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Density Perturbation and Preferential Coordinate Systems in an Expanding Universe

Kazuo SAKAI

Department of Physics, Faculty of Science
University of Tokyo, Tokyo

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Under general coordinate conditions, the equation for density perturbation in an expanding universe has fictitious solutions. In order to exclude the fictitious solutions automatically, we adopt coordinate systems moving with the average distribution of matter; and we obtain some coordinate conditions which provide for these systems. These contain not only the so-called Lagrangian gauge but also new coordinate conditions. Under these conditions, the equation for spatially periodic density perturbation becomes a second-order differential equation with respect to time and reduces to Bessel's differential equation when the equation of state is of the form $p/\varepsilon=\text{const}$ (p=pressure, ε=energy density).

§ 1. Introduction

The gravitational instability of a homogeneous isotropic expanding universe was first investigated by Lifshitz[1] in the general theory of relativity. He considered small perturbation of metric tensor $h_{ij}=\delta g_{ij}$ and solved the gravitational field equations under the coordinate condition $h_0{}^i=0$ $(i=0, 1, 2, 3)$. But these equations had a fictitious solution. So he had to eliminate the fictitious solution by means of an infinitesimal transformation of coordinates. Recently, Field and Shepley[2] emphasized this defect of Lifshitz's formalism. In order to remedy this defect, Nariai[3] adopted a Lagrangian gauge for spherically symmetric perturbation, that is, the perturbed radial velocity is chosen as zero; and he showed that the differential equation for the density contrast $\delta\varepsilon/\varepsilon^{(0)}$ ($\varepsilon^{(0)}$=unperturbed energy density, $\delta\varepsilon$=perturbation of energy density) is of the second order with respect to time and has no fictitious solution both when $p=0$ and when $p=\varepsilon/3$ (p=pressure, ε=energy density). In this paper, we obtain for spatially periodic perturbation, coordinate conditions similar to Nariai's Lagrangian gauge.

Bonnor[4] obtained the same result as Lifshitz's by making use of non-relativistic fluid mechanical equations when $p=0$. Irvine[5] derived general-relativistic fluid mechanical equations imposing the generalized de Donder coordinate condition. In this paper, we derive similar equations without imposing any coordinate conditions.

In § 2, Friedmann's solution for the homogeneous isotropic model of the universe is briefly described. In § 3, we derive fluid mechanical equations. In § 4, the properties of perturbational quantities for the transformation of coordinates are examined, and the coordinate conditions appropriate to describe the density perturbation in an expanding universe are found. Under these coordinate conditions, we solve in § 5 the fluid mechanical equations for the density perturbation, supposing an equation of state $p/\varepsilon=$const. In § 6, solutions for the density contrast under various coordinate conditions are shown in the Table. The reason for the appearance of fictitious solutions is discussed, and the relation of our coordinate conditions to Lifshitz's is examined. In § 7, some concluding remarks are given.

We treat only the flat space model of the universe for the simplicity of analysis.

Notation

We use the same notation as that in the book by Landau and Lifshitz.[6)]

The square of the world interval ds is written in the form

$$ds^2=-g_{ik}dx^i dx^k \tag{1·1}$$

summed over i, $k=0$, 1, 2, 3.

The gravitational field equation is written in the form

$$R_i^k=\kappa(T_i^k-\delta_i^k T/2), \tag{1·2}$$

where R_i^k is the Ricci tensor, δ_i^k is the unit tensor, κ is Einstein's gravitational constant, T_i^k is the energy-momentum tensor, and $T=T_i^i$. For the ideal fluid, T_i^k is given by

$$T_i^k=(\varepsilon+p)u_i u^k+\delta_i^k p, \tag{1·3}$$

where ε is the energy density, p is the pressure, and u^i is the 4-velocity.

