Denotational semantics for programming languages, balanced quasi-metrics and fixed points (SI-CMMSE2006)

Journal:	*International Journal of Computer Mathematics*
Manuscript ID:	GCOM-2006-0163.R1
Manuscript Type:	Original Article
Date Submitted by the Author:	02-Jan-2007
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Denotational semantics for programming languages, balanced quasi-metrics and fixed points

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Abstract

A new mathematical model is introduced for the study of the domain of words. We do it by means of the introduction of a suitable balanced quasi-metric on the set of all words over an alphabet. It will be shown that this construction has better quasi-metric and topological properties than several classical constructions. We also prove a fixed point theorem which allows us to develop an application for the study of Probabilistic Divide and Conquer Algorithms.

Key words: domain of words, Baire metric, balanced quasi-metric, algorithm, recurrence equation

MSC 2000: 68Q55, 68Q25, 54E35

1   Introduction

It is well known that the classical Baire metric (or Baire distance) provides an efficient tool to obtain denotational models for programming languages (see [3], [4], [5]). It is also applied, for instance, to study the representation

of real numbers by means of regular languages [17] and in modelling streams of information in parallel computation [13]. The Baire metric is especially useful for the measurement of distances between totally defined objects (infinite words), and, hence, those computational processes where it is necessary to take into account the information given by the partially defined objects (finite words) are usually modelled by using Baire quasi-pseudo-metrics (see [2], [18], [19], [28], [29], [25], [27], etc.). However, these quasi-pseudo-metrics generally induce only T_0 topologies. Motivated, in part, by this fact, in Section 2 we present a new model for the measurement of distances between words by means of the notion of a balanced quasi-metric. Moreover, our model possesses other rich quasi-metric and topological properties such as right K-completeness, Hausdorffness and uniformizability -equivalently, complete regularity- and it respects several essential aspects of the computational interpretation given by other known Baire quasi-pseudo-metrics. An application of our approach to the analysis of Probabilistic Divide and Conquer Algorithms via techniques of fixed point is also given.

Our basic references for quasi-pseudo-metric spaces are [10] and [16] and for topology is [8].

Let us recall that a quasi-pseudo-metric on a set X is a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

By a quasi-metric on X we mean a quasi-pseudo-metric d on X that satisfies the following condition: (iii) $d(x, y) = 0$ if and only if $x = y$.

A quasi-(pseudo-)metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-(pseudo-)metric on X.

Each quasi-pseudo-metric d on X generates a topology τ_d on X which has as a base the family of open d-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

Observe that if d is a quasi-metric, then τ_d is a T_1 topology.

Given a quasi-(pseudo-)metric d on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-(pseudo-)metric on X, called the conjugate of d. Finally, the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (pseudo-)metric on X.

2

International Journal of Computer Mathematics

2 The Baire balanced quasi-metric on the domain of words

Let Σ be a nonempty alphabet. Denote by Σ^F the set of finite sequences ("finite words") over Σ and by Σ^∞ the set of finite and infinite sequences over Σ. We assume that the empty sequence $\varnothing$ is an element of Σ^F.

Denote by $\sqsubseteq$ the prefix order on Σ^∞, i.e. $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y. If $x \sqsubseteq y$ with $x \neq y$ we write $x \sqsubset y$.

For each $x, y \in \Sigma^\infty$ we define $x \sqcap y$ as the longest common prefix of x and y, and for each $x \in \Sigma^\infty$ we denote by $\ell(x)$ the length of x. Then $\ell(x) \in [1, \infty]$ whenever $x \neq \varnothing$ and $\ell(\varnothing) = 0$.

Remark 1. If $(x_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence in Σ^∞ with respect to $\sqsubseteq$, i.e. $x_n \sqsubset x_{n+1}$ for all $n \in \mathbb{N}$, we shall denote by $\sqcup_k x_k$ the unique element in Σ^∞ such that $x_n \sqcap (\sqcup_k x_k) = x_n$ for all $n \in \mathbb{N}$. Note that, in this case, $x_n \in \Sigma^F$ for all $n \in \mathbb{N}$ and $\ell(\sqcup_k x_k) = \lim_{n \to \infty} \ell(x_n) = \infty$.

