The Annals of Probability
1985, Vol. 13, No. 2, 572–582

DEPENDENCE BY TOTAL POSITIVITY

BY MEI-LING TING LEE[1]

Boston University

A multivariate generalization of Shaked's bivariate families which are dependent by total positivity (DTP) is introduced. Some interrelationships and inequalities are generalized. Monotonicity properties of conditional hazard rate and mean residual life functions of some multivariate DTP families are investigated. Relationships with other positive dependence concepts are given.

0. Introduction. Shaked (1977a) introduced a family of concepts of dependence by total positivity (DTP) for bivariate distribution functions which was motivated by reliability concepts. He showed the equivalence of some of these concepts with some other concepts of positive dependence and characterized some of them by notions from reliability theory. At the time of his investigation, multivariate dependence concepts and multivariate reliability concepts were not fully understood and so the investigation was restricted to the bivariate case.

Although Shaked's definitions have straightforward extensions to the multivariate case, proofs of results analogous to those of Shaked require more care. In this paper these multivariate extensions are attempted. Up-to-date concepts of multivariate dependence are considered and many related questions are studied. In Section 1 we review the bivariate case. The appropriate multivariate DTP families are introduced in Section 2. The major result is Proposition 2.4 which shows that DTP of a fixed order implies DTP of higher orders. Multivariate analogs of Yanagimoto's (1972) bivariate families are introduced in Section 3 and their relationship with the DTP families is discussed using some of the positive dependence concepts of Block, Savits and Shaked (1982). In Section 4 relations with concepts from reliability theory are discussed. Various examples are given in Section 5.

1. Shaked's bivariate family. Define, for $s > 0$

$$\gamma^{(s)}(t) = \begin{cases} (-t)^{s-1}/\Gamma(s), & t \le 0 \\ 0, & t > 0. \end{cases}$$

For $s \ge 1$, $\gamma^{(s)}(x - y)$ is TP_2 in x and y (see Karlin, 1968).

Received October 1982; revised July 1984.

[1] Supported by ONR Contract N00014-76-C-0837 when the author was a research assistant at the University of Pittsburgh.

AMS 1980 *subject classifications*. Primary 60K10, 62H20; secondary 62N05.

Key words and phrases. Positive dependence, total positivity, conditional mean residual life function.

Let (X, Y) be a random vector with joint density $f(x, y)$, and consider

$$\Psi_{m,n}(x, y) = \int_y^\infty \int_{y_{n-1}}^\infty \cdots \int_{y_1}^\infty \int_x^\infty \int_{x_{m-1}}^\infty \cdots \int_{x_1}^\infty f(x_0, y_0)\, dx_0 \cdots dx_{m-1} dy_0 \cdots dy_{n-1}$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \gamma^{(m)}(x - x_0)\gamma^{(n)}(y - y_0)\, dF(x_0, y_0).$$

For the case $m = 0$, $n = 0$, define $\Psi_{0,0}(x, y) = f(x, y)$. Also define $\Psi_{0,n}(x, y) = \int_{-\infty}^\infty \gamma^{(n)}(y - y_0) f(x)\, dG(y_0 \mid x)$, where $G(y \mid x)$ is the conditional distribution function of Y given $X = x$, and f is the density function of X.

DEFINITION 1.1. For $m \geq 0$, $n \geq 0$, the random vector (X, Y), or its distribution function F, is said to be *dependent by total positivity of order* (m, n) (denoted by DTP(m, n)), if $\Psi_{m,n}(x, y)$ is TP_2 in x and y.

2. Dependence by total positivity of multivariate distributions. In this section we introduce the multivariate definition of dependence by total positivity. We give conditions under which a subset of DTP random variables is DTP, and we also notice that the joint distribution of two independent sets of DTP random variables is again DTP. The translation invariant property holds as in the biviarate case. Some interrelationships and inequalities are also generalized. A fundamental property is given in Proposition 2.4.

Let $X_1, \cdots, X_n$ be random variables with joint distribution function F. Let $\gamma^{(k)}(t)$ be defined as in Section 1.

For $k_i > 0$, define the n fold integral $\Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n)$ by

$$\Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n) = \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \gamma^{(k_1)}(x_1 - t_1) \cdots \gamma^{(k_n)}(x_n - t_n)\, dF(t_1, \cdots, t_n)$$

and define $\Psi_{0,\cdots,0}(x_1, \cdots, x_n) = f(x_1, \cdots, x_n)$ if the joint density exists.

Also define $\Psi_{0,\cdots,0,k_{i+1},\cdots,k_n}(x_1, \cdots, x_n)$ to be the $(n - i)$ fold integral

$$\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \gamma^{(k_{i+1})}(x_{k_{i+1}} - t_{i+1}) \cdots \gamma^{(k_n)}(x_{k_n} - t_n) g_i(x_1, \cdots, x_i) \cdot dF(t_{i+1}, \cdots, t_n \mid x_1, \cdots, x_i)$$

where g_i is the joint density of $X_1, \cdots, X_i$, and $F(t_{i+1}, \cdots, t_n \mid x_1, \cdots, x_i)$ is the conditional distribution of $X_{i+1}, \cdots, X_n$ given $X_1 = x_1, \cdots, X_i = x_i$, for $k_{i+1} > 0, \cdots, k_n > 0$. Similarly, we can define $\Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n)$ with any subset of $\{k_1, \cdots, k_n\}$ consisting of zeros.

