DEPTH-ZERO SUPERCUSPIDAL L-PACKETS AND THEIR STABILITY

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In this paper we verify the local Langlands correspondence for pure inner forms of unramified p-adic groups and tame Langlands parameters in "general position". For each such parameter, we *explicitly* construct, in a natural way, a finite set ("L-packet") of depth-zero supercuspidal representations of the appropriate p-adic group, and we verify some expected properties of this L-packet. In particular, we prove, with some conditions on the base field, that the appropriate sum of characters of the representations in our L-packet is stable; no proper subset of our L-packets can form a stable combination. Our L-packets are also consistent with the conjectures of B. Gross and D. Prasad on restriction from SO_{2n+1} to SO_{2n} [24].

These L-packets are, in general, quite large. For example, Sp_{2n} has an L-packet containing 2^n representations, of which exactly two are generic. In fact, on a quasi-split form, each L-packet contains exactly one generic representation for every rational orbit of hyperspecial vertices in the reduced Bruhat-Tits building. When the group has connected center, every depth-zero generic supercuspidal representation appears in one of these L-packets.

We emphasize that there is nothing new about the representations we construct. They are induced from Deligne-Lusztig representations on subgroups of finite index in maximal compact mod-center subgroups, see [42], [44], [61]. The point here is to assemble these representations into L-packets in a natural and explicit way and to verify that these L-packets have the required properties.

To explain further, we need some notation. Let k be a p-adic field of characteristic zero, let K be a maximal unramified extension of k, let $\Gamma = \mathrm{Gal}(K/k)$, and let $\mathrm{Frob} \in \Gamma$ be a Frobenius element. Let $\mathcal{W}_t$, $\mathcal{I}_t$ be the tame Weil group of k and its inertia subgroup. Let $\mathbf{G}$ be a connected reductive k-group which is K-split and k-quasi-split. To simplify the exposition, we assume in this introduction that $\mathbf{G}$ is semisimple. Let $G := \mathbf{G}(K)$, and let F be the action of Frob on G, arising from the given k-structure on $\mathbf{G}$.

In the spirit of local class field theory, we construct both the "geometric" and "p-adic" sides of our local Langlands correspondence, and make an explicit connection between the two sides.

We start with the geometric side. The action of F on the root datum of $\mathbf{G}$ gives rise to an automorphism $\hat{\vartheta}$ of the Langlands dual group $\hat{G}$. The Langlands parameters considered in this paper are continuous homomorphisms

$$\varphi : \mathcal{W}_t \longrightarrow \langle\hat{\vartheta}\rangle \ltimes \hat{G},$$

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(for the discrete topology on $\langle\hat{\vartheta}\rangle \ltimes \hat{G}$) whose centralizer in $\hat{G}$ is finite, and such that $\varphi(\mathrm{Frob})$ is a semisimple element in $\hat{\vartheta}\hat{G}$, and $\varphi(\mathcal{I}_t)$, *a priori* a finite cyclic group, is generated by a regular semisimple element in $\hat{G}$. This latter condition is what we mean by "general position". It implies that $\varphi(\mathcal{I}_t)$ is contained in a unique maximal torus $\hat{T} \subset \hat{G}$. The element $\varphi(\mathrm{Frob})$ normalizes $\hat{T}$, acting via an element of the form $\hat{\vartheta}\hat{w}$, where $\hat{w}$ belongs to the Weyl group of $\hat{T}$ in $\hat{G}$. Moreover, the centralizer of φ is the finite abelian group

$$C_\varphi := \hat{T}^{\hat{\vartheta}\hat{w}}$$

of fixed-points of $\varphi(\mathrm{Frob})$ in $\hat{T}$.

For each irreducible character $\rho \in \mathrm{Irr}(C_\varphi)$, we will define a representation of the group of k-points of a certain inner form of $\mathbf{G}$.

First, we parametrize $\mathrm{Irr}(C_\varphi)$ as follows. The automorphisms $\hat{\vartheta}$ and $\hat{w}$ induce dual automorphisms ϑ and w of the character group $X := X^*(\hat{T})$, and each $\lambda \in X$ determines a character $\rho_\lambda \in \mathrm{Irr}(C_\varphi)$ by restriction from $\hat{T}$ to $\hat{T}^{\hat{\vartheta}\hat{w}}$. Thus we have an isomorphism

$$X/(1-w\vartheta)X \xrightarrow{\sim} \mathrm{Irr}(C_\varphi), \quad \lambda \mapsto \rho_\lambda.$$

Next, for each $\lambda \in X$ we construct an unramified cocycle $u_\lambda \in Z^1(\Gamma, G)$, hence an inner twist of $\mathbf{G}$ with Frobenius $\mathrm{F}_\lambda = \mathrm{Ad}(u_\lambda) \circ \mathrm{F}$, along with an irreducible depth-zero supercuspidal representation π_λ of G^{F_λ}.

The cocycle u_λ is found as follows. Let W be the affine Weyl group of G, acting on the apartment $\mathcal{A} = \mathbb{R} \otimes X$ in the Bruhat-Tits building $\mathcal{B}(G)$ of G. The character $\lambda \in X$ determines a translation $t_\lambda \in W$. Since $\hat{T}^{\hat{\vartheta}\hat{w}}$ is finite, it follows that the operator $t_\lambda w\vartheta$ has a unique fixed-point $x_\lambda \in \mathcal{A}$. If we choose an alcove $C_\lambda \subset \mathcal{A}$ containing x_λ in its closure, we can then uniquely write

$$t_\lambda w\vartheta = w_\lambda y_\lambda \vartheta, \tag{1}$$

where w_λ belongs to the "parahoric subgroup" of W at x_λ and $y_\lambda \in W$ satisfies $y_\lambda\vartheta \cdot C_\lambda = C_\lambda$. The cocycle $u_\lambda : \Gamma \longrightarrow G$ sends Frob to an appropriately chosen representative of y_λ in G.

Now for the representation π_λ. The point x_λ is F_λ-stable, and is in fact a vertex in $\mathcal{B}(G^{\mathrm{F}_\lambda})$. The parahoric subgroup G_λ of G at x_λ is F_λ-stable, and $G_\lambda^{\mathrm{F}_\lambda}$ is a maximal parahoric subgroup of G^{F_λ}. The representation π_λ of G^{F_λ} is compactly-induced from a representation κ_λ of $G_\lambda^{\mathrm{F}_\lambda}$.

