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# Dependence of the Analytical Approximate Solution to the Van der Pol Equation on the Perturbation of a Moving Singular Point in the Complex Domain

**Victor Orlov**

Institute of Digital Technologies and Modeling in Construction, Moscow State University of Civil Engineering, Yaroslavskoye Shosse, 26, 129337 Moscow, Russia; orlovvn@mgsu.ru; Tel.: +7-916-457-4176

**Abstract:** This paper considers a theoretical substantiation of the influence of a perturbation of a moving singular point on the analytical approximate solution to the Van der Pol equation obtained earlier by the author. A priori estimates of the error of the analytical approximate solution are obtained, which allows the solving of the inverse problem of the theory of error: what should the structure of the analytical approximate solution be in order to obtain a result with a given accuracy? Thanks to a new approach for obtaining a priori evaluations of errors, based on elements of differential calculus, the domain, used to obtain an analytical approximate solution, was substantially expanded. A variant of optimizing a priori estimates using a posteriori estimates is illustrated. The results of a numerical experiment are also presented.

**Keywords:** Van der Pol equation; perturbation of a moving singular point; a priori estimate; analytical approximate solution

**MSC:** 34G20; 35A05

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## 1. Introduction

We can see a wide range of applications of the Van der Pol equation to the theory of nonlinear oscillations [1–6], relaxation oscillations [7], and human movement modeling and data transmission synchronization in neural networks [8]. It should be noted that publications on qualitative and asymptotic theories [9–11] do not pay attention to the nonlinear nature of this equation. This situation is reflected in the works of recent years [12–14], in which numerical methods are used without substantiation to obtain a solution to the Van der Pol equation. Only research in the complex domain [15] makes it possible to substantiate the absence of moving singular points [16] in the real domain for a positive value of the parameter of this equation, which justifies the application of numerical methods to its solution. At the moment, there are two options for solving nonlinear differential equations with moving singular points. The first option is related to the solvability in quadratures, which is allowed only in special cases [17–23]. The second option is associated with the author's analytical approximate solution method, successfully tested on a number of classes of nonlinear differential equations [15,24–28]. Therefore, with the exception of special cases, the existing method for finding moving singular points [28] allows only approximate values to be obtained. All of that mentioned above actualizes the presented results. The presentation of the results confirms the novelty of the article material for the first time.

Let us consider the Van der Pol equation in the complex domain

$$w''(z) = -a(w^2 - 1)w'(z) - w(z), \tag{1}$$

where $a$ is the parameter, a real value.

In this paper, we present the results of the development of an analytical approximate method for the considered equation in the complex domain. In [15], the existence of moving singular points was proved and an analytical approximate solution was built,

$$w_N(z) = (z^* - z)^{-1/2} \sum_{n=0}^{N} C_n (z^* - z)^{n/2}, \tag{2}$$

in the neighborhood of these points:

$$|z^* - z| < 1/|a+1|. \tag{3}$$

As a consequence, the absence of moving singular points of the algebraic type was obtained for the case of $a \geq 0$ in the real domain. Presently, existing methods allow obtaining only approximate values of moving singular points [28]. This circumstance confirms the relevance of the problem of studying the effect of an error of the value of a moving singular point on the analytical approximate solution (2). In this case, the latter solution (2) is as follows:

$$\tilde{w}_N(z) = (\tilde{z}^* - z)^{-1/2} \sum_{n=0}^{N} C_n (\tilde{z}^* - z)^{n/2}, \tag{4}$$

where $\tilde{z}^*$ is the approximate value of the moving singular point.

This article has a theoretical substantiation of the effect of perturbation of a moving singular point on the analytical approximate solution (4). An a priori evaluation of the error is obtained. The domain of correct results is maximized using elements of differential calculus. The results of numerical experiments are provided to confirm the theoretical provisions.

## 2. Research Method

Theorem 1 formulated below establishes the influence of the error in the value of the moving singular point $\tilde{z}^*$ on the approximate solution (4) and makes it possible to obtain an a priori estimate of the error.

