DENSITY OF PERIODIC GEODESICS IN THE UNIT TANGENT BUNDLE OF A COMPACT HYPERBOLIC SURFACE

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Abstract

Let S be a compact oriented surface of constant curvature -1 and let T^1S be the unit tangent bundle of S endowed with the canonical (Sasaki) metric. We prove that T^1S has dense periodic geodesics, that is, the set of vectors tangent to periodic geodesics in T^1S is dense in TT^1S.

Let M be a compact Riemannian manifold. M is said to have the DPG property (density of periodic geodesics) if the vectors tangent to periodic geodesics in M are dense in TM, the tangent bundle of M. A compact manifold is known to have this property if, for example, its geodesic flow is Anosov (see [1]), in particular if it is hyperbolic. In this note we prove that the unit tangent bundle of a compact oriented surface of constant curvature -1 shares with the surface the DPG property.

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Theorem *Let S be a compact oriented surface of constant curvature -1 and let T^1S be the unit tangent bundle of S endowed with the canonical (Sasaki) metric. Then T^1S has the DPG property.*

Remarks.
a) Geodesics in T^1S do not project necessarily to geodesics in S.
b) The unit tangent bundle of any compact oriented surface of constant curvature 0 or 1 has also the DPG property.
c) The geodesic flow of T^1S, which is a flow on T^1T^1S, is not Anosov.
d) T^1S may be written as $\Gamma\backslash PSl\,(2,\mathbf{R})$, where Γ is the fundamental group of S. In general, not every compact quotient of a Lie group endowed with a left invariant Riemannian metric has the DPG property.

The proof of the theorem and comments on the remarks can be found at the end of the article. Next, we give some preliminaries. Let H be the hyperbolic plane of constant curvature -1. Any oriented surface S of constant curvature -1 inherits from its universal covering H a canonical complex structure. If V is a smooth curve in TS, then V' will denote the covariant derivative along the projection of V to S. The geodesic curvature of a curve c in S with constant speed $\lambda \neq 0$ is defined by $\kappa\,(t) = \langle \dot{c}'\,(t)\,, i\dot{c}\,(t)\rangle\,/\lambda^3$. We consider on T^1S the canonical (Sasaki) metric, defined by $\|\xi\|^2 = \|\pi_{*v}\xi\|^2 + \|\mathcal{K}(\xi)\|^2$ for $\xi \in T_vT^1S$, $v \in T^1S$, where $\pi : T^1S \to S$ is the canonical projection and $\mathcal{K}$ is the connection operator. Next, we recall from [7] a description of the geodesics of T^1H and some properties of curves in H of constant geodesic curvature.

Proposition 1 *Let V be a geodesic in T^1H and let $c = \pi \circ V$. Then $\|V'\| =$const, $\|\dot{c}\|=$const $=: \lambda$ and one of the following possibilities holds:*

a) If $\lambda = 0$, then V is a constant speed curve in the circle $T^1_{c(0)}H$.

b) If $\lambda \neq 0$, then the geodesic curvature κ of c with respect to the normal $i\dot{c}/\lambda$ is also constant and for $t \in \mathbf{R}$

$$V\,(t) = e^{-2\lambda\kappa t i} z\dot{c}\,(t)\,, \tag{1}$$

where $z \in \mathbf{C}$ is such that $V\,(0) = z\dot{c}\,(0)$.

Conversely, each curve V in T^1H which satisfies (a) or (b) is a geodesic. Moreover, given a constant speed curve c in H with constant geodesic curvature, and $V_0 \in T^1_{c(0)}H$, there is a unique geodesic V in T^1H which projects to c and such that $V\,(0) = V_0$.

We recall from the proof of this proposition that if V is the geodesic in T^1S with initial velocity ξ, then $\lambda = \|\pi_{*V(0)}\xi\|$ and $\mathcal{K}(\xi) = -\lambda\kappa i V(0)$, in particular $\kappa = \pm\|\mathcal{K}(\xi)\|/\lambda$.