From the contracted Bianchi identities, we obtain "fluid mechanical equations"

$$T^k_{i;k}=(1/\sqrt{-g})\partial(\sqrt{-g}\,T_i^k)/\partial x^k-(T^{kl}/2)(\partial g_{kl}/\partial x^i)=0, \tag{1·4}$$

where a semicolon before an index denotes the covariant derivative and $g=\det(g_{ij})$.

§ 2. Homogeneous isotropic universe

Here we consider as the unperturbed state a universe uniformly filled with matter. As is well known, this universe is classified into two models: closed model and open model. In the following, we deal with only the critical model with a flat space for the simplicity of analysis.

As our reference system, we choose a system moving with matter (i.e. comoving reference system), then the interval ds for this model can be written in the form

$$ds^2=-g_{ik}^{(0)}dx^i dx^k=a(\eta)^2[d\eta^2-(dx^2+dy^2+dz^2)]. \tag{2·1}$$

The relation between η and the proper time τ is

$$cd\tau = a(\eta)\, d\eta\,. \tag{2·2}$$

From the gravitational equations, we obtain the equations for the cosmic scale factor $a(\eta)$

$$3\dot{a}^2/a^4 = \kappa\varepsilon^{(0)}, \tag{2·3}$$

$$(\dot{a}^2 - 2\ddot{a}a)/a^4 = \kappa p^{(0)}, \tag{2·4}$$

where dots represent the derivative with respect to the time, $\varepsilon^{(0)}$ is the energy density and $p^{(0)}$ is the pressure in the unperturbed state.

In order to solve Eqs. (3·2) and (2·4), we suppose the equation of state

$$3p/\varepsilon = (1-\nu)/(1+\nu). \qquad (\nu = \text{const},\ 0 \leq \nu \leq 1) \tag{2·5}$$

Here $\nu = 0$ corresponds to $p = \varepsilon/3$ and $\nu = 1$ to $p = 0$. Then we obtain the solution of Eqs. (2·3) and (2·4),

$$a(\eta) = \text{const}\ \eta^{1+\nu}, \qquad \dot{a}/a = (1+\nu)/\eta\,, \qquad \varepsilon^{(0)} = \text{const}\ \eta^{-(4+2\nu)}, \tag{2·6}$$

and the relation of η to the proper time τ,

$$\eta = \text{const}\ \tau^{1/(2+\nu)}. \tag{2·7}$$

Hubble's constant H is written in the form

$$H = (da/d\tau)/a = \frac{1+\nu}{2+\nu} \cdot \frac{1}{\tau}\,. \tag{2·8}$$

§ 3. Fluid mechanical equations and the equation for the gravitational potential

Now we consider a weak gravitational field due to perturbed distribution of matter in an expanding universe. We write the metric tensor, the energy density, the pressure and the 4-velocity in the form

$$\begin{cases} g_{ik} = g_{ik}^{(0)} + h_{ik}\,, \\ \varepsilon = \varepsilon^{(0)} + \delta\varepsilon\,, \\ p = p^{(0)} + \delta p\,, \\ u^i = u^{i(0)} + \delta u^i\,, \end{cases} \tag{3·1}$$

where h_{ik}, $\delta\varepsilon$, δp and δu^i are perturbations. The world interval ds is written with the perturbed metric in the form

$$\begin{aligned} ds^2 &= -g_{ik} dx^i dx^k \\ &= a(\eta)^2 [(1+h_0{}^0) d\eta^2 - h_0{}^\alpha d\eta dx^\alpha - (\delta_\alpha{}^\beta + h_\alpha{}^\beta) dx^\alpha dx^\beta], \end{aligned} \tag{3·2}$$

summed over α, $\beta = 1, 2, 3$. Here $h_i{}^k = g^{kl(0)} h_{il}$.

