \Omega[R_0 \cap R_i] & \longrightarrow & \Omega[R_0] \\ \downarrow & & \downarrow \\ \bigcup_{0 < i \leq j} \Omega[R_i] & \longrightarrow & \bigcup_{0 \leq i \leq j} \Omega[R_i] \end{array}$$

that (2.5) is the composite of two maps in W . This completes the proof of the fact that all the maps of type (2.5) belong to W .

We now turn to the internal faces of the tree S , and let B be the union of $\partial^{\text{ext}}\Omega[S]$ and of a family of internal faces $\Omega[\partial^{a_1} S], \dots, \Omega[\partial^{a_k} S]$, given by internal edges a_1, \dots, a_k , all distinct from another internal edge e . We shall prove that $\partial^{\text{ext}}\Omega[S] \rightarrow B$ belongs to W . Then the composition $\text{Sc}[S] \rightarrow \partial^{\text{ext}}\Omega[S] \rightarrow B$ belongs to W as well, and if B contains all internal faces but the one given by e , we find that the inclusion $\text{Sc}[S] \rightarrow \Lambda^e[S]$ belongs to W . Since $\text{Sc}[S] \rightarrow \Omega[S]$ is in W by assumption, condition (iii) implies that $\Lambda^e[S] \rightarrow \Omega[S]$ is in W . So, to complete the proof of the proposition, it is sufficient to prove that, under our inductive assumption (that the maps in (2.4) all belong to W for smaller trees), the map

$$\partial^{\text{ext}}\Omega[S] \longrightarrow B = \partial^{\text{ext}}\Omega[S] \cup \bigcup_{1 \leq i \leq k} \Omega[\partial^{a_i} S] \tag{2.6}$$

belongs to W for any family of internal edges a_1, \dots, a_k as above.

We proceed by induction on k . If $k = 1$, then S has at least two internal edges, hence $\partial^{a_1} S$ has at least two vertices, so, by assumption on trees smaller than S , the map

$$\partial^{\text{ext}}\Omega[\partial^{a_1} S] \longrightarrow \Omega[\partial^{a_1} S]$$

belongs to W . But

$$\partial^{\text{ext}}\Omega[\partial^{a_1} S] = \Omega[\partial^{a_1} S] \cap \partial^{\text{ext}}\Omega[S],$$

so we have a pushout diagram

$$\begin{array}{ccc} \partial^{\text{ext}}\Omega[\partial^{a_1} S] & \longrightarrow & \Omega[\partial^{a_1} S] \\ \downarrow & & \downarrow \\ \partial^{\text{ext}}\Omega[S] & \longrightarrow & \partial^{\text{ext}}\Omega[S] \cup \Omega[\partial^{a_1} S] \end{array}$$

Thus, the inclusion map $\partial^{\text{ext}}\Omega[S] \rightarrow \partial^{\text{ext}}\Omega[S] \cup \Omega[\partial^{a_1} S]$ is in W . This proves the case $k = 1$.

If $k > 1$, then we have

$$\left(\partial^{\text{ext}}\Omega[S] \cup \bigcup_{1 \leq i < k} \Omega[\partial^{a_i} S] \right) \cap \Omega[\partial^{a_k} S] = \partial^{\text{ext}}\Omega[\partial^{a_k} S] \cup \bigcup_{1 \leq i < k} \Omega[\partial^{a_i} \partial^{a_k} S],$$

so the diagram

$$\begin{array}{ccc} \partial^{\text{ext}}\Omega[\partial^{a_k} S] \cup \bigcup_{1 \leq i < k} \Omega[\partial^{a_i} \partial^{a_k} S] & \longrightarrow & \Omega[\partial^{a_k} S] \\ \downarrow & & \downarrow \\ \partial^{\text{ext}}\Omega[S] \cup \bigcup_{1 \leq i < k} \Omega[\partial^{a_i} S] & \longrightarrow & \partial^{\text{ext}}\Omega[S] \cup \bigcup_{1 \leq i < k} \Omega[\partial^{a_i} S] \end{array}$$

is a pushout. Moreover, the family $\{\partial^{a_i} \partial^{a_k} S\}_{1 \leq i < k}$ of internal faces of $T = \partial^{a_k} S$ misses the edge e , so that, by assumption, all the inclusions in (2.4) belong to W . We conclude that the top arrow in the pushout above belongs to W , whence so does the bottom arrow. By induction on k , the map

$$\partial^{\text{ext}}\Omega[S] \longrightarrow \partial^{\text{ext}}\Omega[S] \cup \bigcup_{1 \leq i < k} \Omega[\partial^{a_i} S]$$

is in W , and as W is closed under composition, we find that (2.6) is in W . □

COROLLARY 2.6. *A dendroidal set X is the nerve of an operad if and only if, for any tree T , the map*

$$X_T = \text{Hom}_{\mathcal{d}\text{Set}}(\Omega[T], X) \longrightarrow \text{Hom}_{\mathcal{d}\text{Set}}(\text{Sc}[T], X)$$

is bijective.

Proof. Assume that $X = N_d(P)$ for a (coloured) operad P . Then, as the functor τ_d sends inner anodyne inclusions to isomorphisms, it follows from Proposition 2.4 that the map $\text{Hom}_{\mathcal{d}\text{Set}}(\Omega[T], X) \rightarrow \text{Hom}_{\mathcal{d}\text{Set}}(\text{Sc}[T], X)$ is bijective. The converse follows easily from Proposition 2.5 and from the characterization of dendroidal nerves given by Moerdijk and Weiss [15, Proposition 5.3 and Theorem 6.1]. □

REMARK 2.7. The preceding corollary gives in particular the well-known characterization of small categories as simplicial sets satisfying the Grothendieck–Segal condition: given a simplicial set X , then, for $T = n$ with $n \geq 1$, we have

$$\text{Hom}_{\mathcal{d}\text{Set}}(\text{Sc}[T], X) = \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$$

3. Equivalences of ∞ -operads

3.1. For an ∞ -category X , we denote by $k(X)$ the maximal Kan complex contained in X ; see [11, Corollary 1.5]. Recall that, if A and X are two dendroidal sets, then $\mathcal{H}om(A, X)$ denotes their internal Hom (with respect to the Boardman–Vogt tensor product of dendroidal sets).

Given two dendroidal sets A and X , we write

$$\text{hom}(A, X) = i^* \mathcal{H}om(A, X).$$

If X is an ∞ -operad, and if A is normal, then $\mathcal{H}om(A, X)$ is an ∞ -operad, so that $\mathit{hom}(A, X)$ is an ∞ -category; see [15, Theorem 9.1].

For an ∞ -operad X and a simplicial set K , we will write $X^{(K)}$ for the subcomplex of $\mathcal{H}om(i_!(K), X)$ which consists of dendrices

$$a : \Omega[T] \times i_!(K) \longrightarrow X,$$

such that, for any edge u in the tree T , the induced map

$$a_u : K \longrightarrow i^*(X)$$

factors through $k(i^*(X))$ (that is, all the 1-cells in the image of a_u are weakly invertible in $i^*(X)$).

For an ∞ -operad X and a normal dendroidal set A , we write $k(A, X)$ for the subcomplex of $\mathit{hom}(A, X)$ which consists of maps

$$u : A \otimes i_!(\Delta[n]) \longrightarrow X,$$

such that, for all vertices a of A (that is, maps $a : \eta \rightarrow A$), the induced map

$$u_a : \Delta[n] \longrightarrow i^*(X)$$

factors through $k(i^*(X))$. So, by definition, for any normal dendroidal set A , any simplicial set K , and any ∞ -operad X , there is a natural bijection:

$$\mathrm{Hom}_{\mathit{sSet}}(K, k(A, X)) \simeq \mathrm{Hom}_{\mathit{dSet}}(A, X^{(K)}). \tag{3.1}$$

Furthermore, by virtue of [7, 6.8], we have the equality:

$$k(A, X) = k(\mathit{hom}(A, X)) \tag{3.2}$$

(in particular, $k(A, X)$ is a Kan complex).

3.2. Recall that, in any model category C , given a cofibrant object A and a fibrant object X , one of the models of the mapping space $\mathrm{Map}(A, X)$ is the simplicial set defined by

$$\mathrm{Map}(A, X)_n = \mathrm{Hom}_C(A, X_n), \tag{3.3}$$

where X_\bullet is a Reedy fibrant resolution of X (where X is seen as a simplicially constant object of $C^{\Delta^{\mathrm{op}}}$); see [10, 5.4.7 and 5.4.9], for instance. In the case where $C = \mathit{dSet}$ and X is an ∞ -operad, then the simplicial dendroidal set $X^{(\Delta[\bullet])}$ is Reedy fibrant, and, for any integer $n \geq 0$, the map $X \rightarrow X^{(\Delta[n])}$ is a weak equivalence (this follows immediately from [7, Corollary 6.9]). In particular, we obtain the following proposition.