This κ_λ is obtained as follows. The element w_λ determines an F_λ-anisotropic torus T_λ of G with $T_\lambda^{\mathrm{F}_\lambda} \subset G_\lambda$. By the depth-zero Langlands correspondence for tori (known, but reproved in Chapter 4.3 below), we can associate to (φ, λ) a depth-zero character χ_λ of $T_\lambda^{\mathrm{F}_\lambda}$, whence a Deligne-Lusztig representation

$$\kappa_\lambda := \pm R_{T_\lambda}^{G_\lambda} \chi_\lambda.$$

Thus, for each $\lambda \in X$, we define

$$\pi_\lambda := \mathrm{ind}_{G_\lambda^{\mathrm{F}_\lambda}}^{G^{\mathrm{F}_\lambda}} \kappa_\lambda,$$

using compact (equivalently, smooth) induction, and prove that π_λ is an irreducible representation of G^{F_λ}. Of course, we now have infinitely many groups G^{F_λ} and representations π_λ, whereas the L-packet $\Pi(\varphi)$ should be parametrized by the finite set $\mathrm{Irr}(C_\varphi)$.

However, according to Vogan's idea of "representations of pure-inner forms" [62], we must take into account the natural G-action on pairs (u, π_u), where $u \in Z^1(\Gamma, G)$ and π_u is a representation of G^{F_u} (here $\mathrm{F}_u = \mathrm{Ad}(u) \circ \mathrm{F}$). We prove that the G-orbit $[u_\lambda, \pi_\lambda]$ is independent of all choices made in the construction, and that for $\lambda, \mu \in X$, we have

$$[u_\lambda, \pi_\lambda] = [u_\mu, \pi_\mu] \quad \Leftrightarrow \quad \rho_\lambda = \rho_\mu \in \mathrm{Irr}(C_\varphi).$$

Thus, our construction leads to an L-packet $\Pi(\varphi)$ in the form of equivalence classes:

$$\Pi(\varphi) = \{[u_\lambda, \pi_\lambda] : \ \rho_\lambda \in \mathrm{Irr}(C_\varphi)\}.$$

We have a partition

$$\Pi(\varphi) = \coprod_{\omega \in H^1(\Gamma, G)} \Pi(\varphi, \omega),$$

where $\Pi(\varphi, \omega)$ consists of the classes $[u_\lambda, \pi_\lambda]$ with $u_\lambda \in \omega$. Let

$$\mathrm{Irr}(C_\varphi) = \coprod_{\omega \in H^1(\Gamma, G)} \mathrm{Irr}(C_\varphi, \omega)$$

be the corresponding partition of $\mathrm{Irr}(C_\varphi)$.

The first expected property of $\Pi(\varphi)$ is that $\mathrm{Irr}(C_\varphi, \omega)$ should be the fiber over ω under the composition

$$\mathrm{Irr}(C_\varphi) \longrightarrow \mathrm{Irr}(\hat{Z}^{\hat{\vartheta}}) \xrightarrow{\sim} H^1(\Gamma, G), \tag{2}$$

where the first map is restriction, the second map is Kottwitz' isomorphism [34], and $\hat{Z}$ is the center of $\hat{G}$. This amounts to proving that the map described in (2) sends $\rho_\lambda \in \mathrm{Irr}(C_\varphi)$ to the class of u_λ in $H^1(\Gamma, G)$. For this, and other purposes, we need a very explicit description of Kottwitz' isomorphism on the level of cocycles. Chapter 2 contains a simple proof of Kottwitz' isomorphism in the form we need, along with related facts used in the proof of stability.

The second expected property of $\Pi(\varphi)$ is that the ratio of formal degrees

$$\frac{\deg(\pi_\lambda)}{\deg(St_\lambda)},$$

where St_λ is the Steinberg representation of G^{F_λ}, should be independent of $\lambda \in X$. This is proved by a direct calculation in Chapter 5.

The third expected property of $\Pi(\varphi)$ is that π_0 (here $\lambda = 0$) should be generic. This is true. In fact, we determine all generic representations in $\Pi(\varphi)$, and show that they are in natural bijection with rational classes of hyperspecial vertices in the reduced building of G. En route, we classify all depth-zero supercuspidal generic representations of unramified groups, see Chapter 6. There is a more general conjecture, due to B. Gross and D. Prasad [23], about which Whittaker models are afforded by the generic representations in $\Pi(\varphi)$. This conjecture is verified for $\Pi(\varphi)$ in [19].

We illustrate the construction and above-mentioned properties, in Chapter 13, with a "canonical example" of L-packets arising from the opposition involution.

The rest of our paper is devoted to the fourth expected property, namely, the stability of $\Pi(\varphi, \omega)$.

We now consider L-packets from the p-adic side. Let $\mathbf{G}$ be any connected reductive K-split k-group with Frobenius automorphism F on G. Take a pair (S, θ), where $S = \mathbf{S}(K)$ is the group of K-points in an unramified k-anisotropic maximal torus $\mathbf{S}$ in $\mathbf{G}$ and θ is a depth-zero character of $S^F = \mathbf{S}(k)$. The group S^F has a unique fixed-point $x \in \mathcal{B}(G^F)$. We have a Deligne-Lusztig virtual character $R_{S,\theta}^{G_x}$ of the parahoric subgroup G_x^F, which we lift to a class function $R(G, S, \theta)$ on the set of regular semisimple elements of G^F, using Harish-Chandra's character integral. One checks that $R(G, S, \theta)$ depends only on the G^F-orbit $\hat{\mathcal{T}}$ of the pair (S, θ). For $(S, \theta) \in \hat{\mathcal{T}}$, we define

$$R(G, \hat{\mathcal{T}}) := R(G, S, \theta).$$

We say that two pairs $(S_1, \theta_1), (S_2, \theta_2)$ as above are *G-stably-conjugate* if there is $g \in G$ such that $\mathrm{Ad}(g)$ sends (S_1^F, θ_1) to (S_2^F, θ_2). Each G-stable class $\hat{\mathcal{T}}_{\mathrm{st}}$ of pairs (S, θ) is a finite disjoint union

$$\hat{\mathcal{T}}_{\mathrm{st}} = \hat{\mathcal{T}}_1 \sqcup \cdots \sqcup \hat{\mathcal{T}}_n$$

of G^F-orbits. We consider the function

$$R(G, \hat{\mathcal{T}}_{st}) := \sum_{i=1}^{n} R(G, \hat{\mathcal{T}}_i).$$

Our aim is to prove that $R(G, \hat{\mathcal{T}}_{st})$ is a stable class-function on the set of strongly regular semisimple elements in G^F.

