Density of Kinks after a Quench: When Symmetry Breaks, How Big are the Pieces?

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Numerical study of order parameter evolution in the course of symmetry breaking transitions with Landau-Ginzburg–like dynamics shows that the density of topological defects, kinks which form during the quench, is proportional to the fourth root of its rate. This is a limited (1D) test of the more general theory of domain-size evolution in the course of symmetry breaking transformations proposed by one of us. Using these ideas, it is possible to compute the density of topological defects from the quench time scale and from the equilibrium scaling of the correlation length and relaxation time near the critical point. [S0031-9007(97)02876-7]

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The dynamics of symmetry breaking phase transitions has been the focus of attention because of its implications for cosmological scenarios [1–5] and its importance in the context of condensed matter physics [6–11]. The likely cosmological setting for this process involves a second order phase transformation associated with a nonconserved order parameter φ. Initially, the system is near equilibrium in the symmetric phase with $\langle\varphi\rangle = 0$. When the effective potential assumes the "sombrero" shape, φ is forced to make a choice of one of many possible broken symmetry true vacua. As pointed out by Kibble [3], in the cosmological context, the finiteness of the speed of light implies that these choices must be independent in sufficiently distant regions of space. Therefore, when the homotopy group, characterizing the relation between the manifold of broken symmetry vacua and the space in which φ evolves, is nontrivial, locally independent choices of the new vacuum will lead to topologically stable objects such as monopoles, cosmic strings, or domain walls. The initial density of such defects is of great interest and cannot be deduced from the scenario sketched above, except for a much too generous lower limit which follows from the light-cone causality alone [12].

As suggested in [1], it is also possible to study "cosmological" mechanisms of defect formation in condensed matter. In this context, the estimate of defect density based on speed of light arguments is considerably less useful than in cosmological situations. Thus, along with the proposal for condensed matter tests of defect formation in cosmological scenarios, a theory aimed at computing the defect density was developed [1,11]. The key idea is to realize that the order parameter can become correlated only through its dynamics. Furthermore, in the vicinity of the critical temperature T_C, order parameters exhibit universal behavior characterized by two simultaneously occurring divergences. That is, when parametrized in terms of the relative temperature $\epsilon = (T_C - T)/T_C$, both the equilibrium *healing length* (also known as the *correlation length*),

$$\xi = \xi_0/|\epsilon|^{\nu}, \tag{1}$$

and the *dynamical relaxation time*,

$$\tau = \tau_0/|\epsilon|^{\mu}, \tag{2}$$

will simultaneously diverge at T_C (see Fig. 1). Above, ξ_0 and τ_0 characterize the low temperature ($T = 0, \epsilon = 1$) healing length and relaxation time, respectively.

The healing length and the dynamical relaxation time have similar physical significance. ξ is the distance over which the order parameter returns to its equilibrium value

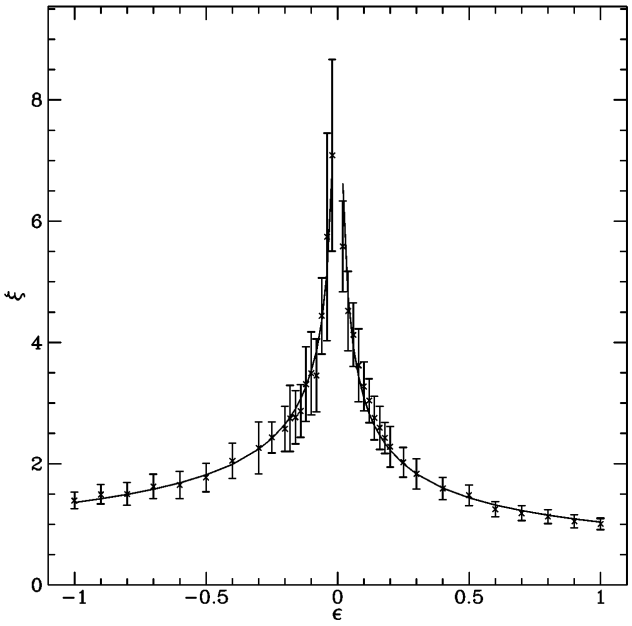

FIG. 1. Characteristic equilibrium correlation (or healing) length and a fit to $\xi = \xi_0/|\epsilon|^{\nu}$ for the system under investigation. The reservoir (noise) temperature here and elsewhere in the paper is kept constant at 0.01 [see Eq. (6)]. The best fitting yields $\xi_0 = 1.38 \pm 0.06$, $\nu = 0.41 \pm 0.03$ ($\chi^2 = 1.2$) above T_C, and $\xi_0 = 1.02 \pm 0.04$, $\nu = 0.48 \pm 0.02$ ($\chi^2 = 3.7$) below T_C, close to the Landau-Ginzburg exponent of $\nu = 1/2$.

