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Densest Packing of Translates of the Union of Two Circles

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Abstract. Let u be the union of two unit circles whose centers have a distance at most 2. Motivated by more general problems it is proved that the density of a packing of translates of u never exceeds the density of the densest lattice-packing.

In the Euclidean plane let w be a domain. Let $d(w)$ be the density of the densest packing of translates of w. Let $\bar{d}(w)$ be the density of the densest lattice-packing of translates of w. It is known [1], [2], [3] that if w is convex then

$$d(w)=\bar{d}(w). \tag{1}$$

An interesting field of research arises by trying to extend (1) to more general domains [4]. From some wider families of domains, which presumably share property (1) with the convex domains, we emphasize

Conjecture 1. *Let u be the union of two convex domains having a point in common. Then $d(u)=\bar{d}(u)$.*

As a modest step in this direction we shall prove the following:

Theorem. *Let u be the union of two unit circles centered at a distance at most 2 from one another. Then $d(u)=\bar{d}(u)$.*

Let u_1 and u_2 be translates of u such that u and u_1 have three points of contact, and u_2 has one point of contact with both u and u_1. The densest lattice-packing of translates of u is generated by the translations $u\to u_1$ and $u\to u_2$. Let $2t$ be the distance of the centers of the two unit circles contained in u. Then $\bar{d}(u)$ is given in terms of t by the function

$$f(t)=\frac{2(\arcsin t+t\sqrt{1-t^2})+\pi}{2t\sqrt{4-t^2}+\sqrt{12}}.$$

For $0<t<1$ we have $f(t)>f(0)=f(1)=\pi/\sqrt{12}$. The maximum of $f(t)$ is attained at a value $t_0\approx 0.582$. Since translates of the union of two disjoint circles cannot be packed with a density greater than $\pi/\sqrt{12}$ the theorem implies that the density of a packing of translates of the union of two equal circles is at most $f(t_0)=0.936599\ldots$.

What can be said about the supremum of $d(u)$ extended over the set S_n of all unions u of n unit circles? We shall return to this question after the proof of the theorem.

The proof of the theorem is based on an idea used in a proof of Thue's well-known theorem which claims that for a circle c we have $d(c)=\bar{d}(c)=\pi/\sqrt{12}$. First we will reproduce this proof [5] in a slightly modified form. We shall denote the area of a domain w by $|w|$. By a circle we shall mean an open circular disc.

Let the unit circles $c_1, c_2, \ldots$ with centers $C_1, C_2, \ldots$ form a packing. Let z_i be the Dirichlet cell of c_i defined as the set of those points which are nearer to C_i than to the center of any other circle. We shall show that the density $|c_i|/|z_i|$ of any circle c_i in its Dirichlet cell is at most $\pi/\sqrt{12}$. Writing $c_i=c$ and $z_i=z$, this means that $|z|\geq|h|$ where h is a regular hexagon circumscribed about c.

We shall prove a sharper inequality: If k is the circumcircle of h then

$$|z\cap k|\geq|h|.$$

Let $s_1,\ldots,s_n$ be the segments cut off from k by the sides of z. Let $F_1,\ldots,F_n$ be the orthogonal projections of C to the respective sides of z. Since the distance between any two of the C_i's is at least 2, the distance between the F_i's is at least 1. It follows that the s_i's do not overlap. Therefore

$$|z\cap k|=|k|-|s_1|-\cdots-|s_n|\geq|k|-n|s|,$$

where s is a segment cut off from k by a tangent of c. For $n\leq 6$ this implies the desired inequality

$$|z\cap k|\geq|k|-6|s|=|h|.$$

The case when $n=7$ can be settled by a very rough estimate which automatically rules out the possibility that $n>7$. Since the length

$$\frac{4}{\sqrt{3}}\sin\frac{\pi}{7}=1.00201\cdots$$

of a side of a regular heptagon inscribed into k is close to 1, it is clear that the points $F_1,\ldots,F_7$ must lie close to the boundary bd k of k, suggesting that the s_i's are very small. Though this is sufficiently convincing, we present the details.