In order to examine the behavior of the density perturbation, it is sufficient to calculate only the fluid mechanical equations and the R_{00} component of the gravitational field equation instead of all components. Before we derive these equations, we define the velocity of fluid v^α by the equation

$$v^\alpha = dx^\alpha/d\eta . \qquad (\alpha = 1, 2, 3) \tag{3·3}$$

The v^α is related to $u^\alpha = \delta u^\alpha$ by the expression

$$u^\alpha = (v^\alpha/a)\,[(1+h_0{}^0) - h_0{}^\beta v^\beta - (\delta_\beta{}^\gamma + h_\beta{}^\gamma)\,v^\beta v^\gamma]^{-1/2}. \tag{3·4}$$

To the first order regarding the small quantities $h_i{}^k$ and v^α, we get the following expressions:

$$g = -a^8(1+h), \qquad (h = h_0{}^0 + h_1{}^1 + h_2{}^2 + h_3{}^3) \tag{3·5}$$

$$\begin{aligned} u^0 &= (1-h_0{}^0/2)/a, & u_0 &= -a(1+h_0{}^0/2),\\ u^\alpha &= v^\alpha/a, & u_\alpha &= a(v^\alpha + h_0{}^\alpha). \end{aligned} \tag{3·6}$$

Substituting Eqs. (3·5) and (3·6) into the fluid mechanical equation (1·4) and picking out the main terms and the terms of the next order, we obtain the equation of continuity and equation of motion,

$$(1/a^3)\,\partial(a^3\varepsilon)/\partial\eta + \partial[(\varepsilon+p)\,v^\gamma]/\partial x + [(\varepsilon+p)/2]\,\partial(h-h_0{}^0)/\partial\eta + 3p(\dot{a}/a) = 0, \tag{3·7}$$

$$(1/a^4)\,\partial[a^4(\varepsilon+p)\,(v^\alpha+h_0{}^\alpha)]/\partial\eta + \partial[(\varepsilon+p)\,v^\gamma(v^\alpha+h_0{}^\alpha)]/\partial x^\gamma + \partial p/\partial x^\alpha + [(\varepsilon+p)/2]\,\partial h_0{}^0/\partial x^\alpha = 0. \tag{3·8}$$

From the R_{00} component of the gravitational field equation, we get the equation

$$(1/2)\,\partial^2(h_0{}^0-h)/\partial\eta^2 + \partial^2 h_0{}^\gamma/(\partial\eta\partial x^\gamma) + (1/2)\,\partial^2 h_0{}^0/(\partial x^\gamma)^2 + (\dot{a}/a)\,\partial h_0{}^\gamma/\partial x^\gamma + (\dot{a}/a)\,\partial(2h_0{}^0 - h/2)/\partial\eta + 3h_0{}^0 d(\dot{a}/a)/d\eta = (3/2)\,(\delta\varepsilon/\varepsilon^{(0)})\,(1+3\delta p/\delta\varepsilon)\,(\dot{a}/a)^2. \tag{3·9}$$

Making use of τ instead of η, we rewrite Eqs. (3·7)~(3·9) in the form

$$(1/a^3)\,\partial(a^3\varepsilon)/\partial\tau + \nabla_\gamma[(\varepsilon+p)\,v^\gamma] + [(\varepsilon+p)/2]\,\partial(h-h_0{}^0)/\partial\tau + p\,(da^3/d\tau)/a^3 = 0, \tag{3·10}$$

$$(1/a^4)\,\partial[a^4(\varepsilon+p)\,(v^\alpha+h_0{}^\alpha)]/\partial\tau + \nabla_\gamma[(\varepsilon+p)\,v^\gamma(v^\alpha+h_0{}^\alpha)] + \nabla_\alpha p + [(\varepsilon+p)/2]\,\nabla_\alpha h_0{}^0 = 0, \tag{3·11}$$

$$(1/2)\,\partial^2(h_0{}^0-h)/\partial\tau^2 + \nabla_\gamma(\partial h_0{}^\gamma/\partial\tau) + \nabla^2 h_0{}^0/2 + (a'/a)\,\nabla_\gamma h_0{}^\gamma + (a'/a)\,\partial(5h_0{}^0/2-h)/\partial\tau + 3h_0{}^0(a''/a) = (\kappa/2)\,(\varepsilon+3p-\varepsilon^{(0)}-3p^{(0)}), \tag{3·12}$$

where $\nabla_\alpha = (1/a)\;(\partial/\partial x^\alpha)$, $a' = da/d\tau$, and we put the velocity of light $c=1$.