when perturbed, for instance, by the boundary conditions. ξ is also the typical scale of the perturbations. That is, to the leading order, the correlation function of *perturbations* away from equilibrium, $\delta\varphi(x,t) = \varphi(x,t) - \langle\varphi\rangle$ (where $\langle\varphi\rangle = 0$ above T_C, but in the broken symmetry state $\langle\varphi\rangle^2 \simeq \epsilon$) behaves as $\langle\delta\varphi(x,t), \delta\varphi(x+\Delta,t)\rangle \sim \exp(-|\Delta|/\xi)$. Below T_C, the spatial order is established on scales much larger than ξ; however, the scale over which the healing occurs, i.e., from the "wounds" inflicted by topological defect, is characterized by ξ. Similarly, τ characterizes the time required for the order parameter to relax to its equilibrium value. During the phase transition, in the immediate vicinity of the critical temperature, the motion will often be overdamped (that is, dominated by the first time derivative of the order parameter) and $\tau = \tau_0/|\epsilon|$.

The estimate of the density of defects put forward in Ref. [1] is based on a linear quench,

$$\epsilon = t/\tau_Q\,, \tag{3}$$

which is expected to be a suitable approximation in the neighborhood of T_C. In (3), τ_Q is the *quench time scale* and t the time before ($t < 0$) and after ($t > 0$) the transition ($\epsilon = 0$). When the critical temperature is crossed at the fixed rate given by Eq. (3), there will come a moment when the order parameter—because of the critical slowing down implied by Eq. (2)—will simply be unable to adjust its actual correlation length to the equilibrium ξ given by Eq. (1). This will occur when the time remaining until the transition equals the relevant dynamical relaxation time, that is, when $\tau(\hat{t}) = \tau_0/(|\hat{t}|/\tau_Q)^\mu = \hat{t}$. This yields a relative temperature $\hat{\epsilon} = \hat{t}/\tau_Q = (\tau_0/\tau_Q)^{1/(1+\mu)}$. At this point, the correlation length of φ will cease to diverge in accordance with Eq. (1), as the phase transition region is traversed. Instead, ξ will reach a value approximately given by

$$\hat{\xi} \approx \xi_0/|\hat{\epsilon}|^\nu = \xi_0(\tau_Q/\tau_0)^{\nu/(1+\mu)}. \tag{4}$$

Regions more distant than $\hat{\xi}$ will be forced to select the new vacuum independently.

The size of independently selected domains of the new vacuum will be given, to the leading order, by $\hat{\xi}$, which will also determine the typical separation of topological defects and, therefore, the initial defect density n. For instance, for monopoles $n \approx 1/(f\hat{\xi})^D$, with D the space dimension and $f \sim \mathcal{O}(1)$, a factor presumably somewhat larger than unity. f takes into account the possibility that the independent choices of the vacuum may be similar anyway, and that the correlations length will slowly grow following the instant $\hat{t}$ due to diffusion, etc.; the key prediction of the theory of [1] is, in any case, the *scaling*, as is usually the case in the critical phenomena.

The aim of the work in this Letter is to test this theory with a computer experiment. The immediate motivation of our research comes from recent superfluid experiments [7,9,10]. These experiments follow the original suggestion in Ref. [1] and appear to support the estimate of defect formation based on $\hat{\xi}$. While the quench-generated density of vortex lines is somewhat uncertain, it is, nonetheless, in accordance with the theory summed up above but in conflict with the ideas based on activation and Ginzburg temperature [12]. However, these experiments were, at least so far, unable to vary the quench time τ_Q. Thus, it was not possible to systematically test the key predictions of Ref. [1], namely, the power law dependence of the size of the fragments of the broken symmetry vacuum on the quench rate τ_Q^{-1} given by Eq. (4), and the complementary dependence of the initial number of defects.