But first, we relate $R(G, \hat{\mathcal{T}}_{st})$ to the L-packets constructed previously on the geometric side. To do this, we must put the representations in $\Pi(\varphi)$ in "normal form", as follows. We fix $\omega \in H^1(\Gamma, G)$, and choose $u \in \omega$. For each $\lambda \in X$, with $u_\lambda \in \omega$, there is m_λ in G such that $\mathrm{Ad}(m_\lambda)$ sends G^{F_λ} to G^{F_u}. For each $\rho \in \mathrm{Irr}(C_\varphi, \omega)$, we define

$$\pi_u(\varphi, \rho) := \mathrm{Ad}(m_\lambda)_* \pi_\lambda$$

where $\lambda \in X$ is such that $\rho_\lambda = \rho$. Then $\pi_u(\varphi, \rho)$ is a representation of G^{F_u} whose isomorphism class is independent of the choices of λ and m_λ. The "normalized" L-packet is then defined as

$$\Pi_u(\varphi) := \{\pi_u(\varphi, \rho) : \ \rho \in \mathrm{Irr}(C_\varphi, \omega)\};$$

it consists of representations of the fixed group G^{F_u}.

The comparison between the p-adic and geometric sides consists in proving that the sum of characters in $\Pi_u(\varphi)$ is, up to a constant factor, a function of the form $R(G, \hat{\mathcal{T}}_{\mathrm{st}})$ for an appropriate $\hat{\mathcal{T}}_{\mathrm{st}}$. This involves an explicit parametrization of the G-stable classes of pairs (S, θ), in terms of characters $\lambda \in X$. This parametrization follows naturally from our study of Kottwitz' isomorphism in Chapter 2.

Now, to prove stability for our L-packets, it remains to prove that the functions $R(G, \hat{\mathcal{T}}_{\mathrm{st}})$ are stable. The first main step is a reduction formula, using the topological Jordan decomposition.

This reduction becomes trivial on the set of strongly regular topologically semisimple elements in G^F, proving stability there without any restrictions on the residue characteristic.

To prove stability everywhere, we must examine the restriction of $R(G, \hat{\mathcal{T}}_{\mathrm{st}})$ to the topologically unipotent set. We are dealing here with a p-adic analogue of a Green function, so we write $Q(G, \hat{\mathcal{T}}_{\mathrm{st}})$ for the restriction of $R(G, \hat{\mathcal{T}}_{\mathrm{st}})$ to the topologically unipotent set in G^F.

To use the reduction formula, we must establish an identity between $Q(G, \hat{\mathcal{T}}_{\mathrm{st}})$ and $Q(G', \hat{\mathcal{T}}'_{\mathrm{st}})$, where $\mathbf{G}'$ is an inner form of $\mathbf{G}$. To prove this identity, we use Murnaghan-Kirillov theory. The idea is to use a logarithm map and Kazhdan's proof of the Springer Hypothesis [31] to express $Q(G, \hat{\mathcal{T}}_{\mathrm{st}})$ as the Fourier transform of a stable orbital integral on the Lie algebra of G^F. We then invoke a deep result of Waldspurger [63], to the effect that the fundamental lemma is valid for inner forms, and this completes the proof.

However, there are two difficulties with this argument, one pleasant, one not. The pleasant difficulty is about a certain sign in Waldspurger's result. It is given in [63] as a ratio of gamma constants. For us, it is necessary that this ratio be equal to Kottwitz' sign $e(\mathbf{G})$ [33]. This equality of signs is a particular case of a conjecture of Kottwitz. Because of its importance, here and elsewhere, we give two proofs, the first using Shalika germs, the second continuing in the combinatorial spirit of [63].

The unpleasant difficulty is about the logarithm map, which is required to satisfy certain compatibility properties with respect to the Moy-Prasad filtrations on G and its Lie algebra. It is at this point that restrictions on k must be imposed. We require that $p \geq (2+e)n$, where p is the residual characteristic of k, e is the ramification degree of $k/\mathbb{Q}_p$, and n is the dimension of a faithful algebraic representation of $\mathbf{G}$ over k.

Finally, some remarks about exhaustion. All depth-zero supercuspidal representations of G^F are constructed in [44]. Many of them do not appear in our L-packets $\Pi(\varphi)$. They should appear in square-integrable L-packets where φ is tame, but has a nontrivial component on $SL_2(\mathbb{C})$ and therefore cannot be in general position. For groups with connected center, such L-packets have been found for unramified φ in [39], [40], [41], [48]. For groups with connected center, the L-packets constructed in this paper should be exactly those depth-zero L-packets which consist entirely of supercuspidal representations. See Chapter 3 for more discussion of this.

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While we were writing the details of our stability proof, D. Kazhdan and Y. Varshavsky announced a similar stability result, also using Murhaghan-Kirillov theory. See [32].

CONTENTS

1. Basic Notation

The cardinality of a finite set X is denoted by $|X|$. We denote the action of a group G on a set X by $g \cdot x$ or ${}^g x$, for $g \in G, x \in X$. The fixed point set of g in X is denoted by X^g, and $X^G := \cap_{g\in G} X^g$. The set of G-orbits in X is denoted by X/G. The centralizer of $g \in G$ is denoted by $C_G(g)$. The conjugation map $g' \mapsto gg'g^{-1}$ on G is denoted by $\mathrm{Ad}(g)$. The normalizer of a subgroup $S \subset G$ is denoted by $N(G, S)$. In this paper, the phrase "representation of a group G" means "equivalence class of complex representations of G". The set of irreducible representations of a finite group G is denoted by $\mathrm{Irr}(G)$.

In this paper, k is a field of characteristic zero with a nontrivial discrete valuation for which k is complete with finite residue field $\mathfrak{f}$. Let $q = |\mathfrak{f}|$, and let p be the characteristic of $\mathfrak{f}$. We fix an algebraic closure $\bar{k}$ of k. Let K be the maximal unramified extension of k in $\bar{k}$, and let $\mathfrak{F}$ denote the residue field of K. Then $\mathfrak{F}$ is an algebraic closure of $\mathfrak{f}$. Until Section 12 there are no restrictions on p or q. We fix an element $\varpi \in k$ of valuation equal to one.