PROPOSITION 3.3. *If A is a normal dendroidal set and X an ∞ -operad, then there is a natural isomorphism (in fact, identity) of simplicial sets*

$$k(A, X) \simeq \mathrm{Map}(A, X).$$

Proof. This follows immediately from the identifications (3.1) and (3.3). □

LEMMA 3.4. *Let X be an ∞ -operad.*

(i) *For any simplicial set K , and for any normal dendroidal set A , there is a natural bijection*

$$\mathrm{Hom}_{\mathbf{Ho}(\mathit{sSet})}(K, k(A, X)) \simeq \mathrm{Hom}_{\mathbf{Ho}(\mathit{dSet})}(A, X^{(K)}).$$

(ii) For any cofibration (respectively, trivial cofibration) between normal dendroidal sets $A \rightarrow B$, the map $k(B, X) \rightarrow k(A, X)$ is a fibration (respectively, a trivial fibration) between Kan complexes.

(iii) For any pushout of normal dendroidal sets

$$\begin{array}{ccc} A & \longrightarrow & A' \\ i \downarrow & & \downarrow i' \\ B & \longrightarrow & B' \end{array}$$

with i a cofibration, the commutative square

$$\begin{array}{ccc} k(B', X) & \longrightarrow & k(A', X) \\ \downarrow & & \downarrow \\ k(B, X) & \longrightarrow & k(A, X) \end{array}$$

is a pullback.

(iv) For any sequence of cofibrations between normal dendroidal sets

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \cdots,$$

the map

$$k\left(\varinjlim_n A_n, X\right) \longrightarrow \varprojlim_n k(A_n, X)$$

is an isomorphism.

Proof. This follows immediately from Proposition 3.3, using the general properties of mapping space functors; see, for instance, [10, Proposition 5.4.1 and Theorem 5.4.9]. \square

THEOREM 3.5. *Let $f : X \rightarrow Y$ be a morphism between ∞ -operads. The following conditions are equivalent.*

- (a) For any integer $n \geq 0$, the map $k(\Omega[C_n], X) \rightarrow k(\Omega[C_n], Y)$ is a simplicial homotopy equivalence, as is the map $k(\eta, X) \rightarrow k(\eta, Y)$.
- (b) For any tree T , the map $k(\Omega[T], X) \rightarrow k(\Omega[T], Y)$ is a simplicial homotopy equivalence.
- (c) For any normal dendroidal set A , the map $k(A, X) \rightarrow k(A, Y)$ is a simplicial homotopy equivalence.
- (d) The map $f : X \rightarrow Y$ is a weak operadic equivalence.

Proof. Assume condition (a). We claim that, for any tree T , the induced map $k(\text{Sc}[T], X) \rightarrow k(\text{Sc}[T], Y)$ is a simplicial homotopy equivalence. Note that, for $T = [0]$, this is a special case of (a). Therefore, to prove this, we may assume that T has at least one vertex. Let v_1, \dots, v_k be the vertices of T , and, for $1 \leq i \leq k$, write n_i for the number of input edges of v_i in T . We then have

$$\text{Sc}[T] = \bigcup_{1 \leq i \leq k} \Omega[C_{n_i}].$$

Moreover, for two indices $i \neq j$, the intersection $\Omega[C_{n_i}] \cap \Omega[C_{n_j}]$ is either empty or isomorphic to η . For $1 \leq j \leq k$, define

$$A_i = \bigcup_{1 \leq i \leq j} \Omega[C_{n_i}].$$

For $1 < i \leq k$, there is a pushout square

$$\begin{array}{ccc} A_{i-1} \cap \Omega[C_{n_i}] & \longrightarrow & \Omega[C_{n_i}] \\ \downarrow & & \downarrow \\ A_{i-1} & \longrightarrow & A_i \end{array}$$

in which the intersection $A_{i-1} \cap \Omega[C_{n_i}]$ is isomorphic to a finite sum of copies of η . By the cube lemma (see the dual version of [10, Lemma 5.2.6]), using properties (ii) and (iii) of Lemma 3.4, we obtain by induction on i that the maps

$$k(A_i, X) \longrightarrow k(A_i, Y)$$

are simplicial homotopy equivalences. In particular, for $i = k$, this means that the map

$$k(\text{Sc}[T], X) \longrightarrow k(\text{Sc}[T], Y)$$

is a simplicial homotopy equivalence. Thus, since the vertical maps in the commutative square

$$\begin{array}{ccc} k(\Omega[T], X) & \longrightarrow & k(\Omega[T], Y) \\ \downarrow & & \downarrow \\ k(\text{Sc}[T], X) & \longrightarrow & k(\text{Sc}[T], Y) \end{array}$$

are simplicial homotopy equivalences as well (by Proposition 2.4 and Lemma 3.4(ii)), this proves (b).

The fact that condition (b) implies condition (c) follows by similar arguments[†] from Lemma 3.4, using the skeletal filtration of normal dendroidal sets [15, section 4].

A reformulation of Lemma 3.4(i) in the particular case where $K = \Delta[0]$ is that, for a normal dendroidal set A and an ∞ -operad X , we have a natural bijection

$$\pi_0(k(A, X)) \simeq \text{Hom}_{\mathbf{Ho}(d\text{Set})}(A, X).$$

It thus follows from the Yoneda lemma that condition (c) implies condition (d).

Finally, the fact that condition (d) implies condition (c) (and, therefore, condition (a)) is obvious: it follows from Lemma 3.4(ii) and from Ken Brown’s Lemma [10, Lemma 1.1.12] that the functor $k(A, -)$ sends weak operadic equivalences between ∞ -operads to simplicial homotopy equivalences. □

3.6. Let X be an ∞ -operad. Given an $(n + 1)$ -tuple of 0-cells (x_1, \dots, x_n, x) in X , the space of maps $X(x_1, \dots, x_n; x)$ is obtained by the pullback below, in which the map p is the map induced by the inclusion $\eta \amalg \dots \amalg \eta \rightarrow \Omega[C_n]$ (with $n + 1$ copies of η , corresponding to the $n + 1$ objects (a_1, \dots, a_n, a) of C_n ; see (2.1)).

$$\begin{array}{ccc} X(x_1, \dots, x_n; x) & \longrightarrow & \text{Hom}(\Omega[C_n], X) \\ \downarrow & & \downarrow p \\ \eta & \xrightarrow{(x_1, \dots, x_n, x)} & X^{n+1} \end{array}$$

Using the identification $s\text{Set} = d\text{Set}/\eta$, we shall consider $X(x_1, \dots, x_n; x)$ as a simplicial set. Observe that $X(x_1, \dots, x_n; x)$ is actually a Kan complex (see [7, Proposition 6.13]).

[†]Another proof consists to see that, by Cisinski and Moerdijk [7, 1.7], for a normal dendroidal set A , the category Ω/A is a regular skeletal category in the sense of [6, 8.2.3], from which we deduce that A is the homotopy colimit of representable objects over A (see [6, 8.2.9]), and we can use Proposition 3.3 to see that the functor $k(-, X)$ turns homotopy colimits into homotopy limits (see, for instance, [5, 6.13]), which implies the result.

DEFINITION 3.7. Let $f : X \rightarrow Y$ be a morphism of ∞ -operads.

The map f is *fully faithful* if, for any $(n + 1)$ -tuple of 0-cells (x_1, \dots, x_n, x) in X , the morphism

$$X(x_1, \dots, x_n; x) \longrightarrow Y(f(x_1), \dots, f(x_n); f(x))$$

is a simplicial homotopy equivalence.

The map f is *essentially surjective* if the functor underlying the morphism of operads $\tau_d(X) \rightarrow \tau_d(Y)$ is essentially surjective.

REMARK 3.8. For an ∞ -operad X , the set of isomorphism classes of objects in $\tau_d(X)$ is in bijection with the set $\pi_0(k(i^*X))$; see [7, 4.1]. The condition of essential surjectivity is thus equivalent to the fact that the map $k(i^*X) \rightarrow k(i^*Y)$ induces a surjection on connected components.

Moreover, by virtue of [7, Proposition 6.14], we have natural bijections

$$\pi_0(X(x_1, \dots, x_n; x)) \simeq \tau_d(X)(x_1, \dots, x_n; x).$$

As a consequence, if $f : X \rightarrow Y$ is fully faithful, so is the induced morphism of operads $\tau_d(f) : \tau_d(X) \rightarrow \tau_d(Y)$. Therefore, if $f : X \rightarrow Y$ is fully faithful and essentially surjective, then the map $k(i^*X) \rightarrow k(i^*Y)$ induces a bijection on connected components.

We recall the following well-known fact.

LEMMA 3.9. A commutative square of simplicial sets

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ p \downarrow & & \downarrow p' \\ Y & \xrightarrow{v} & Y' \end{array}$$

in which p and p' are Kan fibrations is a homotopy pullback square if and only if, for any 0-simplex y of Y , the map between the corresponding fibres

$$p^{-1}(y) = X_y \longrightarrow X'_{v(y)} = p'^{-1}(v(y))$$

is a simplicial homotopy equivalence.

The following lemma is a direct consequence of the preceding lemma.

LEMMA 3.10. Consider a commutative square of simplicial sets.

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ p \downarrow & & \downarrow p' \\ Y & \xrightarrow{v} & Y' \end{array}$$

Assume, furthermore, that p and p' are Kan fibrations, and that the map v is a weak homotopy equivalence. Then the map u is a weak homotopy equivalence if and only if, for any 0-simplex y of Y , the map between the corresponding fibres

$$X_y \longrightarrow X'_{v(y)}$$

is a simplicial homotopy equivalence.