To investigate this issue, we considered the numerical evolution of a 1D system for a real field φ according to the equation of motion derived from the Landau-Ginzburg potential $V(\varphi) = (\varphi^4 - 2\epsilon\varphi^2 + 1)/8$. The system is in contact with a thermal reservoir. Thus, it obeys the Langevin equation,

$$\ddot{\varphi} + \eta\dot{\varphi} - \partial_{xx}\varphi + \partial_\varphi V(\varphi) = \vartheta\,. \tag{5}$$

The noise term ϑ has correlation properties

$$\langle\vartheta(x,t), \vartheta(x',t')\rangle = 2\eta\theta\delta(x'-x)\delta(t'-t)\,. \tag{6}$$

In Eq. (6), η is the overall damping constant which also helps characterize the amplitude of the noise through Eq. (5). The parameter ϵ measures the distance from the phase transition and varies according to Eq. (3), that is, $\epsilon = \min(1, t/\tau_Q)$. θ describes the temperature of the reservoir and is set to 0.01 throughout this work. This separate parametrization of ϵ and θ was adopted to correspond to the situation in the superfluids (especially ^{3}He, because in ^{4}He the Landau-Ginzburg theory is a poor approximation). There the symmetry breaking can be induced by the change of the pressure and occurs with inconsequential adjustments of the absolute temperature [1,7]. The second order time evolution allows us, in principle, to make contact with cosmology, but, in the regime considered here, the evolution is dominated by the dissipative term $\eta\dot{\varphi}$. Hence, in effect, we are dealing with the time dependent Landau-Ginzburg equation.

We investigate the creation of kinks as a function of τ_Q by starting at some $\epsilon < 0$ suitably above the transition, and then gradually adjusting the potential in accordance with Eq. (3). Figure 2 shows a sequence of "snapshots" of φ obtained in the course of such a quench. When $\epsilon < 0$, φ fluctuates around its expectation value $\langle\varphi\rangle = 0$. The same situation initially persists for slightly positive ϵ. However, further below T_C, φ settles locally around one of the two alternatives: $\langle\varphi\rangle = \pm\sqrt{\epsilon}$. Moreover, local choices of one of these two alternatives cannot be easily undone once a certain symmetry breaking is selected, unless the kinks are separated by distances no larger than ξ (see Fig. 2).

To test the predictions of defect density in Ref. [1], we note again that, for a sizeable η and sufficiently small ϵ, the damping term $\eta\dot{\varphi}$ in Eq. (5) is bound to dominate. Such overdamped evolution will take place whenever

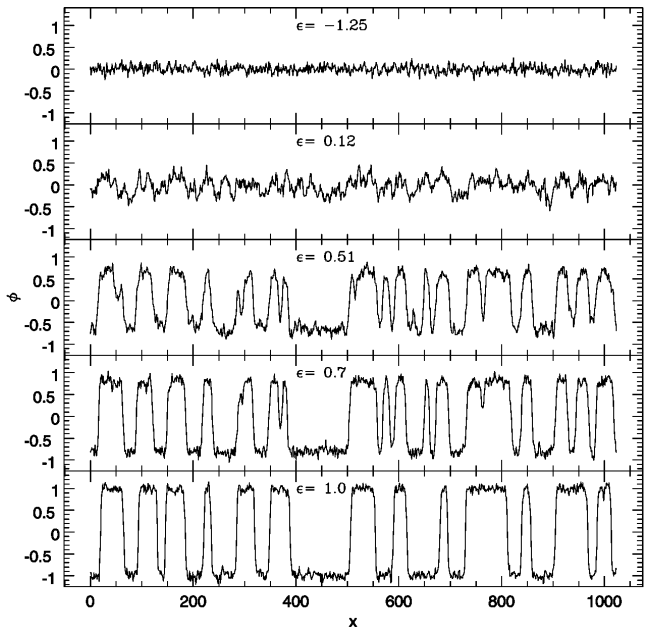

FIG. 2. Snapshots of φ during kink formation with a quench time scale of $\tau_Q = 64$ and damping parameter $\eta = 1$. The figures, from top to bottom, correspond to $t = -80.0$, 7.5, 32.5, 45.0, and 333.0, respectively.