Let $\mathcal{I}$ be the inertia subgroup of the Galois group $\mathrm{Gal}(\bar{k}/k)$, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)/\mathcal{I}$. Then Γ is topologically generated by an element Frob whose *inverse* induces the automorphism $x \mapsto x^q$

on $\mathfrak{F}$. We let Frob, "the Frobenius", denote both this automorphism of K/k and the automorphism of $\mathfrak{F}/\mathfrak{f}$ which it induces. We have $K = \bar{k}^{\mathcal{I}},\ k = K^{\mathrm{Frob}}$.

We use the following conventions for algebraic groups and their groups of rational points. For any k-group $\mathbf{G}$, we identify $\mathbf{G}$ with its group $\mathbf{G}(\bar{k})$ of $\bar{k}$-rational points, and let $G := \mathbf{G}(K) = \mathbf{G}^{\mathcal{I}}$ denote the K-rational points of $\mathbf{G}$. For most of our purposes, the group G will play the role of "algebraic group". The given action of $\mathrm{Gal}(\bar{k}/k)$ on $\mathbf{G}$ restricts to an action of Γ on G, which is completely determined by an automorphism $F \in \mathrm{Aut}(G)$ given by the action of Frob. We have $G^F = \mathbf{G}(k)$. Likewise, we identify $\mathfrak{f}$-groups G with their groups of $\mathfrak{F}$-rational points, and we have $\mathsf{G}^F = \mathsf{G}(\mathfrak{f})$.

The set of irreducible admissible representations of G^F is denoted by $\mathrm{Irr}(G^F)$. The subset of square-integrable representations in $\mathrm{Irr}(G^F)$ is denoted by $\mathrm{Irr}^2(G^F)$.

If $\mathbf{S}$ is a k-torus in $\mathbf{G}$, we say that a character $\theta \in \mathrm{Irr}(S^F)$ is *F-regular* if θ has trivial stabilizer in $[N(G,S)/S]^F$.

Given an element γ in either $\mathbf{G}$ or G, we let $\mathbf{G}_\gamma$ or G_γ denote the *identity component* of the centralizer of γ in $\mathbf{G}$ or G, respectively. If $\gamma \in G$, then we set $G_\gamma := G \cap \mathbf{G}_\gamma$. We say the element γ in $\mathbf{G}$ or G is *regular semisimple* if $\mathbf{G}_\gamma$ or G_γ is a torus. We let G^{rss} denote the set of regular semisimple elements of G. We say that γ in $\mathbf{G}$ or G is *strongly regular semisimple* if $C_{\mathbf{G}}(\gamma)$ or $C_{\mathsf{G}}(\gamma)$ is a torus. We let G^{srss} denote the set of strongly regular semisimple elements of G. If $\mathbf{S}$ is a maximal k-torus in $\mathbf{G}$, then by [8, 1.10] the set $G^{\mathrm{srss}} \cap S^F$ is nonempty.

For two reductive groups $\mathbf{G}_1,\ \mathbf{G}_2$ or $\mathsf{G}_1,\ \mathsf{G}_2$ of respective ranks r_1, r_2 over k or $\mathfrak{f}$, we let

$$\varepsilon(\mathbf{G}_1, \mathbf{G}_2) = (-1)^{r_1 - r_2}, \qquad \varepsilon(\mathsf{G}_1, \mathsf{G}_2) = (-1)^{r_1 - r_2},$$

respectively.

For any torus $\mathbf{S}$ or S, we let $X_*(\mathbf{S})$ or $X_*(\mathsf{S})$ denote the group of algebraic one-parameter subgroups of $\mathbf{S}$ or S. We say an $\mathfrak{f}$-torus $\mathsf{S} \subset \mathsf{G}$ is *F-minisotropic in* G if every $\mu \in X_*(\mathsf{S})^F$ has image contained in the center of G.

The analogous notion for tori in $\mathbf{G}$ has an extra condition: In this paper, an *unramified torus* is a group of the form $S = \mathbf{S}(K)$, where $\mathbf{S}$ is a k-torus which splits over K. These conditions mean that $\mathcal{I}$ acts trivially on $X_*(\mathbf{S})$, and the action of $\mathrm{Gal}(\bar{k}/k)$ on $X_*(\mathbf{S})$ factors through Γ. An *F-minisotropic torus in* G is a group of the form $S = \mathbf{S}(K)$, where $\mathbf{S}$ is a k-torus in $\mathbf{G}$ such that $\mathbf{S}$ is split over K, and the Frobenius F, arising from the given k-structure on $\mathbf{G}$, has the property that every $\mu \in X_*(\mathbf{S})^F$ has image contained in the center of $\mathbf{G}$.

If $\mathbf{S}$ is a K-split k-torus, we let ${}^0S$ denote the maximal bounded subgroup of the unramified torus S. We have an isomorphism

$$K^\times \otimes X_*(\mathbf{S}) \xrightarrow{\sim} S$$

given by evaluation. This restricts to an isomorphism

$$R_K^\times \otimes X_*(\mathbf{S}) \xrightarrow{\sim} {}^0S,$$

where $R_K^\times$ is the group of units in the ring of integers of K.

For this paper, until the appendices, $\mathbf{G}$ denotes a connected reductive k-group which splits over K. Let F be the Frobenius automorphism of G arising from the given k-structure on $\mathbf{G}$.

Let $\mathcal{B}(G),\ \mathcal{B}(G^F)$ denote the Bruhat-Tits buildings of $G,\ G^F$, respectively. The Frobenius F acts naturally on $\mathcal{B}(G)$, and we have $\mathcal{B}(G^F) = \mathcal{B}(G)^F$.

Let $j : \mathbf{G} \to \mathbf{G}_{ad}$ denote the adjoint quotient. Following our conventions, we set $G_{ad} := \mathbf{G}_{ad}(K)$, and denote again by F the action of Frob on G_{ad}.

Via the map j, the group G acts on $\mathcal{B}(G_{ad})$. The latter is sometimes referred to as the "reduced building" of G. Likewise, the reduced building of G^F is $\mathcal{B}(G_{ad}^F) = \mathcal{B}(G_{ad})^F$.