**Theorem 1.** *When the conditions are met:*

1. *$\tilde{z}^*$ is a moving singular point of equation solution (1) with initial conditions*

$$w(z_0) = w_0, w'(z_0) = w_1. \tag{5}$$

2. *An estimate of the error of the moving singular point is known*

$$|\tilde{z}^* - z^*| \leq \Delta\tilde{z}^*,$$

*then for values z from the region*

$$|\tilde{z}^* - z| < \rho,$$

*the estimate of the analytical approximate solution error (4) is true*

$$\Delta\tilde{w}_N(z) \leq \Delta_0 + \Delta_1 + \Delta_2,$$

*where*

$$\Delta_0 \leq \frac{\sqrt{3/(2|a|)}}{2|\tilde{z}^* - z|^{3/2}} \Delta\tilde{z}^*,$$

$$\Delta_1 \leq 4\sqrt{3/(2|a|)}(|a|+1)\Delta\tilde{z}^* \left( \frac{1}{2^{1/2}|\tilde{z}^* - z|^{1/2}} + \frac{(|a|+1)2^{3/2}|\tilde{z}^* - z|^{1/2}}{3(1 - (|a|+1)2|\tilde{z}^* - z|)} \right),$$

$$\Delta_2 \leq \frac{\sqrt{3/(2|a|)}(|a|+1)^{(N+1)/2}|\tilde{z}^*-z|^{N/2}}{N(N-2)/4\left(1-(|a|+1)^{1/2}|\tilde{z}^*-z|^{1/2}\right)},$$

$$\rho = \min\left\{\frac{1}{|a|+1}, \frac{1}{2(|a|+1)}\right\}.$$

**Proof.** Based on the classical approach, we have

$$\Delta\tilde{w}_N(z) = |w(z)-\tilde{w}_N(z)| \leq |w(z)-\tilde{w}(z)| + |\tilde{w}(z)-\tilde{w}_N(z)| =$$

$$\left|\sum_{n=0}^{\infty} C_n(z^*-z)^{(n-1)/2} - \sum_{n=0}^{\infty} C_n(\tilde{z}^*-z)^{(n-1)/2}\right| +$$

$$\left|\sum_{n=0}^{\infty} C_n(\tilde{z}^*-z)^{(n-1)/2} - \sum_{n=0}^{N} C_n(\tilde{z}^*-z)^{(n-1)/2}\right| \leq$$

$$\left|C_0\left((z^*-z)^{-1/2}-(\tilde{z}^*-z)^{-1/2}\right)\right| + \left|\sum_{n=1}^{\infty} C_n\left((z^*-z)^{(n-1)/2}-(\tilde{z}^*-z)^{(n-1)/2}\right)\right| +$$

$$\left|\sum_{N+1}^{\infty} C_n(\tilde{z}^*-z)^{(n-1)/2}\right| = \Delta_0+\Delta_1+\Delta_2.$$

Let us consider $\Delta_0$:

$$\Delta_0 = \left|C_0\left((z^*-z)^{-1/2}-(\tilde{z}^*-z)^{-1/2}\right)\right| = |C_0| \times \left|(\tilde{z}^*-z+\Delta\tilde{z}^*)^{-1/2}-(\tilde{z}^*-z)^{-1/2}\right|.$$

From the latter, after performing a series of transformations, we obtain

$$\Delta_0 \leq |C_0| \frac{\Delta\tilde{z}^*}{(|\tilde{z}^*-z|+\Delta\tilde{z}^*)^{1/2}|\tilde{z}^*-z|^{1/2}\left(|\tilde{z}^*-z|^{1/2}+(|\tilde{z}^*-z|+\Delta\tilde{z}^*)^{1/2}\right)}.$$

Or, taking into account the value of the coefficient $C_0$ [15], it follows

$$\Delta_0 \leq \frac{\sqrt{3/(2|a|)}\Delta\tilde{z}^*}{2|\tilde{z}^*-z|^{3/2}} .$$

Let us estimate $\Delta_1$. It was previously established [15] that all odd coefficients $C_n$ are equal to zero, and for even values $k \geq 2$, the following estimates hold:

$$|C_{2k}| \leq \frac{\sqrt{3/(2|a|)}(|a|+1)^k}{|(2k-1)/2(2k-3)/2|}. \tag{6}$$

Therefore, for $\Delta_1$, we obtain

$$\Delta_1 \leq \left|\sum_{n=1}^{\infty} C_n\left((z^*-z)^{(n-1)/2}-(\tilde{z}^*-z)^{(n-1)/2}\right)\right| \leq$$

$$\left|\sum_{k=1}^{\infty} C_{2k}\left((z^*-z)^{(2k-1)/2}-(\tilde{z}^*-z)^{(2k-1)/2}\right)\right| \leq$$

$$\frac{|C_2|\Delta\tilde{z}^*}{|\tilde{z}^*-z|^{1/2}2^{1/2}} + |C_4|\Delta\tilde{z}^* 2^{3/2}|\tilde{z}^*-z|^{1/2} + |C_6|2^{5/2}\Delta\tilde{z}^*|\tilde{z}^*-z|^{3/2}+$$

$$|C_8|2^{7/2}\Delta\tilde{z}^*|\tilde{z}^*-z|^{5/2}+\ldots+|C_{2k}|2^{(2k-1)/2}\Delta\tilde{z}^*|\tilde{z}^*-z|^{(2k-3)/2}+\ldots$$