THEOREM 3.11. *Let $f : X \rightarrow Y$ be a morphism between two ∞ -operads. Then f is a weak operadic equivalence if and only if it is fully faithful and essentially surjective.*

Proof. Given an ∞ -operad X , and an $(n + 1)$ -tuple (x_1, \dots, x_n, x) of objects of X , that is, a 0-simplex of $k(i^*X)^{n+1} = k(\partial\Omega[C_n], X)$, we have the following diagram in which the right-hand square is a pullback square (see [7, Remark 6.2 and Corollary 6.8]):

$$\begin{array}{ccccc} X(x_1, \dots, x_n; x) & \longrightarrow & k(\Omega[C_n], X) & \longrightarrow & \mathit{hom}(\Omega[C_n], X) \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \xrightarrow{(x_1, \dots, x_n, x)} & k(i^*X)^{n+1} & \longrightarrow & i^*X^{n+1} \end{array}$$

Hence, the left-hand square above is a pullback as well. Assume that f is fully faithful and essentially surjective. We will first prove that the induced morphism $k(i^*X) \rightarrow k(i^*Y)$ is a simplicial homotopy equivalence. As the corresponding map $\pi_0(k(i^*X)) \rightarrow \pi_0(k(i^*Y))$ is bijective (see Remark 3.8), it is sufficient to prove that, for any 0-simplex x of $k(i^*X)$, the map of loop spaces

$$\Omega(k(i^*X), x) \longrightarrow \Omega(k(i^*Y), f(x))$$

is a weak homotopy equivalence. For this purpose, it will be sufficient to prove that the commutative square

$$\begin{array}{ccc} \Omega(k(i^*X), x) & \longrightarrow & \Omega(k(i^*Y), f(x)) \\ \downarrow & & \downarrow \\ X(x; x) & \longrightarrow & Y(f(x); f(x)) \end{array} \tag{3.4}$$

is homotopy cartesian. In general, the set of connected components of the Kan complex $X(x_1, \dots, x_n; x)$ is in bijection with the set $\tau_d(X)(x_1, \dots, x_n; x)$; see [7, Proposition 6.14]. We can thus describe the loop space $\Omega(k(i^*X), x)$ as the disjoint union of the connected components of $X(x; x)$ which correspond to automorphisms of x in the category underlying the operad $\tau_d(X)$. In particular, the inclusion $\Omega(k(i^*X), x) \subset X(x; x)$ is a Kan fibration between Kan complexes. Using the fact that the functor underlying the map $\tau_d(X) \rightarrow \tau_d(Y)$ is fully faithful (whence conservative), we see that the map

$$\pi_1(k(i^*X), x) = \pi_0(\Omega(k(i^*X), x)) \longrightarrow \pi_0(\Omega(k(i^*Y), f(x))) = \pi_1(k(i^*Y), f(x))$$

is bijective. The square (3.4) is thus cartesian, and, as its vertical maps are Kan fibrations, it is homotopy cartesian as well. Therefore, the map $k(i^*X) \rightarrow k(i^*Y)$ is a simplicial homotopy equivalence. As a consequence, for any corolla C_n , we also have simplicial homotopy equivalences

$$k(i^*X)^{n+1} = k(\partial\Omega[C_n], X) \longrightarrow k(i^*Y)^{n+1} = k(\partial\Omega[C_n], Y).$$

By applying Lemma 3.10 to the commutative squares

$$\begin{array}{ccc} k(\Omega[C_n], X) & \longrightarrow & k(\Omega[C_n], Y) \\ \downarrow & & \downarrow \\ k(\partial\Omega[C_n], X) & \longrightarrow & k(\partial\Omega[C_n], Y) \end{array} \tag{3.5}$$

we conclude that the maps $k(\Omega[C_n], X) \rightarrow k(\Omega[C_n], Y)$ are all simplicial homotopy equivalences. The characterization given by condition (a) of Theorem 3.5 thus implies that f is a weak operadic equivalence.

For the converse, we just apply again Lemma 3.10 to the commutative squares (3.5). □

REMARK 3.12. As we have seen implicitly in the proof above, Theorem 3.5 and Lemma 3.9 lead to a characterization of fully faithful maps: a morphism between ∞ -operads $X \rightarrow Y$ is fully faithful if and only if the commutative squares of shape (3.5) are homotopy pullback squares of Kan complexes for any $n \geq 0$.

4. *Locally constant simplicial dendroidal sets*

4.1. Let $sdSet = dSet^{\Delta^{op}} \simeq \widehat{\Delta} \times \Omega$ be the category of simplicial dendroidal sets. The category $dSet$ (respectively, $sSet$) of dendroidal (respectively, simplicial) sets is naturally embedded in $sdSet$, by viewing a dendroidal (respectively, simplicial) set as a constant simplicial (respectively, dendroidal) object in $dSet$ (respectively, in $sSet$). For a simplicial dendroidal set X , a tree T , and an integer $n \geq 0$, the evaluation of X at (T, n) will often be denoted by $X(T)_n$.

Given a simplicial dendroidal set X , we denote by

$$sSet^{op} \longrightarrow dSet, \quad K \longmapsto X^K,$$

the (essentially) unique limit-preserving functor which sends $\Delta[n]$ to $X^{\Delta[n]} := X_n$.

Starting from the model category structure on $dSet$, and using that the category of simplices Δ is a Reedy category, one obtains the Reedy model structure on $sdSet$; see [10, Theorem 5.2.5]. We shall call this structure the *simplicial Reedy model category structure on $sdSet$* . By definition, the weak equivalences are the termwise weak operadic equivalences (by evaluating at simplices), while the fibrations (respectively, the trivial fibrations) are the maps $X \rightarrow Y$ such that, for any integer $n \geq 0$, the map

$$X^{\Delta[n]} \longrightarrow X^{\partial\Delta[n]} \times_{Y^{\partial\Delta[n]}} Y^{\Delta[n]}$$

is a fibration (respectively, a trivial fibration) in $dSet$. In other words, we have the following proposition.

PROPOSITION 4.2. *The simplicial Reedy model structure on $sdSet$ is a cofibrantly generated model category. A generating set of cofibrations of $sdSet$ is given by the inclusions*

$$\partial\Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T] \longrightarrow \Delta[n] \times \Omega[T],$$

for any integer $n \geq 0$ and any tree T , while, if K is a generating set of trivial cofibrations (between normal dendroidal sets) in $dSet$, then a generating set of trivial cofibrations of $sdSet$ is given by the inclusions

$$\partial\Delta[n] \times B \cup \Delta[n] \times A \longrightarrow \Delta[n] \times B,$$

for any integer $n \geq 0$ and any map $A \rightarrow B$ in K .

COROLLARY 4.3. *The cofibrations of the simplicial Reedy model category structure on $sdSet$ are the termwise normal monomorphisms of dendroidal sets.*

Proof. It is well known that any Reedy cofibration is a termwise cofibration. Therefore, it is sufficient to prove that any termwise normal monomorphism is a cofibration of the simplicial Reedy model category structure on $sdSet$. As both Ω and Δ are skeletal categories in the sense of [6, 8.1.1], so is their product [6, 8.1.7]. Moreover, for any integer $n \geq 0$ and any tree T , the boundary of the representable presheaf $\Delta[n] \times \Omega[T]$ is nothing but the presheaf $\partial\Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T]$. Therefore, the Reedy cofibrations of $sdSet$ are the normal monomorphisms in the absolute sense (see [6, 8.1.30 and 8.1.35] for $A = \Delta \times \Omega$). Thus, Reedy cofibrations are precisely the monomorphisms $X \rightarrow Y$ in $sdSet$ such that, for any integer $n \geq 0$ and any

tree T , any non-degenerate element $y \in Y_{n,T}$ which does not belong to the image of $X_{n,T}$ has a trivial stabilizer in $\text{Aut}([n], T) = \text{Aut}(T)$. On the other hand, a monomorphism of simplicial dendroidal sets $X \rightarrow Y$ is termwise normal if and only if, for any integer $n \geq 0$ and any tree T , any element $y \in Y_{n,T}$ which does not belong to the image of $X_{n,T}$ has a trivial stabilizer. Therefore, any termwise normal monomorphism is a Reedy cofibration. \square

4.4. In the sequel, we will simply refer to the cofibrations of the simplicial Reedy model category structure on $sdSet$ as *normal monomorphisms*.

REMARK 4.5. Any fibrant object of the simplicial Reedy model category on $sdSet$ is termwise fibrant (that is, is termwise an ∞ -operad).

DEFINITION 4.6. We define the *locally constant model category structure* on $sdSet$ as the left Bousfield localization of the simplicial Reedy model category structure on $sdSet$ by the set of projections $\Delta[n] \times \Omega[T] \rightarrow \Omega[T]$, for any tree T and any integer $n \geq 0$ (see [9] for the general theory of left Bousfield localization of model categories).

PROPOSITION 4.7. *Let X be a simplicial dendroidal set. Assume that X is fibrant for the simplicial Reedy model category structure. Then the following conditions are equivalent:*

- (i) *the map from X to the terminal object has the right lifting property with respect to the inclusions*

$$\Lambda^k[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T] \longrightarrow \Delta[n] \times \Omega[T],$$

for any tree T and for any integers $n \geq 1$ and $0 \leq k \leq n$;

- (ii) *for any integer $n \geq 0$, the map $X_0 \rightarrow X_n$ is an equivalence of ∞ -operads;*
- (iii) *X is fibrant for the locally constant model category structure on $sdSet$.*

Proof. The equivalence between conditions (ii) and (iii) holds by definition of left Bousfield localizations, as, for any tree T , we have a natural identification in $\mathbf{Ho}(sdSet)$

$$\text{Map}(\Omega[T], X_n) = \text{Map}(\Delta[n] \times \Omega[T], X),$$

for any integer $n \geq 0$.

