DEPTH OF IDEMPOTENT-GENERATED SUBSEMIGROUPS OF A REGULAR RING

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Abstract

If S is an idempotent-generated semigroup, its depth is the minimum number of idempotents needed to express a general element as a product of idempotents. Here we study the depth of S where S is the semigroup generated by all the idempotents of a von Neumann regular ring, and the depth of various subsemigroups of S. For example, if R is directly finite, the depth of S equals the index of nilpotence of R, which considerably extends a result of Ballantine (1978) for matrices over a field. We also answer a query of Professor Howie by supplying a ring-theoretic explanation of Reynolds and Sullivan's (1985) result that the depth is 3 for certain subsemigroups in the infinite-dimensional full linear case.

Introduction

The *depth* $\triangle(S)$ of an idempotent-generated semigroup S is the least positive integer n such that each element of S can be written as a product of n idempotents, if such an n exists, otherwise $\triangle(S) = \infty$. Thus depth is a measure of how far the elements of S are from being idempotent (or how far the semiband S is from being a band). In 1981, Howie [H2] calculated $\triangle(S)$ for the semigroup S generated by the idempotents of a full transformation semigroup, and also the depth of certain of its subsemigroups. Ballantine [B] (see also Dawlings [D1]) and Reynolds and Sullivan [RS] obtained analogous results for the semigroup generated by the idempotent linear transformations of a vector space, in the finite-dimensional and infinite-dimensional cases respectively. In this paper we provide a very much broader setting for the results of Ballantine and Reynolds and Sullivan, by employing ring-theoretic machinery to calculate the depth of idempotent-generated subsemigroups of a (von Neumann) regular ring, in the case where the ring is directly finite or right self-injective. The basic theme is relating depth to the more established ring-theoretic concept of nilpotent index, *index(R)*, of a ring R (or of an ideal of R). Our results rely heavily on the work in [OM] and [HO], which gave purely ring-theoretic characterizations of products of idempotents in regular rings.

Let R be a regular ring and let S be the multiplicative semigroup generated by its idempotents. In §1 we show (Theorem 1.3) that always $\triangle(S) = index(R)$ if R is directly finite. This generalizes Ballantine's 1978 result that $\triangle(S) = n$ if $R = M_n(F)$, the ring of $n \times n$ matrices over a field F. It also establishes that a directly finite regular ring has bounded index of nilpotence exactly when S has finite depth. For directly infinite rings, the connection between depth and index is more subtle. We show in Theorem 1.6, that for a regular, right self-injective ring R, $\triangle(S) = index(R)$ except when R is a direct product $R_1 \times R_2$ of a ring R_1 of bounded index of nilpotence and a nonzero ring R_2 of Type III, in which case

$$\triangle(S) = max\{index(R_1), 3\}.$$

Thus finite depth for this class of rings is equivalent to the ring being a direct product of a ring of bounded index and one of Type III.

Both Howie [H2] and Reynolds and Sullivan [RS] broke up their idempotent-generated semigroups S into a disjoint union of a "finite" part and "infinite" parts, the latter being labelled by infinite cardinals. Each part is in turn a regular, idempotent-generated subsemigroup of S, and the finite part has depth ∞ (for an infinite set or infinite dimensional space). In the full transformation setting of Howie, the infinite parts have depth 4. But in contrast, the infinite parts in the full linear setting of Reynolds and Sullivan turned out to have depth 3. (This phenomenon turned up first in Dawlings' [D2] 1983 study of products of idempotent transformations of a separable Hilbert space.) This prompted Professor Howie to ask (the second author) whether this depth 3 was a quirk or was there a ring-theoretic explanation for it. In §2 we provide just such an explanation. Let S be the semigroup generated by the idempotents of a unit-regular or regular, right self-injective ring R. Then S can be split up into a disjoint union of regular, idempotent-generated subsemigroups S_I labelled by the principal ideals I of R, namely

$$S_I = \{a \in R \mid Rr(a) = \ell(a)R = R(1-a)R = I\}$$

(which is the "slice" consisting of all "balanced elements of R of weight I"). Moreover, if R is regular, right self-injective, the depth of S_I is determined as follows (Theorem 2.12): if I is directly finite, then $\triangle(S_I) = index(I)$; if I is purely infinite, then $\triangle(S_I) = 3$; in the mixed case, if I is directly infinite, then $\triangle(S_I) = max\{index(I_1), 3\}$ where I_1 is the directly finite part of I. These subsemigroups S_I agree with the Reynolds and Sullivan components in the full linear case because of the natural labelling of the ideals of a full linear ring (or any prime, regular, right self-injective ring) by cardinals. Thus their result is the special case where the regular, right self-injective ring R is a full linear ring. However, as a guide to the limit of such results, we give an example of a directly finite regular ring with comparability for which $\triangle(S_I) \neq index(I)$ for the principal ideal $I = R$.

This broader setting for the slices S_I brings to light a phenomenon that was too simple to be worth noticing in Reynolds and Sullivan's full linear ring case, and indeed in Howie's full transformation case, namely, that when R is unit-regular or regular, right self-injective, then S is a semilattice of the semigroups S_I. This last statement means $S = \dot{\cup} S_I$ and $S_I S_J \subseteq S_{I+J}$, where we are viewing the set of principal ideals of R as a semilattice under addition. In particular, S can be mapped homomorphically onto the additive semilattice of principal ideals of R. Even for a general regular ring R, our techniques show that the map $a \mapsto R(1-a)R$ is a semigroup homomorphism from the *multiplicative* semigroup generated by the idempotents of R onto the *additive* semilattice of principal ideals of R. In view of the complexity of this semilattice in general, we find this "duality" in such a general setting a little surprising.

In the final section, §3, we show that for a unit-regular ring or a regular, right self-injective ring R, the "bottom slice" S_R is precisely the subsemigroup generated by the nilpotent elements of R (Corollary 3.4).

Preliminaries

All rings considered here are associative with an identity element. Semigroups, on the other hand, need not have an identity. The unqualified term *ideal* always refers to a two-sided ideal. For a subset X of a ring R, we let $r(X) = \{a \in R \mid Xa = 0\}$ denote the *right annihilator* of X in R, or if the parent ring is in doubt, the notation is $r_R(X)$. Similarly $\ell(X)$ or $\ell_R(X)$ denotes the left annihilator. A module A is *subisomorphic* to a module B, written $A \lesssim B$, if A is isomorphic to a submodule of B. The direct sum of n copies of a module A is denoted nA (for n a positive integer).

A ring (or semigroup) R is (von Neumann) *regular* if for each $x \in R$ there exists $y \in R$ such that $x = xyx$. If y can always be chosen to be a unit (i.e. an invertible element), then R is *unit-regular*. All our rings are regular. Throughout the paper, the reader will frequently be referred to Goodearl's book [G] for properties of regular rings. Any unexplained notation or terms relating to regular rings can be found there. A ring R is *right self-injective* if the module R_R is injective.