DEFINITION 2.1. $X_1, \cdots, X_n$ is said to be *dependent by total positivity with degree* $(k_1, \cdots, k_n)$, denoted by DTP$(k_1, \cdots, k_n)$, if $\Psi_{k_1,\cdots,k_n}(x_1, \cdots, x_n)$ is TP_2 in pairs of $x_1, \cdots, x_n$.

For example, if (X_1, X_2, X_3) is DTP(1, 1, 1), then $\int_{x_1}^{\infty}\int_{x_2}^{\infty}\int_{x_3}^{\infty} dF(t_1, t_2, t_3)$ is TP_2 in pairs of x_1, x_2 and x_3.

We have the following properties for multivariate DTP families.

PROPOSITION 2.2. *Assume* $\mathbf{X} = (X_1, \cdots, X_n)$ *is* $DTP(k_1, \cdots, k_n)$ *with* $k_i = 0$ *or* 1 *for some* $1 \le i \le n$, *then* $\mathbf{X}^{(i)} = (X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$ *is* $DTP(k_1, \cdots, k_{i-1}, k_{i+1}, \cdots, k_n)$.

PROOF. Assume first that $i = 1$, $k_1 = 0$, and $k_j > 0$ for $j = 2, \cdots, n$. Then by the definition of DTP$(0, k_2, \cdots, k_n)$ we have

$$\Psi_{0,k_2,\cdots,k_n}(x)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \gamma^{(k_2)}(x_2 - t_2) \cdots \gamma^{(k_n)}(x_n - t_n) g_i(x_1)\, dF(t_2, \cdots, t_n \mid x_1)$$

is TP_2 in pairs of $x_1, x_2, \cdots, x_n$. Hence by Theorem 5.1, page 123 of Karlin (1968),

$$\Psi_{k_2,\cdots,k_n}(x_2, \cdots, x_n) = \int_{-\infty}^{\infty} \Psi_{0,k_2,\cdots,k_n}(x_1, x_2, \cdots, x_n)\, dx_1$$

is TP_2 in pairs of $x_2, \cdots, x_n$, i.e. $(X_2, \cdots, X_n)$ is DTP$(k_2, \cdots, k_n)$. Now assume $k_1 = 1$, and $k_j > 0$ for $j = 2, \cdots, n$, then by the definition of DTP$(1, k_2, \cdots, k_n)$

$$\Psi_{1,k_2,\cdots,k_n}(\mathbf{x})$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{x_1}^{\infty} \gamma^{(k_2)}(x_2 - t_2) \cdots \gamma^{(k_n)}(x_n - t_n)\, dF(t_1, \cdots, t_n)$$

is TP_2 in pairs of $x_1, x_2, \cdots, x_n$. Thus let x_1 tend to $-\infty$,

$$\Psi_{k_2,\cdots,k_n}(x_2, \cdots, x_n) = \lim_{x_1 \to \infty} \Psi_{1,k_2,\cdots,k_n}(x_1, \cdots, x_n)$$

is TP_2 in pairs of $x_2, \cdots, x_n$, i.e. $(X_2, \cdots, X_n)$ is DTP$(k_2, \cdots, k_n)$. The proof is similar if $i = 2, 3, \cdots, n$, or if there are more than one i such that $k_i = 0$ or 1.

PROPOSITION 2.3. *Let* $(X_1, \cdots, X_m)$ *be independent of* $(Y_1, \cdots, Y_n)$. *Assume* $(X_1, \cdots, X_m)$ *is* $DTP(k_1, \cdots, k_m)$, *and* $(Y_1, \cdots, Y_n)$ *is* $DTP(\ell_1, \cdots, \ell_n)$. *Then* $(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ *is* $DTP(k_1, \cdots, k_m, \ell_1, \cdots, \ell_n)$.

PROOF. This follows from the fact that

$$\Psi_{k_1,\cdots,k_m,\ell_1,\cdots,\ell_n}(x_1, \cdots, x_m, y_1, \cdots, y_n)$$
$$= \Psi_{k_1,\cdots,k_m}(x_1, \cdots, x_m)\Psi_{\ell_1,\cdots,\ell_n}(y_1, \cdots, y_n).$$

Bivariate DTP families of distributions were shown to be closed under linear transformations. A similar result holds for multivariate DTP families; that is if $(X_1, \cdots, X_n)$ is DTP$(k_1, \cdots, k_n)$, then $(a_1X_1 + b_1, \cdots, a_nX_n + B_n)$ is DTP$(k_1, \cdots, k_n)$, for any $a_1 > 0, \cdots, a_n > 0$, and any $b_1, \cdots, b_n$ real.

By using Theorem 5.1 of Karlin (1968), we have the following generalization of Proposition 4.1 of Shaked (1977a).

PROPOSITION 2.4. *Assume* $(X_1, \cdots, X_n)$ *is* $DTP(k_1, \cdots, k_n)$, $k_i \geq 0$ *for* $i = 1, \cdots, n$. *Then* $(X_1, \cdots, X_n)$ *is* $DTP(s_1, \cdots, s_n)$ *for any* $s_i \in \{k_i\} \cup [k_{i+1}, \infty)$, $i = 1, \cdots, n$.

