DEPTH TWO, NORMALITY AND A TRACE IDEAL CONDITION FOR FROBENIUS EXTENSIONS

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ABSTRACT. A ring extension $A \mid B$ is depth two if its tensor-square satisfies a projectivity condition w.r.t. the bimodules ${}_AA_B$ and ${}_BA_A$. In this case the structures $(A \otimes_B A)^B$ and $\mathrm{End}\,{}_BA_B$ are bialgebroids over the centralizer $C_A(B)$ and there is a certain Galois theory associated to the extension and its endomorphism ring. We specialize the notion of depth two to induced representations of semisimple algebras and character theory of finite groups. We show that depth two subgroups over the complex numbers are normal subgroups. As a converse, we observe that normal Hopf subalgebras over a field are depth two extensions. A generalized Miyashita-Ulbrich action on the centralizer of a ring extension is introduced, and applied to a study of depth two and separable extensions, which yields new characterizations of separable and H-separable extensions. With a view to the problem of when separable extensions are Frobenius, we supply a trace ideal condition for when a ring extension is Frobenius.

1. INTRODUCTION AND PRELIMINARIES

In noncommutative Galois theory we have the classical notions of Frobenius extension and separable extension. For example, finitely generated Hopf-Galois extensions are Frobenius extensions via a nondegenerate trace map defined by the action of the integral element t in the Hopf algebra. Also a Hopf-Galois extension is separable if its Hopf algebra is semisimple or if the counit of t is nonzero.

There have been various efforts to define a notion of noncommutative normal extension by Elliger and others. Recently a notion of depth two for Bratteli diagrams of pairs of C^*-algebras has been widened to Frobenius extensions in [9] and to ring extensions in [12] for the purpose of reconstructing bialgebroids of various types depending on the hypotheses that are placed on the ring extension and its centralizer. For example, a depth two balanced Frobenius extension of algebras with trivial centralizer is a Hopf-Galois extension (with normal basis property), since dual Hopf algebras with natural actions are constructible on the step two centralizers in the first levels of the Jones tower. At the other extreme, a depth two balanced extension has dual bialgebroids over the centralizer with Galois actions. In between

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these two types of extensions we have extensions that have natural Galois actions coming from Hopf algebroids or weak Hopf algebras (e.g., groupoid algebras): dual antipodes appearing when we place the Frobenius condition on a depth two extension. In Section 3, we extend the analogy of normal subfields and their Galois correspondence with normal subgroups to show that depth two subgroups of finite groups, over the complex numbers, are normal subgroups. We also observe a converse more generally stated for normal Hopf subalgebras (that they are depth two extensions).

There are intriguing problems in how to draw a Venn diagram for the various ring extensions in noncommutative Galois theory. For example, are split, separable extensions with conditions of finite generation and projectivity automatically Frobenius extensions or quasi-Frobenius (QF) extensions [4, 12]? Or an H-separable extension is separable and of depth two, but what extra condition do we need for a separable depth two extension to be H-separable? In Sections 2, 4, and 5 we work towards a clarification of these two questions by focusing on a generalized Miyashita-Ulbrich action on the centralizer denoted by the module R_T. In Hopf-Galois theory (or normal subgroup theory) for a ring extension (or group ring extension) $A \,|\, B$ with centralizer R, this is a right action defined by

$$r \triangleleft h := t^1 r t^2 := r \cdot t$$

where an element h of the Hopf algebra H corresponds to an element $t := \beta^{-1}(1 \otimes h)$ in $T := (A \otimes_B A)^B$ under the Galois isomorphism $\beta : A \otimes_B A \xrightarrow{\cong} A \otimes H$. The ring T is also an R-bialgebroid in the depth two theory [12]. We will see in sections 3 and 4 that various conditions on the module R_T thus defined and a ternary product isomorphism involving this module lead to depth two, separable or H-separable extensions. In section 5 we begin with a brief introduction to Frobenius extensions and then create a small theory of trace ideal condition and Morita context from the basic idea that an extension $A \,|\, B$ should be Frobenius if there is a bimodule map $E : A \to B$ and Casimir element $e = \sum_i e_i^1 \otimes e_i^2$ such that $\sum_i E(e_i^1) e_i^2 = 1$.

In this paper, a ring extension $A \,|\, B$ is any unital ring homomorphism $B \to A$, called proper if the map is monic, subrings being the most important case. This induces a B-B-bimodule structure on A via pullback denoted by ${}_B A_B$. Given this object of study we fix a series of notations important to the study of depth two. Denote its endomorphism ring $S := \mathrm{End}\, {}_B A_B$. Denote the group of homomorphisms $\mathrm{Hom}\,({}_B A_B, {}_B B_B) := \hat{A}$, a right S-module with respect to ordinary composition of functions, or, if $A \,|\, B$ is proper, identifiable with a right ideal in S.

Recall that the centralizer R of the ring extension $A \,|\, B$ above is defined as the B-central elements of ${}_B A_B$ with notations $R := C_A(B) := A^B$. Then there are two ring mappings of R into S given by $\lambda : R \to S$, $\lambda(r)(a) = ra$, a ring homomorphism, and $\rho : R \to S$, $\rho(r)(a) = ar$, a ring antihomomorphism in our convention. Note that the two mapping commute at all pairs of image points in S:

$$\lambda(r)\rho(r') = \rho(r')\lambda(r) \quad (r, r' \in R) \tag{1}$$

The two commuting maps induce two R-R-bimodule structures on S: a left R^e-structure denoted by ${}_{\lambda,\rho}S$ and given by $(\alpha \in S, r, r' \in R)$

$$ {}_{\lambda,\rho}S : \quad r \cdot \alpha \cdot r' = \lambda(r)\rho(r')\alpha = r\alpha(-)r' \tag{2}$$

and a right R^e-structure on S denoted by $S_{\lambda,\rho}$ and given by

$$S_{\lambda,\rho} : \quad r \cdot \alpha \cdot r' := \alpha\lambda(r')\rho(r) = \alpha(r'(-)r). \tag{3}$$

We note that S itself forms an R^e-R^e-bimodule under these two actions.

In the tensor-square $A \otimes_B A$ we draw the reader's attention to the group of so-called B-casimir elements $T := (A \otimes_B A)^B$ within. T has a natural ring structure induced from the isomorphism $T \cong \mathrm{End}\, {}_A A \otimes_B A_A$ given by

$$t \longmapsto (a \otimes a' \mapsto ata')$$

for each $t \in T, a, a' \in A$ with inverse $F \mapsto F(1 \otimes 1)$: the induced ring structure has unity element $1_T = 1 \otimes 1$ with multiplication

$$tt' = t'^1 t^1 \otimes t^2 t'^2 \tag{4}$$

where we suppress a possible summation $t = \sum_i t_i^1 \otimes t_i^2$. There is a natural T-A^e-bimodule on $A \otimes_B A$ stemming from "inner" multiplication from the left (as above) and "outer" multiplication from the right. We note the right T-module structure on the centralizer R given by

$$R_T : \quad r \cdot t = t^1 r t^2 \quad (r \in R, t \in T),$$

a cyclic module since $1 \otimes r \in T$ for every $r \in R$. Finally the two mappings $\sigma, \tau : R \to T$ given by $\tau(r) = r \otimes 1$, an antihomorphism of rings, and $\sigma(r) = 1 \otimes r$, a ring homomorphism, commute as in Eq. (1), whence they induce two commuting R^e-bimodule structures denoted by ${}_{\sigma,\tau}T$ and $T_{\sigma,\tau}$ given by

$$T_{\sigma,\tau} : \quad r \cdot t \cdot r' := t\sigma(r')\tau(r) = rt^1 \otimes t^2 r' \tag{5}$$

which is the same as the restriction to R of the ordinary bimodule ${}_A A \otimes_B A_A$, and the more exotic bimodule

$$ {}_{\sigma,\tau}T : \quad r \cdot t \cdot r' := \sigma(r)\tau(r')t = t^1 r' \otimes r t^2. \tag{6}$$