A module A is *directly finite* if A is not isomorphic to a proper direct summand of itself. Otherwise A is *directly infinite*. A module A is *purely infinite* if it does not contain any nonzero, fully invariant, directly finite, direct summands. A ring R is *directly finite* if $xy = 1$

implies $yx = 1$ (this is equivalent to the module R_R being directly finite); otherwise R is *directly infinite*. A regular ring R is called *purely infinite* if it contains no nonzero, directly finite, central idempotents (this is equivalent to the module R_R being purely infinite).

An *abelian* regular ring is one in which all idempotents are central. Any property P already defined for regular rings can also be conferred on an idempotent e of a regular ring R by the convention that e has P when the corner ring eRe has P. Thus, for example, we can speak of abelian idempotents, directly finite idempotents, and so on. Incorporating these latter two notions is a well-developed theory of types for regular, right self-injective rings (see [G, Chapter 10]): any such ring R is uniquely a direct product of rings of Types I, II, and III, where (roughly speaking) Type I rings have lots of abelian idempotents, Type II rings have no nonzero abelian idempotents but lots of directly finite ones, and Type III rings have no nonzero directly finite idempotents. The archetypical Type I ring, for example, is a direct product of full linear rings over division rings.

A regular ring R satisfies the *comparability axiom* if for any $x, y \in R$ either $xR \lesssim yR$ or $yR \lesssim xR$, while it satisfies *general comparability* if for any $x, y \in R$ there is a central idempotent $e \in R$ such that $exR \lesssim eyR$ and $(1-e)yR \lesssim (1-e)xR$. By [G, Corollary 9.15] any right self-injective regular ring satisfies general comparability.

The nilpotent *index* of an ideal J of a ring R is the least positive integer n such that $x^n = 0$ for all nilpotent elements $x \in J$, if such an n exists, otherwise the index is ∞. We denote the index of J by $index(J)$. If $index(J) < \infty$, we say J has *bounded index* of nilpotence.

Finally, we recall the *depth* of an idempotent-generated semigroup S (defined in the introduction). It is customary to denote the depth of S by $\triangle(S)$. Occasionally we also refer to the depth of an element x in S, which is the least n that works for this particular x. In what follows the semigroup S will be the multiplicative semigroup generated by all the idempotents of a regular ring R, or various idempotent-generated subsemigroups of this. Here we can take $\triangle(S)$ as a measure of how far this semiband is from being a band (where all elements are idempotent). We remark that the case where S is a band

($\triangle(S) = 1$) occurs exactly when R is an abelian regular ring, because if R is not abelian, it must contain nonzero nilpotent elements [G, Theorem 3.2] and these are in S (see Lemma 1.2).

1 Depth of the full idempotent-generated semigroup

In this section, we examine the depth of the multiplicative semigroup S generated by all the idempotents in a regular ring R. Our goal is to relate the depth of S to the index of nilpotency of R when R is directly finite (Theorem 1.3) or right self-injective (Theorem 1.6). As a first step, we have the following general relationship.

Proposition 1.1 *Let R be any regular ring and let S be the multiplicative semigroup generated by all its idempotents. Then $\triangle(S) \leq index(R)$.*

Proof. If $index(R) = \infty$ there is nothing to prove, so suppose that $index(R) = n < \infty$ and let $a \in R$ be a product of idempotents. We must show that a is a product of (at most) n idempotents. But since R has bounded index, it is unit-regular [G, Corollary 7.11] and so, by [HO, Theorem 1.2], it is enough to show that $(1-a)R \lesssim n(r(a))$. Since R is unit-regular it is thus enough, by [G, Theorem 4.19], to show that for any prime ideal P of R we have

$$(1-a)R/(1-a)P \lesssim n[r(a)/r(a)P].$$

Let P be a prime ideal of R and let $\bar{}$ denote the homomorphism onto $\bar{R} = R/P$. Choose $x \in R$ such that $a = axa$ and so $r(a) = (1-xa)R$. Since $index(R) = n$, $\bar{R}$ is a $k \times k$ matrix ring over a division ring and $k \leq n$ by [G, Theorem 7.9]. Since a is a product of idempotents, so is $\bar{a}$ and hence $\bar{a}$ is a product of $k \leq n$ idempotents by [B] or [HO, Theorem 1.2]. Thus by [HO, Proposition 1.1] we have $\overline{(1-a)R} = (\bar{1}-\bar{a})\bar{R} \lesssim n(r_{\bar{R}}(\bar{a})) = n(\bar{1}-\bar{x}\bar{a})\bar{R} = n(\overline{r(a)})$, which is what we had to show. Hence $\triangle(S) \leq n$, as required. □

In the directly finite case we actually have $\triangle(S) = index(R)$ but to see this we need the following (probably well-known) result.

Lemma 1.2 *Let R be a regular ring and let $a \in R$ with $a^n = 0$. Then a is a product of n idempotents in R.*

Proof. We use induction on n. The case $n = 1$ is trivial, so suppose the result is true for elements satisfying $x^{n-1} = 0$ and let $a \in R$ with $a^n = 0$. By [G, Lemma 7.1] there is a decomposition $R = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ such that $aA_n = 0$ and $aA_i = A_{i+1}$ for $1 \leq i \leq n-1$. Hence there are orthogonal idempotents $e_1, \ldots, e_n$ summing to 1, such that $ae_n = 0$ and $ae_iR = e_{i+1}R$ for $1 \leq i \leq n-1$. Hence $a = ae_1 + \cdots + ae_{n-1} = e_2ae_1 + \cdots + e_nae_{n-1}$. Let $x = e_3ae_2 + \cdots + e_nae_{n-1}$. Then $x \in (1-e_1)R(1-e_1)$ and since the idempotents are orthogonal it is easy to see that $x^{n-1} = 0$. Since $(1-e_1)R(1-e_1)$ is a regular ring, by induction there are idempotents $f_1, \ldots, f_{n-1} \in (1-e_1)R(1-e_1)$ such that $x = f_1 \cdots f_{n-1}$. Hence $e_1 + x = (e_1 + f_1) \cdots (e_1 + f_{n-1})$ is a product of $n-1$ idempotents in R. Since $[e_2ae_1 + (1-e_1)](e_1 + x) = e_2ae_1 + x = a$ and since $e_2ae_1 + (1-e_1)$ is idempotent, it follows that a is a product of n idempotents, as required. □

Theorem 1.3 *Let R be a directly finite regular ring and let S be the multiplicative semigroup generated by all its idempotents. Then $\triangle(S) = index(R)$. (In particular, finite depth for S is equivalent to bounded index of nilpotence for R.)*