In the following we consider the upper half space model $H = \{x + iy \mid y > 0\}$ with the metric $ds^2 = (dx^2 + dy^2)/y^2$.

Lemma 2 *Let c be a complete curve in H of constant geodesic curvature κ. Given $\theta \in (0, \pi)$, let c_θ be the curve in H defined by $c_\theta(t) = e^t e^{i\theta}$.*

a) If $|\kappa| > 1$, the image of c is a geodesic circle of radius $|r|$ and length $|2\pi \sinh r|$, where $\coth r = \kappa$ (this implies that the length is $2\pi/\sqrt{\kappa^2 - 1}$).

b) If $|\kappa| = 1$, the image of c is a horocycle.

c) If $\kappa = \cos\theta$, the image of c is congruent to that of c_θ.

Let $G = PSl(2, \mathbf{R}) = \{g \in M_2(\mathbf{R}) \mid \det g = 1\}/\{\pm I\}$ and let $\mathfrak{g} = \{X \in M_2(\mathbf{R}) \mid \operatorname{tr} X = 0\}$ be its Lie algebra. Via the canonical action of G on H by Möbius transformations, G is the group of orientation preserving isometries of H. Hence, H may be identified with G/K, where $K = PSO(2)$ is the isotropy group at the point $i \in H$.
Consider the Cartan decomposition $\mathfrak{g} = \mathbf{R}Z \oplus \mathfrak{p}$, where $Z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ spans the Lie algebra of K and $\mathfrak{p} = \{X \in \mathfrak{g} \mid X = X^t\}$. As usual we shall identify $T_{eK}H$ with $\mathfrak{p}$. Under this identification, the quasi-complex structure induced on $\mathfrak{p}$ is given by $\operatorname{ad}_Z : \mathfrak{p} \to \mathfrak{p}$ and $X_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{p}$ is a unit vector. One can show that G acts simply transitively and by isometries on T^1H. Hence, the map $\Phi : G \to T^1H$ defined by $\Phi(g) = g_{*K}(X_0)$ is a diffeomorphism which induces in G a left invariant metric. From now on we identify sometimes in this way G with T^1H. In particular, the unit tangent bundle of a surface $\Gamma\backslash S$ may be identified with $\Gamma\backslash G$.
Let S be an oriented surface of constant curvature -1 and let κ be a real number. The κ-flow on T^1S is defined by $\phi_t^\kappa(v) = \dot{c}_v^\kappa(t)$, where c_v^κ is the unique unit speed curve in H with constant geodesic curvature κ and initial velocity v. In particular, the 0-flow is the geodesic flow of S. Next, we obtain the κ-flow on T^1H using the identification $\Phi : G \to T^1H$, taking advantage of the group structure of G. Let L_h, R_h denote left and right multiplication by h, respectively, and set $Y_\kappa = X_0 + \kappa Z$.

Lemma 3 *If φ_t^κ denotes the κ-flow on T^1H, then for all t we have*

$$\varphi_t^\kappa \circ \Phi = \Phi \circ R_{\exp(tY_\kappa)}.$$

Proof. Let $g \in G$ and let $c(t) = \pi(g \exp(tY_\kappa))$. We compute

$$\begin{aligned}\dot{c}(t) &= \left.\tfrac{d}{ds}\right|_t \pi(g\exp(sY_\kappa)) = \pi_*\left(L_{g\exp(tY_\kappa)}\right)_*\left(\left.\tfrac{d}{ds}\right|_0 \exp(sY_\kappa)\right) = \\ &= \left(L_{g\exp(tY_\kappa)}\right)_* \pi_*(Y_\kappa) = (g\exp(tY_\kappa))_*(X_0) = \Phi(g\exp(tY_\kappa)).\end{aligned}$$