PROOF. For $u > 0$, $v > 0$,

$$\gamma^{(u+v)}(x) = \int_{-\infty}^{\infty} \gamma^{(u)}(t)\gamma^{(v)}(x-t)\,dt \quad \text{for all} \quad x,$$

hence

$$\begin{aligned}
&\Psi_{s_1,\cdots,s_n}(x_1, \cdots, x_n)\\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Psi_{k_1,\cdots,k_n}(w_1, \cdots, w_n)\gamma^{(s_1-k_1)}(x_1 - w_1) \\
&\qquad\qquad \cdots \gamma^{(s_n-k_n)}(x_n - w_n)\,dw_1 \cdots dw_n \\
&= \int_{-\infty}^{\infty} \left(\cdots \int_{-\infty}^{\infty} \cdots \left(\int_{-\infty}^{\infty} \Psi_{k_1,\cdots,k_n}(w_1, \cdots, w_n)\gamma^{(s_1-k_1)}(x_1 - w_1)\,dw_1\right) \cdots\right) \\
&\qquad\qquad \cdot\gamma^{(s_n-k_n)}(x_n - w_n)\,dw_n.
\end{aligned}$$

By assumption, $\Psi_{k_1,\cdots,k_n}(w_1, \cdots, w_n)$ is TP_2 in pairs of $w_1, \cdots, w_n$. Now using the fact that $\gamma^{(s_1-k_1)}(x_1 - w_1)$ is TP_2 in w_1 and w_j for any $j = 2, \cdots, n$, we have by Theorem 5.1, page 123 of Karlin, that the integral

$$\int_{-\infty}^{\infty} \Psi_{k_1,\cdots,k_n}(w_1, \cdots, w_n)\gamma^{(s_1-k_1)}(x_1 - w_1)\,dw_1$$

denoted by $\phi_1(x_1, w_2, \cdots, w_n)$, is TP_2 in pairs of $w_2, \cdots, w_n$.

On the other hand, the composition formula implies that $\phi_1(x_1, w_2, \cdots, w_n)$ is TP_2 in x_1 and w_j for any $j = 2, \cdots, n$, with w_k fixed for any $k \neq j$, $k = 2, \cdots, n$. Hence $\phi_1(x_1, w_2, \cdots, w_n)$ is TP_2 in pairs of $x_2, w_2, \cdots, w_n$.

Similarly, $\int_{-\infty}^{\infty} \phi_1(x_1, w_2, \cdots, w_n)\gamma^{(s_2-k_2)}(x_2 - w_2)\,dw_2$, denoted by $\phi_2(x_1, x_2, w_3, \cdots, w_n)$, is TP_2 in pairs of $x_1, x_2, w_3, \cdots, w_n$. Iterating the same procedure gives $\Psi_{s_1,\cdots,s_n}(x_1, \cdots, x_n)$ is TP_2 in pairs of $x_1, x_2, \cdots, x_n$.

COROLLARY 2.5. *If* $(X_1, \cdots, X_n)$ *have density* $f(x_1, \cdots, x_n)$ *which is* TP_2 *in pairs of* $x_1, \cdots, x_n$, *then the joint survival function* $\bar{F}(x_1, \cdots, x_n)$ *is* TP_2 *in pairs of* $x_1, \cdots, x_n$.

PROOF. DTP$(0, \cdots, 0)$ implies DTP$(1, \cdots, 1)$, and DTP$(1, \cdots, 1)$ means that $\Psi_{1,\cdots,1}(x_1, \cdots, x_n) = \bar{F}(x_1, \cdots, x_n)$ is TP_2 in pairs of $x_1, \cdots, x_n$.

PROPOSITION 2.6. *If* $(X_1, \cdots, X_n)$ *is* $DTP(k_1, \cdots, k_n)$, *then* $\Psi_{s_1,\cdots,s_n}(x_1, \cdots, x_n)$ *is* TP_2 *in pairs of* $s_1, \cdots, s_n$, *with* $s_i \geq k_i$, $i = 1, \cdots, n$, *for all* $x_1, \cdots, x_n$.

PROOF. For $s_i > k_i$, $\Psi_{s_1,\cdots,s_n}(x_1, \cdots, x_n)$ can be written as

$$\int_{-\infty}^{\infty} \cdots \left(\int_{-\infty}^{\infty} \rho_1(s_1, w_2, \cdots, w_n)\, \gamma^{(s_2-k_2)}(x_2 - w_2)\, dw_2\right) \cdots \gamma^{(s_n-k_n)}(x_n - w_n)\, dw_n,$$

where

$$\rho_1(s_1, w_2, \cdots, w_n) = \int_{-\infty}^{\infty} \Psi_{k_1,\cdots,k_n}(w_1, \cdots, w_n)\gamma^{(s_1-k_1)}(x_1 - w_1)\, dw_1.$$

If we prove that $\rho_1(s_1, w_2, \cdots, w_n)$ is TP_2 in pairs of $s_1, w_2, \cdots, w_n$, then the result follows by iteration. Now

$$\gamma^{(s_1-k_1)}(x_1 - w_1) = \Gamma(s_1 - k_1)^{-1}(w_1 - x_1)^{s_1-k_1-1} \quad \text{is } TP_2 \text{ in } s_1 \text{ and } w_1,$$

hence by the basic composition formula, $\rho_1(s_1, w_2, \cdots, w_n)$ is TP_2 in s_1, w_j for $j = 2, \cdots, n$. And since $\gamma^{(s_1-k_1)}(x_1 - w_1)$ is also TP_2 in w_1 and w_j, we have $\phi_1(s_1, w_2, \cdots, w_n)$ is TP_2 in pairs of $w_2, \cdots, w_n$ again by Theorem 5.1, page 123 of Karlin. Thus $\rho_1(s_1, w_2, \cdots, w_n)$ is TP_2 in pairs of $s_1, w_2, \cdots, w_n$.