The distance CF_7 attains its minimum $m=1.1383\cdots$ if $CF_1=\cdots=CF_6=2/\sqrt{3}=1.1547\ldots$, and all sides of the heptagon $F_1\cdots F_7$ are equal to 1. It follows that

$$|s_i|\leq\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2(2\omega-\sin 2\omega)=0.00424\cdots,\qquad i=1,\ldots,7,$$

where $\omega = \arccos \sqrt{3}\, m/w$. Hence

$$|s_1| + \cdots + |s_7| < 0.0297 < 6|s| = 0.724\ldots.$$

Turning to the proof of the theorem, we consider a rectangular coordinate system (x, y) in which the centers of the two unit circles contained in u coincide with $(-t, 0)$ and $(t, 0)$. Let c^- and c^+ be the left- and right-sided circles, and C^- and C^+ their centers. Let the translates $u_1, u_2, \ldots$ of u form a packing. Let c_1^-, c_1^+, c_2^-, $c_2^+, \ldots$ be the set of pertaining circles. In this set let z_i^- and z_i^+ be the Dirichlet cells of c_i^- and c_i^+. We define the Dirichlet cell of u_i by $z_i = z_i^- \cup z_i^+$. In order to simplify the notations we suppose that u is an arbitrary member of the packing, and denote its Dirichlet cell by z.

We shall prove the theorem by showing that

$$|u|/|z| \leq \bar{d}(u).$$

We shall prove a sharper inequality: If k^- and k^+ are circles concentric with c^- and c^+ of radius $2/\sqrt{3}$ and $k = k^- \cup k^+$ then

$$|z \cap k| \geq |u|/\bar{d}(u) = |\bar{z}|, \tag{2}$$

where $\bar{z}$ is a Dirichlet cell in the densest lattice-packing.

Let $u_1, \ldots, u_n$ be those domains other than u whose Dirichlet cells intersect k. Let the respective intersections be $s_1, \ldots, s_n$. Since the Dirichlet cells do not overlap neither do the s_i's. Therefore

$$|z \cap k| = |k| - |s_1| - \cdots - |s_n|. \tag{3}$$

We claim that among $s_1, \ldots, s_n$ there are at most two whose areas are greater than $|s|$. We shall prove this by showing that, with the exception of at most two, the sets $s_1, \ldots, s_n$ are circular segments.

Let u_0 be a translate of u not necessarily disjoint of u. Let C_0^- and C_0^+ be the corresponding translates of C^- and C^+. In the arrangement of translates of u, consisting only of u and u_0, let z_0 be the Dirichlet cell of u_0. We say that the intersection $s_0 = k \cap z_0$ is exceptional if it is not empty and not a circular segment. Let p be the set of points defined by the following property: If either $C^- \in p$ or $C^+ \in p$ then s_0 is exceptional.

Let a and b be circles of radius $2/\sqrt{3}$ centered at the points of intersection of $\operatorname{bd} k^-$ and $\operatorname{bd} k^+$ (Fig. 1). The difference $(2k^- \cap 2k^+) \backslash (a \cup b)$ consists of four arc-sided triangles. Let v and w be the triangles bisected by the y-axis. We claim that $p = a \cup b \cup v \cup w$. This can easily be seen by the following alternative definition of p: The set p consists of the centers of those circles of radius $2/\sqrt{3}$ whose boundaries intersect both arcs constituting $\operatorname{bd} k$.

Presently we are only interested in the part of p which lies in the ring r defined by $r = (2k^- \cup 2k^+) \backslash (2c^- \cup 2c^+)$. This part consists of two disjoint components whose diameter is less than or equal to 2 with equality only in the limiting case as $t \to 0$. Since the distance between centers C_i^- or C_i^+ belonging to different u_i's is at least 2, the assertion is proved.