Let us recall that the Baire metric (or Baire distance) on Σ^∞ is the metric d_B defined by

$$d_B(x, x) = 0 \quad \text{for all } x \in \Sigma^\infty,$$
$$d_B(x, y) = 2^{-\ell(x \sqcap y)} \quad \text{for all } x, y \in \Sigma^\infty \text{ with } x \neq y.$$

(We adopt the convention that $2^{-\infty} = 0$).

Smyth introduced in [28] the following quasi-pseudo-metric modification of the Baire metric. For each $x, y \in \Sigma^\infty$ let

$$d_\sqsubseteq(x, y) = 0 \quad \text{if } x \sqsubseteq y,$$
$$d_\sqsubseteq(x, y) = d_B(x, y) \quad \text{otherwise.}$$

Then $d_\sqsubseteq$ is a T_0 quasi-pseudo-metric on Σ^∞ and $(d_\sqsubseteq)^s$ is the Baire metric on Σ^∞.

Observe that condition $d_\sqsubseteq(x, y) = 0$ can be used to distinguish between the case that x is a prefix of y and the remaining cases for $x, y \in \Sigma^\infty$.

Other T_0 quasi-pseudo-metric variants of the Baire metric, with applications, may be found in [15], [18], [19], [22], [25], etc. However, it is well known and easy to see that both $d_\sqsubseteq$ and these quasi-pseudo-metrics do not generate Hausdorff (in fact, not even T_1) topologies. Next we construct a new quasi-metric on Σ^∞ which possesses several interesting properties. In

3

particular, the generated topology will be Hausdorff and completely regular. Furthermore, the information given by condition $d_{\sqsubseteq}(x, y) = 0$ is also preserved by our approach.

Doitchinov introduced in [6] the notion of a balanced quasi-metric space in order to obtain a consistent theory of quasi-metric completion that preserves complete regularity.

Recall that a quasi-metric d on a set X is balanced provided that whenever $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are sequences in X and x, y are points in X such that $d(x, x_n) \leq r_1$, $d(y_m, y) \leq r_2$ for all $n, m \in \mathbb{N}$, and $\lim_{n,m \to \infty} d(y_m, x_n) = 0$, then $d(x, y) \leq r_1 + r_2$. A quasi-metric space (X, d) is called balanced if d is a balanced quasi-metric on X. Each balanced quasi-metric space is Hausdorff and completely regular [6].

Remark 2. It is well known and easy to see that a quasi-metric space (X, d) is balanced if and only if (X, d^{-1}) is balanced. Therefore, if (X, d) is balanced, then (X, d^{-1}) is Hausdorff and completely regular.

Theorem 1. *Let Σ be an alphabet and let $q_b : \Sigma^\infty \times \Sigma^\infty \to [0, 1]$ be the function given by*

$$q_b(x, y) = 2^{-\ell(x)} - 2^{-\ell(y)} \qquad \textit{if } x \sqsubseteq y,$$

$$q_b(x, y) = 1 \qquad \textit{otherwise.}$$

Then, the following statements hold:

(1) q_b is a balanced quasi-metric on Σ^∞.

(2) $\lim_{n \to \infty} q_b(x_n, x) = 0 \Rightarrow \lim_{n \to \infty} d_B(x_n, x) = 0$.

(3) If $(x_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence in Σ^∞ with respect to $\sqsubseteq$, then $\lim_{n \to \infty} q_b(x_n, \sqcup_k x_k) = 0$.

Proof. (1) We first show that q_b is a quasi-metric on Σ^∞.

It is clear that for $x, y \in \Sigma^\infty$ such that $q_b(x, y) = 0$, it follows that $x = y$.