PROPOSITION 2.7. *If $(X_1, \cdots, X_n)$ is $DTP(k_1, \cdots, k_n)$ for $k_\ell > 0$, $\ell = 1, \cdots, n$, then $\Psi_{s_1,\cdots,s_n}(x_1, \cdots, x_n)$ is TP_2 in pairs of s_i, $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ where $s_i \geq k_i$, $s_\ell \in \{k_\ell\} \cup [k_{\ell+1}, \infty)$, $\ell \neq i$, and for any x_i, $i = 1, \cdots, n$.*

The proof is similar to the one in the above proposition.

Shaked showed that if (X, Y) is $DTP(m, n)$ with $m, n = 0, 1, 2$, then $\mathrm{cov}(X, Y) \geq 0$, provided it is defined. In the multivariate case we have the result that if $(X_1, \cdots, X_n)$ is $DTP(k_1, \cdots, k_n)$ with $k_i = 0$ or 1 for $i = 1, \cdots n$, then $\mathrm{cov}(X_i, X_j) \geq 0$ for any $i \neq j$, $i, j = 1, \cdots, n$, provided it is defined.

We give an example such that (X, Y) is DTP(0, 0), (Y, Z) is DTP(0, 0), and (X, Z) is DTP(0, 0), but (X, Y, Z) is not DTP(0, 0, 0).

EXAMPLE 2.8. Let

$$P(X = 0, Y = 1, Z = 0) = 1/8, \qquad P(X = 1, Y = 0, Z = 0) = 1/8,$$
$$P(X = 0, Y = 0, Z = 1) = 1/4, \qquad P(X = 1, Y = 1, Z = 1) = 1/2.$$

Then it is easy to check that X and Y are TP_2, Y and Z are TP_2, Z and X are TP_2. But X, Y, Z are not TP_2 in pairs.

3. Relations with other notions of positive dependence.

Yanagimoto (1972) defined families of positively dependent bivariate distributions $\mathscr{P}(i, j)$, by considering four two-dimensional intervals.

We consider a multivariate analogue of Yanagimoto's families of dependent d.f.'s. First recall that Block, Savits and Shaked (1982) define a measure to be TP_2 in pairs if for all $1 \leq i \neq j \leq n$

$$\begin{aligned} \mu(I_1, \cdots, I_i, \cdots, I_j, \cdots, I_n)\mu(I_1, \cdots, I_i', \cdots, I_j', \cdots, I_n) \\ \geq \mu(I_1, \cdots, I_i', \cdots, I_j, \cdots, I_n)\mu(I_1, \cdots, I_i, \cdots, I_j', \cdots, I_n) \end{aligned} \tag{3.0}$$

for all intervals $I_1, \cdots, I_n, I_i', I_j'$ where $I_i < I_i'$, $I_j < I_j'$. The notation $I < I'$

means $x < y$ for any $x \in I$, $y \in I'$. Restricting the form of I_i, I_i', I_j and I_j' we obtain the following definitions. Among these, the first one is in fact equivalent to the Block, Savits and Shaked definition.

DEFINITION 3.1. Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a probability measure μ.

(1) $\mathbf{X}$ is said to be $\mathscr{P}(3, 3, \cdots, 3)$ if (3.0) holds for all intervals with the restriction that for the components with the primes the intervals are of the form $I = (a, b)$ and $I' = (b, c)$ where $a < b < c$.

(2) $\mathbf{X}$ is said to be $\mathscr{P}(3, \cdots, 3, 2', 3, \cdots, 3)$ with $2'$ in the ith coordinate, if (3.0) holds for all intervals with the restriction that for the components with the primes, the intervals are of the form $I_i = (a, b)$, $I_i' = (b, \infty)$ where $a < b$ and for $j \neq i$, $I_j = (c, d)$ and $I_j' = (d, e)$ where $c < d < e$.

(3) $\mathbf{X}$ is said to be $\mathscr{P}(2', 2', \cdots, 2')$ if (3.0) holds for all intervals with the restriction that for the components with the primes the intervals are of the form $I_i = (a_i, b_i)$, $I_i' = (b_i, \infty)$ where $a_i < b_i$ and $I_j = (a_j, b_j)$, $I_j' = (b_j, \infty)$ where $a_j < b_j$.

REMARK. If μ is TP_2 in pairs according to Block et al. (1982), it implies that $\mathbf{X}$ is $\mathscr{P}(3, \cdots, 3)$. Conversely, if $\mathbf{X}$ is $\mathscr{P}(3, \cdots, 3)$, then μ is TP_2 in the sense of Block et al. See Lee (1982) for a proof.

PROPOSITION 3.2.