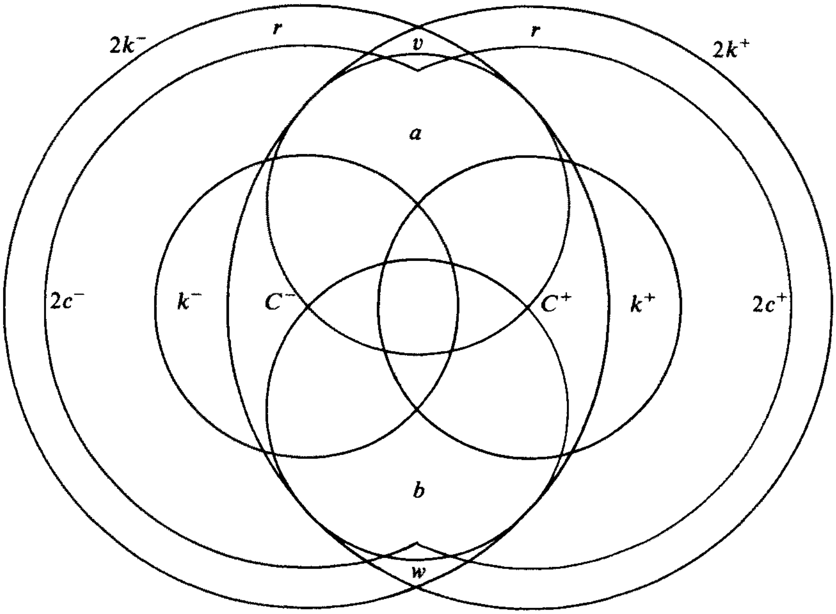

Fig. 1

Now we make use of the easily seen fact that under the sole condition that u and u_0 are disjoint $|s_0|$ attains its maximum if u_0 and u have three points of contact. Denoting s_0 in this position by $\hat{s}$, and assuming that $n \leq 6$, we have, by (3), in accordance with (2),

$$|z \cap k| \geq |k| - 4|s| - 2|\hat{s}| = |\bar{z}|.$$

Now we consider seven pairs of centers $(C_1^-, C_1^+), \ldots, (C_7^-, C_7^+)$ such that of each pair at least one center is in r. Let (C_i^-, C_i^+) be a pair of centers such that C_i^- lies in the half-plane $x \leq 0$ and C_i^+ lies in the half-plane $x \geq 0$. We assume that among the seven pairs of centers there are (i) two such pairs, or (ii) one such pair, or (iii) none.

Case (i). Let C_7^+, C_1^-, C_2^-, C_3^+ be in $x \geq 0$, and let C_3^-, C_4^+, C_5^+, C_6^+, C_7^- be in $x \leq 0$. Translating C_4^+, C_5^+, and C_6^+ through the vector $\mathbf{C^-C^+}$ we obtain, along with C_7^+, C_1^-, C_2^-, and C_3^+ seven points in the ring $2k^+ \backslash 2c^+$ with a distance at least 2 from one another. Referring to the above proof of Thue's theorem we see that these points must be close to bd $2k^+$. Therefore the original points must be close to bd $(2k^- \cup 2k^+)$ so that $|z \cap k|$ is, by far, greater than $|\bar{z}|$.

Case (ii). We can translate the domains $u_1, \ldots, u_7$ continuously "outwards" without overlapping each other until, of each pair of centers, one center lies on bd$(2k^- \cup 2k^+)$, and the other lies either on bd$(2k^- \cup 2k^+)$ or outside $2k^- \cup 2k^+$. We number the centers so that C_1^-, C_2^-, C_3^-, C_4^+, C_5^+, C_6^+, C_7^-, and C_7^+ lie, in