Next, it follows easily from [6, Corollary 2.1.20; 10, Lemma 4.2.4] that the class of monomorphisms $K \rightarrow L$ in $sSet$ such that, for any tree T , the map

$$K \times \Omega[T] \cup L \times \partial\Omega[T] \longrightarrow L \times \Omega[T]$$

is a trivial cofibration of the locally constant model category structure on $sdSet$, contains the class of trivial cofibrations of the usual model category structure on $sSet$. Therefore, condition (iii) implies condition (i). Conversely, as the horn inclusions generate the trivial cofibrations of the usual model category structure on $sSet$, it is clear that condition (i) implies condition (iii). \square

PROPOSITION 4.8. *The inclusion $dSet \subset sdSet$ is a left Quillen equivalence from the model category for ∞ -operads to the locally constant model category structure. Moreover, this inclusion preserves and detects weak equivalences between arbitrary objects.*

Proof. This inclusion functor is left adjoint to the evaluation at zero functor

$$\text{ev}_0 : sdSet \longrightarrow dSet, \quad X \longmapsto X_0.$$

Note first that the inclusion functor $dSet \subset sSet$ is a left Quillen functor which preserves weak equivalences: by virtue of Corollary 4.3, it preserves cofibrations, and it preserves weak equivalences by definition of the locally constant model structure. Thus, to complete the proof, it is sufficient to check the following two properties:

- (a) for any fibrant object X of the locally constant model structure, the natural map $X_0 \rightarrow X$ is a weak equivalence;
- (b) for any fibrant object X in $dSet$, there exists a weak equivalence $X \rightarrow Y$ in $sSet$ with Y fibrant in the locally constant model structure, such that the induced map $X \rightarrow Y_0$ is a weak equivalence of dendroidal sets.

Property (a) follows from the characterization of fibrant objects given by condition (ii) of the previous proposition. Property (b) is a particular instance of the existence of Reedy fibrant resolutions.[†] □

5. Dendroidal Segal spaces

5.1. We shall now consider different model category structures on the category of simplicial dendroidal sets.

Given a simplicial dendroidal set X , let us denote by

$$dSet^{op} \longrightarrow sSet, \quad A \longmapsto X^A,$$

the (essentially) unique limit-preserving functor which sends a tree T to $X^{\Omega[T]} := X(T)$.

Starting from the usual model category structure on the category of simplicial sets, we first have the following proposition.

PROPOSITION 5.2. *The category $sSet$ admits a cofibrantly generated and proper model category structure whose weak equivalences are the termwise simplicial weak homotopy equivalences (that is, the maps $X \rightarrow Y$ such that, for any tree T , the map $X_T \rightarrow Y_T$ is a simplicial weak homotopy equivalence), and whose cofibrations are the normal monomorphisms. Moreover, a morphism of simplicial dendroidal sets $X \rightarrow Y$ is a fibration (respectively, a trivial fibration) if and only if, for any tree T , the map*

$$X^{\Omega[T]} \longrightarrow X^{\partial\Omega[T]} \times_{Y^{\partial\Omega[T]}} Y^{\Omega[T]} \tag{5.1}$$

is a Kan fibration (respectively, a trivial Kan fibration). In other words, a set of generators for cofibrations is provided by the maps

$$\partial\Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T] \longrightarrow \Delta[n] \times \Omega[T],$$

for any tree T and for any integer $n \geq 0$, while a generating set of trivial cofibrations is given by the maps

$$\Lambda^k[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T] \longrightarrow \Delta[n] \times \Omega[T],$$

for any tree T and for any integers $n \geq 1$ and $0 \leq k \leq n$.

Proof. As the category Ω is a generalized Reedy category (see [3, Example 2.8]), the model category above can be obtained as a special case of [3, Theorem 1.6]. □

[†]One can also prove (b) more explicitly from [7, Corollary 6.9]: given an ∞ -operad X , we can consider the simplicial dendroidal set Y defined by $Y_{n,T} = k(\Omega[T], X)_n$ for any integer $n \geq 0$ and any tree T (see Subsection 3.2).

REMARK 5.3. We shall call the model category of Proposition 5.2 the *generalized Reedy model category structure* on $sdSet$.

DEFINITION 5.4. We define the *model category structure for dendroidal Segal spaces* as the left Bousfield localization of the generalized Reedy model category on $sdSet$ by the set of maps $Sc[T] \rightarrow \Omega[T]$, for any tree T (we consider $dSet$ as full subcategory of $sdSet$ via the obvious inclusion $dSet \subset sdSet$). A fibrant object for this model category will be called a *dendroidal Segal space*.

PROPOSITION 5.5. *The model category structure for dendroidal Segal spaces is the left Bousfield localization of the generalized Reedy model category structure on $sdSet$ by the set of maps $\Lambda^e[T] \rightarrow \Omega[T]$, for any tree T with an inner edge e .*

Proof. It follows immediately from Proposition 2.5 that, for any tree T with given inner edge e , the map $\Lambda^e[T] \rightarrow \Omega[T]$ is a weak equivalence of the model category structure for dendroidal Segal spaces. Conversely, let W be the class of maps of dendroidal sets which are weak equivalences in the left Bousfield localization of the generalized Reedy model category on $sdSet$ by the set of maps $\Lambda^e[T] \rightarrow \Omega[T]$, for any tree T with an inner edge e . It is clear that any inner anodyne extension in $dSet$ belongs to W , so that, by virtue of Proposition 2.4, for any tree T , the inclusion $Sc[T] \rightarrow \Omega[T]$ is in W . □

COROLLARY 5.6. *Let X be a simplicial dendroidal set. Assume that X is fibrant for the generalized Reedy model category structure. Then the following conditions are equivalent:*

- (i) X is a dendroidal Segal space;
- (ii) for any tree T , the map $X^{\Omega[T]} \rightarrow X^{Sc[T]}$ is a trivial Kan fibration in $sSet$;
- (iii) for any tree T with a given inner edge e , the map $X^{\Omega[T]} \rightarrow X^{\Lambda^e[T]}$ is a trivial Kan fibration in $sSet$.

Proof. For any normal dendroidal set A , there is a canonical identification

$$\text{Map}(A, X) = X^A.$$

The corollary thus follows from the definition of left Bousfield localizations and from Proposition 5.5. □

PROPOSITION 5.7. *A morphism of dendroidal Segal spaces $X \rightarrow Y$ is a weak equivalence if and only if, its evaluation at T*

$$X(T) = X^{\Omega[T]} \longrightarrow Y^{\Omega[T]} = Y(T)$$

is a simplicial homotopy equivalence for $T = \eta$ as well as for $T = C_n, n \geq 0$.

Proof. A morphism of dendroidal Segal spaces is a weak equivalence of the model structure for dendroidal Segal spaces if and only if it is a weak equivalence of the generalized Reedy model category structure. In other words, a morphism of dendroidal Segal spaces $X \rightarrow Y$ is a weak equivalence if and only if its evaluation at T

$$X^{\Omega[T]} \longrightarrow Y^{\Omega[T]}$$

is a simplicial homotopy equivalence for any tree T . By virtue of condition (ii) of the preceding corollary, we see that evaluating a dendroidal Segal space at a tree T gives the same information as evaluating at $\text{Sc}[T]$. We easily conclude the proof from the fact that $\text{Sc}[T]$ is a (homotopy) colimit of dendroidal sets of shape η or $\Omega[C_n]$, $n \geq 0$. \square

5.8. If X is a dendroidal Segal space, and if (x_1, \dots, x_n, x) is an $(n + 1)$ -tuple of elements of $X(\eta)_0^{n+1}$, then we define $X(x_1, \dots, x_n; x)$ by the following pullback:

$$\begin{array}{ccc} X(x_1, \dots, x_n; x) & \longrightarrow & X(C_n) \\ \downarrow & & \downarrow \\ \Delta[0] & \xrightarrow{(x_1, \dots, x_n, x)} & X(\eta)^{n+1} \end{array}$$

in which the map $X(C_n) \rightarrow X(\eta)^{n+1} = X^{\partial\Omega[C_n]}$ is the map induced by the inclusion $\coprod_{n+1} \eta = \partial\Omega[C_n] \rightarrow \Omega[C_n]$. As $X(C_n) \rightarrow X(\eta)^{n+1}$ is a Kan fibration, the pullback square above is homotopy cartesian, and $X(x_1, \dots, x_n; x)$ is a Kan complex.

DEFINITION 5.9. A morphism of dendroidal Segal spaces $f : X \rightarrow Y$ is *fully faithful* if, for any $(n + 1)$ -tuple of 0-cells (x_1, \dots, x_n, x) in $X(\eta)$, the morphism

$$X(x_1, \dots, x_n; x) \longrightarrow Y(f(x_1), \dots, f(x_n); f(x))$$

is a simplicial homotopy equivalence.