(a) Let $\mathbf{X}$ *be a random vector with a density f with respect to a product measure* $m = m_1 \times \cdots \times m_n$ *of* σ*-finite measures such that f is continuous on the support of m (denoted by S) and zero off S, then* $\mathbf{X}$ *is* $\mathscr{P}(3, \cdots, 3)$ *implies that* $\mathbf{X}$ *is* $DTP(0, \cdots, 0)$. *If we assume that* $\{f > 0\} \cap S = \tilde{S}$ *is a product space, and f is* TP_2 *in pairs on* $\tilde{S}$, *the the converse holds, i.e.* $\mathbf{X}$ *is* $DTP(0, \cdots, 0)$ *implies that* $\mathbf{X}$ *is* $\mathscr{P}(3, \cdots, 3)$.

(b) *Let* $(X_1, \cdots, X_n)$ *be absolutely continuous. If* $\mathbf{X}$ *is* $\mathscr{P}(3, \cdots, 3, 2', 3, \cdots, 3)$ *then* $\mathbf{X}$ *is* $DTP(0, \cdots, 0, 1, 0, \cdots, 0)$.

(c) *If* $\mathbf{X}$ *is* $\mathscr{P}(2', 2', \cdots, 2')$ *then* $\mathbf{X}$ *is* $DTP(1, 1, \cdots, 1)$.

See Lee (1982) for the proof.

Now we investigate the relationships with other multivariate positive dependence concepts discussed, e.g., in Barlow and Proschan (1975), and Block and Ting (1981).

In the bivariate case, the random vector (X, Y) is DTP(1, 1) if and only if (X, Y) is RCSI (see Harris, 1970). However, the multivariate generalization of the "only if" part does not hold without a condition on the support of the joint survival function as we will see in the following proposition.

For random vectors $\mathbf{U} = (U_1, \cdots, U_n)$ and $\mathbf{V} = (V_1, \cdots, V_n)$, we use the following notations. Let $K \subset \{1, 2, \cdots, n\}$ be any subset and $\bar{K} = \{1, 2, \cdots, n\} - K$ be the complement. Denote U_K to be the vector obtained from the set $\{U_i, i \in K\}$ by placing subscripts in ascending order. And denote $V_{\bar{K}}$ to be the vector obtained from $\{V_i, i \in \bar{K}\}$ by placing subscripts in ascending order.

PROPOSITION 3.3 *If* $(X_1, \cdots, X_n)$ *is RCSI, then* $(X_1, \cdots, X_n)$ *is*

DTP(1, ⋯, 1). *Conversely, if* $(X_1, \cdots, X_n)$ *is DTP*(1, ⋯, 1) *and such that* $\bar{F}(x_1, \cdots, x_n)$ *takes positive values on a product space, then* $(X_1, \cdots, X_n)$ *is RCSI.*

PROOF. Theorem 3.2 of Brindley and Thompson (1972) shows that $\mathbf{X} = (X_1, \cdots, X_n)$ is RCSI if and only if for every $K \subset \{1, 2, \cdots, n\}$

$$P\{X_K > x_K + \Delta_K \mid X_K > x_K,\ X_{\bar{K}} > x_{\bar{K}}\} \quad \text{is increasing}$$

in $x_{\bar{K}}$ for all x_K and all $\Delta_K > 0$. It is not hard to show that this is equivalent to

$$\frac{\bar{F}(\mathbf{x})}{\bar{F}(\mathbf{x} \wedge \mathbf{y})} \le \frac{\bar{F}(\mathbf{x} \vee \mathbf{y})}{\bar{F}(\mathbf{y})} \quad \text{for any} \quad \mathbf{x}, \mathbf{y}.$$

Then the result follows from Theorem 1 of Block and Ting (1981).

Relationships with CIS and RTIS will be stated in the next section.

4. Relations with reliability theory. Shaked showed that in the bivariate case some of the DTP distribution functions can be characterized through monotonicity properties of conditional hazard rate or mean residual life functions. In the multivariate case, we have similar but necessarily weaker results. Let $\mathbf{X}$ be an absolutely continuous random vector with distribution F and density f. Let $S_{\mathbf{X}} = \{x\colon f(x) > 0\}$. Denote $\mathbf{X}^{(i)} = (X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$, $S_{\mathbf{X}}(i) = \{\mathbf{x}^{(i)}\colon g(\mathbf{x}^{(i)}) > 0$, where g is the joint density of $\mathbf{X}^{(i)}\}$. Let A be a Borel set in $S_{\mathbf{X}}(i)$, consider the conditional hazard rate defined by

$$r(x_i \mid \mathbf{X}^{(i)} \in A) = \frac{f(x_i \mid \mathbf{X}^{(i)} \in A)}{\bar{F}(x_i \mid \mathbf{X}^{(i)} \in A)},$$

and the mean residual life function defined by

$$m(x_i \mid \mathbf{X}^{(i)} \in A) = \int_{x_i}^{\infty} \frac{\bar{F}(t \mid \mathbf{X}^{(i)} \in A)}{\bar{F}(x_i \mid \mathbf{X}^{(i)} \in A)}\, dt = E(X_i - x_i \mid X_i > x_i, \mathbf{X}^{(i)} \in A).$$

If $\mathbf{X}$ is also nonnegative, denote the conditional hazard function of X_i given $\mathbf{X}^{(i)} \in A$ by

$$R(x_i \mid \mathbf{X}^{(i)} \in A) = \int_0^{x_i} r(u \mid \mathbf{X}^{(i)} \in A)\, du = -\log P(X_i > x_i \mid \mathbf{X}^{(i)} \in A).$$

PROPOSITION 4.1. *Let* $\mathbf{X}$ *be a random vector as above.*

(1) *If* $\mathbf{X}$ *is DTP*(0, 0, ⋯, 0, 1) *then* $r(x_n \mid \mathbf{X}^{(n)} = \mathbf{x}^{(n)})$ *is decreasing in* $\mathbf{x}^{(n)} \in S_{\mathbf{X}}(n)$, *for any* x_n.

