Density of states and order parameter in dirty anisotropic superconductors

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We analyze in detail how the scattering by nonmagnetic impurities affects the shape and amplitude of the order parameter (OP) and the density of states in anisotropic superconductors in the framework of BCS theory. Special attention is paid to the case when the OP is a mixture of d and s waves changing its sign on the Fermi surface. The critical temperature is shown to decay with the increase of the residual resistance according to the power law. At zero temperature impurity scattering gives rise to a peculiar phase transition from a gapless regime to a state with a finite gap in the quasiparticle spectrum. [S0163-1829(96)02342-9]

I. INTRODUCTION

Experimental study of the order parameter (OP) in high-T_c superconductors Y-Ba-Cu-O and Bi-Sr-Ca-Cu-O established firmly two facts:

(1) The OP reverses its sign on the Fermi surface.

(2) The shape of the OP cannot be described exactly as a d wave in a tetragonally symmetric crystal. In particular, its angular average is not zero.

The first statement is supported by the Josephson tunneling experiment in the corner geometry.[1] The second one has been proven by the c-axis Josephson tunneling experiments[2] and by the measurement of the T_c dependence on residual resistivity in the ion and radiation damaged Y-Ba-Cu-O and $Y_{1-x}Pr_x$Ba-Cu-O.[3]

The physical reasons for the formation of such a complex OP are still not clearly understood. The initial idea of the antiferromagnon exchange[4,5] was opposed by Schrieffer[6] who indicated that this interaction is strongly suppressed at the nesting vector. In a model taking into account phonon and Coulomb interactions near extended van Hove singularities Abrikosov[7] has found the sign reversal of the OP. To our knowledge strong-coupling theories do not explain nontrivial properties of the OP.

No matter what mechanism gives rise to this special shape of the OP, its very existence leads to a number of physical phenomena. The purpose of this article is to describe these phenomena in some detail. As it will be demonstrated below, an initially gapless excitation spectrum of a clean superconductor may acquire a gap due to the scattering of electrons by nonmagnetic impurities.[8,9,7,10] Thus, the layered anisotropic superconductors are potentially predisposed to a sort of phase transition at zero temperature. The latter may exhibit itself, e.g., in the quasiparticle tunneling and the temperature correction to the penetration depth λ in single crystals of Y-Ba-Cu-O doped by Pr or subject to the radiation damage. The transmutation of the gapless behavior of these quantities at small residual resistivity to the activated regime as the residual resistivity increases would clearly signal this transition.

We have established a close relationship between this plausible phase transition just described and the breakdown of superconductivity in the purely d-wave case found in Refs. 12 and 13. Namely, for an extended s superconductor close to a d superconductor, the value of the scattering rate $\tau_\star$ at the new transition point is close to the value τ_c at the breakdown point in the d superconductor. We argue below that a weak violation of the d pairing takes place in the Bi-Sr-Ca-Cu-O compound.

Another important phenomenon considered in our work is a powerlike decay of the critical temperature induced by the increase of the impurity scattering rate or the residual resistance. It will be shown that the power exponent is expressed in terms of the anisotropy coefficient (AC) κ which is the ratio of the angular average of the square of the OP to the square of average of the OP. We find that the shape of the OP changes with the impurity concentration.

It is worthwhile mentioning that the density of states (DOS) and the energy gap are nonmonotonous functions of the impurity scattering rate or the impurity concentration: both have a maximum.

Having several overlappings with the above-mentioned works, the present article differs from them by a more general approach: we do not presume any special interaction. This approach enabled us to discover a phenomenon missed by other authors: the change of the OP shape with the impurity concentration. We believe also that a more complete description of the DOS and OP in the entire range of the energy, concentration, and temperature, as well as the derivation of Ginsburg-Landau equation presented below is important for comparison with the experiment. We carried out our analysis within the Born approximation for the individual impurity scattering, but the generalization of our theory to the unitary limit is straightforward.

The main reason for the d-s mixture in high-T_c superconductors is probably a small orthorhombic distortion which is invariably present in the superconducting state (see Refs. 14–17). Although the ratio of the lattice constants in plane is close to 1 the orthorhombic anisotropy is strongly enhanced in the electronic properties. The orthorhombic distortion is especially weak in the Bi 2:2:1:2 compound.[17] Therefore, one can expect that this compound is well described as an

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almost d superconductor with a weak admixture of the s wave. This conclusion is supported by angle-resolved photoemission spectroscopy measurements of the Fermi surface and the gap anisotropy[18] and by measurements of ac conductivity.[19] For this reason we present a thorough analysis of this situation.

In the next section we give the general formulation of the problem. In Sec. III the critical line and the behavior of the OP near it is found. In Sec. IV we derive equations for the amplitude of the OP and solve them in the vicinity of the critical curve. In Sec. V we derive the Ginzburg-Landau equations. In Sec. VI the OP and the DOS are found at zero temperature. In particular we find the critical value of scattering time τ_* at which the gap in the spectrum appears. A brief report on part of the results has been published in Ref. 10.

II. GENERAL RELATIONSHIPS

The OP represents a spin-singlet state either for the d-wave or for the s-wave pairing. We start with the Abrikosov-Gor'kov equations[20] for the electronic matrix Green function (Nambu representation[21]) averaged over the random ensemble of the elastic scatterers:

$$\hat{G}=\begin{pmatrix} G(\mathbf{p},i\epsilon_n) & F^{\dagger}(\mathbf{p},i\epsilon_n) \\ F(\mathbf{p},i\epsilon_n) & -G(-\mathbf{p},-i\epsilon_n)\end{pmatrix}, \quad (1)$$

where $G(\mathbf{p},i\epsilon_n)$ and $F(\mathbf{p},i\epsilon_n)$ are the normal and anomalous Green functions, respectively, $\epsilon_n=(2n+1)\pi T$ stands for the Matsubara frequencies, and $\mathbf{p}$ denotes the momentum. The Dyson equation for $\hat{G}$ reads

$$\hat{G}^{-1}=\hat{G}_0^{-1}-\hat{\Sigma}, \quad (2)$$

where $\hat{G}_0$ is the Green tensor in a clean superconductor

$$\hat{G}_0^{-1}=\begin{pmatrix} i\epsilon_n-\xi & \Delta^* \\ \Delta & i\epsilon_n+\xi\end{pmatrix}, \quad (3)$$

and the self-energy $\hat{\Sigma}$ is given by the following expression:

$$\hat{\Sigma}(\mathbf{p},i\epsilon_n)=N_i\int w(\mathbf{p},\mathbf{p}')\hat{G}(\mathbf{p}',i\epsilon_n)\frac{d^2p}{(2\pi)^2}. \quad (4)$$

Here $\xi=(p^2/2m)-\mu$ is the energy of a normal electron counted from the Fermi level, N_i is the impurity concentration, $w(\mathbf{p},\mathbf{p}')$ is the scattering probability. We consider the two-dimensional system both for the simplicity and keeping in mind applications to the high-T_c superconductivity. We search for a solution of Eq. (2) in the following form:

$$\hat{G}^{-1}=\begin{pmatrix} i\tilde{\epsilon}_n-\xi & \tilde{\Delta}_n^* \\ \tilde{\Delta}_n & i\tilde{\epsilon}_n+\xi\end{pmatrix}, \quad (5)$$

where both $\tilde{\epsilon}_n$ and $\tilde{\Delta}_n$ depend on the polar angle φ and the integer n and obey the following nonlinear integral equations:

$$\tilde{\epsilon}_n(\varphi)-\pi N_i\int w(\varphi,\varphi')\frac{m(\varphi')\tilde{\epsilon}_n(\varphi')d\varphi'}{\sqrt{\tilde{\epsilon}_n^2(\varphi')+\tilde{\Delta}_n^2(\varphi')}}=\epsilon_n, \quad (6)$$

$$\tilde{\Delta}_n(\varphi)-\pi N_i\int w(\varphi,\varphi')\frac{m(\varphi')\tilde{\Delta}_n(\varphi')d\varphi'}{\sqrt{\tilde{\epsilon}_n^{\,2}(\varphi')+\tilde{\Delta}_n^2(\varphi')}}=\Delta(\varphi). \quad (7)$$

Here $m(\varphi)=\sqrt{p_F^2+(dp_F/d\varphi)^2}/v_F$ is the local effective mass. In contrast to the case of the isotropic s pairing, the ratios of the renormalized to the bare frequencies $\tilde{\epsilon}/\epsilon$ and the corresponding order parameters $\tilde{\Delta}_n/\Delta$ are not the same.

Assuming that $m(\varphi)$ is a periodic function of its argument it can be eliminated from the AG equations by means of the following mapping:

$$\tilde{\varphi}(\varphi)=2\pi\int_0^{\varphi}m(\varphi')d\varphi'\left(\int_0^{2\pi}m(\varphi')d\varphi'\right)^{-1}. \quad (8)$$

This mapping is single valued provided $m(\varphi)$ is positive in the whole domain $0\leqslant\varphi<2\pi$. We shall omit the tilde over φ in the subsequent formulas having in mind that the latter no longer represents a real polar angle in the momentum plane. Although this substitution does affect the angular dependence of the OP, it leaves the crystal symmetry intact. In the remaining part of the article we consider the isotropic scattering only: $w(\varphi,\varphi')=\text{const}$, which, of course, is invariant with respect to the above mapping. Then Eqs. (6) and (7) are reduced to the following ones:

$$\tilde{\epsilon}_n-\frac{\tilde{\epsilon}_n}{\tau}\left\langle\frac{1}{\sqrt{\tilde{\epsilon}_n^2+\tilde{\Delta}_n(\varphi)^2}}\right\rangle=\epsilon_n, \quad (9)$$

$$\tilde{\Delta}_n(\varphi)-\frac{1}{\tau}\left\langle\frac{\tilde{\Delta}_n(\varphi)}{\sqrt{\tilde{\epsilon}_n^2+\tilde{\Delta}_n(\varphi)^2}}\right\rangle=\Delta(\varphi). \quad (10)$$

Here angular brackets denote angular averaging: $\langle F\rangle=\int_0^{2\pi}F(\varphi)d\varphi/(2\pi)$. The analysis of the above equations presented below is greatly simplified due to the fact that the renormalized frequency remains angular independent.

The order parameter $\Delta(\varphi)$ satisfies the usual self-consistency condition:

$$\Delta(\mathbf{p})=2T\sum_n\int V(\mathbf{p},\mathbf{p}')F(\mathbf{p}',\epsilon_n)\frac{d^2p'}{(2\pi)^2}, \quad (11)$$

where $V(\mathbf{p},\mathbf{p}')$ is the electronic interaction potential. The integration over ξ in Eq. (11) can be performed explicitly giving the following result (the same mapping eliminating effective mass should be employed here):

$$\Delta(\varphi)=T\sum_n\int V(\varphi,\varphi')\frac{\tilde{\Delta}_n(\varphi')}{\sqrt{\tilde{\epsilon}_n^2+\tilde{\Delta}_n^2(\varphi')}}\frac{d\varphi'}{2\pi}. \quad (12)$$

Equations (9), (10), and (12) form a closed system determining $\Delta(\varphi,T)$.