Within T is the (possibly zero) left ideal of Casimir elements $\mathcal{C} := (A \otimes_B A)^A$ satisfying a type of right integral condition $(e, e' \in \mathcal{C})$,

$$ee' = e\mu(e') = \mu(e')e$$

where $\mu : A \otimes_B A \to A$ denotes the multiplication map $\mu(a \otimes a') := aa'$ an A-B or B-A-split epimorphism. (Indeed, a right integral condition with respect to the R-bialgebroid structure on T for a depth two extension $A \,|\, B$.) We note that the restriction $\mu : T \to R$ has image in the centralizer, the restriction being denoted by the counit ε_T in section 2.

In section 4 we make use of two R-bimodules on $(A \otimes_B A)^A$ and $\hat{A}$. The Casimirs form an R-R-bimodule ${}_R\mathcal{C}_R$ as a submodule of (6) given by

$$r \cdot e \cdot r' = e^1 r' \otimes r e^2. \tag{7}$$

The bimodule dual of A forms an R-R-bimodule ${}_R\hat{A}_R$ as a submodule of (3).

We need a word about general modules as well. Given any ring Υ, a right module M_Υ is isomorphic to a direct summand of another right module N_Υ if we use the suggestive notation $M_\Upsilon \oplus * \cong N_\Upsilon$. A bimodule ${}_\Upsilon M_\Upsilon$ is of course the same as the left or right $\Upsilon^e := \Upsilon \otimes_{\mathbb{Z}} \Upsilon^{\mathrm{op}}$-module M.

Hirata extends Morita theory in an elegant way [5] by defining two modules M_Υ and N_Υ to be *H-equivalent* if both $M_\Upsilon \oplus * \cong N_\Upsilon^n$ (which equals $N \oplus N \oplus \cdots \oplus N$, n times) and $N_\Upsilon \oplus * \cong M_\Upsilon^m$ for some positive integers n and m. A theory is outlined in [5], the main point being that $\operatorname{End} M_\Upsilon$ and $\operatorname{End} N_\Upsilon$ are Morita equivalent rings with Morita context being $\operatorname{Hom}(M_\Upsilon, N_\Upsilon)$ and $\operatorname{Hom}(N_\Upsilon, M_\Upsilon)$ with composition in either order [15].

2. Depth two theory

Any ring extension $A \mid B$ satisfies the property $A \oplus * \cong A \otimes_B A$ as either B-A or A-B-bimodules, since $\mu : A \otimes_B A \to A$ is an epi split by either $a \mapsto 1 \otimes a$ or $a \mapsto a \otimes 1$, respectively. Its converse from the point of view of Hirata's theory is satisfied by special extensions introduced in [12] called depth two extensions (due to their origins in subfactor theory). Thus a *depth two extension* or D2 extension $A \mid B$ is a ring extension where

$$A \otimes_B A \oplus * \cong A^n \tag{8}$$

as natural B-A-bimodules (*left D2*) and as A-B-bimodules (*right D2*). Of course, if some $A \mid B$ satisfies the left and right conditions for different integers m and n, they both hold for the integer $\max\{m, n\}$.

There are several classes of examples of depth two extension $A \mid B$; such as an algebra that is finitely generated projective over its base ring, finite Hopf-Galois extension, H-separable extension or centrally projective extension [9, 12].

How do we recognize depth two in any extension? This is not always easy from the definition we provided above: let us provide a theorem with several equivalent conditions for left depth two extensions (a similar theorem is then easily seen for right D2). The theorem will use the left and right sides of the bimodules ${}_{\lambda,\rho}S$ and $T_{\sigma,\tau}$ respectively, given in eqs. (2) and (5).

Theorem 2.1. *The following are equivalent to the left depth two condition (8) on a ring extension $A \mid B$ in (1)-(4), on a Frobenius extension in (5) and on an algebra A with semisimple subalgebra B in (6):*

1. *the bimodules ${}_B A_A$ and ${}_B A \otimes_B A_A$ are H-equivalent;*

(2) *there are* $\{\beta_j\}_{j=1}^n \subset S$ *and* $\{t_j\}_{j=1}^n \subset T$ *such that*
$$a \otimes a' = \sum_j t_j \beta_j(a) a' \, ; \tag{9}$$
(3) *as natural* B-A-*bimodules* $A \otimes_B A \cong \mathrm{Hom}\,({}_R S, {}_R A)$ *and* ${}_R S$ *is a f.g. projective module;*
(4) *as natural* B-A-*bimodules* $T \otimes_R A \cong A \otimes_B A$ *and* T_R *is f.g. projective;*
(5) *the endomorphism ring Frobenius extension has dual bases elements in* T*;*
(6) *for each simple module* V_B*,*
$$\mathrm{Ind}_B^A \mathrm{Res}_A^B \mathrm{Ind}_B^A V \oplus * \cong \mathrm{Ind}_B^A V \oplus \cdots \oplus \mathrm{Ind}_B^A V \tag{10}$$
(finitely many times on the right side).

Proof. We have already remarked above that the right $B^{\mathrm{op}} \otimes A$-modules A and $A \otimes A$ are H-equivalent iff $A \,|\, B$ is left D2. The second condition is called the D2 quasibase condition and is noted in [12]. It is based on condition (8) and the identifications $\mathrm{Hom}\,({}_B A_A, {}_B A \otimes A_A) \cong T$, $\mathrm{Hom}\,({}_B A \otimes A_A, {}_B A_A) \cong S$ and the existence of $2n$ maps f_j, g_j in these two Hom-groups satisfying $\sum_j f_j \circ g_j = \mathrm{id}_{A \otimes A}$.

Next we note the B-A homomorphism $\chi : A \otimes A \xrightarrow{\cong} \mathrm{Hom}\,({}_R S, {}_R A)$ given by
$$\chi(a \otimes a')(\alpha) = \alpha(a) a'.$$
It is noted in [12] that χ has inverse $\chi^{-1}(f) = \sum_j t_j f(\beta_j)$ if $A \,|\, B$ is left D2. For example, we check that $\chi \circ \chi^{-1} = \mathrm{id}$ on an $\alpha \in S$:
$$\sum_j \alpha(t_j^1) t_j^2 f(\beta_j) = f(\sum_j \alpha(t_j^1) t_j^2 \beta_j) = f(\alpha)$$
for each $\alpha \in S$, since $\alpha(t_j^1) t_j^2 \in R$ for each j and $a \otimes 1 = \sum_j t_j \beta_j(a)$ from Eq. (9) implies $\alpha = \sum_j \alpha(t_j^1) t_j^2 \beta_j$. We also note that ${}_R S$ is finite projective with dual (projective) bases $\{\psi(b_j)\}$, $\{\beta_j\}$ where $\psi : T_R \xrightarrow{\cong} \mathrm{Hom}\,({}_R S, {}_R R)$ is defined by $\psi(t)(\alpha) = \alpha(t^1) t^2$.