(2) *If* $\mathbf{X}$ *is DTP*(0, 0, ⋯, 0, 2), *then* $m(x_n \mid \mathbf{X}^{(n)} = \mathbf{x}^{(n)})$ *is increasing in* $\mathbf{x}^{(n)} \in S_{\mathbf{X}}(n)$, *for any* x_n.

(3) *If* $\mathbf{X}$ *is DTP*(0, 0, ⋯, 0, m), *for* $m > 1$, *then*

$$\frac{E[(X_n - x_n)^{m-1} \mid X_n > x_n, \mathbf{X}^{(n)} = \mathbf{x}^{(n)}]}{E[(X_n - x_n)^{m-2} \mid X_n > x_n, \mathbf{X}^{(n)} = \mathbf{x}^{(n)}]}$$

is increasing in $\mathbf{x}^{(n)} \in S_{\mathbf{X}}(n)$, *for any* x_n.

(4) $\mathbf{X}$ *is* $DTP(1, \cdots, 1)$ *if and only if* $r(x_j \mid \mathbf{X}^{(j)} > \mathbf{x}^{(j)})$ *is decreasing in* $\mathbf{x}^{(j)}$, *for any* $x_j, j = 1, 2, \cdots, n$.

PROOF. By Theorem 1.5, page 158 of Karlin, $\Psi_{\mathbf{k}}(x_1, \cdots, x_n)$ is TP_2 in pairs of $x_1, \cdots, x_n$ is equivalent to the condition that for any $i \neq j$, $1 \leq i, j \leq n$, $(\partial/\partial x_j)\log \Psi_{\mathbf{k}}(x_1, \cdots, x_n)$ is increasing in x_i. Now

$$-\frac{\partial}{\partial x_n} \log \Psi_{0,\cdots,0,1}(x_1, \cdots, x_n) = \frac{f(x_1, \cdots, x_n)}{\int_{x_n}^{\infty} f(x_1, \cdots, x_{n-1}, t)\, dt} = r(x_n \mid \mathbf{X}^{(n)} = \mathbf{x}^{(n)}),$$

and

$$\begin{aligned} -\frac{\partial}{\partial x_n} &\log \Psi_{0,\cdots,0,m}(x_1, \cdots, x_n) \\ &= \frac{\int_{x_n}^{\infty} \frac{(t - x_n)^{m-2}}{(m-2)!} f(x_1, \cdots, x_{n-1}, t)\, dt}{\int_{x_n}^{\infty} \frac{(t - x_n)^{m-1}}{(m-1)!} f(x_1, \cdots, x_{n-1}, t)\, dt} \\ &= \frac{(m-1)E[(X_n - x_n)^{m-2} \mid X_n > x_n, \mathbf{X}^{(n)} = \mathbf{x}^{(n)}]}{E[(X_n - x_n)^{m-1} \mid X_n > x_n, \mathbf{X}^{(n)} = \mathbf{x}^{(n)}]}, \end{aligned}$$

thus assertions (1), (2) and (3) follow. As for assertion (4), since

$$\begin{aligned} -(\partial/\partial x_j)&\log \Psi_{1,\cdots,1}(x_1, \cdots, x_n) \\ &= \frac{\int_{x_n}^{\infty} \cdots \int_{x_{j+1}}^{\infty} \int_{x_{j-1}}^{\infty} \cdots \int_{x_1}^{\infty} f(t_1, \cdots, t_{j-1}, x_j, t_{j+1}, \cdots, t_n) \pi_{i \neq j}\, dt_j}{\int_{x_n}^{\infty} \cdots \int_{x_1}^{\infty} f(t_1, \cdots, t_n)\, dt_1, \cdots, dt_n} \\ &= r(x_j \mid \mathbf{X}^{(j)} > \mathbf{x}^{(j)}) \end{aligned}$$

for any $j = 1, \cdots, n$; hence, the equivalence relation holds.

Denote $r_j(\mathbf{x}) = r(x_j \mid \mathbf{X}^{(j)} > \mathbf{x}^{(j)})$, then $(r_1(\mathbf{x}), \cdots, r_n(\mathbf{x}))$ is the gradient of the hazard function $R(\mathbf{x}) = -\log \bar{F}(x_1, \cdots, x_n)$ as was pointed out by Johnson-Kotz (1975). Johnson-Kotz define that a random vector $\mathbf{X}$ has an increasing hazard rate (IHR) distribution if for $j = 1, \cdots, n$, $r_j(\mathbf{x}) = -(\partial/\partial x_j)\log \bar{F}(\mathbf{x})$ is increasing in x_j. Harris (1970) defines $\mathbf{X}$ to be IHR if

(a) $\dfrac{\bar{F}(x_1 + \delta, \cdots, x_n + \delta)}{\bar{F}(x_1, \cdots, x_n)}$ is decreasing in $x_1, \cdots, x_n$ for any $\delta > 0$,

and (b) $\mathbf{X}$ is RCSI.