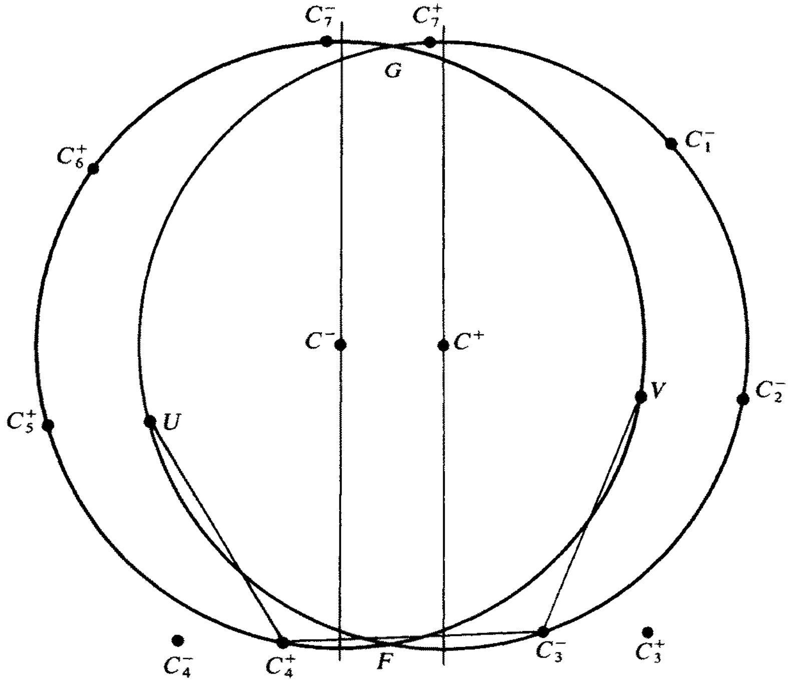

Fig. 2

this cyclic order, on $\mathrm{bd}(2k^- \cup 2k^+)$ (Fig. 2). Translate C_5^+ through the vector $\mathbf{C}^-\mathbf{C}^+$ and C_2^- through $\mathbf{C}^+\mathbf{C}^-$ obtaining the points U and V. Let $\mathrm{bd}\,2k^-$ and $\mathrm{bd}\,2k^+$ intersect at the points F and G, G being in the same half-plane $y>0$ or $y<0$ as C_7^- and C_7^+. Let Σ and σ be the length of the arcs on $\mathrm{bd}\,2k^+$ subtended by the chords of length 2 and $2t$, respectively. Using an obvious notation for the length of an arc on $\mathrm{bd}\,2k^-$ or $\mathrm{bd}\,2k^+$, we have

$$\begin{aligned} 8\pi/\sqrt{3}+2\sigma &= \overset{\frown}{C_7^-G}+\overset{\frown}{GC_7^+}+\overset{\frown}{C_7^+C_2^-}+\overset{\frown}{C_2^-F}+\overset{\frown}{FC_5^+}+\overset{\frown}{C_5^+C_7^-} \\ &= \sigma+\overset{\frown}{C_7^+C_2^-}+\overset{\frown}{VF}+\sigma+\overset{\frown}{FU}+\sigma+\overset{\frown}{C_5^+C_7^-} \geq 4\Sigma+3\sigma+\overset{\frown}{UF}+\overset{\frown}{FV}, \end{aligned}$$

whence

$$\Lambda = \overset{\frown}{UF}+\overset{\frown}{FV}+\sigma+4\Sigma-8\pi/\sqrt{3} \leq 0.$$

For a given value of t, $\overset{\frown}{UF}+\overset{\frown}{FV}$ attains its minimum if, in the inequalities $C_4^+C_3^- \geq 2$, $C_4^+U = C_4^-C_5^+ \geq 2$, $C_3^-V = C_3^+C_2^- \geq 2$, equality holds. Geometrical considerations suggest, and numerical computations, support the conjecture that in the extremal position either C_4^+ or C_3^- coincides with F. From the computed values of Λ belonging to different values of t and different positions of C_4^+ and C_3^- we present, for a few values of t, only the smallest one which belong to the suggested position:

t	0	0.065	0.066	0.1	0.2	0.3	0.4
Λ	−0.0313	−0.0002	0.0004	0.0156	0.0572	0.0944	0.1280
t	0.5	0.6	0.7	0.8	0.9	1	
Λ	0.1588	0.1954	0.2140	0.2390	0.2626	0.2852	

This table shows that there is a value $\bar{t}=0.065\cdots$ such that for $t>\bar{t}$ case (ii) cannot occur.

The same argument shows that case (ii) (for $t\leq\bar{t}$) cannot occur either if, from all seven pairs of centers, at least one center lies in $\lambda c^-\cup\lambda c^+$ where $\lambda=1/\sin(\pi/7)$ is the circumradius of a regular heptagon of side-length 2. But if for some $j\leq 7$ both c_j^- and c_j^+ are outside $\lambda c^-\cup\lambda c^+$ then $|s_j|$ is very small, more exactly less than $3|w|=0.00067\cdots$ where w is a circular segment cut off from k^+ by a tangent of $(\lambda/2)c^+$. On the other hand, u_j prevents the rest of the centers under consideration from getting close to bd $2k^-\cup 2k^+$, so that inequality (2) is again amply fulfilled.

Case (iii) can be settled in a similar manner to (ii). Without going into details we mention that there is a constant $\bar{\bar{t}}=0.034\cdots$ such that for $t>\bar{\bar{t}}$ case (iii) cannot occur.

This ends the proof of the theorem.

We still make some remarks about $d_n=\sup_{u\in S_n} d(u)$, where S_n is the set of all possible unions of n unit circles.