Equations (3·10) and (3·11) are interpreted in terms of the classical fluid mechanical concepts. In these equations $\varepsilon+p$ plays the role of the density of inertial mass in the non-relativistic mechanics. In Eq. (3·10), the first term represents the rate of increase of energy density, the second term the energy flow, the third term the rate of the decrease of the energy density caused by the increase of volume due to the gravitational field, and the last term the work done by the pressure. Namely, Eq. (3·10) expresses the energy conservation law. In Eq. (3·11), $(\varepsilon+p)\times(v^\alpha+h_0{}^\alpha)$, $h_0{}^0/2$ and $(p+\varepsilon)\times\nabla_\alpha h_0{}^0/2$ correspond to the classical momentum, the gravitational potential and the gravitational force, respectively. Equation (3·11), however, represents the conservation law of the angular momentum $a(\varepsilon+p)\times(v^\alpha+h_0{}^\alpha)$ rather than of the momentum.

Moreover, if we choose the coordinate condition $h_0{}^\alpha=0$ and $h-h_0{}^0=0$, in the limits $p\ll\varepsilon$, $L\tau\ll1$ and $L\tau_0\ll1$ where L and τ_0 are a characteristic length and a characteristic time of inhomogeneity, respectively, Eqs. (3·10), (3·11) and (3·12) reduce to the equations[7]

$$(1/a^3)\partial(a^3\rho)/\partial\tau+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})=0\,, \tag{3·13}$$

$$(1/a^4)\partial(a^4\rho\boldsymbol{v})/\partial\tau+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}\boldsymbol{v})+\boldsymbol{\nabla}p+\rho\boldsymbol{\nabla}h_0{}^0/2=0\,, \tag{3·14}$$

$$\nabla^2h_0{}^0/2=(\kappa/2)(\rho-\rho^{(0)}), \tag{3·15}$$

where ρ is the density of matter, $\boldsymbol{\nabla}=(\nabla_1,\nabla_2,\nabla_3)$, and $\boldsymbol{v}=(v^1,v^2,v^3)$.

§ 4. Preferential coordinate systems

In order to solve general relativistic equations, we must impose a coordinate condition. The coordinate condition, in general, does not uniquely determine the coordinate system. There are the transformations of coordinates which conserve the imposed coordinate condition.

In the following, we show that fictitious density and velocity inhomogeneity is produced on account of this arbitrariness of the coordinate system, and that if we impose suitable coordinate conditions, we can limit the arbitrariness and eliminate the fictitious inhomogeneity automatically.

We perform the transformation of coordinates,

$$x^i\longrightarrow x^{i\prime}=x^{i\prime}(x). \tag{4·1}$$

The quantities after the transformation are represented by primes. Since $\varepsilon(x)$, $u^i(x)$ and $g_{ik}(x)$ are scalar, vector and tensor, respectively, they are transformed as follows:

$$\varepsilon'(x')=\varepsilon(x), \tag{4·2}$$

$$u^{i\prime}(x')=(\partial x^{i\prime}/\partial x^l)u^l(x), \tag{4·3}$$

$$g_{ik}(x)=(\partial x^{l\prime}/\partial x^i)(\partial x^{m\prime}/\partial x^k)g'_{lm}(x'). \tag{4·4}$$