A morphism of dendroidal Segal spaces is a *weak equivalence on objects* if its evaluation at η is a simplicial weak equivalence.

COROLLARY 5.10. A morphism of dendroidal Segal spaces is a weak equivalence if and only if it is fully faithful as well as a weak equivalence on objects.

Proof. This follows immediately from Proposition 5.7 and from Lemma 3.10. \square

6. Complete dendroidal Segal spaces

6.1. Recall the dendroidal interval $J_d = i_!(J)$, where $J = N\pi_1(\Delta[1])$ denotes the nerve of the contractible groupoid with two objects 0 and 1.

DEFINITION 6.2. We define the *dendroidal Rezk model category* as the left Bousfield localization of the model category for dendroidal Segal spaces (5.4) by the maps

$$\Omega[T] \otimes J_d \longrightarrow \Omega[T], \quad T \in \Omega,$$

obtained by tensoring with the unique morphism $J_d \rightarrow \eta$, the image under $i_!$ of the unique map $J \rightarrow \Delta[0]$. The fibrant objects of the dendroidal Rezk model category will be called *complete dendroidal Segal spaces*. The weak equivalences of this model category structure will be called the *complete weak equivalences*.

PROPOSITION 6.3. For any normal dendroidal set A , the map $A \otimes J_d \rightarrow A$ induced by the projection $J_d \rightarrow \eta$ is a complete weak equivalence.

Proof. It is sufficient to prove that this map is a weak equivalence for the induced model category structure on $sdSet/A$, the latter being equivalent to the category of presheaves on the category $\Delta \times \Omega/A$ (where Ω/A is the category of elements of A). On the other hand, we know that tensoring by J_d preserves colimits as well as normal monomorphisms. As, by virtue of [7, Corollary 1.7], Ω/A is then a regular skeletal category in the sense of [6, 8.2.3], this proposition is a straightforward application of [6, 8.2.14]. \square

COROLLARY 6.4. *The inclusion $dSet \subset sdSet$ sends the weak operadic equivalences between normal dendroidal sets to complete weak equivalences.*

Proof. By virtue of Propositions 5.5 and 6.3, this inclusion functor sends J -anodyne extensions in the sense of [7, 3.2] to complete weak equivalences. This corollary thus immediately follows from [7, Proposition 3.16] and from Ken Brown's Lemma [10, Lemma 1.1.12]. \square

COROLLARY 6.5. *Let K be a generating set of trivial cofibrations in $dSet$. We assume that all the maps in K are morphisms between normal dendroidal sets (which is a harmless hypothesis, by virtue of [7, Remark 3.15]). A simplicial dendroidal set X is a complete dendroidal Segal space if and only if it is a Segal space, and if the map from X to the terminal simplicial dendroidal set has the right lifting property with respect to the inclusions of shape*

$$\partial\Delta[n] \times B \cup \Delta[n] \times A \longrightarrow \Delta[n] \times B,$$

for $j : A \rightarrow B$ in K and $n \geq 0$.

Proof. Let L be the set of maps $\partial\Delta[n] \times B \cup \Delta[n] \times A \rightarrow \Delta[n] \times B$, for $A \rightarrow B$ in K and $n \geq 0$, and consider the left Bousfield localization of the model category for dendroidal Segal spaces by L . It is clear that the L -fibrant objects are precisely the Segal spaces X such that the map from X to the terminal object has the right lifting property with respect to the elements of L . Therefore, it is sufficient to prove that this left Bousfield localization at L coincides with the dendroidal Rezk model category structure on $sdSet$. Using Corollaries 4.3 and 6.4, we easily see that the elements of L are cofibrations and complete weak equivalences. On the other hand, for any trivial cofibration $X \rightarrow Y$ in $dSet$ and any integer $n \geq 0$ the map $\partial\Delta[n] \times Y \cup \Delta[n] \times X \rightarrow \Delta[n] \times Y$ is a trivial cofibration of the localized model structure at L (this readily follows from [10, Lemma 4.2.4], for instance). In particular, for any tree T , and $\varepsilon \in \{0, 1\}$, the map $\Omega[T] = \Omega[T] \otimes \{\varepsilon\} \rightarrow \Omega[T] \otimes J_d$ is a weak equivalence of the localized model structure at L . Therefore, any complete weak equivalence is a weak equivalence for the left Bousfield localization by L . \square

THEOREM 6.6. *The locally constant model category structure on $sdSet$ (4.6) and the dendroidal Rezk model category structure (6.2) are equal.*

Proof. As these two model category structures on $sdSet$ have the same class of cofibrations, it is sufficient to observe that they have the same class of fibrant objects, which follows from Corollary 6.5 and from the characterization of fibrant objects given by Propositions 4.2, 4.7(i), 5.2, and Corollary 5.6(iii). \square

COROLLARY 6.7. *The inclusion functor $dSet \subset sdSet$ is a left Quillen equivalence from the model category for ∞ -operads to the dendroidal Rezk model category.*

Proof. This follows from the preceding theorem and from Proposition 4.8. □

COROLLARY 6.8. *A morphism of dendroidal sets is a weak operadic equivalence if and only if its image under the inclusion $dSet \subset sdSet$ is a complete weak equivalence.*

Proof. This is a reformulation of the last assertion of Proposition 4.8 through Theorem 6.6. □

COROLLARY 6.9. *Let $X \rightarrow Y$ be a morphism between complete dendroidal Segal spaces. The following conditions are equivalent.*

- (a) *The map $X \rightarrow Y$ is a complete weak equivalence.*
- (b) *For any integer $n \geq 0$, the map $X_n \rightarrow Y_n$ is an equivalence of ∞ -operads.*
- (c) *The map $X_0 \rightarrow Y_0$ is an equivalence of ∞ -operads.*
- (d) *For any tree T , the map $X(T) \rightarrow Y(T)$ is a homotopy equivalence between Kan complexes.*

6.10. Consider the cosimplicial dendroidal set

$$\Delta_J : \Delta \longrightarrow dSet, \tag{6.1}$$

defined by

$$\Delta_J[n] = i_! N\pi_1(\Delta[n]) \tag{6.2}$$

(so that $\Delta_J[1] = J_d$). This cosimplicial object defines a unique colimit-preserving functor

$$sdSet \longrightarrow dSet, \quad X \longmapsto |X|_J, \tag{6.3}$$

such that

$$|\Delta[n] \times \Omega[T]|_J = \Delta_J[n] \otimes \Omega[T]. \tag{6.4}$$

The functor $|-|_J$ has a right adjoint

$$dSet \longrightarrow sdSet, \quad X \longmapsto \text{Sing}_J(X), \tag{6.5}$$

defined by

$$\text{Sing}_J(X)(T)_n = \text{Hom}_{dSet}(\Delta_J[n] \otimes \Omega[T], X). \tag{6.6}$$

PROPOSITION 6.11. *The functor (6.3) is a left Quillen equivalence from the dendroidal Rezk model category to the model category for ∞ -operads.*

Proof. Using the fact that $dSet$ is a monoidal model category, it is easily seen that (6.3) is a left Quillen functor from the generalized Reedy model structure (given by Proposition 5.2) to the model category for ∞ -operads. Therefore, to prove that this is a left Quillen functor for the dendroidal Rezk model structure, it is sufficient to prove that it sends inner horns as well as maps of shape

$$\Omega[T] \otimes J_d \longrightarrow \Omega[T], \quad T \in \Omega,$$

to weak operadic equivalences. But this latter property follows from the fact that the composition of (6.3) with the inclusion $dSet \subset sdSet$ is (isomorphic to) the identity. Similarly, to prove that (6.3) is a left Quillen equivalence, by virtue of Corollary 6.7, it is sufficient to prove

that its composition with the inclusion $dSet \subset sdSet$ is a left Quillen equivalence, which is more than obvious. □

6.12. Let $\infty\text{-Operad}$ be the full subcategory of $dSet$ spanned by ∞ -operads (that is, fibrant objects). We define a functor

$$K : \infty\text{-Operad} \longrightarrow sdSet, \tag{6.7}$$

by the formula below (see 3.1):

$$K(X)(T)_n = k(\Omega[T], X)_n. \tag{6.8}$$

PROPOSITION 6.13. *The functor (6.7) takes its values in the full subcategory of $sdSet$ spanned by complete dendroidal Segal spaces. Moreover, it preserves fibrations as well as weak equivalences between ∞ -operads, and, under the canonical equivalence $\mathbf{Ho}(\infty\text{-Operad}) \simeq \mathbf{Ho}(dSet)$, the corresponding functor*

$$K : \mathbf{Ho}(dSet) \longrightarrow \mathbf{Ho}(sdSet)$$

is canonically isomorphic to the functor

$$\mathbf{R} \text{Sing}_J : \mathbf{Ho}(dSet) \longrightarrow \mathbf{Ho}(sdSet)$$

(which is an equivalence of categories).