In particular, $\dot{c}(0) = \Phi(g)$. It remains only to prove that c has constant geodesic curvature κ. Now, the curve $c_0(t) = \pi \exp(tY_\kappa)$ is an orbit of the one parameter group $\psi_t = \exp(tY_\kappa)$ of isometries of H, hence it has constant geodesic curvature, say κ_0. Since $c = gc_0$, it suffices to show that $\kappa = \kappa_0$. Let W be the Killing field on H associated with ψ_t and let $W_0 = W(c(0)) = \dot{c}_0(0) = X_0$. We have

$$\kappa_0 = \langle \nabla_{W_0} W, iW_0\rangle = -\langle \nabla_{iW_0} W, W_0\rangle = -\tfrac{1}{2}\left.\tfrac{d}{ds}\right|_0 \|W(b(s))\|^2, \tag{2}$$

where b is a curve in H with $\dot{b}(0) = iW_0 = iX_0$, for example $b(s) = \pi\exp(s[Z, X_0])$. Now,

$$\begin{aligned}\|W(b(s))\| &= \|\exp(-s[Z,X_0])_* W(b(s))\| = \\ &= \left\|\exp(-s[Z,X_0])_* \left.\tfrac{d}{dt}\right|_0 \pi\exp(tY_\kappa)\exp(s[Z,X_0])\right\| \\ &= \|\pi_{*e}\operatorname{Ad}[\exp(-s[Z,X_0])](Y_\kappa)\|\end{aligned}$$

Let $\mathfrak{k}$ be the Lie algebra of K. Since π_* is the projection to $\mathfrak{p}$ along $\mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we have

$$\begin{aligned}\kappa_0 &= -\left\langle \left.\tfrac{d}{ds}\right|_0 \pi_* e^{-s\operatorname{ad}[Z,X_0]}(Y_\kappa), \pi_*(Y_\kappa)\right\rangle = \\ &= \langle [[Z,X_0], \kappa Z], X_0\rangle = -\kappa\left\langle (\operatorname{ad}_Z)^2 X_0, X_0\right\rangle = \kappa\end{aligned}$$

(notice that $(\operatorname{ad}_Z)^2 = -\operatorname{id}$ and $\|X_0\| = 1$). □

Lemma 4 *Let $S = \Gamma\backslash H$ be a compact oriented surface of constant curvature -1 and suppose that $|\kappa| < 1$.*

a) Y_κ is conjugate in $\operatorname{Ad}(G)$ to aX_0, where $a = \sqrt{1-\kappa^2}$. In particular, there exists $h \in G$ such that $\exp(atX_0) = h\exp(tY_\kappa)h^{-1}$ for all t.

b) The κ-flow ϕ_t^κ on T^1S is conjugate to a constant reparametrization of the geodesic flow of S. More precisely, if h and a are as above, then $F = \Phi \circ R_h \circ \Phi^{-1}$ is a diffeomorphism of T^1H which induces a diffeomorphism f of the quotient $\Gamma\backslash T^1H \approx T^1S$, satisfying $\phi_t^\kappa = f \circ \phi_{at}^0 \circ f^{-1}$ for all t.

c) The vectors tangent to the periodic orbits of the κ-flow are dense in T^1S.

Proof. a) follows from the fact that $Y_\kappa = \frac{1}{2}\begin{pmatrix} 1 & \kappa \\ -\kappa & -1 \end{pmatrix}$ diagonalizes with eigenvalues $\pm a/2$, since $|\kappa| < 1$.

b) If a, h are as above, then $R_h \circ R_{\exp(atX_0)} = R_{\exp(tY_\kappa)} \circ R_h$ for all t. Therefore, Lemma 3 implies that $\varphi_t^\kappa = F \circ \varphi_{at}^0 \circ F^{-1}$ for all t. One checks that $F(\gamma v) = \gamma F(v)$ for all $\gamma \in \Gamma, v \in T^1 H$ and the existence of f is proved. The last assertion follows from straightforward computations.

c) We have that the κ-flow on $T^1 S$ is conjugate to a constant rate reparametrization of the geodesic flow of S, which is known to be Anosov and has dense periodic orbits by Theorem 3 of [1]. □