PROPOSITION 4.2. *Let* $\mathbf{X}$ *be a random vector with differentiable density function. If* $\mathbf{X}$ *is IHR (Harris) then* $\mathbf{X}$ *is IHR (Johnson-Kotz).*

PROOF. The condition (a) of Harris implies that

$$\left(\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}\right)\log \bar{F}(\mathbf{x}) \quad \text{is decreasing in } x_1, \cdots, x_n,$$

hence for any $j = 1, \cdots n$,

$$\left(\frac{\partial^2}{\partial x_j \partial x_1} + \frac{\partial^2}{\partial x_j \partial x_2} + \cdots + \frac{\partial^2}{\partial x_j \partial x_n}\right)\log \bar{F}(\mathbf{x}) \le 0 \quad \text{for all} \quad \mathbf{x}.$$

Condition (b), RCSI, implies DTP$(1, \cdots, 1)$, thus by (4) in Proposition 4.1, we have

$$(\partial^2/\partial x_i \partial x_j)\log \bar{F}(\mathbf{x}) \ge 0 \quad \text{for all} \quad \mathbf{x}, i \ne j.$$

Therefore $(\partial^2/\partial x_j^2)\log \bar{F}(\mathbf{x}) \le 0$, or $r_j(x)$ is increasing in x_j for all j and all $\mathbf{x}$.

Note: (1) Marshall's (1975) definition of IHR requires that $r_j(\mathbf{x})$ is increasing in $\mathbf{x}$, which implies that $(\partial^2/\partial x_j \partial x_i)\log \bar{F}(\mathbf{x}) \le 0$, hence $\bar{F}(\mathbf{x})$ is SR_2 in pairs but not TP_2 in pairs. Here a function $K(x, y)$ is said to be SR_2 if for all $x_1 < x_2$, $y_1 < y_2$, there exist an ε either $+1$ or -1 such that

$$\varepsilon \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) \\ K(x_2, y_1) & K(x_2, y_2) \end{vmatrix} \ge 0.$$

(2) A similar comment applies to the IHR definition of Block (1977).

By using Proposition 4.1, we have the following results.

PROPOSITION 4.3. *Let* $\mathbf{X}$ *be a nonnegative random vector.*
(1) *If* $\mathbf{X}$ *is DTP*$(0, \cdots, 0, 1)$ *then* $\mathbf{X}$ *is CIS* (see Esary and Proschan, 1968).
(2) *If* $\mathbf{X}$ *is DTP*$(1, \cdots, 1, 1)$ *then* $\mathbf{X}$ *is RTIS* (see Block and Ting, 1981).

PROOF. By Proposition 2.2, if $(X_1, \cdots, X_n)$ is DTP$(0, \cdots, 0, 1)$ then $(X_1, \cdots, X_i)$ is DTP$(0, \cdots, 0)$ for any $2 \le i \le n - 1$. But DTP$(0, \cdots, 0)$ implies DTP$(0, \cdots, 0, 1)$ hence by Proposition 4.1 $r(x_i \mid X_1 = x_1, \cdots, X_{i-1} = x_{i-1})$ is decreasing in $x_1, \cdots, x_{i-1}$ for any x_i and for $2 \le i \le n - 1$. Now assertion (1) follows from the fact that

$$\begin{aligned} R(x_i \mid X_1 = x_1, \cdots, X_{i-1} = x_{i-1}) &= \int_0^{x_i} r(u \mid X_1 = x_1, \cdots, X_{i-1} = x_{i-1})\, du \\ &= -\log P(X_i > x_i \mid X_1 = x_1, \cdots, X_{i-1} = x_{i-1}). \end{aligned}$$

Assertion (2) follows similarly. Since $(X_1, \cdots, X_n)$ is DTP$(1, \cdots, 1)$, this implies that $(X_1, \cdots, X_i)$ is DTP$(1, \cdots, 1)$ for any $2 \le i \le n - 1$ by Proposition 2.2.

5. Examples.

In this section we give examples of the DTP families.

EXAMPLE 5.1. Let $T_1, T_2, \cdots, T_n$ be independently distributed with a common p.d.f. f, such that $P(T_1 \le 0) = 0$. Then the joint probability density of

$X_j = \sum_{i=1}^{j} T_i$, $j = 1, \cdots, n$ is $P_{\mathbf{X}}(x_1, \cdots, x_n) = \prod_{j=1}^{n} f(x_j - x_{j-1})$, where $0 \le x_1 \le \cdots \le x_n$, with $x_0 = 0$. If f is a Polya frequency function of order 2, then $X_1, \cdots, X_n$ is DTP$(0, \cdots, 0)$.