Proof. In view of the identification of Proposition 3.3, this is a straightforward application of the general properties of mapping spaces; see [10, Propositions 5.4.1, 5.4.3, and 5.4.7] (remark that, for any normal dendroidal set A , $\Delta_J[\bullet] \otimes A$ provides a canonical cosimplicial frame of A in the sense of [10, Definition 5.2.7]). □

REMARK 6.14. The Boardman–Vogt tensor product on $dSet$ induces a symmetric monoidal structure on $sdSet$: for two simplicial dendroidal sets X and Y , their tensor product $X \otimes Y$ is simply defined termwise:

$$(X \otimes Y)_n = X_n \otimes Y_n, \quad n \geq 0.$$

Using the fact that $dSet$ is a symmetric monoidal model category with respect to the Boardman–Vogt tensor product (Theorem 1.1), it is easily seen that $sdSet$, endowed with the dendroidal Rezk model structure, is also a symmetric monoidal model category. Moreover, the functor $dSet \subset sdSet$ is a symmetric monoidal left Quillen equivalence.

7. Segal pre-operads

DEFINITION 7.1. A Segal pre-operad is a dendroidal space A such that $A(\eta)$ is a discrete simplicial set (that is, all the simplices of positive dimension in $A(\eta)$ are degenerated). We denote by $PreOper$ the full subcategory of $sdSet$ spanned by Segal pre-operads.

7.2. The category of Segal pre-operads is in fact the category of presheaves on the category $S(\Omega)$, which is obtained as the localization of $\Delta \times \Omega$ by the arrows of shape $([n], \eta) \rightarrow ([m], \eta)$. We denote by

$$\gamma : \Delta \times \Omega \longrightarrow S(\Omega), \tag{7.1}$$

the localization functor. Under the identification $\mathcal{PreOper} \simeq \widehat{S(\Omega)}$, the inverse image functor

$$\gamma^* : \mathcal{PreOper} \longrightarrow \mathit{sdSet} \tag{7.2}$$

is simply the inclusion functor. The inclusion functor γ^* thus has a right adjoint

$$\gamma_* : \mathit{sdSet} \longrightarrow \mathcal{PreOper}, \tag{7.3}$$

as well as a left adjoint

$$\gamma_! : \mathit{sdSet} \longrightarrow \mathcal{PreOper}. \tag{7.4}$$

The explicit description of these adjoints will be needed later on.

The right adjoint, $\gamma_* : \mathit{sdSet} \rightarrow \mathcal{PreOper}$, is defined as follows. Let X be a dendroidal space. Then $\gamma_*(X)$ is the subobject of X given by all the dendrices whose vertices are degenerated. More explicitly, for a tree T , let write $E(T)$ for its set of edges (colours), with the evident inclusion (which is natural in T)

$$v_T : \prod_{e \in E(T)} \eta \longrightarrow \Omega[T].$$

For a simplicial set K , we shall identify the set K_0 with the corresponding discrete simplicial set, and write $s : K_0 \rightarrow K$ for the inclusion. Then $\gamma_*(X)(T)$ is defined as the following pullback of simplicial sets:

$$\begin{array}{ccc} \gamma_*(X)(T) & \longrightarrow & X(T) \\ \downarrow & & \downarrow v_T^* \\ \prod_{e \in E(T)} X(\eta)_0 & \xrightarrow{s} & \prod_{e \in E(T)} X(\eta) \end{array} \tag{7.5}$$

The left adjoint $\gamma_! : \mathit{sdSet} \rightarrow \mathcal{PreOper}$ can also be made explicit as follows. For a simplicial dendroidal set X as above, consider the set $\pi_0 X(\eta)$ of connected components of the simplicial set $X(\eta)$. We have $\gamma_!(X)(T) = X(T)$ for any tree T such that there is no map $T \rightarrow \eta$ in Ω . If there is a map $\varepsilon : T \rightarrow \eta$ in Ω , then it is unique (remember there is a canonical isomorphism $\Omega/\eta = \Delta$), and we can describe $\gamma_!(X)(T)$ as the pushout below.

$$\begin{array}{ccc} X(\eta) & \longrightarrow & X(T) \\ \downarrow & & \downarrow \\ \pi_0 X(\eta) & \longrightarrow & \gamma_!(X)(T) \end{array} \tag{7.6}$$

7.3. A morphism of Segal pre-operads is a monomorphism if and only if its image by γ^* is (because γ^* is a fully faithful limit-preserving functor). We say that a morphism of Segal pre-operads $X \rightarrow Y$ is a *normal monomorphism* if its image by γ^* has the same property (this just means that the map $X_n \rightarrow Y_n$ is a normal monomorphism of dendroidal sets for any integer $n \geq 0$).

A Segal pre-operad X is *normal* if $\emptyset \rightarrow X$ is a normal monomorphism.

A morphism of Segal pre-operads is a *trivial fibration* if it has the right lifting property with respect to the class of normal monomorphisms.

LEMMA 7.4. *If $X \rightarrow Y$ is a normal monomorphism of simplicial dendroidal sets and if $\pi_0 X(\eta) \rightarrow \pi_0 Y(\eta)$ is injective, then $\gamma_!(X) \rightarrow \gamma_!(Y)$ is a normal monomorphism of Segal pre-operads.*

Proof. One sees easily from the explicit description of $\gamma_!$ given by the pushouts (7.6) that, for any tree T above η , the map $\gamma_!(X)(T) \rightarrow \gamma_!(Y)(T)$ is injective. For any tree T which has a non-trivial automorphism in Ω , there is no map from T to η . As, for such a tree T , we have $\gamma_!(X)(T) = X(T)$, it is clear that the map $\gamma_!(X) \rightarrow \gamma_!(Y)$ is a normal monomorphism. \square

PROPOSITION 7.5. *Let I be the set of maps*

$$\gamma_!(\partial\Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T]) \longrightarrow \gamma_!(\Delta[n] \times \Omega[T]), \tag{7.7}$$

for any tree T with at least one vertex, and for any integer $n \geq 0$, together with the map $\emptyset \rightarrow \eta$. Then the smallest class of maps in *PreOper* which is closed under pushouts, transfinite compositions, and retracts, and which contains I , is the class of normal monomorphisms.

Proof. Let us call *I-cofibrations* the elements of the smallest class of maps which contains I and is closed under pushouts, transfinite compositions, and retracts.

If T is a tree with at least one vertex, then, for any integer $n \geq 0$, the evaluation of the map

$$\partial\Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T] \longrightarrow \Delta[n] \times \Omega[T], \tag{7.8}$$

at η is bijective, so that, by virtue of Lemma 7.4, its image by $\gamma_!$ is a normal monomorphism. Hence, any map in I is a normal monomorphism of Segal pre-operads. Therefore, any *I*-cofibration is a normal monomorphism.

Conversely, consider a normal monomorphism of Segal pre-operads $u : A \rightarrow B$. Let A' be the Segal pre-operad obtained from the pushout below.

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & & \downarrow \\ \coprod_{b \in (B(\eta)_0 - A(\eta)_0)} & \xrightarrow{\eta} & A' \end{array}$$

Then the inclusion $A \rightarrow A'$ is certainly an *I*-cofibration, and one checks easily that the canonical map $A' \rightarrow B$ is still a normal monomorphism. Thus, to prove that $u : A \rightarrow B$ is an *I*-cofibration, we may assume, without loss of generality, that the map $A(\eta) \rightarrow B(\eta)$ is bijective on 0-simplices. Applying the small object argument to the map u with the set of maps (7.7) (for any tree T with at least one vertex, and for any integer $n \geq 0$), we obtain a factorization of u of shape

$$A \xrightarrow{v} C \xrightarrow{p} B,$$

in which v is an *I*-cofibration, while p has the right lifting property with respect to maps of shape (7.7) (still for any tree T with at least one vertex, and for any integer $n \geq 0$). Moreover, one checks that v induces a bijection by evaluating at η , which implies that p has the same property. We claim that $\gamma^*(p)$ has the right lifting property with respect to maps of shape (7.8) (for any tree T and any integer n). Indeed, in the case T has at least one vertex this follows by a standard adjunction argument. In the case where $T = \eta$, this lifting property means that the map $C(\eta) \rightarrow B(\eta)$ is a trivial fibration between discrete simplicial sets, that is, is a bijective map on the 0-simplices. Hence, since γ^* is fully faithful, the map p has the right lifting with respect to u . By the retract argument [10, Lemma 1.1.9], this implies that u is a retract of v , whence is an *I*-cofibration. \square

8. Segal operads

DEFINITION 8.1. A Segal operad is a Segal pre-operad X such that, for any tree T , the map

$$X(T) = X^{\Omega[T]} \longrightarrow X^{\text{Sc}[T]}$$

is a trivial fibration of simplicial sets, where, if A is a dendroidal set, then X^A denotes the simplicial set whose n -simplices are the maps of dendroidal sets from A to X_n (with the notation of Subsection 5.1, we thus have $\gamma^*(X)^A = X^A$). We write *SegOper* for the full subcategory of *PreOper* spanned by Segal operads.

A Reedy fibrant Segal operad is a Segal pre-operad whose image by γ^* is fibrant in the model category structure for dendroidal Segal spaces (see Definition 5.4). Note that any Reedy fibrant Segal operad is indeed a Segal operad; see Corollary 5.6.

A morphism of Segal pre-operads is a Segal weak equivalence if its image by γ^* is a complete weak equivalence (6.2).

A morphism between Reedy fibrant Segal operads is fully faithful if its image by the functor γ^* is fully faithful; see Definition 5.9.

PROPOSITION 8.2. Let X be a dendroidal Segal space. Then $\gamma_*(X)$ is a Reedy fibrant Segal operad.