III. THE TRANSITION LINE

A. Linearization

The system Eqs. (9), (10), and (12) are strongly nonlinear. However they may be linearized near the transition line

$T=T_c(\tau)$. Indeed, neglecting $\tilde{\Delta}$ in Eq. (9) and leaving only linear terms in Eq. (10) we get

$$\tilde{\epsilon}_n=\epsilon_n+\mathrm{sgn}(\epsilon_n)/\tau, \qquad (13)$$

$$\tilde{\Delta}_n(\varphi)-\langle\tilde{\Delta}_n(\varphi)\rangle/(|\tilde{\epsilon}_n|\tau)=\Delta(\varphi). \qquad (14)$$

Averaging over angles both sides of Eq. (14) and employing Eq. (13), we derive the following important relationship:

$$\langle\tilde{\Delta}_n\rangle/|\tilde{\epsilon}_n|=\langle\Delta\rangle/|\epsilon_n|. \qquad (15)$$

It is analogous to the Abrikosov-Gor'kov relation[20] for the isotropic case, however it is valid only in the vicinity of transition line and even there for the mean values only. Plugging Eq. (15) into Eq. (14) one can resolve it to find $\tilde{\Delta}$:

$$\tilde{\Delta}_n=\Delta+\langle\Delta\rangle/(|\epsilon_n|\tau). \qquad (16)$$

Then instead of Eq. (12) one obtains

$$\Delta(\varphi)=f(T,\tau)\int V(\varphi,\varphi')\Delta(\varphi')d\varphi'+g(T,\tau)\langle\Delta\rangle\overline{V(\varphi)}, \qquad (17)$$

where $\overline{V(\varphi)}=\int V(\varphi,\varphi')d\varphi'$ and

$$f(T,\tau)=T\sum_n\frac{1}{|\epsilon_n|+1/\tau}; \quad g(T,\tau)=T\sum_n\frac{1}{|\epsilon_n|(1+|\epsilon_n|\tau)}. \qquad (18)$$

Evaluating these functions one must keep in mind that summation is limited by the cutoff energy $|\epsilon|\leqslant\bar{\epsilon}$. Assuming $\bar{\epsilon}\gg T_c$ we find

$$f(T,\tau)=\pi^{-1}[\ln(\bar{\epsilon}/2\pi T)-\psi(x+1/2)], \qquad (19)$$

$$g(T,\tau)=\pi^{-1}[\psi(x+1/2)-\psi(1/2)], \qquad (20)$$

where $x=(2\pi T\tau)^{-1}$ and $\psi(x)$ denotes the digamma function (logarithmic derivative of the Euler gamma function). Below we analyze Eq. (17) for the d-wave and the s-wave pairing in turn.

B. d-wave pairing

In the case of d-wave pairing $\langle\Delta\rangle=0$ due to the symmetry constraint: $\Delta(\varphi+\pi/2)=-\Delta(\varphi)$, leading to a further simplification of Eq. (17),

$$\Delta(\varphi)=f(T,\tau)\int V(\varphi,\varphi')\Delta(\varphi')d\varphi'. \qquad (21)$$

This is a linear homogeneous Fredholm equation which is solvable and its solution is unique if and only if $\Delta(\varphi)$ and $f(T,\tau)$ match one of the eigenfunctions and one of the eigenvalues, respectively, of the linear integral operator:

$$\hat{V}\Psi(\varphi)=\int V(\varphi,\varphi')\Psi(\varphi')d\varphi'. \qquad (22)$$

The minimum of the free energy corresponds to the maximal eigenvalue V_0 and the corresponding wave function $\Psi_0(\varphi)$. Thus, the equation for the transition temperature T_c reads

$$f(T_c,\tau)=V_0^{-1}. \qquad (23)$$

Employing Eq. (23) one gets

$$\ln(T_{c0}/T_c)=\psi(x+1/2)-\psi(1/2), \qquad (24)$$

where

$$T_{c0}=(2\gamma\overline{\epsilon/}\pi)\exp(-\pi/V_0) \qquad (25)$$

is the transition temperature for a clean superconductor and $\gamma=1.78=\exp\mathbf{C}$, **C** is the Euler constant. At small x (small concentration of impurities) T_c is close to T_{c0}.

The critical temperature T_c is a monotonously decreasing function of the scattering rate vanishing at

$$\tau_c=2\gamma(\pi T_{c0})^{-1}=\bar{\epsilon}^{-1}\exp(\pi/V_0). \qquad (26)$$

Indeed, at very small T_c and finite τ the parameter $x=(2\pi\tau T)^{-1}$ becomes large so that the digamma function may be replaced by its logarithmic asymptotics:

$$\psi(x+1/2)=\ln(x)+1/(24x^2); \quad x\gg 1 \qquad (27)$$

and consequently

$$f(T,\tau)=\frac{1}{\pi}\ln(\overline{\epsilon\tau})-\frac{\pi}{6}(T\tau)^2. \qquad (28)$$

Plugging the above asymptotics into the criticality equation (23) which is convenient to cast into the form

$$f(T_c,\tau)=f(T'_c,\tau'),$$

one can analyze the behavior of the critical temperature as the scattering rate increases approaching its critical value. In the immediate vicinity of the latter the drop of T_c is governed by a square root law [this asymptotic is valid in a very narrow interval of its argument. At $(\tau-\tau_c)/\tau_c=0.015$ the correction due to the higher-order terms reaches about 20%]

$$T_c/T_{c0}=\sqrt{6(\tau/\tau_c-1)}/(2\gamma). \qquad (29)$$

At τ smaller than τ_c the d-wave pairing is totally suppressed. The breakdown of the d-wave superconductivity due to elastic scattering has been discussed earlier by Radtke *et al.*,[11] Monthoux and Pines,[12] and by Borkowski and Hirschfeld.[13]

Notice that in the above analysis we did not rely on the d-wave symmetry directly. An essential presumption $\langle\Delta\rangle=0$ is certainly a straightforward consequence of the d-wave symmetry but, generally speaking, does not require the former. On the other hand, one may hardly anticipate that the average of the OP vanishes identically, if not enforced by symmetry.

C. Extended s-wave pairing

In the case of s-wave pairing a formal solution of unreduced Eq. (17) reads

$$\Delta(\varphi,T,\tau)=g(T,\tau)\langle\Delta\rangle(1-f(T,\tau)\hat{V})^{-1}\overline{V(\varphi)}. \qquad (30)$$

Averaging Eq. (30) over the angle one finds the equation for the critical line:

$$1=g(T,\tau)\langle[1-f(T,\tau)\hat{V}]^{-1}\overline{V}\rangle. \qquad (31)$$

Equation (31) can be rewritten in the diagonal representation of the operator $\hat{V}$:

$$1=g(T,\tau)\sum_n \frac{V_n|\langle\Psi_n\rangle|^2}{1-f(T,\tau)V_n}, \tag{32}$$

where V_n and Ψ_n denote the eigenvalues and the corresponding eigenfunctions of the operator $\hat{V}$, respectively.

If impurity concentration is not too high, the sum in Eq. (32) is dominated by a single term corresponding to the maximal eigenvalue V_0. The OP in this range of scattering rate is proportional to the corresponding eigenfunction $\Delta(\varphi)\propto\Psi_0(\varphi)$. The domain of $1/\tau$, in which this single-mode regime holds, extends from a clean limit $1/\tau\ll T_{c0}$ to a relatively high concentration of impurities exceeding the critical value in the purely d-wave case $1/\tau_c=\pi T_{c0}/(2\gamma)$. It is limited by a strong inequality $|\ln\tau T_{c0}|\ll\pi V_0^{-1}$. The dispersion relation in this regime is similar to that in the d-wave case:

$$f(T,\tau)=V_0^{-1}-|\langle\Psi_0\rangle|^2 g(T,\tau). \tag{33}$$

In spite of the similarity, a correction to the d-wave dispersion Eq. (23) represented by the second term in the right-hand side (rhs) of Eq. (33) is extremely important, especially in a vast region of moderately high concentrations starting from $1/\tau\gg T_{c0}$, where in conformity with the dispersion relation Eq. (32) the decrease of T_c is governed by the power law

$$T_c=T_{c0}(\tau/\tau_c)^{\kappa-1} \tag{34}$$

with the exponent determined by the AC $\kappa=|\langle\Psi_0\rangle|^{-2}$. The AC may be expressed in another, more physically meaningful way:

$$\kappa=\langle|\Delta^2|\rangle/|\langle\Delta\rangle|^2. \tag{35}$$

If the admixture of the s-wave component to the dominating d-wave OP is small, the boundary curve between the normal and the superconducting phases clings to the analogous curve in the purely d-wave case. For this type of the OP, the AC $\kappa\gg 1$. As a consequence, the powerlike drop of T_c begins already in the vicinity of the critical value of scattering rate $1/\tau_c$. The dependence of T_c on τ for $T_c\ll T_{c0}$ may be approximated by means of the following equation:

$$\frac{(\pi T_c\tau_c)^2}{6}+|\langle\Psi_0\rangle|^2\ln\left(\frac{T_c}{T_{c0}}\right)=\log(\tau/\tau_c). \tag{36}$$

This equation provides a smooth interpolation between the square root law Eq. (29) and the power law Eq. (34).