EXAMPLE 5.2 As in the bivariate case, the DTP family can be constructed in the following way. Let W be a r.v. with continuous or discrete d.f. $G(w)$ and let $F_i(x_i \mid w)$ be the conditional d.f.'s of X_i given $W = w$ for $i = 1, \cdots, n$. Assume $X_1, \cdots, X_n$ are conditionally independent given $W = w$, then the joint d.f. of $X_1, \cdots, X_n$ is

$$H(x_1, \cdots, x_n) = \int \prod_{i=1}^{n} F_i(x_i \mid w)\, dG(w).$$

Now, if (X_i, W) is DTP$(k_i, 0)$ for $i = 1, \cdots, n$, then $(X_1, \cdots, X_n)$ is DTP$(k_1, \cdots, k_n)$. To prove this, assume W is absolutely continuous with density $k(w)$, then for $k_i > 0$, $i = 1, \cdots, n$

$$\Psi_{\mathbf{k}}(\mathbf{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \gamma^{(k_i)}(x_i - t_i)\, dH(t_1, \cdots, t_n)$$

$$= \int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} \gamma^{(k_i)}(x_i - t_i) k(w)\, dF_i(t_i \mid w) \right) (k(w))^{-(n-1)}\, dw$$

where the inner integral $\int_{-\infty}^{\infty} \gamma^{(k_i)}(x_i - t_i)\, d(w)\, dF_i(t_i \mid w)$ is TP$_2$ in x_i, w for $i = 1, \cdots, n$, hence $\Psi_{\mathbf{k}}(\mathbf{x})$ is TP$_2$ in pairs of $x_1, \cdots, x_n$ by the basic composition formula.

EXAMPLE 5.3. Let $(X_1, \cdots, X_k)$ be RCSI, and let $T_i = \min_{j \in J_i} X_j$, $i = 1, \cdots, n$, where $J_i \subseteq \{1, \cdots, k\}$. Then $(T_1, \cdots, T_n)$ is DTP$(1, \cdots, 1)$. This follows from the fact that sets of minimums of RCSI r.v.'s are RCSI (see Harris, 1970) and the relation that RCSI r.v.'s are DTP$(1, \cdots, 1)$.

Esary and Marshall (1979) give several conditions of multivariate IFRA. Among them, the random vector $(T_1, \cdots, T_n)$ satisfies Condition D if for some independent IHRA r.v.'s $X_1, \cdots, X_k$ and nonempty subsets $S_i \subseteq \{1, \cdots, k\}$, $T_i = \min_{j \in S_i} X_j$, $i = 1, \cdots, n$. Hence it is clear that if $\mathbf{T}$ satisfies the Condition D of MIFRA by Esary and Marshall, then $\mathbf{T}$ is DTP$(1, \cdots, 1)$.

Note: By the way of Example 5.2 we can construct many DTP random vectors which are also mixtures of independent n-variates d.f.'s with equal marginals (called positive dependent by mixture (PDM) by Shaked, 1977b). However, a r.v. $\mathbf{X}$ is DTP$(k, \cdots, k)$ for some $k \ge 0$ does not imply that $\mathbf{X}$ is PDM. See Shaked (1979, page 72) for a counterexample.

Acknowledgement. I am very grateful to my dissertation adviser, Professor Henry W. Block, for his guidance. Also I thank an associate editor and the referees for their helpful comments.

REFERENCES

AHMED, A. N., LANGBERG, N. A., LEON, R. and PROSCHAN, F. (1978). Two concepts of positive dependence with applications in multivariate analysis. Technical Report M486, Department of Statistics, Florida State University.

BARLOW, R. E. and PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing.* Holt, New York.

BLOCK, H. W. (1977). Monotone failure rates for multivariate distributions. *Naval Research Logistics Quarterly* **24** 627–637.

BLOCK, H. W. and TING, M. (1981). Some concepts of multivariate dependence. *Comm. Statist. A* **10(8)** 749–762.

BLOCK, H. W., SAVITS, T. H. and SHAKED, M. (1982). Some concepts of negative dependence. *Ann. Probab.* **10** 765–772.

BRINDLEY, E. C., JR. and THOMPSON, W. A. (1972). Dependence and aging aspects of multivariate survival. *J. Amer. Statist. Assoc.* **67** 822–830.

ESARY, J. D. and MARSHALL, A. W. (1979). Multivariate distributions with increasing hazard rate average. *Ann. Probab.* **7** 359–370.

ESARY, J. D. and PROSCHAN, F. (1968). Generating associated random variables. Boeing Scientific Research Laboratories Document D1-82-0696. Seattle, Washington.

HARRIS, R. (1970). A multivariate definition for increasing hazard rate distributions. *Ann. Math. Statist.* **41** 1456–1465.

JOHNSON, N. and KOTZ, S. (1975). A vector multivariate hazard rate. *J. Multivariate Anal.* **5** 53–66.

KARLIN, S. (1968). *Total Positivity*, Vol. I. Stanford University Press, Stanford, California.

LEE, M.-L. T. (1982). Dependence by total positivity. Technical Report No. 82-17, University of Pittsburgh.

MARSHALL, A. W. (1975). Some comments on the hazard gradient. *Stochastic Process. Appl.* **3** 293–300.

SHAKED, M. (1977a). A family of concepts of positive dependence for bivariate distributions. *J. Amer. Statist. Assoc.* **72** 642–650.

SHAKED, M. (1977b). A concept of positive dependence for exchangeable random variables. *Ann. Statist.* **5** 505–515.

SHAKED, M. (1979). Some concepts of positive dependence for exchangeable bivariate distributions. *Ann. Inst. Statist. Math.* **31** 67–84.

YANAGIMOTO, T. (1972). Families of positively dependent random variables. *Ann. Inst. Statist. Math.* **24** 559–573.

DEPARTMENT OF MATHEMATICS
111 CUMMINGTON STREET
BOSTON UNIVERSITY
BOSTON, MASSACHUSETTS 02215