Proof. By Proposition 1.1 we just have to show that $index(R) \leq \triangle(S)$ so we may assume that $\triangle(S) = n < \infty$. Suppose that $index(R) > n$ so that by [G, Theorem 7.2] R contains $n+1$ independent, pairwise isomorphic, nonzero right ideals. We may write these as $e_1R, \ldots, e_{n+1}R$ where the e_i are orthogonal idempotents. From the isomorphisms we get elements $e_{ij} \in e_iRe_j$ such that $e_{ij}e_{ji} = e_i$ whenever $i \neq j$. Let $y = e_{21} + e_{32} + \cdots + e_{n+1,n}$ and $e = e_1 + \cdots + e_{n+1}$ and consider the element $a = y + (1-e)$. Notice that since $y \in eRe$ satisfies $y^{n+1} = 0$, Lemma 1.2 shows that a is a product of $n+1$ idempotents in R. We show that a cannot be written as a product of n idempotents, and this contradiction of $\triangle(S) = n$ will complete the proof. If a is a product of n idempotents, then $(1-a)R \lesssim n(r(a))$ by [HO, Proposition 1.1]. But it is easy to see that $(1-a)R = (e-y)R = eR$ while $r(a) = e_{n+1}R$. Hence $(1-a)R \cong (n+1)r(a)$ and so $(1-a)R \lesssim n(r(a))$ contradicts the fact that R is directly finite. Thus a is not a product of n idempotents. $\square$

Corollary 1.4 *Let R be a unit-regular ring and let $T = M_n(R)$ be the ring of $n\times n$ matrices over R. Let S and U be respectively the semigroups generated by all the idempotents of R and T. Then*

$$\triangle(U) = n \triangle (S).$$

Proof. By [G, Theorem 7.12], $index(T) = n(index(R))$. Thus since R and T are both directly finite [G, Proposition 5.2], we have by Theorem 1.3 that

$$\triangle(U) = index(T) = n(index(R)) = n \triangle (S).$$

$\square$

To obtain Ballantine's formula for depth in the case of $n \times n$ matrices over a field, notice that when we take R to be a field F (or any abelian regular ring for that matter), the formula in Corollary 1.4 gives the depth of the full idempotent-generated semigroup of $M_n(F)$ as n (because then $\triangle(S) = 1$).

The equation $\triangle(S) = index(R)$ does not hold for all regular rings as we shall now see by calculating $\triangle(S)$ for right self-injective regular rings. Firstly we recall a result proved in [HO, Lemma 2.7].

Lemma 1.5 *Let R be a regular ring satisfying general comparability. If $e, f \in R$ are idempotents such that*

$$eR \lesssim (1-e)R \qquad \text{and} \qquad fR \lesssim (1-f)R$$

then each $a \in eRf$ is a product $e_1e_2e_3$ of three idempotents each of which satisfies $e_iR \lesssim (1-e_i)R$. $\square$

Theorem 1.6 *Let R be a right self-injective regular ring and let S be the semigroup generated by its idempotents. Then*

$$\triangle(S) = index(R)$$

except when R is a direct product $R_1 \times R_2$ of a ring R_1 of bounded index of nilpotence and a nonzero ring R_2 of Type III, in which case

$$\triangle(S) = max\{index(R_1), 3\}.$$

Proof. Firstly suppose R is a nonzero ring of Type III. We shall show by an argument used in [HO] that then $\triangle(S) = 3$ (whereas in this case, $index(R) = \infty$ because R is directly infinite). Since R satisfies $R_R \cong 2R_R$, [HO, Example 2.2] shows that $\triangle(S) \geq 3$. Suppose on the other hand that $a \in S$. We want to show that a is a product of 3 idempotents and by the reduction technique in [HO, Lemma 2.5] (restated below in Lemma 2.2), we may assume that

$$Rr(a) = \ell(a)R = R \tag{1}$$

(since the ring gRg in the Lemma is still of Type III). There are idempotents $e, f \in R$ such that $eR = aR$ and $Rf = Ra$, and so (1) gives us $R(1-f)R = R(1-e)R = R$. Since $a \in eRf$ it is thus enough by Lemma 1.5 to show that $eR \lesssim (1-e)R$ and $fR \lesssim (1-f)R$. Hence we just need to show that $RhR = R$ implies $R \lesssim hR$. But if $RhR = R$ then $R \lesssim n(hR)$ for some integer n (by [G, Corollary 2.23]). Since R is Type III, it is purely infinite and so $nR \cong R \lesssim n(hR)$ (by [G, Theorem 10.16]). Hence $R \lesssim hR$ by [G, Theorem 10.34]. Thus a is a product of 3 idempotents and so $\triangle(S) = 3$ as required.

It now follows from Theorem 1.3 that if $R = R_1 \times R_2$ is a direct product of a ring R_1 of bounded index and a nonzero ring R_2 of Type III, then $\triangle(S) = max\{index(R_1), 3\}$ and certainly $\triangle(S) \neq index(R) = \infty$ in this case.

Conversely, suppose that $\triangle(S) \neq index(R)$. We shall prove that R is a direct product of a ring of bounded index and a nonzero ring of Type III. By Proposition 1.1 we know $\triangle(S) < \infty$, say $\triangle(S) = n$. We first show that R cannot contain $n+1$ independent copies of a nonzero directly finite right ideal e_1R. Suppose, on the contrary that it does. As in the proof of Theorem 1.3 we can then find an element $a \in R$ such that $(1-a)R \cong (n+1)r(a)$ where $r(a) \cong e_1R$, and such that a is a product of $n+1$ idempotents. As before $\triangle(S) = n$ implies that $(1-a)R \lesssim n(r(a))$ by [HO, Proposition 1.1]. Thus $(n+1)e_1R \lesssim n(e_1R)$. Since R is right self-injective and e_1R is directly finite, [G, Corollary 9.19] says that we can cancel e_1R factors in such an embedding and so we get the contradiction $e_1R = 0$.

Now let $R = R_1 \times R_2 \times R_3$ be the decomposition of R into its Type I,II and III parts, respectively [G, Theorem 10.13]. Since every nonzero right ideal of R_1 contains a nonzero abelian idempotent and since any abelian idempotent is directly finite, the previous paragraph shows that R_1 cannot contain $n+1$ independent, pairwise isomorphic, nonzero right ideals. By [G, Theorem 7.2], it follows that $index(R_1) \leq n$. To complete the proof we just have to show that $R_2 = 0$, and then Theorem 1.3 will ensure $R_3 \neq 0$ because $\triangle(S) \neq index(R)$. But if $R_2 \neq 0$ then R contains a nonzero directly finite idempotent e such that eR contains no nonzero abelian idempotents. By [G, Proposition 10.28] applied to the ring eRe there would then be an idempotent $e_1 \in eRe$ such that $eR \cong (n+1)e_1R$. Since this e_1R would also be directly finite, the previous paragraph shows that this cannot happen and so $R_2 = 0$ as required. □

Remark 1.7 *As noted in the proof, if R itself is of Type III, then $\triangle(S) = 3$ while $index(R) = \infty$.* □

Regular right self-injective rings of bounded index, being finite direct products of full $n \times n$ matrix rings over abelian self-injective regular rings [G, Theorem 7.20], are vastly different objects from those of Type III. In a ring R of Type III every nonzero corner ring eRe (where $e = e^2 \in R$) contains an infinite-dimensional matrix ring since it is directly infinite. (Perhaps the simplest example of such a ring is the maximal right quotient ring of a proper factor ring of an infinite dimensional full linear ring.) Thus the joint appearance of bounded index and Type III in the following corollary is unusual.