Or, taking into account (6), after a series of transformations we obtain

$$\Delta_1 \leq \frac{\sqrt{3/(2|a|)}(|a|+1)\Delta\tilde{z}^*}{1/4}\left(\frac{1}{2^{1/2}|\tilde{z}^*-z|^{1/2}}+\frac{(|a|+1)2^{3/2}|\tilde{z}^*-z|^{1/2}}{3(1-(|a|+1)2|\tilde{z}^*-z|)}\right).$$

The expression for $\Delta_1$ is obtained in the area

$$|\tilde{z}^*-z| < \frac{1}{2(|a|+1)}.$$

Let us move on to the estimate $\Delta_2$. Taking into account the estimates for the coefficients $C_n$ (6), we obtain in the case of $N > 2$

$$\Delta_2 \leq \sum_{n=N+1}^{\infty}|C_n||\tilde{z}^*-z|^{(n-1)/2} \leq \frac{\sqrt{3/(2|a|)}(|a|+1)^{(N+1)/2}|\tilde{z}^*-z|^{N/2}}{N(N-2)/4}+$$

$$\frac{\sqrt{3/(2|a|)}(|a|+1)^{(N+2)/2}|\tilde{z}^*-z|^{(N+1)/2}}{(N+1)(N)/4}+$$

$$\frac{\sqrt{3/(2|a|)}(|a|+1)^{(N+3)/2}|\tilde{z}^*-z|^{(N+2)/2}}{(N+1)(N+2)/4}+$$

$$\frac{\sqrt{3/(2|a|)}(|a|+1)^{(N+4)/2}|\tilde{z}^*-z|^{(N+3)/2}}{(N+2)(N+3)/4}+\ldots\leq$$

$$\frac{\sqrt{3/(2|a|)}(|a|+1)^{(N+1)/2}|\tilde{z}^*-z|^{N/2}}{N(N-2)/4\left(1-(|a|+1)^{1/2}|\tilde{z}^*-z|^{1/2}\right)}.$$

The estimate expression for $\Delta_2$ is valid in the domain

$$|\tilde{z}^*-z| < \frac{1}{|a|+1}.$$

Therefore, the theorem is fair in the domain

$$|\tilde{z}^*-z| < \rho,$$

where

$$\rho = \min\left\{\frac{1}{|a|+1}, \frac{1}{2(|a|+1)}\right\}. \tag{7}$$

□

The comparison of the domains, made using Formulas (3) and (7), shows that the domain obtained by applying Formula (7) is substantially smaller than the one obtained using Formula (3). If the classical approach is applied in Theorem 1 in the course of obtaining the a priori estimate of the approximate solution (4), the following Theorem 2, whose proof is based on the elements of differential calculus [29], allows expanding the domain to a substantial extent (7).

**Theorem 2.**

1. *Let us assume that $\tilde{z}^*$ is a moving singular point in the solution to the Cauchy problem (1), (5);*
2. *The value of the perturbation of a moving singular point $\tilde{z}^*$ is available*

$$|\tilde{z}^*-z^*| \leq \Delta\tilde{z}^*,$$

*then the a priori estimate of the error is correct*

$$\Delta\tilde{w}_N(z) \leq \Delta_0+\Delta_1+\Delta_2+\Delta_3$$

*for the analytical approximate solution (4) in the domain*

$$|\tilde{z}^* - z| < \rho,$$

*where*

$$\Delta_0 \le \frac{\sqrt{3/(2|a|)}}{2|\tilde{z}_1^* - z|^{3/2}} * \Delta\tilde{z}^*, \quad \Delta_1 \le 2\sqrt{3/(2|a|)}|a|\Delta\tilde{z}^* \frac{1}{7|\tilde{z}_1^* - z|^{1/2}},$$

$$\Delta_2 \le \frac{2\sqrt{3/(2|a|)}(|a|+1)^2|\tilde{z}_2^* - z|^{1/2}}{3(1-(|a|+1)|\tilde{z}_2^* - z|)},$$