Let u be the union of n unit circles with centers on a line equally spaced at a distance $2t\leq 2$. Then $\bar{d}(u)$ is given in terms of t by the function

$$f_n(t)=\frac{2(n-1)(\arcsin t+t\sqrt{1-t^2})+\pi}{2(n-1)t\sqrt{4-t^2}+\sqrt{12}}.$$

As we have seen, we have $d_2=\max_{0\leq t\leq 1} f_2(t)$. Now we can phrase

Conjecture 2. *For $n=3$, 4, and 5 we have $d_n=\max_{0\leq t\leq 1} f_n(t)$.*

However, passing from $n=5$ to $n=6$ the extremal configuration seems to change drastically. Let v be the union of six unit circles centered at the vertices of a regular hexagon of side-length 1. Then we have

$$\bar{d}(v)=\frac{3\sqrt{3}+2\pi}{(\sqrt{3}/2)(7+3\sqrt{5})}=0.96695\ldots,$$

whilst $\max_{0\leq t\leq 1} f_6(t)=0.96686\ldots$. This phenomenon is similar to the "sausage catastrophe" observed by Wills [6].

Improving a construction suggested by J. Pach we now shall show that

$$\lim_{n\to\infty}(1-d_n)n \le \frac{4\sqrt{3}-2\pi}{3\sqrt{3}}. \tag{4}$$

Let $k>1$ and $l\ge 0$ be integers. In the thinnest lattice-covering of the plane with unit circles the centers of the circles form a lattice generated by the vertices of a regular triangle of side-length $\sqrt{3}$. Let H be a closed regular hexagon of side-length $\sqrt{3}\,k$ containing $3k(k+1)+1$ lattice-points. On bd H we put, on each segment determined by consecutive lattice-points l, new points so as to divide the segment into $l+1$ parts of length $2t=\sqrt{3}/(l+1)$. Now we have in H altogether

$$n=1+3k(k+1)+6kl$$

points. Let U be the union of the unit circles centered at these points. We have

$$\begin{aligned}|U| &= |\text{conv } U|-|\text{conv } U\setminus U|\\ &= \frac{9\sqrt{3}}{2}k^2+6\sqrt{3}\,k+\pi-6k(l+1)(2t-\arcsin t-t\sqrt{1-t^2}),\end{aligned}$$

where conv U denotes the convex hull of U.

In the densest lattice-packing of translates of U each domain is touched by six others so that adjacent domains have $2kl-1$ points of contact. A unit cell C of this lattice is given by joining to H along three consecutive sides parallelograms of side-length 2 and $\sqrt{3}\,k$ including an angle $\arcsin(\frac{1}{2}\sqrt{4-t^2})$ and two equilateral triangles of side-length 2 between the parallelograms. Thus we have

$$|C|=\frac{9\sqrt{3}}{2}k^2+3\sqrt{3}\,k\sqrt{4-t^2}+2\sqrt{3}$$

and

$$|C|-|U|=3\sqrt{3}\,k\left(\sqrt{4-t^2}-\frac{1}{t}\arcsin t-\sqrt{1-t^2}\right)+2\sqrt{3}-\pi.$$

Since

$$\lim_{t\to 0}\frac{1}{t^2}\left(\sqrt{4-t^2}-\frac{1}{t}\arcsin t-\sqrt{1-t^2}\right)=\tfrac{1}{12}$$

there is a positive constant c such that for $0<t\le\sqrt{3}/2$ we have

$$\sqrt{4-t^2}-\frac{1}{t}\arcsin t-\sqrt{1-t^2}<ct^2.$$

It follows that

$$1-d_n \le \frac{|C|-|U|}{|C|} < \frac{3\sqrt{3}\,ckt^2+2\sqrt{3}-\pi}{\frac{9\sqrt{3}}{2}k^2}.$$

Let p be a positive number less than 1/8. Letting n tend to infinity and choosing l so that $n^p < l \le m^{1/2-p}$ we have $\lim_{n\to\infty} n/k^2 = 3$ and $\lim_{n\to\infty} k/l^2 = 0$. Therefore multiplying the last inequality by n and going over to the limiting value as $n \to \infty$ we obtain inequality (4).

It is conjectured that in (4) the sign of equality holds. We formulate the following sharper

Conjecture 3. *Let $D(u)$ be the density of the densest packing of congruent copies of a domain u. Then*

$$\sup_{u\in S_n} D(u) \sim 1-\frac{4\sqrt{3}-2\pi}{3\sqrt{3}\,n}.$$

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