Conversely, if ${}_R S \oplus * \cong {}_R R^m$, then applying the functor $\mathrm{Hom}\,(-, {}_R A)$ from left R-modules into B-A-bimodules, we obtain from the isomorphism of the A-dual of S with the tensor-square that $A \otimes_B A \oplus * \cong {}_B A_A^m$.

The next condition follows from the D2 condition [12], since the B-A-homomorphism
$$m : T \otimes_R A \xrightarrow{\cong} A \otimes_B A, \quad m(t \otimes a) = ta \tag{11}$$
has an inverse $m^{-1}(a \otimes a') = \sum_j t_j \otimes \beta_j(a') a$ according to Eq. (9). By the same token, T_R is finite projective with dual bases $\{t_j\}$, $\{\langle \beta_j \,|\, - \rangle\}$ from the nondegenerate pairing given in Eq. (13) below.

Conversely, if $T_R \oplus * \cong R_R^t$ for some positive integer t, then applying the functor $- \otimes_R A$ from right R-modules into B-A-bimodules results in

${}_BA \otimes_B A_A \oplus * \cong {}_BA^t_A$, after applying the isomorphism of the tensor-square with $T \otimes_R A$.

The next-to-last condition depends on the fact that a Frobenius extension $A \,|\, B$ with Frobenius homomorphism $E : A \to B$ and dual bases x_i, y_i has Frobenius structure on its endomorphism ring extension $\lambda : A \hookrightarrow \mathrm{End}\, A_B \cong A \otimes_B A$ (isomorphism $f \mapsto \sum_i f(x_i) \otimes y_i$ with inverse $a \otimes b \mapsto \lambda(a)E\lambda(b)$). Denote the endomorphism ring structure induced on $A \otimes_B A$ by A_1; if $e_1 := 1 \otimes 1$, its multiplication is the E-multiplication given by

$$(ae_1b)(ce_1d) = aE(bc)e_1d = ae_1E(bc)d.$$

A Frobenius homomorphism $E_A : A_1 \to A$ is given by the multiplication mapping $E_A(ae_1b) = ab$, since x_ie_1, e_1y_i are dual bases for this mapping. Note that $T = C_{A_1}(B)$, the centralizer of B in the composite ring extension $B \to A \hookrightarrow A_1$.

The forward implication is [12, Prop. 6.4]; viz., given a left D2 quasibase as above $t_i \in T, \beta_i \in S$ for $A \,|\, B$, then dual bases for E_A in T are given by

$$\{t_i\}, \{\sum_j \beta_i(x_j)e_1y_j\}$$

as shown in its proof. (Also, $A \,|\, B$ is necessarily right D2.)

For the reverse implication, we start with dual bases $\{c_j\}$ and $\{t_j\}$ in T for E_A, and note that for every $a, b \in A$

$$\begin{aligned} ae_1b &= \sum_j c_j^1 e_1 c_j^2 E_A(t_j a e_1 b) \\ &= \sum_j c_j t_j^1 E(t_j^2 a) b \\ &= \sum_j c_j \eta_j(a) b \end{aligned}$$

where $\eta_j = t_j^1 E(t_j^2 -)$ is clearly in S. Thus, $\{c_j\}$ and $\{\eta_j\}$ form a left D2 quasibase. (Similarly, a right D2 quasibase is given by $\{t_j\}$ and $\{\rho_j := E(-c_j^1)c_j^2\}$.)

The last condition for left depth two follows in the forward implication by tensoring the defining Eq. (8) by V_B from the left and applying the definition of induced module: $\mathrm{Ind}_B^A V := V \otimes_B A$, and Res denoting restriction of an A-module to a B-module. The reverse implication follows from expressing the regular representation as a direct sum of irreducible representations (or simple modules),

$$B_B \cong V_1^{n_1} \oplus \cdots \oplus V_t^{n_t}.$$

If each V_i satisfies Eq. (10) with m_i $\mathrm{Ind}V_i$'s to the right, they satisfy Eq. (10) with $m = \max\{m_i\}$ to the right, and it follows that

$$B \otimes_B A \otimes_B A \oplus * \cong \oplus_{i=1}^t V_i^{n_i} \otimes A \otimes A \oplus * \cong \oplus_{i=1}^t V^{mn_i} \otimes A$$

as B-A-bimodules. Then ${}_BA \otimes_B A_A \oplus * \cong \oplus_{i=1}^m {}_BA_A$. □

In addition to their ring structures, the constructs $S = \mathrm{End}\,{}_BA_B$ and $T = (A \otimes A)^B$ introduced in the preliminaries carry R-bialgebroid structures that are isomorphic to each others R-dual bialgebroids. (See [12, 2.4] for definition of left and right R-dual bialgebroids over any ring R given a finite projectivity condition). First, with respect to the R-R-bimodule ${}_{\lambda,\rho}S$ in (2), S satisfies

$$S \otimes_R S \cong \mathrm{Hom}\,({}_BA \otimes A_B, {}_BA_B)$$

via

$$\alpha \otimes \beta \longmapsto (a \otimes a' \mapsto \alpha(a)\beta(a')).$$

It follows that S is an R-coring with coassociative comultiplication $\Delta : S \to S \otimes_R S$ given by

$$\Delta(\alpha)(a \otimes a') = \alpha(aa')$$

after identification, and counit $\varepsilon_S : S \to R$ given by

$$\varepsilon_S(\alpha) = \alpha(1),$$

(see [2] for the theory of corings). The ring-coring bimodule structure $(S, R, \lambda, \rho, \Delta, \varepsilon)$ satisfies the axioms of a left bialgebroid in [12] among which we find the important axioms

$$\Delta(\alpha\beta) = \Delta(\alpha)\Delta(\beta)$$

since $\mathrm{Im}\,\Delta$ is contained in a subgroup of $S \otimes_R S$ where ordinary tensor product multiplication makes sense; in addition, ε_S satisfies the following alternative homomorphism property:

$$\varepsilon_S(\alpha\beta) = \varepsilon_S(\alpha\lambda(\varepsilon_S(\beta))).$$

A *left integral* ℓ in S is an element in S satisfying

$$\alpha \circ \ell = \lambda(\varepsilon_S(\alpha))\ell = \rho(\varepsilon_S(\alpha))\ell, \quad (\forall \alpha \in S) \tag{12}$$

For example, any element in $\hat{A}$ is a left integral. A left integral ℓ is *normalized* if $\varepsilon_S(\ell) = 1$, which for $E \in \hat{A}$ is the case when $E(1) = 1$, i.e., $E : {}_BA_B \to {}_BB_B$ is a bimodule projection onto B, or *conditional expectation* for the *split extension* $A \,|\, B$. Note the following for any proper ring extension $A \,|\, B$:

Lemma 2.2. *$A \,|\, B$ is a split extension if and only if $\hat{A} \otimes_S A \cong B$ as B-B-bimodules and $\hat{A}_S$ is f.g. projective.*

Proof. ($\Rightarrow$) Define $\hat{A} \otimes_S A \stackrel{\cong}{\to} B$ via $F \otimes a \mapsto F(a)$, with inverse $b \mapsto E \otimes b$. Note that the conditional expectation E is a left identity.