Each unramified torus S in G determines apartments $\mathcal{A}(S) \subset \mathcal{B}(G)$ and $\mathcal{A}_{ad}(S) \subset \mathcal{B}(G_{ad})$; these apartments can be defined as the fixed-point sets of ${}^0S$ in $\mathcal{B}(G)$ and $\mathcal{B}(G_{ad})$, respectively. The Euclidean closure of any subset J of an apartment is denoted by $\bar{J}$.

If J is an F-stable subset of a facet in $\mathcal{B}(G)$ or $\mathcal{B}(G_{ad})$, we let G_J denote the corresponding parahoric subgroup of G, and let G_J^+ denote the pro-unipotent radical of G_J. The quotient $\mathsf{G}_J := G_J/G_J^+$ is the group of $\mathfrak{F}$-points of a connected reductive group over $\mathfrak{f}$. We have $F(G_J) = G_J, F(G_J^+) = G_J^+$, and the induced action of F on G_J agrees with the $\mathfrak{f}$-structure on G_J. We have $\mathsf{G}_J^F = G_J^F/G_J^{+F}$.

Recall that $\mathbf{G}$ is split over K. By [10, 5.1.10], there exists a K-split maximal torus $\mathbf{T} \subset \mathbf{G}$ which is defined over k and maximally k-split. We abbreviate $X := X_*(\mathbf{T})$, $\mathcal{A} := \mathcal{A}(T)$. Let N be the normalizer of T in G. The affine Weyl group of T in G is the quotient

$$W := N/{}^0T.$$

We will use $T = \mathbf{T}(K)$ as a "platonic" unramified torus in G; various unramified tori S as above will arise from twisted embeddings of T in G.

Let $\mathbf{T}_{ad} = j(\mathbf{T})$ denote the image of $\mathbf{T}$ in $\mathbf{G}_{ad}$, and abbreviate $\mathcal{A}_{ad} := \mathcal{A}(T_{ad})$, $X_{ad} := X_*(\mathbf{T}_{ad})$. Let W_{ad} be the affine Weyl group of T_{ad} in G_{ad}. Since $\mathbf{T}$ and $\mathbf{T}_{ad}$ are defined over k, the Frobenius F induces automorphisms of $X, X_{ad}, \mathcal{A}, \mathcal{A}_{ad}, W, W_{ad}$. We write also

$$j : X \to X_{ad}, \quad j : W \longrightarrow W_{ad}$$

for the maps induced by j. These maps are F-equivariant, since j is defined over k. The kernel and image of the latter map are given as follows.

We may identify X with the normal subgroup $T/{}^0T \triangleleft W$, via evaluation at ϖ. If $\lambda \in X$, we let $t_\lambda := \lambda(\varpi)$ denote both the corresponding element of T and its image in W. There is a map $W_{ad} \longrightarrow X_{ad}/jX$, to be defined shortly, which fits into an exact sequence

$$1 \longrightarrow X^W \longrightarrow W \xrightarrow{j} W_{ad} \longrightarrow X_{ad}/jX \longrightarrow 1. \tag{3}$$

Note that the last group X_{ad}/jX is finite. The group X^W acts trivially on $\mathcal{A}_{ad}$.

There exists an F-stable alcove $C \subset \mathcal{A}$. Let W° be the subgroup of W generated by reflections in the walls of C, and let $\Omega_C := \{\omega \in W :\ \omega \cdot C = C\}$. The group Ω_C is abelian, isomorphic to the quotient of X by the co-root sublattice $X^\circ \subset X$. The normal subgroup $W^\circ \triangleleft W$ acts simply-transitively on alcoves in $\mathcal{A}$, so we have a semidirect product expression

$$W = \Omega_C W^\circ.$$

A similar discussion and decomposition holds for W_{ad}.

We have been using F to denote the Frobenius arising from an arbitrary K-split k-structure on G. When this k-structure is in fact k-quasi-split, we denote the Frobenius by F. The key difference in the quasi-split case is the existence of an F-fixed hyperspecial vertex $o \in \mathcal{A}_{ad}$.

In the quasi-split case, we denote by ϑ the automorphisms of $X, X_{ad}, \mathcal{A}, \mathcal{A}_{ad}, W, W_{ad}$ induced by F. Choose a ϑ-fixed hyperspecial vertex $o \in \mathcal{A}_{ad}$. We let W_o be the image of $N_o := N \cap G_o$ in W. We may identify $W_o = N/T$ via the natural maps

$$W_o \hookrightarrow W = N/{}^0T \longrightarrow N/T.$$

The map j is injective on W_o and we identify W_o with $j(W_o)$. We have semidirect product decompositions

$$W = X \rtimes W_o, \qquad W_{ad} = X_{ad} \rtimes W_o,$$

and all factors are preserved by ϑ. The map $W_{ad} \longrightarrow X_{ad}/jX$ in the exact sequence (3) is induced by projection onto the X_{ad} factor in W_{ad}.

Finally, an inner twist of F by a cocycle $u \in Z^1(\mathrm{F}, G)$ (see Section 2) will be denoted by $\mathrm{F}_u := \mathrm{Ad}(u) \circ \mathrm{F}$.

2. Remarks on Galois cohomology

To state the Langlands conjectures at the level of refinement considered in this paper requires some notions from the Galois cohomology of reductive groups over local fields. The central results here are due to Kottwitz [34, 35], who computes $H^1(k, \mathbf{G})$ in terms of the action of $\mathrm{Gal}(\bar{k}/k)$ on the center of the dual group of $\mathbf{G}$, and Bruhat-Tits [11], who compute $H^1(k, \mathbf{G})$ in terms of the building of G. Here we give simple proofs of the above-mentioned results at the level of cocycles. This allows us to construct cocycles in $\mathbf{G}$ from fixed-points in $\mathcal{A}$ of elements in the affine Weyl group. Such fixed-points arise from the Langlands parameters we consider. Thus we can associate an explicit Frobenius to each Langlands parameter. We also use our cocycles to give representatives for various stable and rational classes of tori and semisimple elements in G. These will be used in the proof of stability.