$$\Delta_3 \le \frac{\sqrt{3/(2|a|)}(|a|+1)^{(N+1)/2}|\tilde{z}^* - z|^{(N-2)/2}}{N(N-2)/4\left(1-(|a|+1)^{1/2}|\tilde{z}^* - z|^{1/2}\right)},$$

$$\rho = F_1 \cap F_2 \cap F_3, \quad F_1 = \left\{z : |\tilde{z}_1^* - z| < \frac{1}{|a|+1}\right\},$$

$$F_2 = \left\{z : |\tilde{z}_2^* - z| < \frac{1}{|a|+1}\right\}, F_3 = \left\{z : |\tilde{z}^* - z| < \frac{1}{|a|+1}\right\},$$

$$|\tilde{z}_1^*| = |\tilde{z}^*| - \Delta\tilde{z}^*, |\tilde{z}_2^*| = |\tilde{z}^*| + \Delta\tilde{z}^*, \arg\tilde{z}_1^* = \arg\tilde{z}_2^* = \arg\tilde{z}^*.$$

**Proof.** Proof is obtained according to the definition

$$\Delta\tilde{w}_N(z) = |w(z) - \tilde{w}_N(z)| \le |w(z) - \tilde{w}(z)| + |\tilde{w}(z) - \tilde{w}_N(z)|.$$

Given that this approach, consisting of applying elements of differential calculus to the estimate of the first summand of the above expression [29], was successfully used to solve a number of non-linear differential equations [27], we obtain

$$|w(z) - \tilde{w}(z)| \le \sup_G \left|\frac{d\tilde{w}}{d\tilde{z}^*}\right| \Delta\tilde{z}^* \le$$

$$\sup_G \left|\sum_0^\infty C_n \left(\frac{n-1}{2}\right)(\tilde{z}^* - z)^{(n-3)/2}\right| \Delta\tilde{z}^* \le$$

$$\sum_0^\infty \sup_G |\tilde{z}^* - z|^{(n-3)/2} |C_n| \left|\frac{n-1}{2}\right| \Delta\tilde{z}^*,$$

where

$$G = \{z : |\tilde{z}^* - z| \le \Delta\tilde{z}^*\}.$$

Furthermore,

$$\sup_G |\tilde{z}^* - z|^{(n-3)/2} = \begin{cases} |\tilde{z}_1^* - z|^{(n-3)/2}, n = 0, 1, 2, \\ |\tilde{z}_2^* - z|^{(n-3)/2}, n = 3, 4, \ldots. \end{cases}.$$

In this case,

$$|\tilde{z}_1^*| = |\tilde{z}^*| - \Delta\tilde{z}^*, |\tilde{z}_2^*| = |\tilde{z}^*| + \Delta\tilde{z}^*, \arg\tilde{z}_1^* = \arg\tilde{z}_2^* = \arg\tilde{z}^*.$$

As a consequence,

$$|w(z) - \tilde{w}(z)| \le |C_0|\Delta\tilde{z}^* \frac{1}{2|\tilde{z}_1^* - z|^{3/2}} + |C_2| \frac{\Delta\tilde{z}^*}{2|\tilde{z}_1^* - z|^{1/2}} +$$

$$\Delta\tilde{z}^* \sum_3^\infty |C_n| \left|\frac{n-3}{2}\right| |\tilde{z}_2^* - z|^{(n-3)/2}.$$

Therefore,

$$\Delta\tilde{w}_N(z) = |w(z) - \tilde{w}_N(z)| \leq |C_0|\Delta\tilde{z}^* \frac{1}{2|\tilde{z}_1^* - z|^{3/2}} + |C_2| \frac{\Delta\tilde{z}^*}{2|\tilde{z}_1^* - z|^{1/2}} +$$

$$\Delta\tilde{z}^* \sum_3^\infty |C_n| \left|\frac{n-3}{2}\right| |\tilde{z}_2^* - z|^{(n-3)/2} + \sum_{N+1}^\infty |C_n||\tilde{z}^* - z|^{(n-3)/2} =$$

$$\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3.$$

For $\Delta_0$ and $\Delta_1$, respectively, we obtain

$$\Delta_0 \leq \frac{\sqrt{3/(2|a|)}}{2|\tilde{z}_1^* - z|^{3/2}} * \Delta\tilde{z}^*, \quad \Delta_1 \leq 2\sqrt{3/(2|a|)}|a|\Delta\tilde{z}^* \frac{1}{7|\tilde{z}_1^* - z|^{1/2}}$$