$\eta^2 > |\epsilon|$ [13]. In our case, $\eta = 1$ and $|\hat{\epsilon}| \ll 1$. The characteristic relaxation time is then given by $\tau = \eta/|\epsilon|$. Consequently, $\mu = 1$ in Eq. (2), and

$$\hat{t} = \sqrt{\eta\,\tau_Q}\,. \tag{7}$$

The corresponding value of the relative temperature immediately obtains $\hat{\epsilon} = \hat{t}/\tau_Q = \sqrt{\eta/\tau_Q}$. Hence

$$\hat{\xi} = \xi_0(\tau_Q/\eta)^{1/4}, \tag{8}$$

where we have adopted $\nu = 1/2$ in accordance with the Landau-Ginzburg theory and in agreement with the critical exponents inferred from the behavior of the healing length in Fig. 1. Equations (7) and (8) are expected to be applicable as long as the condition $\eta^2 > |\epsilon|$ holds at $\hat{\epsilon}$, which, in turn, implies $\eta/\tau_Q > 1$.

Figure 3 is the principal result of our paper. It illustrates the density of kinks obtained in a sequence of quenches with $\eta = 1$ for various τ_Q values. For each τ_Q, the phase transition was simulated 15 times, starting at $\epsilon = -1$ (except for the shortest and longest τ_Q, which were initiated at $\epsilon = -10/\sqrt{\tau_Q}$). Our computational domain had periodic boundary conditions. Production runs were carried out with the resolution of 16 384 gridpoints. Convergence was checked by comparing results at different resolutions. The "physical" size of the computational ring in the production runs was 2048 units. At this scale, the ring is large enough, so boundary effects are avoided.

The number of kinks produced by the quench was obtained by counting the number of zeros of φ. Above and immediately below T_C, there is a significant number of zeros which have little to do with the kinks (see Fig. 2). However, as the quench proceeds, the number of zeros quickly evolves towards an "asymptotic" value (see Fig. 4). This change of the density of zeros and its eventual stabilization is associated with the obvious change of the character of φ and with the appearance

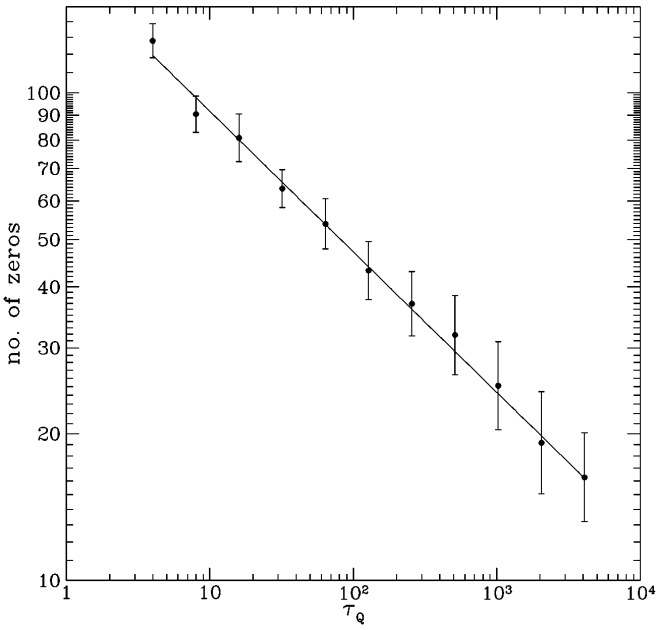

FIG. 3. Number of defects as a function of quench time scale. The plot is obtained at $t/\tau_Q = 4$ (see Fig. 4). The straight line is the best fit to $N = N_0\tau_Q^{-a}$ with $a = 0.28 \pm 0.02$ and $N_0 = 178 \pm 14$ ($\chi^2 = 1.96$). This exponent compares favorably with the theoretical prediction of $1/4$ based on the theory in Ref. [1].