Further increase of the scattering rate is accompanied by a gradual transition from a single-mode to a multimode regime and increasing deviation of the T_{c0} dependence on τ from the power law Eq. (34). Nevertheless, the critical temperature may be expressed through the scattering time explicitly everywhere in the dirty limit. By dirty limit we mean that the scattering rate exceeds its critical value substantially. According to Eqs. (18) and (19) the function $f(\tau,T)$ increases monotonously with τ and at $T\tau\ll 1$ it does not depend on temperature. It can be easily verified that the condition $\tau^{-1}\gg T_c$ implies $\tau^{-1}\gg T_{c0}$. Thus, in the dirty limit one gets

$$T_c=(2\pi\tau)^{-1}\exp\left[-\left(\sum_n \frac{|\langle\Psi_n\rangle|^2}{V_n^{-1}-\ln(\overline{\epsilon\tau})/\pi}\right)^{-1}\right]. \tag{37}$$

Finally, τ reaches what we call the extra-dirty limit

$$1<\ln\overline{\epsilon\tau}\ll\pi V_0^{-1}. \tag{38}$$

Even in this limit the OP does sustain its anisotropy, although its profile is determined by the interaction in the system without impurities:

$$\Delta(\varphi)\propto\overline{V(\varphi)}, \tag{39}$$

whereas T_c is given by the following asymptotic expression:

$$T_c=(2\gamma/\pi)\overline{\epsilon}\exp[-\pi/\langle\overline{V(\varphi)}\rangle](\overline{\epsilon\tau})^{\kappa-1}, \tag{40}$$

where κ is given by Eq. (35). To obtain the last result one should expand Eq. (37) up to terms linear in $V\ln(\overline{\epsilon\tau})$. In conformity with what was discussed above concerning a weak deviation from the d-wave symmetry, the AC κ becomes infinite when $\langle\Delta\rangle\to 0$. Notice, however, that the AC κ in two formulas Eqs. (34) and (40) do not coincide, although they are determined by the same expression in terms of the OP. The profile of the latter, however, varies strongly as the scattering increases. If the admixture of the s wave is small (large κ) the powerlike decay of T_c in the single-mode regime [Eq. (35)] does exist, if $\langle\Psi_0\rangle^2\gg\overline{\epsilon\tau}$. Otherwise the extra-dirty regime starts at $(\tau_c-\tau)/\tau_c\gg\langle\Psi_0\rangle^2\ln\overline{\epsilon\tau_c}$.

The powerlike tail Eq. (40) was derived by Hohenberg[22] under the assumption of weak anisotropy. In this case a single-mode regime prevails at an arbitrary scattering rate. Another peculiarity of a weakly anisotropic system is that the power law Eq. (40) spreads to the whole strong scattering range: $\tau^{-1}\gg\tau_c^{-1}$.

Summarizing, a critical temperature decreases with the increase of the scattering rate in anisotropic layered superconductors. This suppression of superconductivity is more pronounced the greater the anisotropy. The critical temperature lessens like a power of the scattering rate for $\tau^{-1}\gg\tau_c^{-1}$ and for small and moderate values of the anisotropy which may be characterized by the exponent $\kappa-1$, Eq. (35). It drops rapidly in the vicinity of $\tau^{-1}=\tau_c^{-1}$ in an almost d-wave superconductor. In the extra-dirty limit T_c also obeys the powerlike asymptotics Eq. (40). This tail expands into the whole domain of a moderate and a strong scattering for weakly anisotropic superconductors.

Equation (40) may be interpreted as a direct relation between critical temperature and residual resistivity in the normal state provided the impurity doping or the radiation damage does not significantly influence the number of carriers. Accepting the Drude law for the residual resistivity one finds $T_c(\rho)\propto\rho^{1-\kappa}$. We have analyzed the experimental data for $T_c(\rho)$ in the Pr-doped and ion-damaged Y-Ba-Cu-O.[3] Both sets of data being in a reasonable agreement with each other show that $T_c(\tau)$ does not vanish up to the Ioffe-Regel limit $\epsilon_F\tau\approx 1$, strongly implying that the average of the OP is finite. The value of AC evaluated on the basis of the above-mentioned data is $\kappa=2\pm 0.3$ meaning that $\langle\Delta\rangle\approx 0.7\sqrt{\langle\Delta^2\rangle}$. Thus the shape of the OP deviates significantly from the tetragonal d wave.

IV. THE ORDER PARAMETER

In the weak-coupling approximation the shape of the OP can be determined from the linearized self-consistency equation whereas its amplitude can, in principle, be found from the system of nonlinear AG equations and from the orthogonality condition applied to the self-consistency equation.[28,29] Let us start with an easier case of the purely d-wave pairing.

A. d-wave pairing

One can rewrite Eq. (12) in the following way:

$$\Delta(\varphi)-f(T,\tau)\int V(\varphi,\varphi')\Delta(\varphi')d\varphi' = T\sum_n \int V(\varphi,\varphi')\left[\frac{1}{\sqrt{\tilde{\epsilon}_n^2+\Delta^2(\varphi')}}-\frac{1}{|\epsilon_n|+1/\tau}\right]\times\Delta(\varphi')\frac{d\varphi'}{2\pi}, \tag{41}$$

where $f(T,\tau)$ is defined by Eq. (18). Since the latter factor is logarithmically large it compensates the smallness of $|V|$ making two terms in the lhs of the same order. On the other hand the sum in the rhs of Eq. (41) is convergent which allows to extend the summation from $-\infty$ to ∞. Consequently a characteristic magnitude of this term is $|V|\ll 1$ times smaller than Δ. Therefore in the leading approximation Eq. (50) becomes linear. Moreover, it coincides with Eq. (17) for the critical line. Its solution reads

$$\Delta(\varphi)=Q_0(T,\tau)\Psi_0(\varphi), \tag{42}$$

where $\Psi_0(\varphi)$ is a normalized eigenfunction of the operator $\hat{V}$ determining the order parameter on the critical line. Averaging both sides of Eq. (41) weighed with the same function $\Psi_0(\varphi)$ one gets in the next approximation:

$$f(T_c,\tau)-f(T,\tau)=T\sum_n\left[\left\langle\frac{\Psi_0^2(\varphi)}{\sqrt{\tilde{\epsilon}_n^2+Q^2\Psi_0(\varphi)^2}}\right\rangle-\frac{1}{|\epsilon_n|+1/\tau}\right], \tag{43}$$

which should be combined with a system

$$\tilde{\epsilon}_n-\left\langle\frac{\tilde{\epsilon}_n}{\tau\sqrt{\tilde{\epsilon}_n^2+Q^2\Psi_0^2(\varphi)}}\right\rangle=\epsilon_n \tag{44}$$

in order to determine $Q(T,\tau)$. In the vicinity of the critical line this system can be solved explicitly yielding:

$$\tilde{\epsilon}_n=\epsilon_n+\frac{\mathrm{sgn}(\epsilon_n)}{\tau}\left(1-\frac{Q^2}{2\,(|\epsilon_n|+1/\tau)^2}\right). \tag{45}$$

Plugging this into Eq. (43) one obtains

$$f(T,\tau)-f(T_c,\tau)=\frac{Q^2}{2}T\sum_n\left\{\frac{\langle\Psi_0^4(\varphi)\rangle}{(|\epsilon_n|+1/\tau)^3}-\frac{1}{\tau(|\epsilon_n|+1/\tau)^4}\right\}, \tag{46}$$

which can be further rewritten using polygamma functions as follows:

$$\frac{Q^2}{(4\pi T_c)^2}=-\frac{\ln(T_c/T)+\psi(x_c+1/2)-\psi(x+1/2)}{x_c\psi^{(3)}(x_c+1/2)/3+\langle\Psi_0^4(\varphi)\rangle\psi^{(2)}(x_c+1/2)}, \tag{47}$$

where $x=(2\,\pi T\tau)^{-1}$ and $x_c=(2\,\pi T_c\tau)^{-1}$, respectively, and $\psi^{(n)}$ denotes the nth derivative of the digamma function. If the concentration of impurities is sufficiently far from the critical value, this expression is reduced to the following one:

$$\frac{Q^2}{(4\pi T_c)^2}=-\frac{T_c-T}{T_c}\times\frac{1-x_c\psi^{(1)}(x_c+1/2)}{x_c\psi^{(3)}(x_c+1/2)/3+\langle\Psi_0^4(\varphi)\rangle\psi^{(2)}(x_c+1/2)}. \tag{48}$$

Care must be taken, however, when the scattering rate approaches the superconductivity breaking limit. Since $T_c\to 0$ the expansion (27) should be used together with the corresponding expansions for the higher polygamma functions. The result of this calculation accounting for the formula (29) reads

$$\frac{Q^2}{(4\pi)^2}=\frac{T_c^2(\tau)-T^2}{24(\langle\Psi_0^4(\varphi)\rangle-2/3)}. \tag{49}$$

The latter expression at $T_c-T\ll T_c$ matches the asymptotics of Eq. (48) at $x_c\gg 1$. Note that $\langle\Psi_0^4(\varphi)\rangle\geq 1$.

B. Extended s-wave pairing

It is convenient to rewrite Eq. (12) in the following form:

$$\hat{L}(T,\tau)\Delta(\varphi)=T\sum_n\int V(\varphi,\varphi')\frac{d\varphi'}{2\pi}\left\{\frac{\sigma_n}{\sqrt{\tilde{\epsilon}_n^2+\tilde{\Delta}_n^2(\varphi')}}-\frac{\langle\Delta\rangle}{|\epsilon_n|(\tau|\epsilon_n|+1)}+\left[\frac{1}{\sqrt{\tilde{\epsilon}_n^2+\tilde{\Delta}_n^2(\varphi')}}-\frac{1}{|\epsilon_n|+1/\tau}\right]\Delta(\varphi')\right\}. \tag{50}$$

Here we have introduced a linear operator

$$\hat{L}(T,\tau)=\hat{I}-f(T,\tau)\hat{V}-g(T,\tau)\hat{V}\hat{P}, \tag{51}$$

where $\hat{I}$ and $\hat{P}$ are the identity operator and the projector to the rotational-invariant state $|0\rangle$, respectively. The scalar factors $f(T,\tau)$ and $g(T,\tau)$ are defined by Eq. (18). Just as in the d-wave case those are logarithmically large. Therefore one can seek a solution of Eq. (50) perturbatively starting with a linear approximation corresponding to Eq. (22)

$$\Delta(\varphi)=Q(T,\tau)\chi(\varphi)+\Delta^{(1)}(\varphi), \tag{52}$$

where $\chi(\varphi)$ is a null vector of the linear operator $\hat{L}(T_c,\tau)$ proportional to expression (30). The amplitude $Q(T,\tau)$ can be found from the orthogonality condition. To this end let us project both parts of Eq. (50) onto the null vector of the conjugated operator

$$L_c^\dagger(\tau)=\hat{I}-f_c(\tau)\hat{V}-g_c(\tau)\hat{P}\hat{V}.$$

Here and in what follows the subscript c denotes corresponding quantity evaluated at the critical temperature $T_c(\tau)$. The above-mentioned null vector up to normalization is given by the following expression:

$$|\widetilde{\chi}\rangle=[\hat{I}-f_c(\tau)\hat{V}]^{-1}|0\rangle \tag{53}$$

and must be orthogonal to $|\Delta^{(1)}(\varphi)\rangle$. This condition combined with a useful relation between the conjugated null vectors

$$|\chi\rangle=\hat{V}|\widetilde{\chi}\rangle, \tag{54}$$

leads to the following transcendental equation for the amplitude Q:

$$\begin{aligned}(f_c-f)\langle\chi^2(\varphi)\rangle+(g_c-g)\langle\chi(\varphi)\rangle^2\\ =T\sum_n\left\{\left\langle\frac{\chi^2}{\sqrt{\widetilde{\epsilon}_n^2+\widetilde{\Delta}_n^2}}\right\rangle-\frac{\langle\chi^2\rangle}{|\epsilon_n|+1/\tau}\right.\\ \left.+\left(\widetilde{\sigma}_n\left\langle\frac{\chi}{\sqrt{\widetilde{\epsilon}_n^2+\widetilde{\Delta}_n^2}}\right\rangle-\frac{\langle\chi\rangle^2}{|\epsilon_n|(\tau|\epsilon_n|+1)}\right)\right\}.\end{aligned} \tag{55}$$

A new notation $\widetilde{\sigma}_n$ has been introduced for the renormalized σ_n as follows:

$$\sigma_n=Q(T,\tau)\widetilde{\sigma}_n. \tag{56}$$

After this rescaling the only explicit dependence on $Q(T,\tau)$ left in Eq. (55) is due to $\widetilde{\Delta}_n(\varphi)=Q(T,\tau)[\chi(\varphi)+\widetilde{\sigma}_n]$. However, one should keep in mind that the same dependence persists in the AG equations for $\widetilde{\epsilon}_n$ and $\widetilde{\sigma}_n$.