In order to examine the properties of the small perturbations for the transformation of coordinates, we consider the infinitesimal transformation

$$\eta\longrightarrow\eta'=\eta+\eta^0,\qquad x^\alpha\longrightarrow x^{\alpha\prime}=x^\alpha+\xi^\alpha. \tag{4·5}$$

where (η, x^α) is the coordinate systems moving with the average distribution of matter and is defined by means of Eq. (2·1). From the transformation laws (4·2)$\sim$(4·4), we obtain the expressions

$$\delta\varepsilon'(x) = \delta\varepsilon(x) - \dot{\varepsilon}^{(0)}\eta^0, \tag{4·6}$$

$$u^{\alpha\prime}(x) = u^\alpha(x) + (\partial\xi^\alpha/\partial\eta)\, u^0(x), \tag{4·7}$$

$$h_0^{0\prime}(x) = h_0^0(x) - 2(\partial\eta^0/\partial\eta) - 2(\dot{a}/a)\eta^0, \tag{4·8}$$

$$h_0^{\alpha\prime}(x) = h_0^\alpha(x) + \partial\eta^0/\partial x^\alpha - \partial\xi^\alpha/\partial\eta\,, \tag{4·9}$$

$$h_\alpha^{\beta\prime}(x) = h_\alpha^\beta(x) - \partial\xi^\alpha/\partial x^\beta - \partial\xi^\beta/\partial x^\alpha - 2(\dot{a}/a)\eta^0\delta_\alpha^\beta. \tag{4·10}$$

Now we consider the infinitesimal transformation of coordinates in the unperturbed isotropic universe, where the density and velocity inhomogeneity is zero in the comoving reference system (η, x^α). If the transformation (4·5) is performed, then the inhomogeneity

$$\delta\varepsilon = -\dot{\varepsilon}^{(0)}\eta^0, \qquad u^\alpha = (1/a)\,\partial\xi^\alpha/\partial\eta$$

is produced. This inhomogeneity is, of course, a fictitious one. In § 6, we illustrate this fact in the case of Lifshitz's coordinate condition.

If we impose the coordinate conditions which permit only the infinitesimal transformation of coordinates with vanishing η^0 and $\partial\xi^\alpha/\partial\eta$, we can eliminate this fictitious inhomogeneity automatically. So, these conditions are preferential in comparison with those which allow nonvanishing η^0 or $\partial\xi^\alpha/\partial\eta$. They provide for families of the coordinate systems moving with the average distribution of matter.

It is very easy to see that the coordinate conditions

$$\text{a)}\quad h_0^\alpha = 0 \quad \text{and} \quad \partial u^\gamma/\partial x^\gamma = 0\,, \tag{4·11}$$

$$\text{b)}\quad u^\alpha = 0 \quad \text{and} \quad \partial h_0^\gamma/\partial x^\gamma = 0 \tag{4·12}$$

and

$$\text{c)}\quad u_\alpha = 0 \quad \text{and} \quad h - h_0^0 = 0 \tag{4·13}$$

satisfy the above request to the preferential coordinate conditions. Here spatially periodic perturbations are supposed. The condition a), for example, permits only the infinitesimal transformations of coordinates which satisfy the equations

$$\partial\eta^0/\partial x^\alpha - \partial\xi^\alpha/\partial\eta = 0\,, \qquad \partial^2\xi^\alpha/(\partial x^\alpha\partial\eta) = 0\,.$$

Therefore, η^0 and $\partial\xi^\alpha/\partial\eta$ have to vanish.

The condition b) is a Lagrangian gauge, but a) and c) are not Lagrangian.

In the next section, we show that the conditions a), b) and c) give the same linearized equation for the density contrast, which is a second-order differential equation with respect to time. Therefore, so far as we treat the density perturbation, a), b) and c) are preferential to the same extent.