2.1. **Unramified cohomology.** Let U be a group and let F be an endomorphism of U. For an integer $d \geq 1$ and $g \in U$, define

$$N_d(F)(g) := gF(g)\cdots F^{d-1}(g) \in U.$$

Note that

$$N_{dm}(F) = N_m(F^d) \circ N_d(F). \tag{4}$$

Assume that every element of U is fixed by some power of F. Giving U the discrete topology, this means that the group $\hat{\mathbb{Z}}$ of profinite integers, with topological generator F, acts continuously on U. We denote by

$$H^1(F, U) = H^1(\hat{\mathbb{Z}}, U)$$

the continuous (nonabelian) cohomology of U. Any cocycle is determined by its value on F, which is an element of the set

$$Z^1(F,U) := \{u \in U : \; N_m(u) = 1 \quad \text{for some } m \geq 1\}.$$

Thus we view cocycles as elements of U, and $H^1(F,U)$ is the quotient of $Z^1(F,U)$ under the U-action: $g * u = guF(g)^{-1}$. Note that if $N_m(u) = 1$ and $F^d(g) = g$, then $N_{md}(F)(g * u) = 1$.

If U is nonabelian, the set $Z^1(F,U)$ of cocycles is not closed under multiplication. However, if $u, v \in Z^1(F,U)$ and $d \geq 1$ we have

$$N_d(F)(vu) = N_d(F_u)(v) \cdot N_d(F)(u), \tag{5}$$

where

$$F_u := \mathrm{Ad}(u) \circ F \in \mathrm{End}(U).$$

From Equations (4) and (5) we conclude:

Lemma 2.1.1. *If two of the following hold, then so does the third:*

(1) $u \in Z^1(F,U)$,
(2) $v \in Z^1(F_u,U)$,
(3) $vu \in Z^1(F,U)$.

Lemma 2.1.2. *If the fixed-point group U^{F^d} is finite for each $d \geq 1$, then $Z^1(F,U) = U$.*

Proof. Fix $d \geq 1$ and suppose that $g^m = 1$ for each $g \in U^{F^d}$. From Equation (4), we have

$$N_{dm}(F)(g) = N_m(F^d)\left(N_d(F)(g)\right) = \left(N_d(F)(g)\right)^m = 1.$$

□

Lemma 2.1.3. *Suppose U is a compact group with endomorphism F and a decreasing filtration $U = U_0 \supset U_1 \supset U_2 \supset \cdots$ by open normal F-stable subgroups U_n such that $\bigcap_n U_n = \{1\}$. Assume that $H^1(F, U_n/U_{n+1}) = 1$ for all $n \geq 0$. Then $H^1(F,U) = 1$.*

Proof. Let $u \in Z^1(F,U)$, so that $u \in U_n$ for some $n \geq 0$. By the vanishing assumption and normality, there are $g_0 \in U_n$ and $u_1 \in U_{n+1}$ such that $u = g_0 * u_1$. Then $u_1 = g_0^{-1} * u \in Z^1(F,U)$. Repeating, we have elements $g_k, u_k \in U_{n+k}$ for all $k \geq 1$, such that $u = (g_0 g_1 \cdots g_{k-1}) * u_k$. Since U is compact, the limit $g := \lim_k g_0 g_1 \cdots g_k$ exists, and $u = g * 1$. □

2.2. **Steinberg's vanishing theorem.** In this section, $\mathbf{G}$ is only required to be a connected k-group, with Frobenius automorphism F on G. At several points we use the following consequence of a well-known result of Steinberg [56, Thm. 1.9]:

Theorem 2.2.1. $H^1(K,\mathbf{G}) = 1$.

One consequence of Theorem 2.2.1 is that the natural surjection $\mathrm{Gal}(\bar{k}/k) \to \Gamma$ induces an isomorphism

$$H^1(F,G) \simeq H^1(k,\mathbf{G}).$$

Each cocycle $u \in Z^1(F, G)$ arises from a twisted k-structure on $\mathbf{G}$, under which Frob acts on G via the automorphism

$$F_u := \mathrm{Ad}(u) \circ F \in \mathrm{Aut}(G),$$

so that G^{F_u} is the group of k-rational points under this twisted k-structure [51, III.1.3]. Note that for $g \in G$, we have

$$\mathrm{Ad}(g) \circ F_u = \mathrm{F}_{g*u} \circ \mathrm{Ad}(g),$$

so $\mathrm{Ad}(g)$ induces an isomorphism

$$\mathrm{Ad}(g) : G^{F_u} \xrightarrow{\sim} G^{F_{g*u}}.$$

Thus, the isomorphism class of G^{F_u} depends only on the class of u in $H^1(F, G)$. However, the dependence is non-canonical, in the sense that a class in $H^1(F, G)$ does not determine a unique twist of F; one must choose a cocycle in the class. We must therefore accept a wide range of Frobenius endomorphisms F_u giving rise to the same k-isomorphism class of groups.

2.3. **Explicit cocycles.** For the rest of this chapter, $\mathbf{G}$ is a connected reductive k-group with Frobenius automorphism F on G. To keep things as simple and clear as possible, we assume that $\mathbf{G}$ is K-split and k-quasi-split, even though these assumptions are not necessary until later in the paper. The following result is a special case of [64, Prop. 2.3]. We give a direct proof, in our context.

Lemma 2.3.1. *For each $x \in \mathcal{B}(G)^F$ we have $H^1(F, G_x) = 1$, where G_x is the parahoric subgroup attached to x.*

Proof. If $u \in Z^1(F, G_x)$, then $u \in G_x^{F^d}$ for some $d \geq 1$. We want to apply 2.1.3 to the compact group $U = G_x^{F^d}$. Let $G_{x,r}$, $r \in \mathbb{R}_{\geq 0}$, be the Moy-Prasad-Yu filtration of G_x [65]. There is an increasing sequence $\{r_n : n = 0, 1, 2, \dots\} \subset \mathbb{R}_{\geq 0}$ such that for every $r \geq 0$ we have $G_{x,r} = G_{x,r_n}$ for a unique n. These filtration subgroups are F-stable; we set $U_n := G_{x,r_n}^{F^d}$.

Each quotient group U_n/U_{n+1} is the group of $\mathfrak{f}_d$-rational points in a connected $\mathfrak{f}$-group U_n. Here $\mathfrak{f}_d$ denotes the degree d extension of $\mathfrak{f}$. By the Lang-Steinberg theorem, we have $H^1(\mathfrak{f}, \mathsf{U}_n) = 1$ for all $n \geq 0$. Since the natural map

$$H^1(\mathfrak{f}_d/\mathfrak{f}, \mathsf{U}_n(\mathfrak{f}_d)) \longrightarrow H^1(\mathfrak{f}, \mathsf{U}_n)$$

is injective [51, I.5.8], we have $H^1(\mathfrak{f}_d/\mathfrak{f}, \mathsf{U}_n(\mathfrak{f}_d)) = 1$ for all $n \geq 0$.