Again the solution in the vicinity of the critical line can be found explicitly:

$$\widetilde{\epsilon}_n=\epsilon_n+\frac{\mathrm{sgn}(\epsilon_n)}{\tau}\left(1-\frac{Q^2}{2|\epsilon_n|^2}\frac{\langle\chi\rangle^2(1+2\epsilon_n\tau)+\langle\chi^2\rangle\epsilon_n^2\tau^2}{(1+\epsilon_n\tau)^2}\right), \tag{57}$$

$$\widetilde{\sigma}_n=\frac{\langle\chi\rangle}{|\epsilon_n|\tau}-\frac{Q^2}{2\tau|\epsilon_n|^3}\frac{\langle\chi\rangle^3+2\langle\chi\rangle\langle\chi^2\rangle\tau|\epsilon_n|+\langle\chi^3\rangle\tau^2|\epsilon_n|^2}{(1+\tau|\epsilon_n|)^2}. \tag{58}$$

Plugging this into Eq. (55) one gets

$$\begin{aligned}(f_c-f)\langle\chi^2\rangle+(g_c-g)\langle\chi\rangle^2\\ =\frac{Q^2T\tau^3}{2}\sum_n\left\{\frac{\langle\chi^4\rangle}{(1+\tau|\epsilon_n|)^3}-\frac{\langle\chi^2\rangle^2}{(1+\tau|\epsilon_n|)^4}\right.\\ +\frac{4\langle\chi\rangle\langle\chi^3\rangle}{\tau|\epsilon_n|(1+\tau|\epsilon_n|)^3}+\frac{2\langle\chi\rangle^2\langle\chi^2\rangle(2+\tau|\epsilon_n|)}{\tau^2|\epsilon_n|^2(1+\tau|\epsilon_n|)^4}\\ \left.+\frac{\langle\chi\rangle^4}{\tau^3|\epsilon_n|^3(1+\tau|\epsilon_n|)^3}\right\}.\end{aligned} \tag{59}$$

Let us denote

$$h_{k,l}(x)=\sum_{n=0}^{\infty}(n+1/2+x)^{-k}(n+1/2)^{-l}. \tag{60}$$

These functions can be easily expressed in terms of polygamma functions. In particular $h_{k,0}(x)=(-1)^k\psi^{(k-1)}(x)/(k-1)!$. Using these functions Eq. (59) can be rewritten in a more concise fashion

$$\begin{aligned}\langle\chi^2\rangle\ln\left(\frac{T}{T_c}\right)-(\langle\chi^2\rangle-\langle\chi\rangle^2)[g_c-g(x)]\\ =\frac{Q^2}{8(\pi T_c)^2}\{\langle\chi^4\rangle h_{3,0}-\langle\chi^2\rangle^2x_ch_{4,0}+4\langle\chi^3\rangle\langle\chi\rangle x_ch_{3,1}\\ +2\langle\chi^2\rangle\langle\chi\rangle^2(x_c^3h_{4,2}+x_c^2h_{3,2})+\langle\chi\rangle^4x_c^3h_{2,3}\}.\end{aligned} \tag{61}$$

In the above formula we have omitted the argument of all the functions $h_{k,l}(x_c)$. In the isotropic limit ($\chi=1$) this equation reduces to the following one:

$$\frac{Q^2}{(4\pi T_c)^2}=-\frac{2}{\psi^{(2)}(1/2)}\frac{T_c-T}{T_c}\approx0.12\frac{T_c-T}{T_c}. \tag{62}$$

Here we have expanded the logarithm in $(T_c-T)/T_c$. The same behavior is characteristic for the large scattering rate regime provided that the angular average is not too small. If, however, the d-wave symmetry breaking is weak, the asymptotics are preceded by a significant crossover domain for the scattering rates relatively close to $1/\tau_c$. Namely,

$$\frac{Q^2}{(4\pi T_c)^2}=\frac{\langle\chi\rangle^2\ln(x_c/x)-\langle\chi^2\rangle(x^{-2}-x_c^{-2})/24}{(\langle\chi^4\rangle/2-\langle\chi^2\rangle^2/3)x_c^{-2}+\langle\chi^2\rangle\langle\chi\rangle^2 2\pi^2x_c^{-1}-\langle\chi\rangle^4\psi^{(2)}(1/2)/2}. \tag{63}$$

If $x_c \ll (\langle\chi^2\rangle)^{1/2}/\langle\chi\rangle$, the first term in the numerator and the last two in the denominator may be neglected. In this fashion Eq. (63) is reduced to the d-wave-like behavior Eq. (49). In the opposite limit, however, the terms with the maximal powers of $\langle\chi\rangle$, both in the numerator and the denominator, should only be retained. Note, that even in the extra-dirty limit the OP does not become isotropic

$$\Delta(\varphi,T,\tau)\approx 4.35\sqrt{(T_c-T)T_c}\,\overline{V}(\varphi)\langle\overline{V}\rangle^{-1}. \tag{64}$$

V. GINSBURG-LANDAU EQUATION

Several workers discussed a phenomenological Ginzburg-Landau (GL) equation for the d wave and mixed state.[23] In this section we derive the GL equation in the framework of the BCS theory for an anisotropic superconductor with impurities. An analogous derivation for a clean anisotropic superconductor has been presented by Gor'kov and Melik-Barkhudarov.[24]

In the previous section we have already completed the necessary calculations concerning the fourth-order terms. Here we present the results for the anisotropic gradient terms. We begin with the self-consistency equation in the coordinate representation:

$$\Delta(\mathbf{x},\mathbf{y})=V(\mathbf{x},\mathbf{y})\overline{F(\mathbf{x},\mathbf{y},t=0)}, \tag{65}$$

where the overline means the averaging over the random impurity configurations. We need this average calculated in the lowest (linear) order in the amplitude of the OP and up to the second order in the center-of-mass momentum. In the linear approximation and for a fixed configuration of impurities

$$F^\dagger=\tilde{G}_0\Delta^\dagger G_0. \tag{66}$$

Here $\tilde{G}_0(\mathbf{p},\epsilon)=G_0(-\mathbf{p},-\epsilon)$. The averaging over the random field can be performed employing the Abrikosov-Gor'kov technique. In particular, leaving only linear in Δ terms, we get

$$\overline{F^\dagger}(\mathbf{x},\mathbf{x}',\epsilon_n)=\int\overline{\tilde{G}_0(\mathbf{x},\mathbf{y},\epsilon_n)\Delta^\dagger(\mathbf{y},\mathbf{z})G_0(\mathbf{z},\mathbf{x}',-\epsilon_n)}\,d\mathbf{y}d\mathbf{z}. \tag{67}$$

After averaging over the random impurity potential, the Green functions and their products do not depend on the center-of-mass coordinate, but $\Delta^\dagger$ does in general. It is convenient to consider the Fourier transform of $\overline{F^\dagger}(\mathbf{x},\mathbf{x}',\epsilon_n)$ which depends on the relative momentum $\mathbf{p}$ and the center-of-mass momentum $\mathbf{q}$. Keeping only the ladderlike graphs, we obtain

$$\overline{F^\dagger}(\mathbf{p},\mathbf{q},\epsilon_n)=G(\mathbf{p},\epsilon_n)G(-\mathbf{p}+\mathbf{q},-\epsilon_n)\Lambda(\mathbf{p},\mathbf{q},\epsilon_n), \tag{68}$$

where $G(\mathbf{p},\epsilon_n)=(i\tilde{\epsilon}_n-\xi)^{-1}$ is the averaged Green function and $\tilde{\epsilon}_n=\epsilon_n+\mathrm{sgn}(\epsilon_n)/\tau$. The reduced vertex Λ obeys the following equation:

$$\Lambda(\mathbf{p},\mathbf{q},\epsilon_n)=\Delta^\dagger(\mathbf{p},\mathbf{q})+\frac{1}{4\pi\nu\tau}\int G(\mathbf{p}',\epsilon_n)G(-\mathbf{p}'+\mathbf{q},-\epsilon_n)\Lambda(\mathbf{p}',\mathbf{q},\epsilon_n)\frac{d^3p'}{(2\pi)^3}, \tag{69}$$

where $\nu=\int\delta(\xi)d^3p/(2\pi^3)$ is the normal density of states on the Fermi surface. The integration over ξ in Eq. (69) is readily performed with the following result:

$$\int G(\mathbf{p},\epsilon_n)G(-\mathbf{p}+\mathbf{q},-\epsilon_n)d\xi=\frac{\pi}{|\tilde{\epsilon}_n|-i\mathbf{v}\mathbf{q}/2}, \tag{70}$$

where $\mathbf{v}$ is the local velocity on the Fermi surface. We derive the following equation for Λ:

$$\Lambda(\mathbf{p},\mathbf{q},\epsilon_n)=\Delta^\dagger(\mathbf{p},\mathbf{q})+\frac{1}{\tau\nu}\int\frac{\Lambda(\mathbf{p}',\mathbf{q},\tilde{\eta}_n)}{|\tilde{\epsilon}_n|-i\mathbf{v}'\mathbf{q}/2}\frac{d\Omega'}{v'}. \tag{71}$$