§ 5. The density perturbation under our preferential conditions

We consider spatially periodic perturbation such as $\propto \exp(ik_\gamma x^\gamma)$, supposing the equation of state (2·5) and imposing the coordinate condition a), b) or c). Moreover, we linearize Eqs. (3·7)∼(3·9) regarding the small perturbations. Then we obtain the equation for the density contrast $\delta\varepsilon/\varepsilon^{(0)} = K\exp(ik_\gamma x^\gamma)$: for $0 \leq \nu < 1$ (ν is defined by Eq. (2·5)),

$$d^2K/d\zeta^2 + (2\nu/\zeta)(dK/d\zeta) + [1 - 2(1+2\nu)/\zeta^2]K = 0\,, \tag{5·1}$$

and for $\nu = 1$,

$$d^2K/d\eta^2 + (2/\eta)(dK/d\eta) - 6K/\eta^2 = 0\,, \tag{5·2}$$

where $\zeta = \omega\eta$, $\omega^2 = [(1-\nu)/(1+\nu)](a^2k^2/3)$ and $k^2 = k_\gamma k^\gamma$.

At first, we solve the differential equation (5·1). Put $K = \zeta^{1/2-\nu}Z$ and Eq. (5·1) becomes

$$d^2Z/d\zeta^2 + Z/\zeta + [1 - (\nu+3/2)^2/\zeta^2]Z = 0\,. \tag{5·3}$$

This equation is Bessel's differential equation and has the solution $Z = C_1J_{-(\nu+3/2)}(\zeta) + C_2J_{(\nu+3/2)}(\zeta)$. Therefore, we have

$$K = \zeta^{1/2-\nu}[C_1J_{-(\nu+3/2)}(\zeta) + C_2J_{(\nu+3/2)}(\zeta)]. \tag{5·4}$$

When $\omega\eta \ll 1$, this becomes

$$K \propto C_1\eta^{-(2\nu+1)} + C_2\eta^2. \qquad (0 \leq \nu < 1) \tag{5·5}$$

Next, we solve Eq. (5·2). We obtain the solution for all the values of η

$$K \propto C_1\eta^{-3} + C_2\eta^2. \qquad (\nu = 1) \tag{5·6}$$

When $p = 0$ ($\nu = 1$), this solution agrees with Lifshitz's but when $p = \varepsilon/3$ ($\nu = 0$), the first term η^{-1} of the solution (5·5) does not agree with Lifshitz's result. The relation of our result to Lifshitz's is shown in the next section.

§ 6. Density perturbation under various coordinate conditions

We can solve the linearized form of Eqs. (3·7)∼(3·9) for the density perturbation, imposing various coordinate conditions. The results of the calculation are shown in the table in the cases $p = 0$ and $p = \varepsilon/3$. In this table, the figures with bracket imply the exponents μ of the power series which are of the form $\sum_{n=0}^{\infty} C_n\eta^{\mu+n}$, the figures without bracket represent solutions composed only of one term, and the symbol stars denote fictitious solutions.

Equations for the density contrast are in general fourth-order*) differential equations with respect to the time, which have two physical solutions and two fictitious ones. A fictitious solution is the solution that can be eliminated by

*) Arons and Silk[8] have shown that under the harmonic condition, the equation is of the sixth-order.

means of an appropriate transformation of coordinates. Conversely, we can obtain the fictitious solution by means of the inverse transformation from the unperturbed state. For example, we consider Lifshitz's coordinate condition $h_0{}^\alpha=0$ and $h_0{}^0=0$ in the case $p=\varepsilon/3$. We suppose (η, x^1, x^2, x^3) as the comoving coordinate system in the unperturbed state. Then we transform it into a new coordinate system $(\eta', x^{1\prime}, x^{2\prime}, x^{3\prime})$, where $\eta'=\eta+\eta^0$, $x^{\alpha\prime}=x^\alpha+\xi^\alpha$. Under the condition $h_0{}^i=0$, η^0 and ξ^α satisfy the equations

$$\partial\eta^0/\partial\eta+\eta^0/\eta=0\,, \qquad \partial\eta^0/\partial x^\alpha-\partial\xi^\alpha/\partial\eta=0\,. \tag{6·1}$$

These equations have the solution $\eta^0=C/\eta$, where C is a constant. Therefore, the fictitious solution $K=-\eta^0\dot{\varepsilon}^{(0)}/\varepsilon^{(0)}=4C/\eta^2$ is obtained from Eq. (4·6).