Now let $x, y, z \in \Sigma^\infty$. If $q_b(x, z) = 1$ or $q_b(z, y) = 1$, it is obvious that $q_b(x, y) \leq q_b(x, z) + q_b(z, y)$. Assume then that $q_b(x, z) < 1$ and $q_b(z, y) < 1$. Thus $x \sqsubseteq z$ and $z \sqsubseteq y$, so $x \sqsubseteq y$. Hence

$$
\begin{aligned}
q_b(x, y) &= 2^{-\ell(x)} - 2^{-\ell(y)} = 2^{-\ell(x)} - 2^{-\ell(z)} + 2^{-\ell(z)} - 2^{-\ell(y)} \\
&= q_b(x, z) + q_b(z, y).
\end{aligned}
$$

We have shown that q_b is a quasi-metric on Σ^∞.

4

In order to prove that q_b is balanced, let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be sequences in Σ^∞, and x, y points in Σ^∞ such that $q_b(x, x_n) \leq r_1$ for all $n \in \mathbb{N}$, $q_b(y_m, y) \leq r_2$ for all $m \in \mathbb{N}$, and $\lim_{n,m \to \infty} q_b(y_m, x_n) = 0$. We wish to show that $q_b(x, y) \leq r_1 + r_2$. If $r_1 = 1$ or $r_2 = 1$, obviously $q_b(x, y) \leq r_1 + r_2$. Then, we suppose that $r_1 < 1$ and $r_2 < 1$. Thus $x \sqsubseteq x_n$ for all $n \in \mathbb{N}$, and $y_m \sqsubseteq y$ for all $m \in \mathbb{N}$.

We shall distinguish two cases: i) $\ell(x) = \infty$, ii) $\ell(x) < \infty$.

In case i), from the fact that $x \sqsubseteq x_n$, it follows that $x = x_n$ for all $n \in \mathbb{N}$. Thus $\lim_{m \to \infty} q_b(y_m, x) = 0$, and hence $\lim_{m \to \infty} \ell(y_m) = \infty$. Since $y_m \sqsubseteq y$ for all $m \in \mathbb{N}$, it follows that $\ell(y) = \infty$. Therefore

$$\lim_{m \to \infty} d_B(x, y_m) = \lim_{m \to \infty} d_B(y, y_m) = 0,$$

so that $x = y$, i.e. $q_b(x, y) = 0$.

In case ii), since $\lim_{n,m \to \infty} q_b(y_m, x_n) = 0$, there is $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$, $q_b(y_m, x_n) < 1$, so that $y_m \sqsubseteq x_n$ whenever $n, m \geq n_0$. Assume first that $\ell(y_m) < \ell(x)$ for all $m \geq n_0$. Since there is $k \geq n_0$ such that

$$q_b(y_k, x_k) < 2^{-\ell(x)},$$

we deduce that

$$
\begin{aligned}
2^{-\ell(y_k)} \quad &< \quad 2^{-\ell(x_k)} + 2^{-\ell(x)} \\
&\Rightarrow \quad 2^{\ell(x)} < 2^{\ell(y_k)}[2^{\ell(x)-\ell(x_k)} + 1] \\
&\Rightarrow \quad 2^{\ell(x)} < 2^{\ell(y_k)+1} \\
&\Rightarrow \quad \ell(x) < \ell(y_k) + 1,
\end{aligned}
$$

which provides a contradiction. Consequently, there is $m_0 \geq n_0$ with $\ell(x) \leq \ell(y_{m_0})$; but $x \sqsubseteq x_{m_0}$ and $y_{m_0} \sqsubseteq x_{m_0}$, so we deduce that $x \sqsubseteq y_{m_0}$, and hence

$$q_b(x, y_{m_0}) = 2^{-\ell(x)} - 2^{-\ell(y_{m_0})} \leq 2^{-\ell(x)} - 2^{-\ell(x_{m_0})} = q_b(x, x_{m_0}).$$

Therefore

$$
\begin{aligned}
q_b(x, y) \quad &\leq \quad q_b(x, y_{m_0}) + q_b(y_{m_0}, y) \\
&\leq \quad q_b(x, x_{m_0}) + q_b(y_{m_0}, y) \leq r_1 + r_2.
\end{aligned}
$$

We conclude that the quasi-metric q_b is balanced.