Lemma 5 *Let S be a compact oriented surface of constant curvature -1. Let c be a periodic constant speed curve in S of constant geodesic curvature κ_0, with $|\kappa_0| < 1$. Then, for each κ with $|\kappa| < 1$, there exists a periodic constant speed curve c_κ in S, of constant geodesic curvature κ, such that*

a) $c_{\kappa_0} = c$,

b) $c_\kappa(0)$ converges to $c(0)$ and $\dot{c}_\kappa(0)$ converges to $\dot{c}(0)$ for $\kappa \to \kappa_0$,

c) the function $\kappa \mapsto \kappa \operatorname{length}(c_\kappa)$ is continuous, odd and strictly increasing.

Proof. We may suppose that c has unit speed and that $S = \Gamma \backslash H$, where Γ is a uniform subgroup of G which acts freely and properly discontinuously on H. Suppose that t_0 is the period of c and that C is a lift of c to H. Then there exists $g \in \Gamma$ such that $g_* \dot{C}(0) = \dot{C}(t_0)$. Since G acts transitively on $T^1 H$, by conjugating Γ by an element of G if necessary, we may suppose without loss of generality, by Lemma 2 (c), that $C(t) = e^{t \sin\theta_0} e^{i\theta_0}$ with $\cos\theta_0 = \kappa_0$, $0 < \theta_0 < \pi$ and, additionally, that $g(z) = az$, where $a = e^{t_0 \sin\theta_0}$.

For $|\kappa| < 1$, let c_κ be the projection to S of the curve $C_\kappa(t) := e^{t\sin\theta} e^{i\theta}$, where $\cos\theta = \kappa$, $0 < \theta < \pi$. Clearly, c_κ satisfies the first two conditions. By Lemma 2 (c), c_κ has constant geodesic curvature κ. Since $g_* \dot{C}_\kappa(0) = \dot{C}_\kappa(t_0 \sin\theta_0 / \sin\theta)$ and C_κ has unit speed, then c_κ is periodic and

$$\text{length } (c_\kappa) = t_0 \frac{\sin\theta_0}{\sin\theta} = t_0 \sqrt{\frac{1-\kappa_0^2}{1-\kappa^2}}$$

Thus, the function $\kappa \mapsto \kappa \operatorname{length}(c_\kappa)$ has the required properties. □

Comments on the remarks.

(a) follows from Proposition 1. Next we comment on (b). If S is flat, then S is covered by a flat torus (see [8]) whose unit tangent bundle is again a flat torus and clearly has the DPG property. On the other hand, if S has constant curvature 1, then S is covered by a sphere, whose unit tangent bundle is isometric to $SO(3)$ endowed with a bi-invariant metric, all of whose geodesics are periodic (see also [4]). (c) is a consequence of [5], since T^1H has conjugate points. This follows from the proof of Myers' Theorem (see [2]), since if γ is the geodesic in $G \approx T^1H$ with initial velocity Z, then $\text{Ricci}(\dot{\gamma})$ is constant and positive. Indeed, γ is the orbit of the one-parameter group $t \mapsto \exp(tZ)$ of isometries of G, and $\text{Ricci}(Z) > 0$ by Theorem 4.3 of [6]. Finally, a counterexample for (d) can be found for example in [3].

Proof of the Theorem.