($\Leftarrow$) If $\hat{A}_S \oplus * \cong S^n_S$, apply the functor $- \otimes_S A_{B^e}$, obtaining

$${}_BB_B \oplus * \cong {}_BA^n_B.$$

Then there are $E_i \in \hat{A}$, $\lambda(r_i) \in \mathrm{Hom}\,({}_BB_B, {}_BA_B) \cong R$ such that $E := \sum_i E_i \circ \lambda(r_i)$ is id_B when restricted to B. $\square$

Dually, we have

$$S \cong \mathrm{Hom}\,(T_R, R_R)$$

via the nondegenerate pairing

$$\langle \alpha \,|\, t \rangle = \alpha(t^1)t^2 \tag{13}$$

for depth two extension $A \,|\, B$. Consistent with this duality, T has the right bialgebroid structure $(T, R, \sigma, \tau, \Delta_T, \varepsilon_T)$ coming from its ring structure and maps $\sigma, \tau : R \to T$ introduced in the preliminaries and R-coring structure $(T, \Delta_T, \varepsilon_T)$ with respect to the bimodule $T_{\sigma,\tau}$ given in (5), where comultiplication

$$\Delta(t) = t^1 \otimes 1_A \otimes t^2$$

under the identification $T \otimes_R T \cong (A \otimes A \otimes A)^B$ by $t \otimes t' \mapsto t^1 \otimes t^2 t'^1 \otimes t'^2$, and counit

$$\varepsilon_T(t) = t^1 t^2.$$

The details may be found in [12], but the bialgebroid structures are not very important in the present paper.

A normalized right integral in T is an element $u \in T$ such that for every $t \in T$, $ut = u\sigma(\varepsilon_T(t)) = u\tau(\varepsilon_T(t))$ with normalization condition $\varepsilon_T(u) = 1_A$. We note the following easy proposition based on using the separability element viewed in T or a conditional expectation viewed in $\hat{A} \subset S$:

Proposition 2.3. *If $A \,|\, B$ is a (resp., split) separable extension, then T (resp., S) has a normalized right (resp., left) integral.*

Given a depth two extension $A \,|\, B$, there are two measuring actions of S and T. First, T acts on $\mathcal{E} := \mathrm{End}_B A$ via the right action $f \triangleleft t = t^1 f(t^2 -)$. The invariants $\mathcal{E}^T$ under $\triangleleft$ are defined to be

$$\{f \in \mathcal{E} \,|\, \forall\, t \in T, f \triangleleft t = \lambda(\varepsilon_T(t)) \circ f\}$$

It is shown in [12] that $\mathcal{E}^T = \rho(A)$, the right multiplications by elements of A (which is clear in one direction). Forming the smash product $T \ltimes \mathcal{E}$, one may show that it is isomorphic as rings to $\mathrm{End}_A A \otimes_B A$ [7]. It is interesting to note that if A additionally has a separability element $e \in T$, then the mapping $\mathcal{E} \to A^{\mathrm{op}}$ defined by $f \mapsto f \triangleleft e$ is a conditional expectation (a known fact recovered).

Second, there is a very natural action of S on A given by $\alpha \triangleright a := \alpha(a)$. Using a similar definition of invariants, we easily see that $A^S \supseteq B$. In addition to the depth two condition on $A \,|\, B$, we need an extra condition on A_B to prove that $B = A^S$: in [12] this is done by assuming A_B a balanced module, i.e., $B \cong \mathrm{End}_{\mathcal{E}'} A$ where $\mathcal{E}' = \mathrm{End}\, A_B$ and we consider only natural actions. (For example, if A_B is a generator, it is balanced.) The smash product $A \rtimes S$ is isomorphic to $\mathcal{E}'$ [12].

We end this section with a necessary condition for an extension $A \,|\, B$ to be left D2. Recall the left T-module $A \otimes_B A$ and the module R_T both defined in the preliminaries. We let $Z(A)$ denote the center subring of any ring A.

Theorem 2.4. *Suppose $A\,|\,B$ is D2. Then $R \otimes_T (A \otimes_B A) \cong A$ as A-A-bimodules and* $\mathrm{End}\, R_T \cong Z(A)$.

Proof. The mapping $\gamma : R \otimes_T (A \otimes_B A) \to A$ given by

$$\gamma(r \otimes a \otimes a') = ara' \tag{14}$$

is an A-A-bimodule epimorphism for every ring extension $A\,|\,B$ (a type of ternary product through which the multiplication mapping $\mu : A \otimes_B A \to A$ factors).

We work with the bimodule ${}_T T_R$ given by

$$t \cdot t' \cdot r := tt'\tau(r) = {t'}^1 t^1 \otimes t^2 {t'}^2 r$$

derived from the structures introduced in the preliminaries. Note that the diagram on this page below is a commutative square, where the left vertical and bottom horizontal arrows are both induced from canonical maps of the type $R \otimes_R A \cong A$, $r \otimes a \mapsto ra$, while the top horizontal mapping is induced from $m : T \otimes_R A \cong A \otimes A$ given in Eq. (11). Both sides of the square send $r \otimes t \otimes a$ in the upper left-hand corner into $t^1 r t^2 a \in A$. It follows that the fourth map γ is an isomorphism.

$$\begin{tikzcd}
R \otimes_T T \otimes_R A \arrow[r, "\cong", "\mathrm{id} \otimes m"'] \arrow[d, "\cong"'] & R \otimes_T (A \otimes_B A) \arrow[d, "\gamma"] \\
R \otimes_R A \arrow[r, "\cong"] & A
\end{tikzcd}$$

It follows that $\gamma^{-1}(a) = 1 \otimes_T (1 \otimes a)$, a fact that is directly checked as follows:

$$1 \otimes_T (1 \otimes ara') = 1 \otimes (1 \otimes \sum_j t_j^1 r t_j^2 \beta_j(a) a') = \sum_j t_j^1 r t_j^2 \otimes_T (1 \otimes \beta_j(a) a')$$

$$= \sum_j r \otimes_T (t_j \beta_j(a) a') = r \otimes (a \otimes a').$$

Clearly, $Z(A) \hookrightarrow \mathrm{End}\, R_T$ via λ. Conversely, if $f \in \mathrm{End}\, R_T$, we note that $f(1)$ satisfies a type of left integral condition,

$$t^1 t^2 f(1) = f(1) \cdot (t^1 t^2 \otimes 1) = f(1 \cdot t) = f(1) \cdot t = t^1 f(1) t^2.$$

Next we recall the right action of T on $\mathcal{E} = \mathrm{End}\,{}_B A$ outlined above: we see that $R_T \hookrightarrow \mathcal{E}_T$ is a submodule via λ since for every $t \in T$,

$$\lambda(r) \triangleleft t = \lambda(t^1 r t^2) = \lambda(r \cdot t).$$

Then $f(1) \in R$ is in the invariants $\mathcal{E}^T = \rho(A)$, since

$$\lambda(f(1)) \triangleleft t = \lambda(t^1 f(1) t^2) = \lambda(\varepsilon_T(t)) \lambda(f(1)),$$

whence $\lambda(f(1)) = \rho(a) \in \mathcal{E}^T$. Now $a = f(1)$ by evaluating at 1, and $a \in Z(A)$ by evaluating at any $b \in A$. □

Example 2.5. *We have identified examples of depth two extensions coming from centrally projective, H-separable or finite Hopf-Galois extensions of rings or algebras. Let us see by means of the theorem an easy example of an algebra extension which is not D2. If A is the algebra of upper triangular 2×2 matrices over any field, B the subalgebra of diagonal matrices, then $R = B$, T is spanned by $\{e_{11} \otimes e_{11}, e_{22} \otimes e_{22}\}$ in terms of matrix units e_{ij} and it is easy to see that*

$$\dim \mathrm{End}\,(R_T) = \dim \mathrm{End}\, R_R = \dim R = 2$$

while $\dim Z(A) = 1$. *There are also nonexamples of depth two extensions coming from various natural subalgebras of finite dimensional exterior algebras (where condition (4) in Theorem 2.1 is useful as well as the fact that split depth two extensions are f.g. projective by eq. 9, therefore free over local rings).*

3. Depth two as normality for subgroups and Hopf subalgebras

In this section we show that depth two subgroup algebras and Hopf subalgebras are closely related to normal subgroups and normal Hopf subalgebras. For subgroups of finite groups over the complex numbers we have a definitive result in Theorem 3.1 and its corollary.