We have shown that the groups U_n satisfy the conditions of Lemma 2.1.3, which implies that the cocycle u is a coboundary in $H^1(F, U_0)$, hence also in $H^1(F, G_x)$. $\square$

Recall that $\mathbf{T}$ is a K-split maximal k-torus in $\mathbf{G}$, such that $\mathbf{T}$ contains a maximal k-split torus in $\mathbf{G}$, and N is the normalizer of T in G. The affine Weyl group of T in G is the quotient $W := N/{}^0T$, where ${}^0T$ is the maximal bounded subgroup of T. The apartment of T in $\mathcal{B}(G)$ is denoted by $\mathcal{A}$, and the N-action on $\mathcal{A}$ factors through a faithful action of W on $\mathcal{A}$.

To describe $H^1(\mathrm{F}, G)$ on the level of cocycles, the first step is to reduce the group in which the cocycles live. Let C be an F-stable alcove in $\mathcal{A}$ (see [60, 3.4.3]). Let G_C be the Iwahori subgroup of G attached to C. The normalizer in G of G_C is the group

$$G_C^\star := \{g \in G : \ g \cdot C = C\}.$$

We have $N \cap G_C = {}^0T$, and we set

$$N_C := N \cap G_C^\star.$$

Then the group

$$\Omega_C := \{\omega \in W : \ \omega \cdot C = C\}$$

is the image of N_C in W. The inclusion $N_C \hookrightarrow G_C^\star$ induces an isomorphism

$$d : \Omega_C \xrightarrow{\sim} G_C^\star / G_C.$$

Since $\mathrm{F} \cdot C = C$, we have $\mathrm{F}(G_C^\star) = G_C^\star$, so we may define $H^1(\mathrm{F}, G_C^\star)$ as in 2.2, and similarly for $H^1(\mathrm{F}, N_C)$. The first reduction relies on the existence and conjugacy of rational alcoves, already used above.

Lemma 2.3.2. *The inclusion $G_C^\star \hookrightarrow G$ induces an isomorphism*

$$H^1(\mathrm{F}, G_C^\star) \xrightarrow{\sim} H^1(\mathrm{F}, G).$$

Proof. We first prove surjectivity. Let $u \in Z^1(\mathrm{F}, G)$. By [60, 1.10.3] there is an F_u-stable alcove $C_u \subset \mathcal{B}(G)$. We have $g \cdot C_u = C$ for some $g \in G$. Since $\mathrm{F}_u \cdot C_u = C_u$, we have $u\,\mathrm{F}(g^{-1}) \cdot C = g^{-1} \cdot C$, i.e., $g * u \in G_C^\star$.

For injectivity, suppose $u, v \in Z^1(\mathrm{F}, G_C^\star)$, and $g * u = v$ for some $g \in G$. Then

$$\mathrm{F}_v \cdot g \cdot C = v\,\mathrm{F}(g) \cdot C = gu \cdot C = g \cdot C.$$

Thus $g \cdot C$ and C are two F_v-stable alcoves in $\mathcal{B}(G)$. By [60, §2.5] there is $h \in G^{\mathrm{F}_v}$ such that $hg \cdot C = C$, so $hg \in G_C^\star$. However, $h = \mathrm{F}_v(h)$ implies

$$(hg)u\,\mathrm{F}(hg)^{-1} = hv\,\mathrm{F}(h)^{-1} = v,$$

so $[u] = [v]$ in $H^1(\mathrm{F}, G_C^\star)$. □

To go further, we need another vanishing result. The image of ${}^0T$ in G_C is a maximal $\mathfrak{f}$-torus T in G_C. We let ${}^0T^+$ be the kernel of the natural map ${}^0T \longrightarrow \mathsf{T}$. Then ${}^0T^+$ is the pro-unipotent radical of ${}^0T$.

Recall that $\Gamma = \mathrm{Gal}(K/k)$. A *topological* Γ*-module* [50, XIII, p.188] is a Γ-module in which every element is fixed by some power of Frob.

Lemma 2.3.3. *For any $n \in N$, letting* Frob *act on ${}^0T$ via $\mathrm{F}_n := \mathrm{Ad}(n) \circ \mathrm{F}$ makes ${}^0T$ a topological* Γ*-module for which $H^2(\mathrm{F}_n, {}^0T) = 0$.*

Proof. As endomorphisms of T, we have $\mathrm{F}_n = \mathrm{F}_w$, where w is the image of n in N/T. Now

$$\mathrm{F}_w^k = \mathrm{Ad}[w\,\mathrm{F}(w) \cdots \mathrm{F}^{k-1}(w)] \circ \mathrm{F}^k, \quad \text{for all} \quad k \geq 1.$$

Since N/T is finite, the term in brackets is 1 for some k (cf. Lemma 2.1.2) and multiples thereof. Also every $t \in T$ is fixed by some F^m and multiples thereof. The first assertion now follows.

The exact sequence of topological Γ-modules

$$1 \longrightarrow {}^0T^+ \longrightarrow {}^0T \longrightarrow \mathsf{T} \longrightarrow 1$$

gives an exact sequence [51, §2.2, p.10] in Galois cohomology

$$\cdots \longrightarrow H^2(\mathrm{F}_w, {}^0T^+) \longrightarrow H^2(\mathrm{F}_w, {}^0T) \longrightarrow H^2(\mathrm{F}_w, \mathsf{T}) \longrightarrow \cdots .$$

Since T is a torsion group, we have $H^2(\mathrm{F}_w, \mathsf{T}) = 0$ by [50, Proposition 2, p.189]. Since ${}^0T^+$ is the union of an inverse limit of torsion groups, from [50, Lemma 3, p.185] we have $H^2(\mathrm{F}_w, {}^0T^+) = 0$. □

Consider now the following commutative diagram, where the horizontal maps are inclusions, and the vertical maps are the natural projections.

$$\begin{tikzcd} N_C \arrow[r, "b"] \arrow[d, "a"'] & G_C^\star \arrow[d, "c"] \\ \Omega_C \arrow[r, "d"] & G_C^\star/G_C \end{tikzcd}$$

Lemma 2.3.4. *The maps a, b, c, d in the above diagram induce isomorphisms a_*, b_*, c_*, d_* on $H^1(\mathrm{F}, \cdot)$.*

Proof. The map d is already an isomorphism. The map a_* is surjective by Lemma 2.3.3 and [51, Corollary, p.54]. Since the induced diagram on cohomology is commutative, the map c_* is also surjective.