5

(2) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Σ^∞ and $x \in \Sigma^\infty$ such that $\lim_{n\to\infty} q_b(x_n, x) = 0$. If $x \in \Sigma^F$, then $x_n = x$ eventually. If $\ell(x) = \infty$, then there is $n_0 \in \mathbb{N}$ such that

$$q_b(x_n, x) = 2^{-\ell(x_n)} = 2^{-\ell(x_n \sqcap x)} = d_B(x_n, x),$$

for all $n \geq n_0$. Consequently $\lim_{n\to\infty} d_B(x_n, x) = 0$.

(3) If $(x_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence in Σ^∞ with respect to $\sqsubseteq$, then $x_n \sqsubset x_{n+1}$ for all $n \in \mathbb{N}$, and $\sqcup_k x_k$ the unique element in Σ^∞ such that $x_n \sqcap (\sqcup_k x_k) = x_n$ for all $n \in \mathbb{N}$. Hence $x_n \in \Sigma^F$ for all $n \in \mathbb{N}$ and $\ell(\sqcup_k x_k) = \lim_{n\to\infty} \ell(x_n) = \infty$ (see Remark 1). Therefore

$$q_b(x_n, \sqcup_k x_k) = 2^{-\ell(x_n)} - 2^{-\ell(\sqcup_k x_k)} = 2^{-\ell(x_n)},$$

for all $n \in \mathbb{N}$, and thus, $\lim_{n\to\infty} q_b(x_n, x) = 0$. $\square$

Remark 3. It is clear that the measurement of the distance from a word x to another word y by means of the Baire metric does not permit us to know if x is a prefix of y or not. This inconvenience is saved by using the Baire quasi-pseudo-metric $d_{\sqsubseteq}$. However, if $x, y, z \in \Sigma^\infty$ satisfy $x \sqsubset y \sqsubset z$, one obtains $d_{\sqsubseteq}(x, z) = d_{\sqsubseteq}(y, z) = 0$, and we can not decide which word of the two, x or y, gives a better approximation of z. For instance, if we consider the totally defined object π and the partially defined ones $x = 3.1415$ and $y = 3.141592$, then it is clear that y contains more information on π than x, but the quasi-metric $d_{\sqsubseteq}$ is not sensitive to this amount of information. The quasi-metric q_b avoids this disadvantage because $q_b(3.1415, \pi) = 2^{-5}$ and $q_b(3.141592, \pi) = 2^{-7}$ as is desirable. In general, for $x, y, z \in \Sigma^\infty$ with $x \sqsubset y \sqsubset z$, we have $\ell(x) < \ell(y) < \ell(z)$ and thus $q_b(x, z) > q_b(y, z)$. This last inequality allows the word with smaller information x to be distinguished from the word y which captures greater information of the derived one from z. Furthermore $q_b(x, y) < 1$ if and only if $x \sqsubseteq y$ with $x \neq \varnothing$. Other useful properties of q_b and $(q_b)^{-1}$ are given in Remark 4 (c) and Theorem 2 below.

Remark 4. (a) We first note that Theorem 1(1) and Remark 2 imply that both (Σ^∞, q_b) and $(\Sigma^\infty, (q_b)^{-1})$ are Hausdorff and completely regular (actually $(q_b)^{-1}$ is also a balanced quasi-metric). In fact, topological properties of $(q_b)^{-1}$ are used in Theorem 4 below.

(b) From Theorem 1(2) it immediately follows that $\tau_{(q_b)^{-1}}$ is finer than τ_{d_B}, i.e. $\tau_{d_B} \subseteq \tau_{(q_b)^{-1}}$, and thus $(\Sigma^\infty, \tau_{(q_b)^{-1}})$ is submetrizable.

6

(c) Theorem 1(3) provides an appropriate computational interpretation of the fact that each infinite word (totally defined object) can be expressed as the supremum of a strictly increasing sequence of finite words (partially defined objects) and it captures the notion of the amount of information defined by such a sequence. This fact can be mathematically modelled in the realm of right K-sequentially complete quasi-metric spaces as we show in the following.