Proof. Since, for any tree T , the evaluation of the map $\text{Sc}[T] \rightarrow \Omega[T]$ at η is bijective, the commutative square

$$\begin{array}{ccc} \gamma^*\gamma_*(X)^{\Omega[T]} & \longrightarrow & X^{\Omega[T]} \\ \downarrow & & \downarrow \\ \gamma^*\gamma_*(X)^{\text{Sc}[T]} & \longrightarrow & X^{\text{Sc}[T]} \end{array}$$

is cartesian. Therefore, the functor $\gamma^*\gamma_*$ preserves dendroidal Segal spaces. In other words, the functor γ_* sends dendroidal Segal spaces to Reedy fibrant Segal operads. \square

LEMMA 8.3. With the notation of Subsection 6.12, for any ∞ -operad X , the natural map $X \rightarrow K(X)$ is a complete weak equivalence.

Proof. As $X = K(X)_0$, this is a reformulation of the fact that $K(X)$ is a complete dendroidal Segal space; see Propositions 4.7, 6.13, and Theorem 6.6. \square

PROPOSITION 8.4. For any dendroidal Segal space X , the map $X_0 \rightarrow X$ is a complete weak equivalence.

Proof. Let X be a dendroidal Segal space. Given a bisimplicial object U , we write $\text{diag}(U)$ for the simplicial object defined by $\text{diag}(U)_n = U_{n,n}$. We define the bisimplicial dendroidal set V by $V_{m,n} = X_n^{\Delta[m]}$ (see Subsection 3.1), and we put $W = \text{diag}(V)$. The maps $\Delta[m] \rightarrow \Delta[0]$ induce embeddings $X_n = V_{0,n} \subset V_{m,n}$, and thus a monomorphism $X \rightarrow W$. Recall that $K(X_0) = V_{\bullet,0}$ is a fibrant resolution of X_0 in the dendroidal Rezk model structure;

see Proposition 6.13. We have a canonical commutative square of the following form:

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ K(X_0) & \longrightarrow & W \end{array}$$

in which the map $X_0 \rightarrow K(X_0)$ is a complete weak equivalence (by Lemma 8.3). It is thus sufficient to prove that the maps $X \rightarrow W$ and $K(X_0) \rightarrow W$ are complete weak equivalences.

By virtue of Lemma 8.3, the inclusion $X_n \rightarrow V_{\bullet,n} = K(X_n)$ is a weak equivalence for any integer $n \geq 0$. Using Theorem 6.6 (so that we can compute homotopy colimits in $sSet^{\Omega^{op}}$ in the usual way), this implies that the induced map

$$X = \operatorname{hocolim}_{\Delta[n] \in \Delta^{op}} X_n \longrightarrow \operatorname{hocolim}_{\Delta[n] \in \Delta^{op}} K(X_n) = W$$

is a complete weak equivalence.

If we work with the projective model structure on $sdSet = dSet^{\Delta^{op}}$ associated to the model structure on $dSet$ (that is, the model category whose weak equivalences (or fibrations) are the maps whose evaluation at each object of Δ is a weak equivalence (or a fibration, respectively) in $dSet$), then, for any tree T , the functor

$$\Delta[n] \longmapsto \Delta[n] \times \Omega[T]$$

is a cosimplicial resolution of $\Omega[T]$, while

$$\Delta[m] \longmapsto X^{(\Delta[m])} = V_{m,\bullet}$$

is a simplicial resolution of X . Therefore, by virtue of [10, Proposition 5.4.7], for any tree T , we can identify the simplicial set $W(T)$ with the mapping space $\operatorname{Map}(\Omega[T], X)$. On the other hand, as the evaluation at zero functor $X \mapsto X_0$ is a right Quillen functor from $dSet^{\Delta^{op}}$ to $dSet$, we have the following natural identifications in the homotopy category of Kan complexes:

$$\operatorname{Map}(\Omega[T], X_0) \simeq \operatorname{Map}(\Omega[T], X).$$

In other words, with the notation introduced in Subsection 6.12, by virtue of Proposition 3.3, the map $K(X_0) \rightarrow W$ induces a canonical isomorphism

$$K(X_0) \simeq W,$$

in the homotopy category of $sSet^{\Omega^{op}}$ (corresponding to termwise weak equivalences of $sSet$). Therefore, the map $K(X_0) \rightarrow W$ is a complete weak equivalence, and this ends the proof. \square

REMARK 8.5. If we keep the notation used in the proof above, then we may use the simplicial dendroidal set W to obtain a canonical resolution of X by a complete dendroidal Segal space: one may consider a fibrant resolution $W \rightarrow Y$ for the generalized Reedy model structure on $sSet^{\Omega^{op}}$. Then the map $X \rightarrow Y$ is a complete weak equivalence, and Y is a complete dendroidal Segal space.

COROLLARY 8.6. *The functor γ_* sends complete weak equivalences between dendroidal Segal spaces to Segal weak equivalences, and, for any dendroidal Segal space X , the map $\gamma^* \gamma_*(X) \rightarrow X$ is a complete weak equivalence.*

Proof. It is clearly sufficient to prove the last assertion, which follows from the fact that, by virtue of Propositions 8.2 and 8.4, for any dendroidal Segal space X , there exists

a commutative square

$$\begin{array}{ccc}
 \gamma^*\gamma_*(X)_0 & \longrightarrow & \gamma^*\gamma_*(X) \\
 \parallel & & \downarrow \\
 X_0 & \longrightarrow & X
 \end{array}$$

in which three hence all maps are weak equivalences. □

COROLLARY 8.7. *A morphism between dendroidal Segal spaces $X \rightarrow Y$ is a complete weak equivalence if and only if $\gamma_*(X) \rightarrow \gamma_*(Y)$ is a Segal weak equivalence of Segal operads.*

8.8. Given a dendroidal Segal space X , there is a canonical operad $\text{ho}(X)$ associated to it, whose set of colours is $X(\eta)_0$, and whose sets of maps are given by $\pi_0(X(x_1, \dots, x_n; x))$ (the fact that this defines an operad can be proved using the explicit description of the operad associated to an ∞ -operad (see [7, Proposition 6.14]), Corollary 8.7 (to reduce to the case of a complete dendroidal Segal space), Proposition 3.3, as well as the Quillen equivalence of Proposition 6.11; however, it is not difficult to understand this construction in elementary terms).

DEFINITION 8.9. A morphism of dendroidal Segal spaces $X \rightarrow Y$ is *essentially surjective* if the morphism of operads $\text{ho}(X) \rightarrow \text{ho}(Y)$ is essentially surjective.

REMARK 8.10. A morphism of dendroidal Segal spaces $X \rightarrow Y$ is fully faithful (see Definition 5.9) and essentially surjective if and only if the induced morphism $\gamma^*\gamma_*(X) \rightarrow \gamma^*\gamma_*(Y)$ has the same property.

THEOREM 8.11. *Let $f : X \rightarrow Y$ be a morphism of dendroidal Segal spaces. The following conditions are equivalent.*

- (a) *The map f is a complete weak equivalence.*
- (b) *The map $\gamma_*(f) : \gamma_*(X) \rightarrow \gamma_*(Y)$ is a weak equivalence of Segal operads.*
- (c) *The map $\gamma^*\gamma_*(f) : \gamma^*\gamma_*(X) \rightarrow \gamma^*\gamma_*(Y)$ is fully faithful and essentially surjective.*
- (d) *The map f is fully faithful and essentially surjective.*

Proof. The equivalence between (a) and (b) follows from Corollary 8.7, while the equivalence between (c) and (d) is a tautology. To prove the remaining equivalences, we will use Theorem 3.11 to deduce the equivalence between conditions (a) and (d). Indeed, we may assume that X and Y are complete (by Corollary 8.7), so that the evaluated maps $X(T) \rightarrow Y(T)$ may be identified with the maps of mapping spaces $\text{Map}(\Omega[T], X) \rightarrow \text{Map}(\Omega[T], Y)$; up to the Quillen equivalence $d\text{Set} \subset sd\text{Set}$ (see Corollary 6.7), and using Proposition 3.3, condition (d) (respectively, (a)) may be interpreted by saying that the map between the corresponding ∞ -operads is fully faithful and essentially surjective (respectively, satisfies condition (b) of Theorem 3.5). This completes the proof of the theorem. □

LEMMA 8.12. *If a morphism of Segal pre-operads has the right lifting property with respect to normal monomorphisms, then it is a Segal weak equivalence.*

Proof. Consider first a normal resolution E_∞ of the terminal dendroidal set (that is, a cofibrant resolution of the terminal dendroidal set for the model category structure of Theorem 1.1), considered as a simplicially constant simplicial dendroidal set. We may see E_∞ as a Segal pre-operad, and it follows immediately from Theorem 6.6 that, for any Segal pre-operad X , the projection $E_\infty \times X \rightarrow X$ is a Segal weak equivalence. Moreover, as E_∞ is even a normal Segal pre-operad, $E_\infty \times X$ is always normal. Therefore, it is sufficient to prove that, if $p : X \rightarrow Y$ is a morphism of normal Segal pre-operads which has the right lifting property with respect to normal monomorphisms of pre-operads, then it is a Segal weak equivalence. Note that J_d may be seen as Segal pre-operad, so that, for any normal pre-operad A , $J_d \otimes A$ is still a pre-operad, and, whenever A is normal, the map

$$A \amalg A = (\{0\} \amalg \{1\}) \otimes A \longrightarrow J \otimes A$$

is a normal monomorphism; see Corollary 4.3 and [7, Proposition 1.9]. We may now complete the proof in the standard way: as Y is normal, the map p admits a section $s : Y \rightarrow X$, and the commutative square

$$\begin{array}{ccc} X \amalg X & \xrightarrow{(1_X, sp)} & X \\ \downarrow & \nearrow h & \downarrow p \\ J_d \otimes X & \xrightarrow{p\pi} & Y \end{array}$$

admits a lifting h (where $\pi : J_d \otimes X \rightarrow X$ denotes the map induced by the map $J_d \rightarrow \eta$), which means that the map p is a J_d -homotopy equivalence, whence a weak equivalence. \square