The momentum $\mathbf{p}$ in Eq. (71) belongs to the Fermi surface and the integration is carried out over the Fermi surface as well. The reduced vertex Λ is analogous to the modified gap $\tilde{\Delta}^\dagger$ introduced by Abrikosov and Gor'kov. Expanding Green functions up to the second order in $\mathbf{q}$, we deduce a modified equation for Λ:

$$\Lambda(\mathbf{p},\mathbf{q},\epsilon_n)=\Delta^\dagger(\mathbf{p},\mathbf{q})+\frac{1}{\tau\nu|\tilde{\epsilon}_n|}\int\Lambda(\mathbf{p}',\mathbf{q})\frac{d\Omega'}{v'}-\frac{q_\alpha q_\beta}{4\tau\nu|\tilde{\epsilon}_n|^3}\int\Lambda(\mathbf{p}',\mathbf{q},\epsilon_n)v'_\alpha v'_\beta\frac{d\Omega'}{v'}. \tag{72}$$

In the lowest approximation in $\mathbf{q}$ we recover the previously found solution:

$$\Lambda^0(\mathbf{p},\epsilon_n)=\Delta^\dagger(\mathbf{p})+\frac{\langle\Delta^\dagger\rangle}{\tau\epsilon_n}. \tag{73}$$

With the accuracy up to the second order in $\mathbf{q}$ it follows,

$$\Lambda(\mathbf{p},\mathbf{q},\epsilon_n)=\Delta^\dagger(\mathbf{p})+\frac{\langle\Delta^\dagger\rangle}{\tau\epsilon_n}-\frac{q_\alpha q_\beta}{4\tau|\tilde{\epsilon}_n|^2\epsilon_n}\left(\langle v_\alpha v_\beta\Delta^\dagger\rangle+\frac{\langle v_\alpha v_\beta\rangle\langle\Delta^\dagger\rangle}{\epsilon_n\tau}\right). \tag{74}$$

Equation (65) can be transformed into the following form:

$$\Delta^\dagger=f(T,\tau)\hat{V}\Delta^\dagger+g(T,\tau)\overline{V}\langle\Delta^\dagger\rangle-\frac{q_\alpha q_\beta}{16\pi^3T^2}[h_{3,0}\hat{V}(v_\alpha v_\beta\Delta^\dagger)+x_ch_{3,1}\hat{V}(v_\alpha v_\beta)\langle\Delta^\dagger\rangle+x_ch_{3,1}\overline{V}\langle v_\alpha v_\beta\Delta^\dagger\rangle+x_c^2h_{3,2}\overline{V}\langle v_\alpha v_\beta\rangle\langle\Delta^\dagger\rangle], \tag{75}$$

where f, g, and $h_{k,l}$ have been defined earlier by Eqs. (19), (20), (60). Let us note that we omit the arguments of the functions $h_{k,l}(x_c)$. The solution of this equation can be factorized as before:

$$\Delta^\dagger(\mathbf{p},\mathbf{q},T,\tau)=\Phi(\mathbf{q},T,\tau)\chi(\mathbf{p}). \tag{76}$$

Here $\Phi(\mathbf{q},T,\tau)$ is the center-of-mass wave function, which is the wave function in the GL theory, and $\chi(\mathbf{q})$ is the solution of the zero-approximation linear equation, defined by the formulas (53) and (54). We would like to emphasize that the GL wave function Φ feels the symmetry of the function χ only in a rather implicit way.

For T slightly smaller than T_c from the orthogonality condition we find

$$[(f-f_c)\langle\chi^2\rangle+(g-g_c)\langle\chi\rangle^2-\zeta_{\alpha\beta}q_\alpha q_\beta]\Phi=0, \tag{77}$$

where the tensor $\zeta_{\alpha\beta}$ is defined as follows:

$$\zeta_{\alpha\beta}=\frac{1}{16\pi^3T^2}(h_{3,0}\langle v_\alpha v_\beta\chi^2\rangle+2x_c h_{3,1}\langle v_\alpha v_\beta\chi\rangle\langle\chi\rangle+x_c^2h_{3,2}\langle v_\alpha v_\beta\rangle\langle\chi\rangle^2). \tag{78}$$

The tensor $\zeta_{\alpha\beta}$ up to a factor coincides with the inverse effective mass tensor in the GL equation $(m^{-1})_{\alpha,\beta}$, which in turn is associated with the Pippard kernel $Q_{\alpha\beta}$ connecting the electric current j_α and the vector potential A_α:

$$j_\alpha=-Q_{\alpha\beta}A_\beta;\quad Q_{\alpha,\beta}=\frac{4e^2}{c}(m^{-1})_{\alpha\beta}|\Phi|^2. \tag{79}$$

In the clean limit only the first term in Eq. (78) is substantial and we arrive at the result by Gor'kov and Melik-Barkhudarov:[24] $\langle v_\alpha v_\beta\chi^2\rangle$. In the extra-dirty limit only the term with $\langle\chi\rangle^2$ matters. Then $(m^{-1})_{\alpha\beta}\propto\langle v_\alpha v_\beta\rangle$. Finally, combining the results of the previous section with the results of the calculations just completed we can write down the full GL equation as follows:

$$a\Phi+b\Phi|\Phi|^2+\zeta_{\alpha\beta}\partial_\alpha\partial_\beta\Phi=0, \tag{80}$$

where $\zeta_{\alpha\beta}$ is defined above and

$$a=\frac{T-T_c}{T_c}\left[1+\left(1-\frac{1}{\kappa}\right)\psi_c^{(1)}\right], \tag{81}$$

$$b=\frac{1}{8(\pi T_c)^2\langle\chi^2\rangle}\{\langle\chi^4\rangle h_{3,0}-\langle\chi^2\rangle^2x_c h_{4,0}+4\langle\chi^3\rangle\langle\chi\rangle x_c h_{3,1}+2\langle\chi^2\rangle\langle\chi\rangle^2(x_c^3h_{4,2}+x_c^2h_{3,2})+\langle\chi\rangle^4x_c^3h_{2,3}\}. \tag{82}$$

Here $\kappa=\langle\chi^2\rangle/\langle\chi\rangle^2$ is the anisotropy coefficient.

VI. THE ORDER PARAMETER AND THE DENSITY OF STATES AT $T=0$

A. Sign reversal of the order parameter and phase transition in scattering rate

Let us study Eqs. (9)–(11) at a temperature equal to zero. In this case the summation in Eq. (11) should be replaced by integration over the continuous variable $\epsilon_n\to\eta$ and the domain of functions $\tilde\eta(\eta)$ and $\tilde\Delta(\varphi,\eta)$ extends into the whole positive half-axis of the same variable. We are interested in the behavior of the above functions at $\eta\to0$. The values $\tilde\eta_0\equiv\tilde\eta(0)$ and $\tilde\Delta_0(\varphi)\equiv\tilde\Delta(\varphi,0)$ determine the density of states (DOS) on the Fermi surface, vanishing, if $\tilde\eta(0)=0$. Namely,

$$\nu(\epsilon,\varphi)=\frac{m(\varphi)}{\pi}\mathrm{Im}\int G^R(\epsilon,\xi,\varphi)d\xi. \tag{83}$$

The retarded Green function is the analytic continuation of the Matsubara Green function defined by Eq. (5):

$$G^R(\epsilon,\xi,\varphi)=\frac{i\tilde\epsilon_R(\epsilon)}{\tilde\epsilon_R^2(\epsilon)-\tilde\Delta_R^2(\epsilon,\varphi)-\xi^2}, \tag{84}$$

where $\tilde\epsilon_R(\epsilon)$ and $\tilde\Delta_R(\epsilon,\varphi)$ are related to the above Matsubara functions

$$\tilde\epsilon_R(i\eta)=i\tilde\eta(\eta);\quad \tilde\Delta_R(i\eta,\varphi)=\tilde\Delta(\eta,\varphi), \tag{85}$$

by virtue of the analytic continuation from the imaginary axis to the real one in the complex plane of the variable ϵ. The analyticity of the retarded Green function Eq. (84) is guaranteed provided

$$\mathrm{Im}\{\tilde\epsilon_R(\epsilon)\pm\tilde\Delta_R(\epsilon,\varphi)\}>0. \tag{86}$$

After the integration over ξ in Eq. (83) with the Green function given by Eq. (84) one finds

$$\nu_s(\epsilon,\varphi)=\nu_n(\epsilon,\varphi)\mathrm{Re}\left\{\frac{\tilde\epsilon}{\sqrt{\tilde\epsilon^2-\tilde\Delta^2(\epsilon,\varphi)}}\right\}. \tag{87}$$

In particular, the following relation is valid for $\epsilon=0$ (on the Fermi level):

$$\nu_0(\varphi)=\nu_n(\varphi)\frac{\tilde\eta_0}{\sqrt{\tilde\eta_0^2+\tilde\Delta_0^2(\varphi)}}. \tag{88}$$

Here the subscript 0 denotes $\eta=0$. Averaging the above relation over the angle φ and using the first of the AG equations (9) one gets [let us remind the reader that averaging in the AG equations (9),(10) is performed with the weight $m(\varphi)$]

$$\langle\nu_s(\varphi)\rangle=\tilde\eta_0\tau\langle\nu_n(\varphi)\rangle. \tag{89}$$

Now we are in position to prove the following three statements concerning the DOS on the Fermi surface:

Proposition 1: Let $\langle\Delta(\varphi)\rangle\neq0$ and $\Delta(\varphi)$ does not change its sign on the Fermi surface. Then $\tilde\eta_0=0$ for any τ.

Proposition 2: Let $\langle\Delta(\varphi)\rangle\neq0$ but $\Delta(\varphi)$ is a sign alternating function of the angle φ. Then $\tilde\eta_0=0$ for $1/\tau>1/\tau_\star$ but $\tilde\eta_0>0$ for $1/\tau<1/\tau_\star$. The equation for $\tau_\star$ as a functional of $\Delta(\varphi)$ is derived below and reads

$$\langle\Delta(\varphi)/(\Delta(\varphi)+1/\tau_\star)\rangle=0. \tag{90}$$

The above equation for $\tau_\star$ should be supplemented by the self-consistency and the AG equations (11,93,94) with $\tau=\tau_\star$.

Proposition 3: If the OP possesses d symmetry, then $\tilde\eta_0>0$ for any $1/\tau\leqslant1/\tau_c=2e^\gamma(\pi T_{c0})^{-1}$. Before proceeding to the proof let us remark that, according to Eq. (9), the following separation of variables takes place:

$$\tilde\Delta(\varphi,\eta)=\Delta(\varphi)+\sigma(\eta). \tag{91}$$

Let us define a function

$$F(\tilde{\eta},\sigma,\tau)\equiv 1-\frac{1}{\tau}\langle\{[\Delta(\varphi)+\sigma(\eta)]^2+\tilde{\eta}(\eta)^2\}^{-1/2}\rangle \quad (92)$$

in order to rewrite the AG equations in a more convenient form:

$$\eta=\tilde{\eta}F(\tilde{\eta},\sigma,\tau), \quad (93)$$

$$\sigma F(\tilde{\eta},\sigma,\tau)=\frac{1}{\tau}\langle\Delta(\varphi)\{[\Delta(\varphi)+\sigma]^2+\tilde{\eta}^2\}^{-1/2}\rangle. \quad (94)$$

The first proposition stems from Eqs. (92),(93),(94) straightforwardly. Indeed, let $\tilde{\eta}_0\equiv\tilde{\eta}(0)\neq 0$, then it follows from Eq. (93) that $F(\tilde{\eta}_0,\sigma_0,\tau)=0$ where $\sigma_0\equiv\sigma(\eta=0)$. The latter implies that the rhs of Eq. (94) should vanish for $\tilde{\eta}=\tilde{\eta}_0,\sigma=\sigma_0$. This is impossible, however, if $\Delta(\varphi)$ does not reverse its sign. Hence, even if $\Delta(\varphi)$ has nodes but is non-negative in the whole domain $0\leqslant\varphi<2\pi$ (the case studied in Ref. 13), $\tilde{\eta}(0)=0$.