Let us recall [20], that a sequence $(x_n)_{n \in \mathbb{N}}$ in a quasi-metric space (X, d) is right K-Cauchy if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ whenever $m \geq n \geq n_0$. The quasi-metric space (X, d) is said to be right K-sequentially complete if every right K-Cauchy sequence is convergent with respect to τ_d.

We point out that right K-sequential completeness provides an efficient tool for the study of function spaces and hyperspaces for quasi-metric spaces (see, for instance, Section 9 of [16]).

Theorem 2. *The quasi-metric space* $(\Sigma^\infty, (q_b)^{-1})$ *is right K-sequentially complete.*

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a right K-Cauchy sequence in $(\Sigma^\infty, (q_b)^{-1})$. Assume without loss of generality that this sequence is strictly increasing with respect to $\sqsubseteq$. It then follows from Theorem 1 (3) that $\lim_{n \to \infty} q_b(x_n, \sqcup_k x_k) = 0$. Therefore $(\Sigma^\infty, (q_b)^{-1})$ is right K-sequentially complete. $\square$

3   A fixed point theorem on $(\Sigma^\infty, (q_b)^{-1})$ with application to the complexity analysis of algorithms

In this section we obtain a fixed point theorem for certain self-maps on the balanced right K-sequentially complete quasi-metric space $(\Sigma^\infty, (q_b)^{-1})$. We apply such a result to deduce that for several recurrence equations based on the recursive structure of Probabilistic Divide and Conquer Algorithm [9], the associated functionals have a unique fixed point which is the solution for the corresponding recurrence equation. However, in the case that our self-map f satisfies the condition that fx is not a prefix of fy, f cannot be

7

a contraction on Σ^∞ because $q_b(x, y) = 1$ whenever x is not a prefix of y. For this reason we will establish our result only assuming that the self-map satisfies a contraction property on the orbit of a point of Σ^∞, which will be sufficient for our purposes in this context. Actually, we will prove a more general result in the framework of Hausdorff right K-sequentially complete quasi-metric spaces.

If (X, d) is a quasi-metric space we say that a self-map $f : X \to X$ is continuous if it is continuous from (X, τ_d) to (X, τ_d).

Theorem 3. *Let (X, d) be a Hausdorff right K-sequentially complete quasi-metric space and let $f : X \to X$ be a continuous self-map. If there are $x_0 \in X$ and $k \in (0, 1)$ such that*

$$d(f^{n+1}x_0, f^n x_0) \leq k d(f^n x_0, f^{n-1}x_0) \qquad (\star)$$

for all $n \in \mathbb{N}$, then f has a fixed point.

Proof. By our contraction condition, for each $n \in \mathbb{N}$ we obtain

$$d(f^{n+1}x_0, f^n x_0) \leq k^n d(fx_0, x_0).$$

By the triangle inequality it easily follows that for each $m, n \in \mathbb{N}$,

$$d(f^{n+m}x_0, f^n x_0) \leq \frac{k^n}{1-k} d(fx_0, x_0),$$

and, consequently $(f^n x_0)_{n \in \mathbb{N}}$ is a right K-Cauchy sequence. Thus there is $y \in X$ such that $\lim_{n \to \infty} d(y, f^n x_0) = 0$. By continuity of f it follows that $\lim_{n \to \infty} d(fy, f^{n+1}x_0) = 0$, so $y = fy$ by Hausdorffness of (X, d). This concludes the proof. $\qquad \square$

The following easy example shows that uniqueness of the fixed point in the preceding theorem does not hold, in general.

Example. Let $X = a, b$ and let d be the discrete metric on X. Define $f(a) = a$ and $f(b) = b$, and note that the inequality $(\star)$ is obviously satisfied both for a and b.

Now suppose that Σ is an alphabet and let q_b the balanced quasi-metric on Σ^∞ constructed in Theorem 1. Then $(\Sigma^\infty, (q_b)^{-1})$ is a right K-sequentially

8

complete balanced (hence, Hausdorff) quasi-metric space by Theorem 2 and Remark 4 (a). Hence, we derive from Theorem 3 the following.