Corollary 1.8 *Let S be the semigroup generated by the idempotents of a regular right self-injective ring R. Then $\triangle(S) < \infty$ if and only if R is a direct product of a ring of bounded index of nilpotence and a ring of Type III.* □

2 Depth of the slices S_I

Both Howie [H2] and Reynolds and Sullivan [RS] study the depth of their idempotent-generated semigroups in more detail by decomposing the semigroups into disjoint unions of a "finite" part and "infinite" parts, the latter labelled by infinite cardinal numbers. Each of these parts is in turn an idempotent-generated regular subsemigroup of the original semigroup and so each has a depth in its own right. In [HO, Theorem 2.8] we showed that in the context of regular rings, Reynolds and Sullivan's characterization of products of idempotent linear transformations, in terms of vector space dimension of certain associated subspaces, could be replaced by the condition

$$(*) \qquad Rr(a) = \ell(a)R = R(1-a)R$$

on the associated principal ideals in the ring (for $a \in R$). This suggests that we look at subsemigroups of S consisting of elements associated with a common principal ideal of R.

Definition 2.1 *Let R be a regular ring. Given any principal ideal I of R, we set*

$$S_I = \{a \in R \mid Rr(a) = \ell(a)R = R(1-a)R = I\}.$$

□

Thus S_I consists of all elements of R for which the ideals in condition $(*)$ take the common value I. Following Howie [H2] we say that $a \in R$ is *balanced* if it satisfies $(*)$ and that its *weight* is I if $a \in S_I$. In the case where R is the full linear ring $End_F(V)$ these sets of balanced elements correspond to the components of the decomposition considered by Reynolds and Sullivan in [RS]. Their finite part $\mathcal{F}$ consists of those transformations a on V for which

$$0 < n(a) = d(a) \leq s(a) < \aleph_0$$

where $n(a) = dim(ker(a)), d(a) = codim(Im(a)), s(a) = codim\{v \in V : av = v\}$ and this is precisely the set S_I where $I = soc(R)$ is the ideal of all transformations of finite rank (see [OM, Corollary 12]). On the other hand their infinite parts are defined as the subsemigroups

$$\mathcal{I}(\epsilon) = \{a \in R : n(a) = d(a) = s(a) = \epsilon\}$$

where $\aleph_0 \leq \epsilon \leq dim\, V$, and these are just the sets S_I where I is the ideal of R consisting of all transformations of rank $\leq \epsilon$ (again see [OM, Corollary 12]). Notice that with our labelling of the components S_I by the principal ideals I of R, the finite part no longer needs to be singled out for a separate notation.

In this section we show that these "slices" S_I are regular subsemigroups of R, which are idempotent-generated if R is unit-regular or right self-injective. Our principal result (Theorem 2.8) is a precise determination of $\triangle(S_I)$ when R is regular, right self-injective. This yields the Reynolds and Sullivan full linear case as a special instance.

The following "corner reduction" technique was introduced in [OM] and also used in [HO, Lemma 2.5]. Here we restate it in terms of S_I. It simplifies many situations by allowing us to pass from a balanced element $a \in R$ of weight I (that is, $a \in S_I$) to an element y in some corner A of I such that, in the ring A, y is a balanced element of maximum weight (that is, $y \in S_A$).

Lemma 2.2 *Let I be a principal ideal of a regular ring R and let $a \in S_I$. Then there exists an idempotent $g \in I$ such that $I = RgR$ and an element $y \in A \equiv gRg$ such that*

$$a = y + (1 - g),$$

and in the ring A, $y \in S_A$. Conversely, if $g \in I$ is an idempotent with $I = RgR$ (that is, $1 - g \in S_I$) and $y \in S_A$, then $y + (1 - g)$ is in S_I. □

Proposition 2.3 *For any principal ideal I of a regular ring R, the set S_I is a regular subsemigroup of the multiplicative semigroup of R. If R is unit-regular or regular, right self-injective, then S_I is idempotent-generated.*

Proof. Since I is a principal ideal in a regular ring, $I = RgR$ for some $g = g^2 \in R$. Then $1 - g \in S_I$, which establishes that S_I is nonempty. Let $a, b \in S_I$. Noting that $1 - ab = a(1 - b) + (1 - a)$, we have

$$Rr(ab) \supseteq Rr(b) = I \supseteq R(1 - b)R + R(1 - a)R \supseteq R(1 - ab)R$$

and so $Rr(ab) = R(1 - ab)R = I$. Similarly $\ell(ab)R = I$ and therefore $Rr(ab) = \ell(ab)R = R(1 - ab)R = I$. Thus $ab \in S_I$ and hence S_I is a subsemigroup of R.

To show S_I is regular, it is sufficient by Lemma 2.2 to consider the case $I = R$. Let $a \in S_R$. Because R is a regular semigroup, there exists $b \in R$ with $a = aba$ and $b = bab$. Observing that $\ell(a) = R(1 - ab)$ and $r(b) = (1 - ab)R$, we see that $Rr(b) = \ell(a)R = R$. Similarly $\ell(b)R = R$. Hence $Rr(b) = \ell(b)R = R(1 - b)R = R$, which shows b is the desired "inverse" in S_R of a.

Now assume R is unit-regular. Let $a \in S_I$. By [HO, Corollary 1.4], a is a product $f_1 f_2 \cdots f_k$ of idempotents of R. By [HO, Remark 1.3], the f_i can be chosen such that $f_i R \cong aR$. Then, because R is unit-regular, $(1 - f_i)R \cong r(a)$ and hence $R(1 - f_i)R = Rr(a) = I$. Thus each $f_i \in S_I$ which shows that S_I is idempotent-generated.

It will be shown in Theorem 2.12 that S_I is also idempotent-generated when R is regular, right self-injective. □

When $(*)$ characterizes products of idempotents in a regular ring R, then the S_I give a disjoint covering of the semigroup S generated by all the idempotents of R. Our next proposition gives a coarse description of the products ab for $a \in S_I$ and $b \in S_J$.

Proposition 2.4 *Let R be a regular ring.*
(1) If I and J are principal ideals of R, then

$$S_I S_J \subseteq S_{I+J}.$$

(2) If the condition

$$(*) \qquad Rr(a) = \ell(a)R = R(1-a)R$$

characterizes products of idempotents in R (for example, if R is unit-regular or regular right self-injective), then the multiplicative semigroup S generated by all the idempotents of R is a semilattice of the semigroups S_I, where I ranges over the additive semilattice of principal ideals of R.

Proof. (1) Observe that if $x \in R$, then $\ell(x)R \subseteq RyR$ for any $y \in R$ for which $xR+yR = R$. For if we choose $e = e^2 \in R$ with $xR = eR$ and $(1-e)R \subseteq yR$, then $\ell(x)R = R(1-e)R \subseteq RyR$.