To prove the second statement we study, first, the limit of small impurity concentration ($\langle\Delta\rangle\gg 1/\tau$). After rewriting Eq. (94) as

$$\sigma=\frac{1}{\tau}\langle[\Delta(\varphi)+\sigma]\{[\Delta(\varphi)+\sigma]^2+\tilde{\eta}^2\}^{-1/2}\rangle, \quad (95)$$

it becomes evident that $\sigma=O(1/\tau)$. (The integral over φ is always finite even at $\tilde{\eta}_0=0$. Moreover the result is always less than unity). In other words, renormalization of the OP is small. In particular, its nodes remain almost at their original locations. However, the frequency renormalization is not small due to the logarithmic divergence of the term proportional to $1/\tau$ in the rhs of Eq. (92) at $\tilde{\eta}(\eta)=0$. A closer examination of the function $\eta(\tilde{\eta})$ [inverse to $\tilde{\eta}(\eta)$], which can be extracted from Eq. (93), shows that it departs from the coordinate origin with an infinite negative derivative and, after reaching its minimum, crosses the abscissa axis at $\tilde{\eta}_0=\Delta_L'\exp(-\tau\Delta_I'/2)$. Here

$$1/\Delta_I'=1/|\Delta_1'|+1/|\Delta_2'|;$$

$$\ln\Delta_L'=\frac{|\Delta_1'|\ln|\Delta_1'|+|\Delta_2'|\ln|\Delta_2'|}{|\Delta_1'|+|\Delta_2'|}$$

and $\Delta_{1,2}'$ denote the derivatives of the order parameter at its nodes. Among the two roots, only $\tilde{\eta}_0\neq 0$ resides on the physical sheet.[25] Thus the first part of statement (2) is proved.

In the opposite, dirty limit ($1/\tau\gg T_{c0}$) the asymptotic solution of Eqs. (93),(94) reads

$$\tilde{\eta}(\eta)=\eta\left\{1+\frac{1}{\tau}[\langle\Delta\rangle^2+\eta^2]^{-1/2}\right\}, \quad (96)$$

$$\sigma(\eta)=\frac{1}{\tau}\langle\Delta\rangle[\langle\Delta\rangle^2+\eta^2]^{-1/2}. \quad (97)$$

This quasi-isotropic solution reminding us of Anderson theorem is basically the leading term in the expansion of the AG equations in small amplitude of the OP. However in the vicinity of $\eta=0$ this expansion is tricky. In fact it is possible at all due to a strong renormalization of the OP: $\sigma(\eta=0)=1/\tau$. With increasing η the value σ drops sharply reaching its asymptotics $\langle\Delta\rangle/(\tau\eta)$ at $\langle\Delta\rangle\ll\eta\ll 1/\tau$ whereas $\tilde{\eta}(\eta)$ grows rapidly in the same interval of η reaching the level of $1/\tau$. The rapid variations of these two parameters compensate each other in such a way that

$$\rho^2\equiv\tilde{\eta}^2+\langle\tilde{\Delta}^2\rangle\geqslant 1/\tau^2.$$

Assuming that the variable part of the OP $\Delta_{\rm var}(\varphi)\equiv\Delta(\varphi)-\langle\Delta\rangle$ is small with respect to scattering rate

$$1/\tau\gg|\Delta_{\rm var}(\varphi)|,$$

in the whole domain of φ one arrives at the following equation for ρ:

$$\rho-\frac{1}{\tau}\left[1+\frac{\langle\Delta_{\rm var}^2\rangle}{2\rho^2}\left(1-3\frac{\langle\tilde{\Delta}\rangle^2}{\rho^2}\right)\right]=\sqrt{\eta^2+\langle\Delta\rangle^2}. \quad (98)$$

At this point we neglect the contribution proportional to the $\langle\Delta_{\rm var}^2\rangle$ assuming that the scattering rate is large enough so that the following strong inequality holds:

$$\langle\Delta\rangle\gg\langle\Delta_{\rm var}^2\rangle\tau. \quad (99)$$

The latter conjecture can be verified straightforwardly as follows. Plugging the solution in question into Eq. (11) at $T=0$ we derive the following quasilinear equation:[28]

$$\pi\Delta(\varphi)=\ln(\tau\overline{\epsilon)}\hat{V}\Delta(\varphi)-\ln(\tau\langle\Delta\rangle)\overline{V(\varphi)}\langle\Delta\rangle. \quad (100)$$

Assuming that $V_0\ln(\tau\overline{\epsilon)}\ll 1$, where V_0 is the the maximal eigenvalue of the operator $\hat{V}$, as previously, its solution can be found explicitly:

$$\Delta(\varphi)=\langle\Delta\rangle\frac{\overline{V(\varphi)}}{\langle\overline{V}\rangle};\quad\langle\Delta\rangle=\frac{2}{\tau}(\tau\overline{\epsilon)}^{\kappa}\exp(-\pi/\langle\overline{V}\rangle). \quad (101)$$

From Eqs. (96),(97) it follows that $\tilde{\eta}_0=0$ and $\sigma(0)=1/\tau\gg\Delta$. Since the renormalized OP $\tilde{\Delta}(\varphi)$ does not have any nodes, no divergence can arise in Eq. (93). In this way self-consistency of the obtained solution is guaranteed.

Let us prove the third statement. Due to d symmetry $\sigma\equiv 0$ and only one of the AG equations remains:

$$\tilde{\eta}\left(1-\frac{1}{\tau}\left\langle\frac{1}{\sqrt{\tilde{\eta}^2+\Delta^2}}\right\rangle\right)=\eta. \quad (102)$$

Consider η as a function of $\tilde{\eta}$. According to the prescription Eq. (85) we are interested in analytic continuation of the reciprocal function $\tilde{\eta}(\eta)$. The latter must be single valued and span the half-axis $0<\eta<\infty$. It is straightforward to check that the derivative of this function is equal to $-\infty$ at $\tilde{\eta}=0$ and is equal to $+1$ at $\tilde{\eta}\to\infty$. Therefore, $\eta(\tilde{\eta})$ has at least one positive root $\tilde{\eta}_0$. Performing the analytical continuation one should start from a large positive η corresponding to the upper half-plane of energy and move continuously along the curve $\eta(\tilde{\eta})$ (imaginary axis of energy) until the value $\eta=0$ is reached. The corresponding value $\tilde{\eta}_0\equiv\tilde{\eta}(0)$ has been proven to be positive. This concludes the proof of proposition (3).

Summarizing, $\tilde{\eta}_0\neq 0$ in a clean superconductor and it vanishes when a scattering rate substantially exceeds T_{c0}.

Hence, there should exist some special value of $\tau=\tau_\star$ at which $\tilde{\eta}_0$ first turns into zero. Since $\sigma(0)|_{\tau=\tau_\star}=1/\tau_\star$, the value of $\tau_\star$ can be found as a functional of $\Delta(\phi)$ by means of Eq. (90). (This general proof has been given in our work.[10] For special situations the fact of appearance of the gap at a finite impurity concentration has been found earlier in Refs. 8, 9, and 7. We became aware of these articles after the submission of our work[10] and did not cite them. We regret this omission and use this opportunity to restore the priority.)

B. Weakly broken *d*-wave symmetry

Here we consider the case of a small admixture of the s-wave to the dominant d-wave OP. It means that the ratio $\langle\Delta\rangle^2/\langle\Delta^2\rangle$ is small. As we argued above, we expect that the gap appears at $\tau_\star$ close to the critical value for the pure d-superconductor τ_c. Therefore, we study the behavior of the OP and DOS in the vicinity of τ_c. We conjecture that $1/\tau^2\gg\langle\Delta^2\rangle\gg\langle\Delta\rangle$. More specifically, suppose that $\langle\Delta\rangle$ is of the same order or less than $\langle\Delta^2\rangle\tau$. In this situation $\tilde{\eta}^2+\sigma^2\geqslant 1/\tau^2$ and, though each of these quantities may vary rapidly in the vicinity of $\eta=0$, the sum of their squares remains almost constant. This motivates us to introduce new variables as follows:

$$\tilde{\eta}=(W+1/\tau)\sin\alpha;\quad \sigma=(W+1/\tau)\cos\alpha. \tag{103}$$

The AG equations rewritten in terms of these new variables and expanded over a small ratio $\Delta(\varphi)/(W+1/\tau)$ take the following form:

$$\sin\alpha\left(W+\frac{\langle\Delta\rangle\cos\alpha}{(1+W\tau)}-\frac{\langle\Delta^2\rangle\tau(1+3\cos2\alpha)}{4(1+W\tau)^2}\right)=\eta, \tag{104}$$

$$\cos\alpha\left(W+\frac{\langle\Delta\rangle\cos\alpha}{(1+W\tau)}-\frac{\langle\Delta^2\rangle\tau(1+3\cos2\alpha)}{4(1+W\tau)^2}\right)=\frac{\langle\Delta\rangle}{(1+W\tau)}-\frac{\langle\Delta^2\rangle\tau\cos\alpha}{(1+W\tau)^2}. \tag{105}$$

These equations can be solved approximately at small and large η and two asymptotic solutions can be matched in the crossover region. At $\eta\gg\langle\Delta^2\rangle\tau$ the variable W coincides with η in the leading approximation. Another variable $\alpha\to\pi/2$ asymptotically. Up to the first approximation one finds

$$\alpha=\pi/2-\frac{\langle\Delta\rangle}{W(1+W\tau)}, \tag{106}$$

$$\eta=W+\frac{\langle\Delta^2\rangle\tau}{2(1+W\tau)^2}. \tag{107}$$

At small $\eta\ll 1/\tau$ the variable W becomes small, and once again the solution can be found in a parametric form

$$\eta=\langle\Delta\rangle\tan\alpha-\langle\Delta^2\rangle\tau\sin\alpha, \tag{108}$$

$$W=\sin\alpha[\langle\Delta\rangle\tan\alpha-(3/2)\langle\Delta^2\rangle\tau\sin\alpha]. \tag{109}$$

It is interesting to note that whereas equation $\eta=0$ can always be satisfied with $\alpha=0$ corresponding to $\tilde{\eta}=0$ another nontrivial solution

$$\cos\alpha=\frac{\langle\Delta\rangle}{\langle\Delta^2\rangle\tau} \tag{110}$$

arises provided the lhs of the above equation is less than unity. This second solution provides a clear manifestation of the Gor'kov-Kalugin phenomenon.