Character theory [1]**.** Let $H \leq G$ denote a subgroup pair of finite index. We are interested in when the corresponding group algebras over the complex numbers are of depth two. First, we note the fact in [12, 3.9] that a normal subgroup is D2. If ψ is a character of H and ϕ is a character of G, we let ψ^G denote its induced character $\mathrm{Ind}_H^G(\psi)$ and ϕ_H the character $\mathrm{Res}_H^G \phi$ restricted to H. For example, if H is normal in G, then $\psi^G(g) = 0$ if $g \notin H$ and $\phi_H^G(g) = [G : H]\phi_H(g)$ if $g \in H$, since for general subgroups we have

$$\psi^G(g) = \sum_{t^{-1}gt \in H}^{t \in T} \psi(t^{-1}gt)$$

for T a left transversal of H in G. By evaluating again on $g \notin H$ and $g \in H$, we see that $((\psi^G)_H)^G = [G : H]\psi^G$. If V denotes the underlying H-module of ψ, then equality of characters implies isomorphism of modules as follows:

$$\mathrm{Ind}_H^G \mathrm{Res}_G^H \mathrm{Ind}_H^G V \cong \oplus_{i=1}^{[G:H]} \mathrm{Ind}_H^G V.$$

Hence, $A := \mathbb{C}\,[G]$ is depth two extension over $B := \mathbb{C}\,[H]$ if $H \triangleleft G$.

Suppose $\chi_1, \ldots, \chi_n$ are the irreducible characters of G, while $\psi_1, \ldots, \psi_m$ are the irreducible characters of a subgroup H. Suppose a_j^i are nonnegative integers in an induction-restriction table for $H \leq G$ such that

$$\psi_i^G = \sum_{j=1}^n a_j^i \chi_j,$$

and c_s^r are nonnegative integers satisfying

$$(((\psi_r)^G)_H)^G = \sum_{s=1}^n c_s^r \chi_s.$$

Then the group algebra extension $A \mid B$ is D2 if and only if there is a positive integer N such that $c_j^i \leq N a_j^i$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

In particular, $A \mid B$ is not D2 if $a_j^i = 0$ while $c_j^i \neq 0$. For example, the induction-restriction table (based on Frobenius reciprocity $(\psi_i^G, \chi_j)_G = (\psi_i, (\chi_j)_H)_H$) for the permutation groups $S_2 \leq S_3$ is given by

$S_2 \leq S_3$	χ_1	χ_2	χ_3
ψ_1	1	0	1
ψ_2	0	1	1

where ψ_1, χ_1 denote the trivial characters, ψ_2, χ_2 the sign character, and χ_3 the two-dimensional irreducible character of S_3. It follows that $\psi_1^G = \chi_1 + \chi_3$, while

$$((\psi_1^G)_H)^G = 2\psi_1^G + \psi_2^G = 2\chi_1 + \chi_2 + 3\chi_3.$$

Hence, $((\phi_1^G)_H)^G$ is not even a subcharacter of ψ_1^G or any of its integral multiples.

The last example and results from operator algebras raise the question whether subgroups of finite groups that are depth two as complex group subalgebras are necessarily normal subgroups. The next theorem answers this affirmatively.

Theorem 3.1. *Let H be a subgroup of a finite group G such that $\mathbb{C}\,G \mid \mathbb{C}\,H$ is a depth two ring extension. Then H is normal in G.*

Proof. The depth two hypothesis implies that there exists a positive integer n such that

$$\langle \mathrm{Ind}_H^G(\mathrm{Res}_H^G(\mathrm{Ind}_H^G(\psi))) \mid \chi\rangle_G \leq n \langle \mathrm{Ind}_H^G(\psi) \mid \chi\rangle_G$$

for $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$. We choose $\chi = 1_G$ and $1_H \neq \psi \in \mathrm{Irr}(H)$. Then, by Frobenius reciprocity, we have

$$\langle \mathrm{Ind}_H^G(\psi) \mid 1_G\rangle_G = \langle \psi \mid 1_H\rangle_H = 0.$$

Hence, we have

$$\begin{aligned} 0 &= \langle \mathrm{Ind}_H^G(\mathrm{Res}_H^G(\mathrm{Ind}_H^G(\psi))) \mid 1_G\rangle_G = \langle \mathrm{Res}_H^G(\mathrm{Ind}_H^G(\psi)) \mid 1_H\rangle_H \\ &= \sum_{HgH \in H\backslash G/H} \langle \mathrm{Ind}_{H\cap gHg^{-1}}^H(\mathrm{Res}_{H\cap gHg^{-1}}^{gHg^{-1}}({}^g\psi)) \mid 1_H\rangle_H, \end{aligned}$$

by Frobenius reciprocity and Mackey's formula. We conclude that, for $g \in G$,

$$\begin{aligned} 0 &= \langle \mathrm{Ind}_{H\cap gHg^{-1}}^{H}(\mathrm{Res}_{H\cap gHg^{-1}}^{gHg^{-1}}({}^{g}\psi)) \mid 1_H\rangle_H \\ &= \langle \mathrm{Res}_{H\cap gHg^{-1}}^{gHg^{-1}}({}^{g}\psi) \mid 1_{H\cap gHg^{-1}}\rangle_{H\cap gHg^{-1}} \\ &= \langle \mathrm{Res}_{g^{-1}Hg\cap H}^{H}(\psi) \mid 1_{g^{-1}Hg\cap H}\rangle_{g^{-1}Hg\cap H} \\ &= \langle \psi \mid \mathrm{Ind}_{g^{-1}Hg\cap H}^{H}(1_{g^{-1}Hg\cap H})\rangle_H, \end{aligned}$$

again by Frobenius reciprocity and conjugation. On the other hand, we have

$$\langle 1_H \mid \mathrm{Ind}_{g^{-1}Hg\cap H}^{H}(1_{g^{-1}Hg\cap H})\rangle_H = \langle 1_{g^{-1}Hg\cap H} \mid 1_{g^{-1}Hg\cap H}\rangle_{g^{-1}Hg\cap H} = 1,$$

using Frobenius reciprocity one more time. Thus

$$\mathrm{Ind}_{g^{-1}Hg\cap H}^{H}(1_{g^{-1}Hg\cap H}) = 1_H.$$

Comparing degrees we get $H = g^{-1}Hg \cap H$. Hence H is normal in G. □

We summarize the theorem with its converse proven above using characters (and group-theoretically in [12] for finite index normal subgroups over general fields).

Corollary 3.2. *Let H be a subgroup of a finite group G. Then $\mathbb{C}\,G \mid \mathbb{C}\,H$ is a depth two ring extension if and only if H is a normal subgroup of G.*

It is well-known that a finite group algebra is a Hopf-Galois extension of a normal subgroup algebra: see our next subsection for an exposition of this fact in a more general setting. Theorem 3.1 implies a converse, since a finite Hopf-Galois extension is a depth two extension.