Now let $a \in S_I$ and $b \in S_J$. We first show $\ell(ab)R = Rr(ab)$ using the corresponding properties of a and b. Since R is regular, there is a decomposition $r(ab) = r(b) \oplus fR$ for some $f \in R$. Then $bfR = r(a) \cap bR$ and so $fR \cong r(a) \cap bR$. Hence

$$Rr(ab) = Rr(b) + R(r(a) \cap bR). \tag{1}$$

Similarly

$$\ell(ab)R = \ell(a)R + (\ell(b) \cap Ra)R. \tag{2}$$

Also by regularity, there exists $h \in R$ such that $R = (r(a) + bR) \oplus hR$. Note that $RhR = \ell(r(a) + bR)R = (\ell(b) \cap Ra)R$. Using our initial observation, we have from $R = bR + (r(a) + hR)$ and $r(b) \subseteq \ell(b)R$ that $r(b) \subseteq Rr(a) + RhR$, and hence $r(b) \subseteq$

$Rr(a) + (\ell(b) \cap Ra)R$. Hence from (1) and (2) and the fact that $Rr(a) = \ell(a)R$ (because $a \in S_I$), we have

$$\begin{aligned} Rr(ab) &= Rr(b) + R(r(a) \cap bR) \\ &\subseteq Rr(a) + (\ell(b) \cap Ra)R \\ &= \ell(a)R + (\ell(b) \cap Ra)R \\ &= \ell(ab)R \end{aligned}$$

which establishes that $Rr(ab) \subseteq \ell(ab)R$. The reverse containment follows from symmetry, and so $Rr(ab) = \ell(ab)R$.

To complete the proof that $ab \in S_{I+J}$, note that from $1 - ab = (1-a) + a(1-b)$ we have $R(1-ab)R \subseteq R(1-a)R + R(1-b)R = I + J$. Also $I = \ell(a)R \subseteq \ell(ab)R$ and $J = Rr(b) \subseteq Rr(ab)$, hence $I + J \subseteq \ell(ab)R + Rr(ab) = Rr(ab)$. Since $Rr(ab) \subseteq R(1-ab)R$, we have

$$Rr(ab) = \ell(ab)R = R(1-ab)R = I + J$$

which says $ab \in S_{I+J}$. Therefore $S_I S_J \subseteq S_{I+J}$.

(2) When $(*)$ characterizes products of idempotents, S is the disjoint union of the semigroups S_I as I ranges over the principal ideals of R. The set of principal ideals of R forms a semilattice (commutative semigroup of idempotents) under addition, whence by (1) we have that S is a semilattice of the S_I (see [H1,p89] for this concept). □

Remark 2.5 Because of (1), and the fact that $(*)$ is a necessary condition for an element to be a product of idempotents (see [HO, Proposition 2.3]), the map

$$a \mapsto R(1-a)R$$

is always a semigroup homomorphism from the *multiplicative* semigroup S generated by the idempotents of R onto the semilattice of principal ideals of R *under addition* - an unexpected duality in such a general setting. In particular, if we choose generators from the set $\{1 - a \mid a \in S\}$, then principal ideals add according to the rule

$$RxR + RyR = R(x + y - xy)R,$$

which extends the well-known rule for adding central idempotents. What (2) is saying is that when (∗) characterizes products of idempotents, the congruence classes of the kernel of the above homomorphism are precisely the semigroups S_I, as I ranges over the principal ideals of R. □

The semilattice structure of the semigroup S was of course present in Reynolds and Sullivan's full linear ring case, and indeed in Howie's full transformation semigroup. However in both cases the semilattice was too simple to be worth noticing, being essentially a set of infinite cardinal numbers under addition (see [H2, Lemma 2.10] for example). For general regular rings, though, the semilattice can be quite complicated.

Example 2.6 Let B be any Boolean ring and n a positive integer, and let $R = M_n(B)$. Then R has index n and so is unit regular [G, Theorem 7.12 and Corollary 7.11]. Hence by Proposition 2.4 the semigroup S generated by the idempotents of R is a semilattice of the subsemigroups S_I. On the other hand, each principal ideal of R is generated by a central idempotent of B because finitely generated ideals of B are generated by central idempotents. Thus we see that the semilattice of principal ideals of R (under $+$) is isomorphic to the semilattice of idempotents of B (under $\vee$). In particular, by choosing B to have no atoms we see that the semilattice need not be a product of well-ordered chains. Of course, the example would be simpler if $n = 1$, in which case $S = R = B$, but then the subsemigroups S_I would be trivial, consisting of just one idempotent element each. It is perhaps worth noticing that in this case, the homomorphism $a \mapsto R(1 - a)R$ mentioned in Remark 2.5 from the semigroup S onto the semilattice of principal ideals is essentially the complement map $e \mapsto 1 - e$ of the Boolean algebra structure on B which maps the multiplicative structure to the additive structure ($(e \wedge f)' = e' \vee f'$). The next lemma, applied to the present example, shows for a general n that if I is a nonzero principal ideal of R, then

S_I is idempotent-generated with $\triangle(S_I) = index(I) = n$ (so here these slices have uniform depth). By Corollary 1.4, S itself also has depth n. Also, if $n > 1$ and B is infinite, then the S_I are infinite. $\square$

We now aim to describe the depth of S_I when R is a regular, right self-injective ring (Theorem 2.12). As preliminaries we need a number of lemmas.

Lemma 2.7 *If I is a principal ideal of bounded index in a regular ring R, then S_I is idempotent-generated with*

$$\triangle(S_I) = index(I).$$

Proof. Let $a \in S_I$. By Lemma 2.2 there exists $g = g^2 \in R$ with $RgR = I$ such that

$$a = y + (1 - g)$$

for some $y \in S_A$, where $A = gRg$. Certainly $index(A) < \infty$ because $A \subseteq I$. By [G, Corollary 7.11], A is unit-regular. Hence by Propositions 1.1, 2.3 and [HO, Remark 1.3], y is a product of $k = index(A)$ idempotents in S_A, say $y = g_1 g_2 \cdots g_k$. Then $a = [g_1 + (1-g)][g_2 + (1-g)] \cdots [g_k + (1-g)]$ is a product of k idempotents in S_I. Since $k = index(A) \leq index(I)$, this shows that S_I is idempotent-generated with $\triangle(S_I) \leq index(I)$.

To show that $\triangle(S_I) = index(I)$ we just need to find an element of S_I which is a product of $n = index(I)$ idempotents in R but no fewer. By [G, Lemma 7.17] there is a central idempotent $e \in I$ such that $eR \cong M_n(A)$ for some abelian regular ring A. Thus eR is a directly finite regular ring and is isomorphic to a direct sum of n isomorphic right ideals. As in the proof of Theorem 1.3 there is some $y \in eR$ such that $(e - y)R = eR$ and y is a product of n idempotents in eR but no fewer. Since I is principal there is an idempotent $f \in (1 - e)R$ such that $I = eR + RfR$. Now let $x = y + (1 - e - f)$ so that x is a product of n idempotents in R but no fewer (since e is central). Also $R(1 - x)R = I$ and so $x \in S_I$ as required. $\square$

For regular rings (with identity), and modules over such rings, the concepts of directly finite, directly infinite, and purely infinite are well established. We now formulate their appropriate analogues for any principal ideal $I = RwR$ of a regular, right self-injective ring R, in terms of the R-module wR. In this situation, the result is independent of the generator w. However, in view of the open problem [G, Problem 1,p344] of whether $M_n(R)$ is always directly finite for a general directly finite regular ring R, we would be unable to make this claim of independence even for the concept of a directly finite principal ideal in an arbitrary regular ring.