Table. The solutions for the density contrast under various coordinate conditions.

coordinate conditions		$p=0$				$p=\varepsilon/3$			
$h_0{}^\alpha=0$	$h-h_0{}^0=\lambda h_0{}^0$ $(\lambda\neq0,\ \lambda\neq-3)$	$(-3+6/\lambda)^*$	(-1)	$(0)^*$	(2)	$(-2+3/\lambda)^*$	(1)	$(0)^*$	(2)
	$\lambda=0$		(-1)	$(0)^*$	(4)		(1)	$(0)^*$	(4)
	$\lambda=-3$	$(-5)^*$	(-1)	$(0)^*$	(2)	$(-3)^*$	(3)	$(0)^*$	(2)
	$h_0{}^0=0$	-3^*	-3		2	-2^*	(1)		(2)
	$\partial u^\alpha/\partial x^\alpha=0$		-3		2		(-1)		(2)
$u^\alpha=0$	$h_0{}^0=0$	-3^*	-3		2	$(-2)^*$	(1)		(2)
	$\lambda=0$			$(0)^*$				$(0)^*$	
	$\partial h_0{}^\alpha/\partial x^\alpha=0$		-3		2		(-1)		(2)
$u_\alpha=0$	$\lambda=0$		-3		2		(-1)		(2)
	$\partial h_0{}^\alpha/\partial x^\alpha=0$		-3		2		(-1)		(2)

Next, we examine the relation of our coordinate condition (4·11) to Lifshitz's when $p=\varepsilon/3$. Under the condition (4·11), we have the solution

$$K=C_1[\sin\zeta+(\cos\zeta)/\zeta]+C_2[(\sin\zeta)/\zeta-\cos\zeta], \tag{6·2}$$

$$h_0{}^0=-K/2\,. \tag{6·3}$$

We perform the infinitesimal transformation which satisfies the equations

$$K/2+2\partial\eta^0/\partial\eta+2\eta^0/\eta=0\,, \qquad \partial\eta^0/\partial x^\alpha-\partial\xi^\alpha/\partial\eta=0\,. \tag{6·4}$$

As a result of this transformation, we obtain the Lifshitz formalism. Equation (6·4) has the solution

$$\eta^0=C_3/(\omega\zeta)-(1/4\omega\zeta)\int_0^\zeta \zeta' K d\zeta'. \tag{6·5}$$

From Eq. (4·6), we get the transformed density contrast

$$K = C_1[-2(\sin\zeta)/\zeta^2 + 2(\cos\zeta)/\zeta + \sin\zeta] + C_2[-2(1-\cos\zeta)/\zeta^2 + 2(\sin\zeta)/\zeta - \cos\zeta] + C_3(4/\zeta^2). \quad (6\cdot6)$$

When $\zeta \ll 1$, this becomes

$$K \propto C_1\eta + C_2\eta^2 + C_3/\eta^2. \quad (6\cdot7)$$

The solutions $C_1\eta$ and $C_2\eta^2$ agree with Lifshitz's. The solution C_3/η^2 is a fictitious solution caused by the transformation (6·5).

§ 7. Concluding remarks

As we have shown, it is natural and suitable to adopt the coordinate systems moving with the average distribution of matter in order to deal with the density perturbation in an expanding universe. We have obtained some of the coordinate conditions which provide for these systems; they contain not only the so-called Lagrangian gauge but also new conditions. Under these coordinate conditions, the equation for the density perturbation becomes a second-order differential equation, and therefore, fictitious solutions are eliminated automatically. Moreover, we have shown that our coordinate conditions are transformed into Lifshitz's by means of an appropriate infinitesimal transformation of coordinates.

Though we have solved the equation for the density perturbation, supposing the equation of state $p/\varepsilon = \text{const}$, it is desirable to solve it by making use of a more realistic equation of state. We can use fluid mechanical equations to treat rotational perturbation and to discuss non-linear growth rate of density inhomogeneity.[7]

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