Corollary 3.3. *Let H be a subgroup of a finite group G. If $\mathbb{C}\,G \mid \mathbb{C}\,H$ is a Hopf-Galois extension, then H is a normal subgroup of G.*

Normal Hopf subalgebras [16] are depth two. Consider any finite-dimensional Hopf algebra H over a field k with antipode $S : H \to H$, counit $\varepsilon : H \to k$ and comultiplication notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for each $a \in H$. A Hopf subalgebra K is *normal* if $S(a_{(1)})xa_{(2)} \in K$ and $a_{(1)}xS(a_{(2)}) \in K$ for each $a \in H$ and $x \in K$. Form the subset $K^+ := \ker \varepsilon \cap K$ and the left ideal HK^+ in H. Note that $HK^+ = K^+H$ since given $x \in K^+$ and $a \in H$

$$xa = \varepsilon(a_{(1)})xa_{(2)} = a_{(1)}S(a_{(2)})xa_{(3)} \in HK^+,$$

the converse $HK^+ \subseteq K^+H$ being shown similarly. It follows that HK^+ is a Hopf ideal and $\overline{H} := H/HK^+$ is a Hopf algebra. There is the natural surjective Hopf algebra homomorphism $H \to \overline{H}$ denoted by $a \mapsto \overline{a}$.

Proposition 3.4. *Let H be a finite-dimensional Hopf algebra and K a normal Hopf subalgebra in H. Then $H \mid K$ is an $\overline{H}$-Galois extension, therefore of depth two.*

Proof. The natural coaction $H \to H \otimes_k \overline{H}$ given by $a \mapsto a_{(1)} \otimes \overline{a_{(2)}}$ makes H into an $\overline{H}$-comodule algebra with coinvariants K [16, 3.4.3]. The $\overline{H}$-extension $A \,|\, B$ is Galois [16, p. 30] (with Galois mapping $\beta : H \otimes_K H \to H \otimes \overline{H}$ given by $\beta(x \otimes y) = xy_{(1)} \otimes \overline{y_{(2)}}$).

Finally recall from [12, 8] that any finite Hopf-Galois extension $A \,|\, B$ is a depth two extension. □

For example, let G be a finite group with normal subgroup N. Then $H = kG$ and $K = kN$ is a normal Hopf subalgebra pair with Galois mapping $\beta : H \otimes_K H \to H \otimes k[G/N]$ given by $\beta(g \otimes g') = gg' \otimes g'N$ for $g, g' \in G$.

We ask whether a converse of the proposition holds; e.g. if a depth two Hopf subalgebra or depth two semisimple Hopf subalgebra pair is normal.

4. A fresh look at separability

In this section we meet a new criterion for separability which is very close to the necessary condition in Theorem 2.4 for depth two extensions. We will work with a *separable extension* $A \,|\, B$ which is defined to be a ring extension where $\mu : A \otimes_B A \to A$ is split as an A-A-epimorphism. The image of 1_A under any splitting is called a separability element $e \in A \otimes_B A$ which of course satisfies $\mu(e) = 1$ and $e \in (A \otimes_B A)^A$. Fix this (nonunique) separability element $e = e^1 \otimes e^2$ (suppressing a possible summation) for any separable extension below.

Theorem 4.1. *A ring extension $A \,|\, B$ is separable if and only if R_T is f.g. projective and $R \otimes_T (A \otimes_B A) \cong A$ as natural A-A-bimodules.*

Proof. ($\Rightarrow$) An inverse to the A-A-homomorphism γ defined in Eq. (14) is given by $\gamma^{-1}(a) = 1 \otimes ea \in R \otimes_T (A \otimes_B A)$. We compute $\gamma \circ \gamma^{-1} = \mathrm{id}$ from $e^1 e^2 a = a$, and $\gamma^{-1} \circ \gamma = \mathrm{id}$ from

$$1 \otimes_T eara' = 1 \otimes_T er \cdot (a \otimes a') = 1 \cdot er \otimes (a \otimes a') = r \otimes (a \otimes a').$$

R_T is finite projective $\Leftrightarrow$ there are finite dual bases $\{r_i \in R\}$, $\{f_i \in \mathrm{Hom}\,(R_T, T_T) \cong T'\}$ via $f \mapsto f(1)$ and where the subring T' are R-centralizing left integral-type elements, i.e.,

$$T' = \{p \in T \,|\, p \cdot t = t^1 p t^2 = p t^1 t^2 = t^1 t^2 p, \ \forall t \in T\} \tag{15}$$

$\Leftrightarrow$ there are $\{p_i \in T'\}$, $\{r_i \in R\}$ such that

$$r = \sum_{i=1}^{n} p_i^1 r_i p_i^2 r = \sum_i r p_i^1 r_i p_i^2 \quad (r \in R) \tag{16}$$

But $n = 1$, $r_1 = 1$ and $p_1 = e$ will do.

($\Leftarrow$) If $R_T \oplus * \cong T_T^m$, then tensoring by $- \otimes_T (A \otimes_B A)$, we obtain ${}_A A_A \oplus * \cong {}_A A \otimes_B A_A^m$. It follows by a simple argument using split epimorphism, canonical maps between a product and its components, and the identifications $\mathrm{Hom}\,({}_A A_A, {}_A A \otimes_B A_A) \cong (A \otimes_B A)^A$ and $\mathrm{Hom}\,({}_A A \otimes_B A_A, {}_A A_A) \cong R$ (via $f \mapsto f(1 \otimes 1)$) that there are m Casimir elements e_i and m elements $r_i \in R$

such that $\sum_{i=1}^m r_i \cdot e_i = 1_A$. Then $e := \sum_{i=1}^m e_i^1 \otimes r_i e_i^2$ is a separability element. $\square$

We next recall a special case of separable extension which has been extensively studied by Ikehata, Sugano, Szeto and others. A ring extension $A \,|\, B$ is said to be *H-separable* (after Hirata [5]) if $A \otimes_B A \oplus * \cong A^n$ as A^e-modules. Equivalently, there are n Casimir elements e_i and n elements $r_i \in R$, called an *H-separability system*, such that $1 \otimes 1 = \sum_{i=1}^n r_i e_i$. Hirata [5] shows that R is f.g. projective over $Z := Z(A)$, therefore R_Z is a generator module,

$$A \otimes_Z A^{\mathrm{op}} \cong \mathrm{Hom}\,(R_Z, A_Z) \tag{17}$$

via $a \otimes a' \mapsto \lambda(a)\rho(a')$, and so a separability element for $A \,|\, B$ corresponds to a splitting map for $Z \hookrightarrow R$, which exists by the generator property. It follows that A and $A \otimes_B A$ are H-equivalent A-A-bimodules, another characterization of an H-separable extension $A \,|\, B$.

Theorem 4.2. *A ring extension $A \,|\, B$ is H-separable if and only if R_T is a generator and $R \otimes_T (A \otimes_B A) \cong A$ as natural A-A-bimodules.*

Proof. ($\Rightarrow$) Since $A \,|\, B$ is also a separable extension, $R \otimes_T (A \otimes_B A) \cong A$ by Theorem 4.1. Also R_T is f.g. projective, but we need to see that it is a generator (of the category of T-modules). But R_T generator $\Leftrightarrow$ there are finite $\{r_i \in R\}$, $\{g_i \in \mathrm{Hom}\,(R_T, T_T) \cong T'\}$ where T' is given by (15) such that

$$\sum_i g_i(r_i) = 1_T \quad \Leftrightarrow \quad \sum_i q_i r_i = r_i q_i = 1 \otimes 1 \quad (q_i = g_i(1)) \tag{18}$$

which is satisfied by an H-separability system $q_i = e_i$, $\{r_i\}$ as defined above.