THEOREM 8.13. *The category of Segal pre-operads is endowed with a left proper cofibrantly generated model category structure whose weak equivalences are the Segal weak equivalences, and whose cofibrations are the normal monomorphisms.*

Proof. The preceding lemma tells us that any trivial fibration of Segal pre-operads is a Segal weak equivalence. On the other hand, by virtue of Proposition 7.5, the class of normal monomorphism is generated by a small set of maps. The existence of this model category is thus a particular case of J. Smith’s Theorem; see [1, Theorem 1.7 and Proposition 1.18]. The left properness property follows immediately from its counterpart for the Rezk model category structure. \square

8.14. The model category structure above will be called the *Reedy–Segal model category structure* on *PreOper*. We will always consider the category of simplicial dendroidal sets as a model category with the Rezk model structure (6.2). By construction, the functor $\gamma^* : \text{PreOper} \rightarrow \text{sdSet}$ is a left Quillen functor. Our purpose is to prove that it is in fact a left Quillen equivalence, and that the fibrant pre-operads are precisely the Reedy fibrant Segal operads.

THEOREM 8.15. *The functor $\gamma^* : \text{PreOper} \rightarrow \text{sdSet}$ is a left Quillen equivalence from the model category for Segal operads to the model category for complete dendroidal Segal spaces.*

Proof. As γ^* is a fully faithful left Quillen functor which preserves and detects weak equivalences, this follows immediately from Corollary 8.6. \square

REMARK 8.16. Note that Segal pre-operads are closed under tensor product (as defined in Remark 6.14), and that the model category of Theorem 8.13 is symmetric monoidal, in such a way that the left Quillen functor of Theorem 8.15 is symmetric monoidal as well (this is immediate from Remark 6.14).

THEOREM 8.17. *Let X be a Segal pre-operad. The following conditions are equivalent:*

- (a) X is fibrant in the model structure of Theorem 8.13;
- (b) X is a Reedy fibrant Segal operad;
- (c) X is a retract of $\gamma_*(Y)$ for some complete dendroidal Segal space Y ;
- (d) X is a retract of $\gamma_*(Y)$ for some dendroidal Segal space Y .

Proof. Condition (a) implies condition (b) because the inclusions $\text{Sc}[T] \rightarrow \Omega[T]$ are trivial cofibrations in the symmetric monoidal model category structure of Theorem 8.13.

Let us prove that condition (b) implies condition (c). If X is Reedy fibrant, then we can choose a trivial cofibration $\gamma^*(X) \rightarrow Y$ with Y a complete dendroidal Segal space. By virtue of Corollary 8.6, we may assume that the map $\gamma^*X \simeq \gamma^*\gamma_*\gamma^*(X) \rightarrow \gamma^*\gamma_*(Y)$ is a weak equivalence between fibrant objects in the model category for dendroidal Segal spaces. As γ^* is fully faithful, to prove that X is a retract of $\gamma_*(Y)$, it is sufficient to prove that the map $\gamma^*X \rightarrow \gamma^*\gamma_*(Y)$ is a cofibration (that is, a normal monomorphism): this follows from the fact that, by assumption, for any tree T , the group $\text{Aut}(T)$ of automorphisms of T in Ω acts freely on $Y(T) - X(T)$, and that we have an $\text{Aut}(T)$ -equivariant inclusion of $\gamma_*(Y)(T) - X(T)$ in $Y(T) - X(T)$; see the cartesian square (7.5).

Condition (c) implies condition (a): as γ_* is a right Quillen functor (Corollary 8.15), $\gamma_*(Y)$ is fibrant for any complete dendroidal Segal space Y , and the class of fibrant objects of any model category is closed under retracts.

It is clear that condition (c) implies condition (d).

Finally, the fact that condition (d) implies condition (b) follows from the fact that γ_* sends dendroidal Segal spaces to Reedy fibrant Segal operads (see Proposition 8.2). □

REMARK 8.18. Recall from Subsection 6.12 the canonical functor

$$K : \infty\text{-Operad} \longrightarrow \text{sdSet}.$$

We know that K sends ∞ -operads to complete dendroidal Segal spaces, so that, by virtue of Proposition 8.2, we obtain a functor

$$\gamma_*K : \infty\text{-Operad} \longrightarrow \text{SegOper}.$$

We also know from Proposition 6.13 and from Theorem 8.15 that γ_*K sends weak equivalences of ∞ -operads to weak Segal equivalences, and that the induced functor

$$\gamma_*K : \mathbf{Ho}(\infty\text{-Operad}) \longrightarrow \mathbf{Ho}(\text{SegOper})$$

is an equivalence of categories.

Remark as well that any dendroidal set is a pre-operad, so that the inclusion $d\text{Set} \subset \text{sdSet}$ factors through an inclusion $d\text{Set} \subset \text{PreOper}$ which happens to be a left Quillen equivalence (this follows immediately from Corollary 6.7 and from Theorem 8.15). If X is an ∞ -operad, seen as a Segal pre-operad, then $\gamma_*K(X)$ is a canonical fibrant replacement of X in the model category of Theorem 8.13.

REMARK 8.19. The identification $dSet/\eta = sSet$ allows us to deduce the Joyal model category structure for quasi-categories from the model category structure for ∞ -operads; see [7, Corollary 2.10]. Similarly, the dendroidal Rezk model structure of Definition 6.2 induces Rezk’s original model structure for complete Segal spaces [16, Section 12], while the model category structure for Segal categories can be obtained from the model category structure of Theorem 8.13 by slicing over η as well. The Quillen equivalences relating the homotopy theories of Segal categories and of complete Segal spaces, proved by Joyal and Tierney [12] are deduced immediately from their dendroidal analogues, namely Corollary 6.7 and Proposition 6.11, while the Quillen equivalence from the model category for Segal categories to the model category for complete Segal spaces, proved by Bergner [2], is a direct consequence of Theorem 8.15.

References

1. T. BEKE, ‘Sheafifiable homotopy model categories’, *Math. Proc. Cambridge Philos. Soc.* 129 (2000) 447–475.
2. J. BERGNER, ‘Three models for the homotopy theory of homotopy theories’, *Topology* 46 (2007) 397–436.
3. C. BERGER and I. MOERDIJK, ‘On an extension of the notion of Reedy category’, *Math. Z.* 269 (2011) 977–1004.
4. J. M. BOARDMAN and R. M. VOGT, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics 347 (Springer, Berlin, 1973).
5. D.-C. CISINSKI, ‘Images directes cohomologiques dans les catégories de modèles’, *Ann. Math. Blaise Pascal* 10 (2003) 195–244.
6. D.-C. CISINSKI, *Les préfaisceaux comme modèles des types d’homotopie*, Astérisque 308 (Soc. Math. France, Paris, 2006).
7. D.-C. CISINSKI and I. MOERDIJK, ‘Dendroidal sets as models for ∞ -operads’, *J. Topol.* 4 (2011) 257–299.
8. D.-C. CISINSKI and I. MOERDIJK, ‘Dendroidal sets and simplicial operads’, *J. Topol.* 6 (2013) 705–756.
9. P. S. HIRSCHHORN, *Model categories and their localizations*, Mathematical Surveys and Monographs 99 (American Mathematical Society, Providence, RI, 2003).
10. M. HOVEY, *Model categories*, Mathematical Surveys and Monographs 63 (American Mathematical Society, Providence, RI, 1999).
11. A. JOYAL, ‘Quasi-categories and Kan complexes’, *J. Pure Appl. Algebra* 175 (2002) 207–222.
12. A. JOYAL and M. TIERNEY, ‘Quasi-categories vs Segal spaces’, *Categories in algebra, geometry and physics*, Contemporary Mathematics 431 (American Mathematical Society, Providence, RI, 2007) 277–326.
13. J. LURIE, *Higher topos theory*, Annals of Mathematics Studies 170 (Princeton University Press, Princeton, NJ, 2009).
14. I. MOERDIJK and I. WEISS, ‘Dendroidal sets’, *Algebr. Geom. Topol.* 7 (2007) 1441–1470.
15. I. MOERDIJK and I. WEISS, ‘On inner Kan complexes in the category of dendroidal sets’, *Adv. Math.* 221 (2009) 343–389.
16. C. REZK, ‘A model for the homotopy theory of homotopy theory’, *Trans. Amer. Math. Soc.* 353 (2001) 973–1007.
17. C. T. SIMPSON, *Homotopy theory of higher categories*, New Mathematical Monographs 19 (Cambridge University Press, Cambridge, 2012).

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