Let Γ be the fundamental group of S and suppose that $S = \Gamma \backslash H$. Let $P : TT^1H \to T^1H$ and $\pi : TH \to H$ be the canonical projections. By abuse of notation we call also π the restriction of the latter to T^1H. Let $T'T^1H = \{\xi \in TT^1H \mid \pi_*\xi \neq 0\}$ and let $T'T^1S = \Gamma\backslash T'T^1H$. These are open dense subsets of TT^1H and TT^1S, respectively. Let now

$$F : T'T^1H \to \left\{(v, Y, \kappa) \in T^1H \times TH \times \mathbf{R} \mid Y \neq 0 \text{ and } \pi(v) = \pi(Y)\right\}$$

be defined by $F(\xi) = (P\xi, \pi_*\xi, \kappa(\xi))$, where $\kappa(\xi)$ is the (constant by Proposition 1) geodesic curvature of πV, V being the unique geodesic in G with initial velocity ξ. F is a diffeomorphism since it is differentiable and so is the inverse $F^{-1}(v, Y, \kappa) = \dot{V}(0)$, where V is the unique geodesic in T^1H such that $V(0) = v$, and $C := \pi V$ has constant geodesic curvature κ and satisfies $\dot{C}(0) = Y$ (see Proposition 1).

Fix $v_0 \in T^1H$ and $\eta \in T'_{\Gamma v_0}T^1S$. Suppose that η lifts to $\xi \in T'_{v_0}T^1H$ and that $F(\xi) = (v_0, Y_0, \kappa_0)$. We have to show that given $\varepsilon > 0$ and open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of v_0 and Y_0, in T^1H and TH respectively, then there exist κ with $|\kappa - \kappa_0| < \varepsilon$, $v \in \mathcal{U}$ and $0 \neq Y \in \mathcal{V}$, with the same footpoint, such that the geodesic V in T^1H with initial velocity $F^{-1}(v, Y, \kappa)$ projects to a periodic geodesic in T^1S. By the expression (1), it suffices to show that $c := \Gamma\pi V$ is periodic and $2\lambda\kappa t_0 \in 2\pi\mathbf{Q}$ for some positive number t_0 such that $\dot{c}(0) = \dot{c}(t_0)$, where λ, κ are as in Proposition 1.

Suppose that $|\kappa_0| \geq 1$. In this case choose $v = v_0$, $Y = Y_0$ and κ such that $|\kappa - \kappa_0| < \varepsilon$, $|\kappa| > 1$ and $2\kappa/\sqrt{\kappa^2 - 1} \in \mathbf{Q}$ (such a κ exists since the function $\kappa \mapsto 2\kappa/\sqrt{\kappa^2 - 1}$ is odd and strictly monotonic for $\kappa > 1$). Indeed, by Lemma 2 (a), $\dot{c}(0) = \dot{c}(t_0)$ holds for $t_0 = 2\pi/\lambda\sqrt{\kappa^2 - 1}$, since c has constant speed λ. Hence, $2t_0\kappa\lambda \in 2\pi\mathbf{Q}$ by the choice of κ.

If $|\kappa_0| < 1$, then by Lemma 4 there exists $(v_1, Y_1) \in \mathcal{U} \times \mathcal{V} \subset T^1H \times TH$ close to (v_0, Y_0), with $\pi(v_1) = \pi(Y_1)$, such that Γc_1 is periodic, where c_1 is the constant speed curve in H of constant geodesic curvature κ_0 with $\dot{c}_1(0) = Y_1$. By Lemma 5, since $\mathcal{V}$ is open, there exist κ with $|\kappa| < 1$, $|\kappa - \kappa_0| < \varepsilon$, and $(v, Y) \in \mathcal{U} \times \mathcal{V}$ close to (v_1, Y_1), with $Y \neq 0$ and $\pi(v) = \pi(Y)$, such that C projects to a periodic curve c in $\Gamma \backslash H$ with length ℓ satisfying $2\kappa\ell \in 2\pi\mathbf{Q}$, where C is the constant speed curve in H with constant geodesic curvature κ and initial velocity Y. If $t_0 = \ell/\lambda$, then $\dot{c}(0) = \dot{c}(t_0)$ and $2\lambda\kappa t_0 = 2\kappa\ell \in 2\pi\mathbf{Q}$. Consequently, for v, Y and κ as above, the geodesic in G with initial velocity $F^{-1}(v, Y, \kappa)$ projects to a periodic geodesic in $T^1 S$. This completes the proof of the theorem. $\square$

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