Definition 2.8 *Let $I = RwR$ be a principal ideal of a regular, right self-injective ring R. Then I is called* directly finite, directly infinite, *or* purely infinite *according to whether the R-module wR is directly finite, directly infinite, or purely infinite, respectively.* □

Lemma 2.9 *The definition in 2.8 is independent of the generator w for I.*

Proof. Assume $I = RwR = RvR$ for some $w, v \in R$. Then by [G, Corollary 2.23], there are positive integers k and m such that $wR \lesssim k(vR)$ and $vR \lesssim m(wR)$.

Firstly suppose wR is directly finite. Then so is $m(wR)$ by [G, Corollary 9.20] because wR is a nonsingular injective module. Hence, since $vR \lesssim m(wR)$, we have that vR is directly finite. This argument shows that wR is directly finite (respectively, directly infinite) if and only if vR is directly finite (respectively, directly infinite).

Next suppose wR is purely infinite. Since wR is nonsingular and injective, by [G, Proposition 10.33] this means $wR \cong n(wR)$ for all positive integers n. Hence $k(wR) \cong wR \lesssim k(vR)$, which implies $wR \lesssim vR$ by [G, Theorem 10.34] because wR and vR are nonsingular injective modules. Also $vR \lesssim m(wR) \cong wR$ yields $vR \lesssim wR$, whence by the injectivity of wR and vR we have $wR \cong vR$ [G, Theorem 10.14]. Thus vR is purely infinite.

□

The following lemma generalizes the fact that any regular, right self-injective ring is uniquely a direct product of a directly finite ring and a purely infinite ring [G, Proposition 10.21].

Lemma 2.10 *Let I be a principal ideal of a regular, right self-injective ring R. Then I is uniquely a direct sum*

$$I = I_1 \oplus I_2$$

of a directly finite principal ideal I_1 and a purely infinite principal ideal I_2. Moreover, there is a central idempotent e of R such that $I_1 = eI$ and $I_2 = (1-e)I$.

Proof. Suppose $I = RwR$ where $w \in R$. Applying [G, Theorem 10.32] to the nonsingular injective module wR, we see that there exists a central idempotent $e \in R$ such that ewR is directly finite and $(1-e)wR$ is purely infinite. Then

$$I = eI \oplus (1-e)I$$

where $eI = RewR$ is a directly finite principal ideal and $(1-e)I = R(1-e)wR$ is a purely infinite principal ideal.

For the uniqueness, suppose also $RwR = J_1 \oplus J_2$ where J_1 is directly finite and J_2 is purely infinite. Write $w = w_1 + w_2$ where $w_i \in J_i$. Then $J_i = Rw_iR$. Hence we have a decomposition $wR = w_1R \oplus w_2R$ of the nonsingular injective module wR as a direct sum of a fully invariant directly finite submodule w_1R and a fully invariant purely infinite submodule w_2R. By the uniqueness of such decompositions [G, Theorem 10.31], we have $w_1R = ewR$ and $w_2R = (1-e)wR$ where e is the central idempotent above. Thus $J_1 = eI$ and $J_2 = (1-e)I$. □

Lemma 2.11 *Let I be a principal ideal of a regular ring R and let e be a central idempotent of R. Then*

$$S_I = S_{eI} \times S_{(1-e)I}.$$

Proof. It is easily verified that the map

$$a \mapsto (ea + (1-e), e + (1-e)a)$$

provides a semigroup isomorphism from S_I onto $S_{eI} \times S_{(1-e)I}$. Internally, $S_I = S_{eI} S_{(1-e)I}$ and each $a \in S_I$ has a unique representation as a product, namely, $a = [ea + (1-e)][e + (1-e)a]$. Moreover, these factors commute and so we are justified in writing S_I as an internal direct product of S_{eI} and $S_{(1-e)I}$ (despite the fact that these semigroups do not necessarily have an identity, and so do not have natural copies inside S_I). □

We are now in a position to present the main result of this section. It is worth noticing that the formulae given in the following theorem for the depth of the slices S_I in a regular, right self-injective ring R are much "cleaner" than the corresponding formulae for the depth of the full idempotent-generated semigroup S in Theorem 1.6. For instance, if R is purely infinite, S may have infinite depth (for example, if R is an infinite-dimensional full linear ring) or finite depth (for example, R of Type III), whereas for a purely infinite ideal I, S_I always has depth 3.

Theorem 2.12 *Let I be a principal ideal of a regular, right self-injective ring R. Then S_I is a regular, idempotent-generated semigroup whose depth is determined as follows:*

(1) If I is directly finite, then

$$\triangle(S_I) = index(I).$$

(2) If I is purely infinite, then

$$\triangle(S_I) = 3.$$

(3) If I is directly infinite, and $I = I_1 \oplus I_2$ is the unique decomposition of I as a direct sum of a directly finite ideal I_1 and a purely infinite ideal I_2 , then

$$S_I = S_{I_1} \times S_{I_2}$$

and

$$\triangle(S_I) = max\{index(I_1), 3\}.$$

Proof. By Lemma 2.3, S_I is a regular semigroup. In the course of proving (1),(2), and (3), we show that S_I is idempotent-generated.

(1) We consider first the special case $I = R$. Then by Lemma 2.9 R is directly finite, and hence unit-regular by [G, Theorem 9.17]. Let $n \leq index(R)$ be a positive integer. Then by [G, Theorem 7.2] there exist nonzero orthogonal idempotents $e_1, \cdots, e_n \in R$ such that $e_iR \cong e_jR$ for all i, j. Let $e = e_1 + e_2 + \cdots + e_n$. Since R is directly finite and right self-injective, by [G, Theorem 9.25] the ideal ReR contains a nonzero central idempotent f of R. Let $B = fR$. In the ring B,

$$feB = fe_1B \oplus \cdots \oplus fe_nB$$

with $fe_iB \cong fe_jB$ for all i, j. Hence by the argument of Theorem 1.3, there exists $x \in B$ with $r_B(x) = fe_nB$ and $(f - x)B = feB \cong n(r_B(x))$. Now $Br_B(x) = Bfe_nB = BfeB = B$ whence $x \in S_B$ (relative to the ring B), and $Rr(x) = R$ which implies $x \in S_R$. Because B is unit-regular and $(f - x)B \cong n(r_B(x))$, we have by [HO, Theorem 1.2 and Remark 1.3] that x is a product of n idempotents in S_B. But, as B is directly finite, we cannot have $(f - x)B \lesssim m(r_B(x))$ for any $m < n$, whence [HO, Proposition 1.1] shows that x has depth exactly n in S_B. It follows that x also has depth n in S_R. From Proposition 2.3 we already know that S_R is idempotent-generated and we have just shown that $n \leq \triangle(S_R)$. Thus $index(R) \leq \triangle(S_R)$. By Lemma 2.7 we have $\triangle(S_R) \leq index(R)$, hence $\triangle(S_R) = index(R)$, completing the special case.