in the domain

$$F_1 = \left\{ z : |\tilde{z}_1^* - z| < \frac{1}{|a| + 1} \right\}.$$

Let us estimate $\Delta_2$, given that $C_{2n+1} = 0$ [15]:

$$\Delta_2 \leq \Delta\tilde{z}^* \sum_3^\infty |C_n|\left|\tfrac{n-3}{2}\right||\tilde{z}_2^* - z|^{(n-3)/2} \leq \Delta\tilde{z}^* \sum_2^\infty |C_{2n}|\left|\tfrac{2n-3}{2}\right||\tilde{z}_2^* - z|^{(2n-3)/2} \leq$$

$$\Delta\tilde{z}^* \sum_2^\infty \frac{\sqrt{3/(2|a|)}(|a|+1)^n}{\left(\frac{2n-1}{2}\right)} |\tilde{z}_2^* - z|^{(2n-3)/2} \leq$$

$$\Delta\tilde{z}^* \frac{\sqrt{3/(2|a|)}(|a|+1)^2}{\left(\frac{3}{2}\right)} |\tilde{z}_2^* - z|^{1/2} \frac{1}{\left(1 - (|a|+1)|\tilde{z}_2^* - z|\right)}.$$

The estimate of $\Delta_2$ is correct in the domain

$$F_2 = \left\{ z : |\tilde{z}_2^* - z| < \frac{1}{|a| + 1} \right\}.$$

It is also correct for $\Delta_3$. In compliance with the findings of [15], we have

$$\Delta_3 \leq \sum_{N+1}^\infty |C_n||\tilde{z}^* - z|^{(n-3)/2} \leq \frac{\sqrt{3/(2|a|)}(|a| + 1)^{(N+1)/2}|\tilde{z}^* - z|^{(N-2)/2}}{N(N-2)/4\left(1 - (|a| + 1)^{1/2}|\tilde{z}^* - z|^{1/2}\right)}$$

in the domain

$$F_3 = \left\{ z : |\tilde{z}^* - z| < \frac{1}{|a| + 1} \right\}.$$

Consequently, the theorem will be correct in the domain

$$|\tilde{z}^* - z| < \rho,$$

where

$$\rho = F_1 \cap F_2 \cap F_3.$$

□

## 3. Results Discussion

For numerical experiment 1, let us consider the Cauchy problem (1), (5):

$$a = 2; \quad w(0) = i; \quad w'(0) = -i; \quad \tilde{z}^* = 0.85717; \quad \Delta\tilde{z}^* \leq 0.74 * 10^{-5}; \quad z_1 = 0.83157.$$

According to Theorem 1, $\rho = 0.166667$.

For the structure of the analytical approximate solution (4) $\tilde{w}_6(z_1)$, we have the following expressions for the coefficients $C_n$:

$$C_0 = i\sqrt{\frac{3}{2a}}, \quad C_2 = -a \cdot \frac{1}{4}C_0, \quad C_4 = -C_0\left(\frac{1}{160}a^2 - 1\right),$$

$$C_6 = \frac{1}{15}\left(a\left(10C_0C_2C_4 - \frac{5}{4}C_4 + \frac{5}{2}C_2^3\right) - C_2\right).$$

The calculations were carried out in Matlab. The calculation results are presented in Tables 1 and 2.

Notations: $\tilde{w}_6(z_1)$ is the analytical approximate solution (4); $\Delta\tilde{w}_6(z_1)$ is an a priori estimate of the error obtained by the theorem; and $\Delta_4$ is an a posteriori error estimate.

**Table 1.** Characteristics of the calculations of the first stage.

| $z_1$ | $\tilde{w}_6(z_1)$ | $\Delta\tilde{w}_6(z_1)$ | $\Delta_4$ |
|---|---|---|---|
| 0.83157 | 5.346829i | 0.001329 | 0.00005 |

Algorithm for solving the inverse problem of the theory of error:

1. We start with the analysis of the a priori error value $\Delta\tilde{w}_6(z_1)$, and check the condition $\Delta\tilde{w}_6(z_1) < \Delta_4$. If it is satisfied, we go to step 7. Otherwise, we go to the next step of the algorithm.
2. We check the condition $\Delta_0 + \Delta_1 + \Delta_2 < \Delta_3$. If it is accomplished, we go to the next step. Otherwise, we go to step 6.
3. By the value for $\Delta_3$, we determine the minimum value of $N$, for which the condition $\Delta_3 < \Delta_4$ will be fulfilled. Let us go to the next point.
4. We check the condition $\Delta_0 + \Delta_1 + \Delta_2 < \Delta_3$. If it is satisfied, then we go to step 7. Otherwise, we go to the next step.
5. We correct the value $\Delta_3$ according to the formula $\Delta_3 = \Delta_3 + \Delta_0 + \Delta_1 + \Delta_2$ and we go to step 3.
6. We increase the accuracy of the approximate value of the moving singular point $\tilde{z}^*$, decrease the value of $\Delta\tilde{z}^*$, and go to step 2 of the algorithm.
7. Completion of the algorithm. An approximate solution $\tilde{w}_6(z_1)$ with a given accuracy is obtained.

Applying this algorithm to the input data of the example at the second step, we find that in order to solve the problem, it is required to reduce the magnitude of the perturbation of the moving singular point. We correct the approximate value of the moving singular point and the value of its accuracy:

$$\tilde{z}^* = 0.857177; \quad \Delta\tilde{z}^* \leq 0.4 * 10^{-6}.$$

The calculations of the second stage are presented in Table 2.

**Table 2.** Characteristics of the calculations of the second stage.

| $z_1$ | $\tilde{w}_6(z_1)$ | $\Delta\tilde{w}_6(z_1)$ | $\Delta_4$ |
|---|---|---|---|
| 0.83157 | 5.346829i | 0.0001634 | 0.00005 |

For the value $z_1$, for a posteriori estimation $\Delta_3$ in the structure of the analytical approximate solution (4), the value $N = 12$ is required. The sum of the components from 8 to 12 does not exceed the required accuracy. Therefore, the analytical approximate solution $\tilde{w}_6(z_1)$ has an accuracy of $\varepsilon = 0.00005$.

For numerical experiment 2, let us consider the Cauchy problem (1), (5):

$$a = 2; \quad w(0) = i; \quad w'(0) = -i; \quad \tilde{z}^* = 0.85717; \quad \Delta\tilde{z}^* = 0.74 * 10^{-5}.$$

In the first case, the point $z_1 = 0.83157$ is within the domains of Theorems 1 and 2. Calculation results are provided in Table 3. In the second case, $z_2 = 0.67821$ is within the domain of Theorem 2. According to Theorem 2, the value is $\rho = 0.333333$. Calculations are provided in Table 4.

**Table 3.** Comparative characteristics of calculations.

| $z_1$ | $\tilde{w}_6(z_1)$ | $\Delta_5$ | $\Delta_6$ |
|---|---|---|---|
| 0.83157 | 5.346829i | 0.001329 | 0.000861 |

where $\tilde{w}_6(z_1)$ is the value of the analytical approximate solution (4); $\Delta_5$ is the a priori estimate of the error according to Theorem 1; and $\Delta_6$ is the a priori estimate of the error according to Theorem 2. Values of a priori estimates confirm the convergence of results obtained using Theorems 1 and 2.

**Table 4.** Characteristics of calculations made for the second case.

| $z_2$ | $\tilde{w}_6(z_2)$ | $\Delta_7$ | $\Delta_8$ |
|---|---|---|---|
| 0.67821 | 1.927126i | 0.22893 | 0.005 |

where $\tilde{w}_6(z_2)$ is the value of the analytical approximate solution (4) and $\Delta_7$ is the a priori estimate of the error according to Theorem 2.

For the value $z_2$, for a posteriori estimation $\Delta_8 = 0.005$ in the structure of the analytical approximate solution (4), the value $N = 12$ is required. The sum of the components from 8 to 12 does not exceed the required accuracy. Therefore, the analytical approximate solution $\tilde{w}_6(z_2)$ has an accuracy of $\varepsilon = 0.005$.

## 4. Conclusions

The presented results conclude the studies published in [15]. The dependence of the structure of the analytical approximate solution for the Van der Pol equation in the neighborhood of a moving singular point on the magnitude of the perturbation of the moving singular point itself is established. The proven theorems make it possible to solve the inverse problem of the theory of error, as well as to determine the magnitude of the perturbation of a moving singular point in order to obtain an analytical approximate solution with a given accuracy. Theoretical positions are confirmed by numerical experiment. A variant of the algorithm for optimizing a priori estimates using a posteriori estimates is presented.

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