Theorem 4. *Let* $\Phi : \Sigma^\infty \to \Sigma^\infty$ *be a continuous self-map on* $(\Sigma^\infty, (q_b)^{-1})$. *If there are* $w_0 \in \Sigma^\infty$ *and* $k \in (0,1)$ *such that*

$$q_b(\Phi^n w_0, \Phi^{n+1} w_0) \leq k q_b(\Phi^{n-1} w_0, \Phi^n w_0),$$

for all $n \in \mathbb{N}$, *then* Φ *has a fixed point.*

When discussing the analysis of Probabilistic Divide and Conquer Algorithms by means of recurrences, the following general recurrence equation is obtained (see, for instance, Section 4 of [9]):

$$(*) \qquad T(n) = p(n) + \sum_{k=1}^{n-1} q(n,k)T(k),$$

for $n \geq 2$, where $T(1) \geq 0$, $p(n) > 0$, and $q(n,k) > 0$ is proportional to the splitting probabilities that express the changes that a task of size n involve a subtask of size $k < n$.

A paradigmatic example of the above recurrence equation is the sorting algorithm used in the Unix system. The average case analysis of this algorithm is discussed in [14], where the following recurrence equation is obtained:

$$
\begin{aligned}
T(1) &= 0, \quad \text{and} \\
T(n) &= \frac{2(n-1)}{n} + \frac{n+1}{n}T(n-1), \quad n \geq 2.
\end{aligned}
$$

We conclude the paper by applying Theorem 4 to prove that each recurrence equation of type $(*)$ has a unique solution.

To this end, consider as an alphabet Σ the set of nonnegative real numbers. Let T be a recurrence equation of type $(*)$. We associate to T the functional $\Phi_T : \Sigma^\infty \to \Sigma^\infty$ given by $(\Phi_T w)_1 = T(1)$ and

$$(\Phi_T w)_n = p(n) + \sum_{k=1}^{n-1} q(n,k)w_k$$

for all $n \geq 2$ (if $w \in \Sigma^\infty$ has length $n < \infty$, we write $w := w_1 w_2 ... w_n$, and if w is an infinite word we write $w := w_1 w_2 ...$).

9

It is clear by the construction that if $\ell(w) = n$ then $\ell(\Phi_T w) = n + 1$ (in particular, $\ell(\Phi_T w) = \infty$ whenever $\ell(w) = \infty$).

Furthermore, we obtain

$$(**) \qquad v \sqsubseteq w \Rightarrow \Phi_T v \sqsubseteq \Phi_T w.$$

So for $v \sqsubseteq w$ with $\ell(w) = \infty$ we deduce

$$q_b(\Phi_T v, \Phi_T w) = 2^{-\ell(\Phi_T v)} \leq 2^{-\ell(v)},$$

which implies that Φ_T is continuous from $(\Sigma^\infty, (q_b)^{-1})$ into itself, because if $\lim_{n \to \infty} (q_b)^{-1}(w, v_n) = 0$, with $w \neq v_n$, then $\ell(w) = \infty$ and $\lim_{n \to \infty} \ell(v_n) = \infty$, so $\lim_{n \to \infty} (q_b)^{-1}(\Phi_T w, \Phi_T v_n) = 0$.

Now let w be the element of Σ^∞ given by $w := T(1)$. Then $\ell(w) = 1$. Since $w \sqsubseteq \Phi_T w$, it follows from condition $(**)$ that

$$
\begin{aligned}
q_b(\Phi_T^n w, \Phi_T^{n+1} w) &= 2^{-\ell(\Phi_T^n w)} - 2^{-\ell(\Phi_T^{n+1} w)} \\
&= 2^{-1}(2^{-\ell(\Phi_T^{n-1} w)} - 2^{-\ell(\Phi_T^n w)}) = 2^{-1} q_b((\Phi_T^{n-1} w, \Phi_T^n w),
\end{aligned}
$$

for all $n \in \mathbb{N}$. Consequently, we can apply Theorem 4 (with $k = 1/2$), and thus Φ_T has a fixed point w' which is clearly unique by the construction of Φ_T. We conclude that w' is the unique solution of the recurrence T.

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10

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11

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12

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13