As mentioned earlier two asymptotic solutions match each other in the crossover domain

$$\langle\Delta\rangle\ll\eta\ll 1/\tau.$$

The relation between two parameters within this interval reads

$$(W+\langle\Delta^2\rangle\tau/2)(\pi/2-\alpha)=\langle\Delta\rangle. \tag{111}$$

The expansion of the self-consistency equation up to the second order in the amplitude of the OP gives

$$\begin{aligned}\Delta(\varphi)=&\overline{V(\varphi)}\int_0^\epsilon\cos\alpha(\eta)d\eta+\hat{V}\Delta(\varphi)\\&\times\int_0^\epsilon\sin^2\alpha(\eta)[W(\eta)+1/\tau]^{-1}d\eta-(3/2)\hat{V}\Delta^2(\varphi)\\&\times\int_0^\epsilon\sin^2\alpha(\eta)\cos\alpha(\eta)[W(\eta)+1/\tau]^{-2}d\eta\\&+(1/2)\hat{V}\Delta^3(\varphi)\\&\times\int_0^\epsilon\sin^2\alpha(\eta)-1+5\cos^2(2\varphi)[W(\eta)+1/\tau]^{-3}d\eta.\end{aligned} \tag{112}$$

Employing Eqs. (106), (107), (108), (109), and (111) the integration over η in Eq. (112) can be performed and explicitly with the following result:

$$\begin{aligned}\Delta(\varphi)-\frac{1}{\pi}\ln(\overline{\epsilon\tau})\overline{V\Delta}(\varphi)=&\frac{1}{\pi}\overline{V(\varphi)}\left[\langle\Delta\rangle\ln\left|\frac{2\tan(\beta/2)}{\langle\Delta\rangle\tau}\right|-\frac{\langle\Delta^2\rangle\tau}{2}\times\left(\beta-\frac{\sin(2\beta)}{2}\right)\right]\\&-\frac{1}{\pi}\hat{V}\Delta\left[\langle\Delta\rangle\tau\beta-\langle\Delta\rangle^2\tau^2\left(\frac{5}{6}-\cos\beta+\frac{1}{3}\cos^3\beta\right)\right]\\&-\frac{1}{4\pi}\tau^2\hat{V}\Delta^3,\end{aligned} \tag{113}$$

where

$$\beta=\begin{cases}\pi/2 & \text{if } \zeta\geqslant 1\\ \arcsin\zeta & \text{if } \zeta<1\end{cases} \tag{114}$$

and

$$\zeta=\langle\Delta\rangle/\langle\Delta^2\rangle\tau. \tag{115}$$

If the scattering rate is close to its pair-breaking value in the purely d-wave case one can use a single-mode approximation.

$$\Delta(\varphi,\tau)=Q(\tau)\Psi_0(\varphi), \tag{116}$$

where $\Psi_0(\varphi)$ is a normalized eigenfunction of the operator $\hat{V}$ corresponding to its maximal eigenvalue. Then Eq. (113) is reduced to the following:

$$\ln(\tau/\tau_c)=-\langle\Psi_0\rangle^2\ln\left|\frac{2\tan(\beta/2)}{\langle\Psi_0\rangle Q\tau}\right|-Q\tau\langle\Psi_0\rangle\left(\frac{3}{2}\beta-\frac{\sin 2\beta}{2}\right)+Q^2\tau^2\left(\frac{5}{6}-\frac{1}{4}\langle\Psi_0^4\rangle-\cos\beta+\frac{1}{3}(\cos\beta)^3\right), \tag{117}$$

where β is defined by Eq. (114) with

$$\zeta=\frac{\langle\Psi_0\rangle}{Q\tau}. \tag{118}$$

The scattering rate $1/\tau_\star$, where the gapless regime terminates, corresponds to $\zeta=1$. Thus,

$$\ln(\tau_\star/\tau_c)=-\langle\Psi_0\rangle^2\left(\ln(2/\langle\Psi_0\rangle^2)-\frac{1}{4}\langle\Psi_0^4\rangle-\frac{3\pi}{4}+\frac{5}{6}\right). \tag{119}$$

This value separates two different regimes of the OP behavior with respect to the scattering rate. In the close vicinity of this point the amplitude Q is a linear function of $\tau-\tau_\star$:

$$Q-Q_\star=\frac{1}{\mathcal{A}\tau_\star\langle\Psi_0\rangle}\,\frac{\tau-\tau_\star}{\tau_\star}, \tag{120}$$

where $Q_\star=\langle\Psi_0\rangle/\tau_\star$ and

$$\mathcal{A}=\frac{1}{2}\langle\Psi_0^4\rangle+\frac{3\pi}{4}-\frac{2}{3}.$$

Surprisingly, despite different functional dependence of $Q(\tau)$ above and below the transition point $\tau_\star$, this amplitude is continuous together with its derivative.

Since the amplitude of the OP grows rapidly when the scattering rate diminishes, the nonlogarithmic term in Eq. (117) becomes important already in the nearest vicinity of the critical point $1/\tau_\star$. The logarithmic and quartic terms balance each other exactly at the scattering rate equal to $1/\tau_c$. Below this value of $1/\tau$ the quartic term dominates over the logarithmic one and the OP soon follows the square root law, characteristic for the purely d-wave case. At $\tau<\tau_\star$ and $(\tau_\star-\tau)/\tau_\star\gg\langle\Psi_0\rangle^2\ln\langle\Psi_0\rangle^{-2}$ the logarithmic term becomes dominant resulting in the powerlike dependence:

$$Q=\frac{\langle\Psi_0\rangle}{\tau}\left(\frac{\tau}{\tau_\star}\right)^{\langle\Psi_0\rangle^{-2}}. \tag{121}$$

In the strong-scattering regime the single-mode approximation breaks down. Fortunately, it may be replaced by a quasilinear approximation. The only term in the rhs of Eq. (113) essential in this regime is a logarithmic one. According to Eqs. (114), (115) $\beta=\pi/2$ for small τ. Let us introduce the "form factor"

$$\chi(\varphi,\tau)=\left(\hat{I}-\frac{1}{\pi}\ln(\overline{\epsilon\tau})\hat{V}\right)^{-1}\overline{V(\varphi)} \tag{122}$$

which determines the angular dependence of the OP:

$$\Delta(\varphi,\tau)=\tilde{Q}(\tau)\chi(\varphi,\tau). \tag{123}$$

One can easily derive the following expression for its amplitude [the notation for the amplitude has been modified since $\chi(\varphi,\tau)$ is not normalized]:

$$\tilde{Q}(\tau)=\frac{2}{\tau}\langle\chi\rangle^{-1}\exp\left(-\frac{\pi}{\langle\chi\rangle}\right). \tag{124}$$

The form factor may be expressed in terms of the eigenvalues V_n and the orthonormal eigenfunctions $\Psi_n(\varphi)$ of the operator $\hat{V}$ as follows:

$$\chi(\varphi)=\sum_n\left(V_n^{-1}-\frac{1}{\pi}\ln(\overline{\epsilon\tau})\right)^{-1}\langle\Psi_n\rangle\Psi_n(\varphi). \tag{125}$$

In the vicinity of $\tau=\tau_c$ one can retain the only pole corresponding to the maximal eigenvalue V_0. Then,

$$\chi(\varphi)=\frac{\pi\langle\Psi_0\rangle\Psi_0(\varphi)}{\ln(\tau_c/\tau)};\quad \langle\chi\rangle=\frac{\pi\langle\Psi_0\rangle^2}{\ln(\tau_c/\tau)};$$

$$\langle\chi^2\rangle=\frac{\pi^2\langle\Psi_0\rangle^2}{[\ln(\tau_c/\tau)]^2}. \tag{126}$$

Plugging this approximation into Eq. (124), one returns to Eq. (121). In the extra-dirty regime $\ln(\overline{\epsilon\tau})\ll V_0^{-1}$ and the form factor reaches an independent on τ asymptotics:

$$\chi(\varphi,\tau)=\overline{V(\varphi)}. \tag{127}$$

Both the mean value of the OP and its quadratic variation are exponentially small in this limit but their ratio is equal to $\langle\overline{V(\varphi)}\rangle/\sqrt{\langle\overline{V^2(\varphi)}\rangle}$.

C. Angular and energy dependence of the DOS

We concentrate our analysis of the DOS dependence on energy and angle on the most interesting situation when the OP in the clean superconductor reverses its sign. In the interval $\epsilon\ll\Delta_{\max}$ the function $\tilde{\eta}$ may be fairly approximated by the following expression:

$$\tilde{\epsilon}(\epsilon)=i\,\dot{\eta}_0\exp(-i\theta+\theta\tan\theta), \tag{128}$$

where θ is defined by equation

$$\theta\exp(\theta\tan\theta)/\cos\theta=\pi\tau\Delta_0'\epsilon/(4\eta_0). \tag{129}$$

This result matches another approximate expression valid in the interval $1/\tau\ll\epsilon$:

$$\tilde{\epsilon}=\epsilon+i\sigma(\epsilon);\quad \sigma(\epsilon)=\frac{4\epsilon}{\pi\tau}\int_{\phi_0}^{\phi(\epsilon)}\frac{d\phi}{\sqrt{\epsilon^2-\Delta^2(\phi)}}, \tag{130}$$

where ϕ_0 denotes the position of the OP node, $\phi(\epsilon)$ corresponds to the value of $\Delta(\phi)$ equal to ϵ at $\epsilon<\Delta_{\max}$ and is equal to $\phi_{\max}$ otherwise. Further we consider a simplest $\Delta(\phi)$ with only one extremum in the interval $0<\phi<\pi/4$.