($\Leftarrow$) If $T_T \oplus * \cong R_T^n$, which is the generator condition in one of its equivalent forms [15], then tensoring by $- \otimes_T (A \otimes_B A)$ and applying the isomorphism, we obtain ${}_A A \otimes_B A_A \oplus * \cong {}_A A_A^n$, which shows $A \,|\, B$ is H-separable. $\square$

Corollary 4.3. *If $A \,|\, B$ is H-separable, then ${}_Z R_T$ is a faithfully balanced bimodule with Z and T Morita equivalent rings.*

Proof. By restricting A-modules to B-modules (or pullback in case of ring arrow $B \to A$) we see from the defining conditions for H-separability and depth two that an H-separable extension $A \,|\, B$ is depth two. Then $\mathrm{End}\, R_T \cong Z$ from Theorem 2.4 and $\mathrm{End}\, {}_Z R \cong T$ from taking the B-centralizer of (17). Note that we have already seen in this section that ${}_Z R$ and R_T are progenerators. It follows that the rings T and Z are Morita equivalent.

We may alternatively see this last point from the fact noted above that A and $A \otimes_B A$ are H-equivalent A^e-modules, whence their endomorphism rings are Morita equivalent. Hirata theory also shows handily that the Morita context bimodules here are R as before, $(A \otimes_B A)^A$ with T-module as left ideal in T and trivial Z-module, and associative pairings $\langle e \,|\, r\rangle = er \in T$, $[r \,|\, e] = e^1 r e^2 \in Z$. $\square$

Example 4.4. *Suppose $A \mid B$ is an H-separable right K-Galois extension for some Hopf algebra K. Then $T^{\mathrm{op}} \cong R \rtimes K^{\mathrm{op}}$ as algebras, where K acts on the centralizer R via the Miyashita-Ulbrich action* [8]. *Then the module $R_T \cong {}_{R \rtimes K^{\mathrm{op}}} R$ is a generator. But this module is a generator iff R is a right $K^{\mathrm{op}\,*}$-Galois extension over $R^K = Z$ by* [16, p. 133], *which shows* [6, 3.1] *by other means.*

Example 4.5. *A result of Sugano states that given an algebra Λ over a commutative base ring R with center $C = Z(\Lambda)$, then Λ is H-separable over R iff Λ is separable over C and $C \otimes_R C \cong C$ via $c_1 \otimes c_2 \mapsto c_1 c_2$. In other words, Azumaya algebras are H-separable over their centers, while H-separable algebras are Azumaya algebras with center not much larger than the base ring. A depth two, separable extension $A \mid B$ that is not H-separable is then given by letting K be any field, the direct product algebra $A = K^n$ for integer $n > 1$, $B = K1_A$.*

5. A trace ideal condition for Frobenius extensions

Recall that a ring extension $A \mid B$ is *Frobenius* if ${}_B A_A \cong {}_B\mathrm{Hom}\,(A_B, B_B)_A$ (via a *Frobenius isomorphism*, say ψ) and A_B is f.g. projective. Thus the right B-dual A^* of A is a free rank one right A-module. A free generator $E : A \to B$ is called a *Frobenius homomorphism*, and $E \in \hat{A}$ since it is an image of an invertible element in R under ψ: of course, we take $E = \psi(1)$. Projective bases $\{x_i \in A\}_{i=1}^n$, $\{f_i \in A^*\}_{i=1}^n$ for A_B convert to dual bases $\{x_i\}$, $\{y_i := \psi^{-1}(f_i)\}$ satisfying

$$\sum_{i=1}^{n} x_i E(y_i a) = a = \sum_{i=1}^{n} E(a x_i) y_i. \tag{19}$$

The first equation follows immediately from the dual bases equation for A_B, while the second will follow from the first and injectivity of ψ by evaluating $\psi(a - \sum_i E(a x_i) y_i)$ on any $x \in A$. From the equations above, one computes that $\sum_{i=1}^n x_i \otimes y_i$ is a Casimir element:

$$\sum_i a x_i \otimes y_i = \sum_{i,j} x_j \otimes E(y_j a x_i) y_i = \sum_j x_j \otimes y_j a.$$

There are various questions dating back to Nakayama and Eilenberg of the type, to what extent separable extensions are Frobenius [4]? The most important theorem in this area to date is the one of Endo-Watanabe establishing that a separable algebra A over a commutative ring k is a symmetric algebra (in particular, Frobenius algebra) if A is faithful, f.g. projective as k-module. This theorem has been brought to bear on the centralizer by Sugano to show that various H-separable and centrally projective separable extensions are Frobenius extensions.

The next proposition gives necessary conditions for an extension to be Frobenius, properties that could rule out various separable extensions from being Frobenius.

Proposition 5.1. *Suppose $A \,|\, B$ is a Frobenius extension. Then $\hat{A}_R$ and $(A \otimes_B A)^A$ are both free right R-modules of rank one.*

Proof. First apply the functor $(-)^B$, the B-centralizer to both sides of B-A-isomorphism $A \cong A^*$, obtaining $R \cong \hat{A}$ (since the B-A-bimodule structure on A^* is given by $b \cdot f \cdot a = \lambda(b) \circ f \circ \lambda(a)$, so $f \in (A^*)^B \Leftrightarrow \lambda(b)f = f\lambda(b)$ for each $b \in B$). This isomorphism of right R-modules (cf. preliminaries and module structure (3)) is given by $r \mapsto E \cdot r$, with inverse $F \mapsto \sum_i F(x_i)y_i$.

We next note that

$$A \otimes_B A \xrightarrow{\cong} \operatorname{End} A_B, \quad a \otimes a' \mapsto \lambda(a) \circ E \circ \lambda(a') \tag{20}$$

Its inverse is given by $f \mapsto \sum_i f(x_i) \otimes y_i$ where E, x_i, y_i are defined above.

Taking the A-centralizer of this A-A-isomorphism, we see that

$$(A \otimes_B A)^A \cong \operatorname{End}{}_A A_B \cong R^{\mathrm{op}} \tag{21}$$

where the composed isomorphism comes out as $e \mapsto e^1 E(e^2)$ with inverse $r \mapsto \sum_i x_i r \otimes y_i$. These maps are right R-module isomorphisms with respect to the natural right R-module and module (7). $\square$

Example 5.2. *The full $n \times n$ matrix algebra A over a field is a separable algebra, therefore Frobenius algebra, and so $(A \otimes A)^A \cong A$. Indeed, $\sum_i e_{ij} \otimes e_{ri}$ where $j, r = 1, \dots, n$ form a basis of n^2 Casimir elements. (n of these are separability elements in $\mu^{-1}(1)$, and their average, if the characteristic does not divide n, is the unique symmetric separability element.)*

We recall from section 1 that $\mathcal{C} := (A \otimes_B A)^A$ denotes the Casimir elements of $A \,|\, B$, viewed below as a submodule of the R-R-bimodule ${}_{\sigma,\tau}T$; also recall the B-valued bimodule homomorphisms $\hat{A}$, viewed below as a submodule of $S_{\lambda,\rho}$. Next define two R-R-homomorphisms on the tensor product of the R-R-bimodules $\hat{A}$ and $\mathcal{C}$ in either order:

$$\Psi : \ \hat{A} \otimes_R \mathcal{C} \to R, \quad \Psi(F \otimes e) = e^1 F(e^2), \tag{22}$$

a well-defined R-R-homomorphism since $r \cdot F \cdot r' = F \circ \lambda(r') \circ \rho(r)$ $(F \in \hat{A})$ and $r \cdot e \cdot r' = e^1 r' \otimes r e^2$ $(e \in (A \otimes A)^A, r, r' \in R)$, whence

$$\Psi(rF \otimes er') = e^1 r' F(e^2 r) = r e^1 F(e^2) r'.$$

Similarly,

$$\Phi : \ \mathcal{C} \otimes_R \hat{A} \longrightarrow R, \quad \Phi(e \otimes F) = F(e^1)e^2 \tag{23}$$

defines an R-R-homomorphism.