For the general case we use the corner reduction technique. By Lemma 2.2, if $a \in S_I$ then for some idempotent $g \in I$ with $RgR = I$, a takes the form

$$a = y + (1 - g)$$

where, if $A \equiv gRg, y \in S_A$. Conversely, any a of this form is in S_I. Since I is a directly finite ideal, A is a directly finite, regular right self-injective ring. From the above special case applied to $y \in S_A$, we see that a is therefore a product of idempotents in S_I. Suppose such a product involves k idempotents. Then $(1-a)R \lesssim k(r(a))$ by [HO, Proposition 1.1], which implies $(g-y)A \lesssim k(r_A(y))$ because $r_A(y) = r(a)g$. In turn, since A is unit-regular, this implies by [HO, Theorem 1.2 and Remark 1.3] that y is a product of k idempotents in S_A. It follows that

$$\text{depth of } a \text{ in } S_I = \text{depth of } y \text{ in } S_A.$$

Now by the earlier special case, we have

$$\begin{aligned}\triangle(S_I) &= sup\{\triangle(S_A) \mid A = gRg, g^2 = g, RgR = I\} \\ &= sup\{index(A) \mid A = gRg, g^2 = g, RgR = I\} \\ &= index(I)\end{aligned}$$

because $I = \bigcup\{gRg \mid g^2 = g \in R, RgR = I\}$ (by [HO, Lemma 2.4] for example).

(2) Again we first treat the special case $I = R$, and show $\triangle(S_R) \leq 3$. Thus by Lemma 2.9 we are supposing R is purely infinite. Let $a \in S_R$ and let $e, f \in R$ be idempotents with $aR = eR$ and $Ra = Rf$. Then $\ell(a)R = R = Rr(a)$ implies $R(1-e)R = R = R(1-f)R$, which as in the proof of Theorem 1.6 yields $R \lesssim (1-e)R$ and $R \lesssim (1-f)R$ because R is purely infinite. Hence by Lemma 1.5, $a = e_1e_2e_3$ for some idempotents $e_i \in R$ satisfying $e_iR \lesssim (1-e_i)R$. In particular, $R(1-e_i)R = R$, placing each e_i in S_R. Thus S_R is idempotent-generated with $\triangle(S_R) \leq 3$.

For the general case, if $a \in S_I$ then by Lemma 2.2 there is an idempotent $g \in R$ such that $RgR = I$ and a has the form

$$a = y + (1-g)$$

for some $y \in S_A$, where $A = gRg$. Since I is a purely infinite ideal, gR is a purely infinite R-module and so gRg is a purely infinite ring. Invoking the special case, we know $\triangle(S_A) \leq 3$ and

$$y = g_1g_2g_3 \qquad \text{with each } g_i \in S_A.$$

Now

$$a = [g_1 + (1-g)][g_2 + (1-g)][g_3 + (1-g)]$$

is a product of 3 idempotents in S_I. Thus S_I is idempotent-generated with $\triangle(S_I) \le 3$.

To show $\triangle(S_I) = 3$, fix an idempotent $h \in R$ with $RhR = I$, and let $B = hRh$. Since B is a purely infinite, regular, right self-injective ring, $B \cong 2B$. Hence by [HO, Example 2.2] there exists $z \in S_B$ with $r_B(z) \subseteq zB$ and $z^2 \ne z^3$. The element $x = z + (1 - h)$ is now in S_I but is not a product of two idempotents (even in R) by [HO,Lemma 2.1], because $r(x) \subseteq xR$ whereas $x^2 \ne x^3$. Thus $\triangle(S_I) = 3$.

(3) This follows directly from Lemmas 2.10 and 2.11 together with (1) and (2). □

Corollary 2.13 (Reynolds and Sullivan [RS]) *Let V_F be an infinite-dimensional vector space and let $L = L(V)$ be the full linear semigroup.*

(1) The set $\mathcal{F} = \{a \in L \mid 0 < n(a) = d(a) \le s(a) < \aleph_0\}$ is a regular, idempotent-generated subsemigroup of L of infinite depth.

(2) For $\aleph_0 \le \aleph \le dim\ V$, the set $\mathcal{I}(\aleph) = \{a \in L \mid n(a) = d(a) = s(a) = \aleph\}$ is a regular, idempotent -generated subsemigroup of depth 3.

(3) The semigroup S generated by all the proper ($\ne 1$) idempotent linear transformations is the union of $\mathcal{F}$ and the $\mathcal{I}(\aleph)$ for $\aleph_0 \le \aleph \le dim\ V$.

Proof. Let $R = End_F(V)$ be the full linear ring, which is a regular, right self-injective ring [G, Theorem 9.12]. Then $\mathcal{F} = S_J$ where $J = soc(R)$ is the ideal of all transformations of finite rank, while $\mathcal{I}(\aleph) = S_I$ where I is the ideal of all transformations of rank $\le \aleph$. Thus $\mathcal{F}$ and the $\mathcal{I}(\aleph)$ are regular, idempotent-generated subsemigroups by Proposition 2.3. Also $S = \mathcal{F} \cup \{\mathcal{I}(\aleph) \mid \aleph_0 \le \aleph \le dim\ V\}$ by Proposition 2.4(2) and the fact that a nonzero principal ideal of R is either J or of the form $\{a \in R \mid rank(a) \le \aleph\}$ for some $\aleph_0 \le \aleph \le dim\ V$. Inasmuch as J is a directly finite ideal of infinite index, $\triangle(\mathcal{F}) = \triangle(S_J) = \infty$ by Theorem 2.12(1). On the other hand, for $\aleph_0 \le \aleph \le dim\ V$, the ideal $I = \{a \in R \mid rank(a) \le \aleph\}$ is purely infinite, whence Theorem 2.12(2) says $\triangle(\mathcal{I}(\aleph)) = \triangle(S_I) = 3$. □

Remark 2.14 *Since R (above) is a prime ring, it has no nontrivial central idempotents and so the mixed case (3) in Theorem 2.12 does not occur.* $\square$

Although the relationship $\Delta(S) = index(R)$ holds for any directly finite, regular ring (Theorem 1.3), the connection in part (1) of Theorem 2.12 between depth and index at the subsemigroup level S_I breaks down in general, even for directly finite, regular rings with comparability, as the following example illustrates.