Equation (130) is invalid in a close vicinity of the point $\epsilon=\Delta_{\max}$, where instead of the logarithmic singularity displayed by this equation a finite maximum of $\sigma(\epsilon)$ of the magnitude $\ln(\tau\Delta''_{\max})$ occurs. Plugging these results into Eq. (87) one calculates the DOS:

$$\frac{\nu_s(\epsilon,\phi)}{\nu_n(\epsilon,\phi)}=\begin{cases}\sigma(\epsilon)\Delta^2(\phi)/[\Delta^2(\phi)-\epsilon^2]^{3/2}, & 0<\epsilon<\Delta(\phi)\\ \epsilon/\sqrt{\epsilon^2-\Delta(\phi)^2}, & \Delta(\phi)<\epsilon.\end{cases} \tag{131}$$

Its analysis shows that the DOS is of the order of $\nu_s\sim\nu_n/\tau\Delta'_0$ for $\epsilon<\Delta(\phi)$. It possesses a peak of the width $1/\tau\Delta'_0$ and of the height $\sim(\tau\Delta'_0)^{1/2}$ in the vicinity of $\epsilon=\Delta(\phi)$. Far from the peak it decreases as $\epsilon/\sqrt{\epsilon^2-\Delta^2}$. Note that $\nu_n(\phi)$ has its own anisotropy. When $\tau\Delta'_0$ reaches the order of unity the peak becomes broader and gradually decreases.

Now we will find the DOS in the extra-dirty limit. Plugging the analytical continuation of Eqs. (96), (97) into general Eq. (87) we find that formally the ratio ν_s/ν_n depends on ϕ, but this dependence is very weak in the extra-dirty limit, since $\Delta(\phi)\ll\langle\Delta\rangle/(\tau\sqrt{\epsilon^2-\langle\Delta\rangle^2})$ as soon as $\epsilon\tau\ll 1$. In the leading order over $\Delta\tau$ this ratio coincides with its isotropic limit:

$$\frac{\nu_s(\epsilon,\phi)}{\nu_n(\epsilon,\phi)}=\mathrm{Re}\left\{\frac{\epsilon}{\sqrt{\epsilon^2-\langle\Delta\rangle^2}}\right\}. \tag{132}$$

Thus, the ratio of superconducting and normal densities of states does not depend on angle in the extra-dirty limit. This is a generalization of the Anderson theorem[30] for an anisotropic normal DOS. We also have found the criterion of validity of the Anderson isotropization: it works in the extra-dirty limit.

It is clear now that the thermodynamics of the s superconductor in the extra-dirty limit will be the same as the thermodynamics of the isotropic superconductor, though the density of states remains anisotropic.

VII. CONCLUSIONS

We developed a consistent BCS theory of the OP and DOS for essentially anisotropic superconductors with the emphasis on the d-wave superconductors and the extended s-wave superconductors with elastic impurities. Our interest in this class of superconducting materials was stimulated by experiments. The OP anisotropy, usually weak in natural low-temperature superconductors, becomes strong in high-T_c superconductors because the coherence length is small in them. When the OP reverses its sign on the Fermi surface a new class of impurity-driven phase transitions emerges at zero temperature. In d superconductors the OP vanishes above a certain value of the scattering rate $1/\tau_c$. In extended s superconductors the OP is not totally suppressed, instead the energy gap opens at some critical scattering rate $1/\tau_\star$. For a special case of an extended s-wave system close to a d-wave one, $\tau_\star$ is close to τ_c. We argued that this situation is relevant to Bi2:2:1:2 and presented a thorough description of the temperature, the energy, and the scattering rate dependence of the OP and the DOS. In particular we found that the OP is continuous together with its derivative at $\tau=\tau_\star$.

In extended s superconductors T_c decreases with the scattering rate, and its decay is governed by the power law $\tau^{\kappa-1}$. This behavior is consistent with the experiments.[3] Analyzing these experiments we were able to estimate the AC $\kappa\approx 2$ for Y-Ba-Cu-O. It would be interesting to perform analagous measurements for other copper-oxide superconductors. In the extra-dirty limit the energy gap becomes isotropic in accordance with the Anderson theorem whereas the OP remains anisotropic.

In the BCS approximation the OP is factorizable. In a pure d superconductor its shape is invariant at any available temperature and scattering rate and only its amplitude changes with these parameters. In an anisotropic s superconductor the OP shape does depend on the impurity concentration in general, but if the latter is fixed, the former does not depend on the temperature. When the impurity scattering rate is either very small or very large the OP profile becomes independent on scattering, but the two limiting shapes are different.

We studied the angular, energy and scattering rate dependence of the DOS. The latter, being finite at the Fermi level for $\tau>\tau_\star$, turns to zero for $\tau<\tau_\star$. For clean superconductors at low temperatures the DOS has a sharp peak at $\epsilon=\Delta(\varphi)$. This peak is smeared by the temperature and scattering, but it sharpens again in extended s superconductors at $\tau<\tau_\star$.

A GL equation for an arbitrary value of scattering rate and general shape of the OP in a clean system was derived. The impurity-dependent part of the coefficient a at the linear term is proportional to $1-1/\kappa$, providing an alternative opportunity to find the value κ from the experiment. The cubic term has a complicated impurity dependence. It grows rapidly in dirty superconductors ($\tau T_{c0}\ll 1$) being proportional to τ^{-1}. The gradient term is characterized by the inverse effective mass tensor $\zeta_{\alpha\beta}$ which posesses the full crystal symmetry. In the clean superconductors it is determined by the average $\langle\Delta^2 v_\alpha v_\beta\rangle$ in accordance with the old result by Gor'kov and Melik-Barkhudarov.[24] In the dirty case it decreases as $\zeta_{\alpha\beta}\propto\tau\langle v_\alpha v_\beta\rangle\langle\chi^4\rangle/\langle\chi^2\rangle$.

Since the interaction in the high-T_c superconductors is not weak it is interesting to discuss to what extent the results of our work are model dependent. It is clear that the factorizability of the OP is tightly associated with the weak coupling. On the other hand, if the OP is small, our results are stable since the interaction is effectively small. This remark applies, in particular, to the impurity-driven phase transitions, especially for an extended s superconductor close to a d superconductor, and to the powerlike tails of the critical curve $T_c(1/\tau)$, and to the energy gap isotropization.

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[1] D.A. Wollman *et al.*, Phys. Rev. Lett. **71**, 2134 (1993); D.A. Brawner and H.R. Ott, Phys. Rev. B **50**, 6530 (1994); A. Mathai, Y. Gim, R.C. Black, A. Amar, and F.C. Wellstood, Phys. Rev. Lett. **74**, 4523 (1995).
[2] A.G. Sun *et al.*, Phys. Rev. Lett. **72**, 2267 (1994).

[3] J.M. Valles *et al.*, Phys. Rev. B **39**, 11 599 (1989); M.B. Maple *et al.*, J. Alloys Compounds **181**, 135 (1992); A.G. Sun *et al.*, Phys. Rev. B **50**, 3266 (1994).
[4] D.J. Scalapino, E. Loh, and J. Hirsch, Phys. Rev. B **35**, 6694 (1987).
[5] D. Thelen, D. Pines, and J.P. Liu, Phys. Rev. B **47**, 9151 (1993).
[6] J.R. Schrieffer, Bull. Am. Phys. Soc. **41**, 74 (1996).
[7] A.A. Abrikosov, Physica C **244**, 243 (1995).
[8] D.S. Hirashima, Phys. Rev. B **50**, 10 142 (1994).
[9] R. Fehrenbacher and M.R. Norman, Physica C **235-240**, 2407 (1994).
[10] S.V. Pokrovsky and V.L. Pokrovsky, Phys. Rev. Lett. **75**, 1150 (1995).
[11] R.J. Radtke *et al.*, Phys. Rev. B **48**, 653 (1993).
[12] P. Monthoux and D. Pines, Phys. Rev. B **49**, 4261 (1994).
[13] L.S. Borkowski and P.J Hirschfeld, Phys. Rev. B **49**, 15 404 (1994).
[14] V.J. Emery, Nature (London) **370**, 598 (1994).
[15] V.J. Emery and S.A. Kivelson, Physica C **235**, 198 (1995).
[16] V.L. Pokrovsky (unpublished).
[17] S.A. Sunshine *et al.*, Phys. Rev. B **38**, 893 (1988).
[18] E.R. Ratner *et al.*, Phys. Rev. B **48**, 10 482 (1993); D.S. Dessau *et al.*, Phys. Rev. Lett. **71**, 2781 (1993); P. Aebi *et al.*, *ibid.* **72**, 2757 (1994); H. Ding *et al.*, Phys. Rev. B **50**, 1333 (1994); D.M. King *et al.*, Phys. Rev. Lett. **73**, 3298 (1994); D.S. Marshall *et al.*, Phys. Rev. B **52**, 12 548 (1995).
[19] M.A. Quijada, D.B. Tanner, R.J. Kelley, and M. Onellion, Z. Phys. B **94**, 255 (1994); Physica C **235-240**, 1123 (1994).
[20] A.A. Abrikosov and L.P. Gor'kov, Zh. Éksp. Teor. Fiz. **39**, 1781 (1960) [Sov. Phys. JETP **10**, 593 (1960)].
[21] Y. Nambu, Phys. Rev. **117**, 648 (1960).
[22] P.C. Hohenberg, Zh. Éksp. Teor. Fiz. **45**, 1208 (1963) [Sov. Phys. JETP **18**, 834 (1964)].
[23] R. Joynt, Phys. Rev. B **41**, 4271 (1990); P.I. Soinen, K. Callin, and A.J. Berlinsky, *ibid.* **50**, 13 883 (1994); A.J. Berlinsky, A.L. Fetter, M. Franz, C. Kallin, and P.I. Soinen, Phys. Rev. Lett. **75**, 2200 (1995).
[24] L.P. Gorkov and Melik-Barkhudarov, Zh. Éksp. Teor. Fiz. **45**, 1493 (1963) [Sov. Phys. JETP **18**, 1031 (1964)].
[25] The root $\tilde{\epsilon}_0$ was discovered by Gor'kov and Kalugin (Ref. 26). It has been indicated recently (Ref. 27) that the AG equations are insufficient in the immediate vicinity of $\epsilon=0$ and nonladder diagrams should be taken into account, resulting in nullification of DOS on the Fermi surface. However, the DOS grows rapidly with energy in the cited model, reaching Gor'kov-Kalugin's value at some very small scale. It is not clear whether this pseudogap may be detected experimentally.
[26] L.P. Gorkov and P.A. Kalugin, Pis'ma Zh. Éksp. Teor. Fiz. **41**, 208 (1985) [JETP Lett. **41**, 263 (1985)].
[27] A.A. Nersesyan, A.M. Tsvelik, and F. Wenger, Phys. Rev. Lett. **72**, 2628 (1994).
[28] V.L. Pokrovsky, Zh. Éksp. Teor. Fiz. **40**, 641 (1961) [Sov. Phys. JETP **13**, 447 (1961)].
[29] A.A. Abrikosov, L.P. Gor'kov, and I.E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Mechanics* (Dover, New York, 1975).
[30] P.W. Anderson, J. Phys. Chem. Solids **11**, 26 (1959).