We first see that a necessary condition for Φ and Ψ to be surjective is that $\hat{A}$ and $\mathcal{C}$ be left and right R-generators (by the trace ideal characterization of generator).

Theorem 5.3. *The datum $(R,\, R,\, \hat{A},\, \mathcal{C},\, \Phi,\, \Psi)$ is a Morita context* [15] *where Φ and Ψ are surjective if any one of the conditions below are satisfied*

(1) *$A \,|\, B$ is a Frobenius extension;*
(2) *R is a simple ring and either $\Phi \neq 0$ or $\Psi \neq 0$;*

(3) *one of $\hat{A}$ and $\mathcal{C}$ is an R-progenerator while the other is isomorphic to its corresponding R-dual with Ψ and Φ corresponding to evaluation and co-evaluation.*

Proof. We first show that the two associativity squares below are commutative. First, the commutativity of the square,

$$\begin{tikzcd} \mathcal{C} \otimes_R \hat{A} \otimes_R \mathcal{C} \arrow[r, "1 \otimes \Psi"] \arrow[d, "\Phi \otimes 1"'] & \mathcal{C} \otimes_R R \arrow[d, "\mathrm{can}"] \\ R \otimes_R \mathcal{C} \arrow[r, "\mathrm{can}"'] & \mathcal{C} \end{tikzcd}$$

corresponds to the equality ($e, f \in (A \otimes A)^A$; $E, F \in \hat{A}$)

$$e \cdot \Psi(F \otimes f) = e \cdot (f^1 F(f^2)) = e^1 f^1 F(f^2) \otimes e^2$$

$$= f^1 \otimes F(f^2 e^1) e^2 = (F(e^1)e^2) \cdot f = \Phi(e \otimes F) \cdot f.$$

The commutativity of the other square,

$$\begin{tikzcd} \hat{A} \otimes_R \mathcal{C} \otimes_R \hat{A} \arrow[r, "1 \otimes \Phi"] \arrow[d, "\Psi \otimes 1"'] & \hat{A} \otimes_R R \arrow[d, "\mathrm{can}"] \\ R \otimes_R \hat{A} \arrow[r, "\mathrm{can}"'] & \hat{A} \end{tikzcd}$$

follows from:

$$E \cdot \Phi(e \otimes F) = F(e^1)E(e^2 -) = F(-e^1)E(e^2) = \Psi(E \otimes e) \cdot F.$$

If $A \,|\, B$ is a Frobenius extension, then there is $E \in \hat{A}$ and $e \in \mathcal{C}$ such that $\Phi(e \otimes E) = 1 = \Psi(E \otimes e)$, as outlined above. But $\mathrm{Im}\,\Psi$ and $\mathrm{Im}\,\Phi$ are both two-sided ideals in R, since both maps are R-R-homomorphisms.

By the same token, if either map is nonzero and R is simple, the map is surjective. The following is an elementary fact from [13]:

$$\Phi(e \otimes \hat{A}) \neq 0 \Leftrightarrow \Psi(\hat{A} \otimes e) \neq 0$$

and similarly

$$\Psi(E \otimes \mathcal{C}) \neq 0 \Leftrightarrow \Phi(\mathcal{C} \otimes E) \neq 0.$$

The last item is a consequence of the well-known theorem of Morita that if P_R is a progenerator, then R and $\mathrm{End}\, P_R$ are Morita equivalent with context P and its R-dual P^* with evaluation and coevaluation as the associative bimodule isomorphisms. $\square$

If we consider Picard groups for noncommutative rings as the group of auto-equivalences of the corresponding full module category, we see via Morita theory that Φ and Ψ are surjective iff $\hat{A}$ and $(A \otimes A)^A$ represent inverse elements in the Picard group of R, which coalesce by Proposition 5.1 to the neutral element in case $A \mid B$ is Frobenius.

The theorem provides a criterion for ring extensions to be Frobenius in analogy with the trace ideal condition for modules to be generators. The images of Ψ or Φ are two-sided ideals, which we check to see if the maps are surjective; if so, we can apply the next Frobenius result to right f.g. projective extensions with centralizer R satisfying $xy = 1$ for $x, y \in R$ implies $yx = 1$ (i.e., R is *Dedekind-finite* [15]).

Theorem 5.4. *Suppose $A \mid B$ is a ring extension where A_B is f.g. projective and its centralizer is Dedekind-finite. If the R-bimodule pairings Ψ and Φ on $\hat{A}$ and $\mathcal{C}$ defined above are surjective, then A is a Frobenius extension of B.*

Proof. If the pairings above are surjective, there are $e, f \in \mathcal{C}$ and $E, F \in \hat{A}$ such that

$$E(e^1)e^2 = 1 = f^1 F(f^2).$$

Then the homomorphism ${}_BA_A \to {}_BA^*_A$ given by $a \mapsto Ea$ (where $Ea(x) = E(ax)$ as usual) is injective, for

$$Ea = 0 \Rightarrow 0 = (Ea)(e^1)e^2 = E(e^1)e^2 a = a.$$

But the homomorphism ${}_BA_A \to {}_BA^*_A$ given by $a \mapsto Fa$ is onto since

$$a = af^1F(f^2) = f^1F(f^2a) \;\Rightarrow\; \forall \phi \in A^* :\; \phi(a) = \phi(f^1)F(f^2a) = Fx(a)$$

where $x = \phi(f^1)f^2$. It follows that $b = E(f^1)f^2 \in R$ satisfies $Fb = E$. Then

$$1 = E(e^1)e^2 = F(be^1)e^2 = F(e^1)e^2b$$

implies b is invertible by Dedekind-finiteness of R. Hence our map $a \mapsto Ea$ is 1-1 and onto, since $x \mapsto Fx$ is onto and $Fx = Eb^{-1}x$ for any $x \in A$. It follows that ${}_BA_A \cong {}_B\mathrm{Hom}\,(A_B, B_B)_A$ and A_B is finite projective, whence $A \mid B$ is a Frobenius extension. □

Let k be a commutative ring below. Note that Ψ is surjective iff the ideal $\{e^1F(e^2) \mid F \in \hat{A}, e \in \mathcal{C}\} = R$.

Corollary 5.5. *If $A \mid B$ is a right free k-algebra extension of finite rank where A and B are f.g. projective as k-modules, and Ψ is surjective, then $A \mid B$ is a Frobenius extension.*

Proof. From the proof above we see that there is an epi $\vartheta : {}_BA_A \to {}_B\mathrm{Hom}\,(A_B, B_B)_A$ given by $a \mapsto Fa$ for some $F \in \hat{A}$. But ϑ is an epi between f.g. projective k-modules of the same P-rank with respect to localizations at prime ideals P in k, since A_B is free. It follows from a well-known fact that ϑ is bijective. (For similar reasons, a f.g. projective k-algebra is Dedekind-finite.) Hence $A \mid B$ is a free Frobenius extension. □

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