Example 2.15 *There is a directly finite ring R satisfying the comparability axiom and such that the subsemigroup S_R has depth 2 but R has index ∞ (cf. Theorem 2.12(1)).*

Proof. (See [G, Example 16.23].) Let V be a countable-dimensional vector space over a field F and let $Q = End_F(V)$, the elements of Q being viewed as column-finite $\aleph_0 \times \aleph_0$ matrices over F. For each integer n let J_n be the set of all $x \in Q$ of the form

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & 0 \end{array}\right)$$

where $*$ indicates an $n \times n$ block, and let $R_n = F + J_n$. Thus R_n is a subring of Q such that $R_n \cong F \times M_n(F)$ and so R_n is unit-regular. Also $R_1 \subseteq R_2 \subseteq \cdots$ so that if we let $R = \bigcup R_n$ then R is a unit-regular, and hence directly finite, subring of Q. It is easy to see that the only ideals of R are $0, R$ and $J = \bigcup J_n$. Also R satisfies the comparability axiom and has nilpotent index ∞. To see that $\Delta(S_R) = 2$ consider any $a \in S_R$. Then $Rr(a) = R$ implies that $a \in J$ and so $a \in J_n$ for some n. Hence there is an idempotent $e \in J_n$ with $a \in eRe$ and $eR \lesssim (1-e)R$. Then $e = yz$ for some $y \in eR(1-e)$ and $z \in (1-e)Re$. We now have the factorization (cf. Lemma 1.5)

$$a = [e + y][e + z(a - e)] \equiv e_1 e_2 \qquad \text{(say)}$$

where the idempotents e_1, e_2 satisfy $e_1 R = eR$ and $Re_2 = Re$ and hence both lie in S_R because $e \in S_R$. (Notice that this shortens the factorization given in [OM, Proposition 2].)

In matrix form this factorization is

$$a = \left[\begin{array}{cc|c} A & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} I & I & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cc|c} I & 0 & 0 \\ A-I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

where the entries in the top left-hand block are $n \times n$ matrices (see [L]). Hence $\triangle(S_R) = 2$.

Of course, by Theorem 1.3, we must have $\triangle(S) = index(R) = \infty$. What has happened here is that all the elements of S needing more than 2 idempotents have been gathered into the subsemigroup S_J. (By Proposition 2.4, $S = S_0 \cup S_J \cup S_R = \{1\} \cup S_J \cup S_R$.) □

A similar example, using the ring R of $\aleph_1 \times \aleph_1$ column-finite matrices which have an arbitrary countable-by-countable top left-hand block, a scalar down the rest of the diagonal, and zeroes elsewhere, produces a regular ring with comparability and a purely infinite ideal $I = R$ such that S_I is idempotent-generated with $\triangle(S_I) = 2$. Thus even parts (2) and (3) of Theorem 2.12 break down here.

3 Subsemigroup generated by the nilpotent elements

We have seen that any nilpotent element in a regular ring R is a product of idempotents (Lemma 1.2). Since $1 - a$ is a unit whenever a is nilpotent it follows that the nilpotent elements of R all belong to the subsemigroup S_R. In this section we characterize those idempotents which are products of nilpotent elements and show that as long as R is unit-regular or regular, right self-injective, then S_R is in fact the subsemigroup of R generated by all the nilpotent elements of R. We begin with a lemma which may be of independent interest.

Lemma 3.1 *Suppose A, B are finitely generated projective modules over a regular ring R and that k is a positive integer. Then $A \lesssim kB$ if and only if there is a decomposition $A = A_1 \oplus \cdots \oplus A_k$ such that*

$$A_1 \lesssim A_2 \lesssim \cdots \lesssim A_k \lesssim B.$$

Proof. It is easy to see that $A \lesssim kB$ is necessary for such a decomposition. We prove the converse by using induction on k, the case $k = 1$ being trivial. Suppose $A \lesssim kB$ where $k \geq 2$. By [G, Corollary 2.9] there is a decomposition $A = U \oplus W$ where $U \lesssim (k-1)B$ and $W \lesssim B$. By the induction hypothesis, there is a chain

$$C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{k-1}$$

of submodules of B such that $U \cong C_1 \oplus C_2 \oplus \cdots \oplus C_{k-1}$. Let $D_1, \cdots, D_k$ be submodules of B such that $D_1 = C_1, C_i = C_{i-1} \oplus D_i$ for $2 \leq i \leq k-1$, and $B = C_{k-1} \oplus D_k$. Then $B = D_1 \oplus \cdots \oplus D_k$. Inasmuch as $W \lesssim B$, [G, Corollary 2.9] shows that there are submodules $X_i \subseteq D_i$ such that $W \cong X_1 \oplus \cdots \oplus X_k$. Hence

$$\begin{aligned} A &= U \oplus W \\ &\cong C_1 \oplus \cdots \oplus C_{k-1} \oplus X_1 \oplus \cdots \oplus X_k \\ &= X_1 \oplus (C_1 \oplus X_2) \oplus (C_2 \oplus X_3) \oplus \cdots \oplus (C_{k-1} \oplus X_k) \end{aligned}$$

where

$$X_1 \subseteq C_1 \oplus X_2 \subseteq C_2 \oplus X_3 \subseteq \cdots \subseteq C_{k-1} \oplus X_k \subseteq B$$

and so the induction step is complete. □

Remark 3.2 *An equivalent formulation of the lemma would be that $A \lesssim kB$ if and only if there are decompositions*

$$A = A_1 \oplus \cdots \oplus A_k$$

$$\text{and} \qquad B = B_1 \oplus \cdots \oplus B_k \oplus B_{k+1}$$

such that $A_i \cong iB_i$ for $i = 1, \cdots, k$. □

Proposition 3.3 *Let R be any regular ring and let $e = e^2 \in R$. Then the following conditions are equivalent:*

(i) $e \in S_R$

(ii) $R(1-e)R = R$

(iii) *e is a product of 2 nilpotent elements of R*

(iv) *e is a product of nilpotent elements of R*

Proof. The equivalence of (i) and (ii) follows from the definition of S_R since $r(e) = (1-e)R$ and $\ell(e) = R(1-e)$. Also (iii) $\Rightarrow$ (iv) is trivial, while (iv) $\Rightarrow$ (i) because nilpotent elements all belong to S_R and S_R is closed under multiplication (Proposition 2.3). This leaves only (ii) $\Rightarrow$ (iii). So suppose $R(1-e)R = R$. By [G, Corollary 2.23] we have $eR \subseteq R \lesssim k(1-e)R$ for some integer k. By Lemma 3.1 there are orthogonal idempotents $e_1, \cdots, e_k$ such that $e = e_1 + \cdots + e_k$ and $e_1R \lesssim \cdots \lesssim e_kR \lesssim (1-e)R$. Let $e_{k+1} = 1-e$ and for $1 \leq i \leq k$ pick $x_i \in e_iRe_{i+1}$ and $y_i \in e_{i+1}Re_i$ such that $x_iy_i = e_i$. Then $x = x_1 + \cdots + x_k$ and $y = y_1 + \cdots + y_k$ satisfy $x^{k+1} = 0 = y^{k+1}$ and $xy = e$. Thus e is a product of two nilpotent elements. □

Corollary 3.4 *Let R be a regular ring which is unit-regular or right self-injective. Then the subsemigroup*

$$S_R = \{a \in R \mid Rr(a) = \ell(a)R = R\}$$

(which is regular and idempotent-generated) is precisely the subsemigroup of R generated by the nilpotent elements of R.

Proof. This follows immediately from Proposition 3.3 because S_R is idempotent-generated (Proposition 2.3). □

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