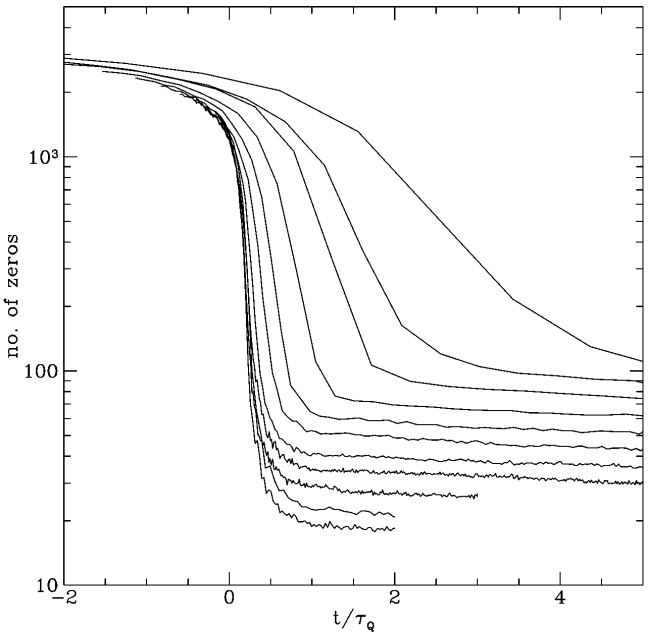

FIG. 4. Average number of zeros as a function of time in units of τ_Q; from top to bottom $\tau_Q = 4, 8, \ldots, 2048, 4096$. The number of kinks used in Fig. 3 were obtained at $t/\tau_Q \sim 4$, except in the large, computationally expensive, τ_Q cases where an extrapolated value was used.

of the clearly defined kinks. By then, the number of kinks is nearly constant in the runs with long τ_Q, and, correspondingly, their kink density is low. Even in the runs with the smallest τ_Q, there is still a clear break between the postquench rates of the disappearance of zeros and the long-time, relatively small rate at which the kinks annihilate.

In the regime investigated here, the theoretical scaling relation for the number density of kinks, $n \approx (\eta/\tau_Q)^{1/4} \approx 1/f\hat{\xi}$, appears to be well satisfied, with $f \sim 8$. We find a kink density $n = (0.087 \pm 0.007)\,\tau_Q^{-0.28\pm0.02}$, when the kinks are counted at approximately the same t/τ_Q value. A similar scaling is also obtained for kinks counted at equal t times after the quench.

We note that there are of course no "true" phase transitions in 1D [14] and that instead of vortices we are counting kinks. These and the other differences between our simulations and 3D superfluid experiments imply that their direct comparison would be dangerous. On the other hand, our 1D system exhibits some of the salient symptoms of an almost well-defined phase transition, such as scaling (Fig. 1), formation of large domains with broken symmetry (Fig. 2), asymptotic number of zeros and an abrupt change between the two regimes (Fig. 4). This is a consequence of the smallness of our thermal reservoir's temperature, which, in turn, implies that the fluctuation region $[0 < \epsilon < \epsilon_G \sim (\theta/\xi_0)^{2/3}]$ is narrow—Ginzburg temperature $\epsilon_G \leq 1/20$. Scaling in Fig. 1 suggests in fact an even narrower fluctuation region. In our simulations the estimates of $\hat{\xi}$ obtained from Eq. (8) fall within (or, in any case, close) to the scaling range shown in Fig. 1, where the scaling arguments relied upon are valid. Hence, our reliance on scaling is both justified *a priori* and seems to be borne out by the resulting scaling of the density of kinks with the quench rate, Fig. 3.

In summary, our numerical experiment appears to provide a strong confirmation of the theoretical predictions given by one of us [1]. As expected, the density of topological defects is somewhat less than the inverse of $\hat{\xi}$ but of the right order of magnitude. Most importantly, the scaling of the kink density with τ_Q follows closely theoretical expectations. These results are also supported by the dependence of the number of kinks on the damping parameter η, as well as by the preliminary results of the computer experiments involving complex order parameter and/or more than 1D space [15].

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[12] Kibble has also suggested [3] that thermal activation in the correlation-sized volumes, which is thought to occur below T_C but above the Ginzburg temperature T_G, will play a crucial role in determining the density of the defects. This would lead to the separation of defects of the order of the correlation length at T_G. Much of the past cosmological literature was based on this assumption, which differs from the nonequilibrium paradigm of Ref. [1]. It now appears that, at least in the cases investigated to date, the association between the T_G and the density of defects has not only been ruled out by superfluid experiments [7,9,10] but also—as we show in this paper—the alternative of Ref. [1] is suggested by numerical simulations and it is consistent with the experimental data. See also A. J. Gill and T. W. B. Kibble, J. Phys. A **29**, 4289 (1996) for related discussion.

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