If $u \in Z^1(\mathrm{F}, G_C^\star)$, then from [51, Corollaries 1 and 2, p.52] the fiber of c_* through $[u]$ is in bijection with $\ker[H^1(\mathrm{F}_u, G_C^\star) \to H^1(\mathrm{F}_u, G_C^\star/G_C)]$. By the exact sequence

$$\cdots \longrightarrow H^1(\mathrm{F}_u, G_C) \longrightarrow H^1(\mathrm{F}_u, G_C^\star) \longrightarrow H^1(\mathrm{F}_u, G_C^\star/G_C)$$

in nonabelian cohomology [51, Proposition 38, p.51] and the vanishing of $H^1(\mathrm{F}_u, G_C)$ by Lemma 2.3.1, the above kernel is trivial. Hence c_* is injective. A similar argument shows that a_* is injective, which completes the proof. □

2.4. **Kottwitz' Theorem.** In this section we will recover Kottwitz' theorem on the level of cocycles. First we need an elementary result.

Lemma 2.4.1. *Let A be a finitely generated abelian group, and let $\sigma \in \mathrm{Aut}(A)$ be an automorphism of finite order. Define*

$$A_1 := \{a \in A : \ (1 + \sigma + \cdots + \sigma^{n-1})a = 0 \quad \textit{for some} \quad n \geq 1\},$$

$$A_2 := \{a \in A : \ ma \in (1 - \sigma)A \quad \textit{for some} \quad m \geq 1\}.$$

Then $A_1 = A_2$.

Proof. For $p \geq 1$ let $N_p = 1 + \sigma + \cdots + \sigma^{p-1} \in \mathrm{End}(A)$. Then

$$N_{pq} = N_p + \sigma^p N_p + \cdots + \sigma^{p(q-1)} N_p = (1 + \sigma^p + \cdots + \sigma^{p(q-1)})N_p.$$

Hence if $N_p(a) = N_q(b) = 0$, then $N_{pq}(a+b) = 0$. That is, A_1 is a subgroup of A. Also, since A_{tor} is finite, every element of A_{tor} is fixed by some power of σ. If $qa = 0$, then $\sigma^p(a) = a$ for some $p \geq 1$, so $N_{pq}(a) = qN_p(a) = N_p(qa) = 0$. Thus, $A_{\mathrm{tor}} \subseteq A_1$.

Set $\bar{A} = A_1/A_{\mathrm{tor}}$, $V = \mathbb{Q} \otimes \bar{A}$. The latter is a finite-dimensional $\mathbb{Q}$-vector space, to which σ and N_p extend for all p. We claim that $V^\sigma = \{0\}$. If $0 \neq v \in V^\sigma$, we may assume, by clearing denominators, that $v \in \bar{A}$. Then $N_p(v) = pv \neq 0$ for all $p \geq 1$, a contradiction. Hence $1 - \sigma$ is invertible on V. Let $a \in A_1$ have image $\bar{a} \in \bar{A}$. Write $\bar{a} = (1-\sigma)\bar{b}$, for some $\bar{b} \in V$. Clearing denominators, we have $m\bar{a} = (1-\sigma)\bar{c}$ for some $m \in \mathbb{Z}$, $c \in A_1$. So $ma = (1-\sigma)c + z$, where $z \in A_{\mathrm{tor}}$. Say $qz = 0$. Then $qma = (1-\sigma)qc \in (1-\sigma)A$, showing that $A_1 \subseteq A_2$.

The other containment is easy: If $qa = (1-\sigma)b$, and p is the order of σ, then

$$N_{pq}(a) = qN_p(a) = N_p(qa) = N_p(1-\sigma)b = (1-\sigma^p)b = 0.$$

□

Let $X = X_*(\mathbf{T})$, and let W° be the subgroup of W generated by reflections in the walls of an alcove in $\mathcal{A}$. Evaluation at ϖ identifies $\lambda \in X$ with the operator $t_\lambda \in W$ of translation by λ on $\mathcal{A}$. Under this identification, $X \cap W^\circ =: X^\circ$ is the co-root lattice of $\mathbf{T}$. We set $\bar{X} := X/X^\circ$. The group W° acts simply-transitively on alcoves, hence we have the semidirect product decomposition

$$W = W^\circ \rtimes \Omega_C.$$

The automorphism F preserves T, hence induces an automorphism ϑ of W, which preserves X, W°, Ω_C. If $\mathbf{G}$ is actually k-split then ϑ is trivial. In general, ϑ has finite order.

For $\lambda \in X$, let ω_λ be the unique element of $t_\lambda W^\circ \cap \Omega_C$. Then $\omega_\lambda = 1$ exactly when λ belongs to X°; the map $\lambda \mapsto \omega_\lambda$ induces a ϑ-equivariant group isomorphism $\bar{X} \xrightarrow{\sim} \Omega_C$.

Corollary 2.4.2. *The map $\lambda \mapsto \omega_\lambda$ induces an isomorphism*

$$\big[\bar{X}/(1-\vartheta)\bar{X}\big]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}, \Omega_C).$$

Proof. Apply Lemma 2.4.1 to the abelian group $A = \Omega_C$, $\sigma = \vartheta$. Since $(1-\vartheta)\Omega_C \subset (\Omega_C)_2$, we have

$$H^1(\mathrm{F}, \Omega_C) = (\Omega_C)_1/(1-\vartheta)\Omega_C = (\Omega_C)_2/(1-\vartheta)\Omega_C = \big[\Omega_C/(1-\vartheta)\Omega_C\big]_{\mathrm{tor}}.$$

The isomorphism $\bar{X} \simeq \Omega_C$ finishes the proof. □

Combining 2.3.4 and 2.4.2, we can express Kottwitz' isomorphism in the following form.

Corollary 2.4.3. *The composition*

$$\big[\bar{X}/(1-\vartheta)\bar{X}\big]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}, \Omega_C) \xrightarrow{a_*^{-1}} H^1(\mathrm{F}, N_C) \xrightarrow{b_*} H^1(\mathrm{F}, G)$$

is a bijection. A class $[\lambda] \in \big[\bar{X}/(1-\vartheta)\bar{X}\big]_{\mathrm{tor}}$, represented by $\lambda \in X$, corresponds to the class $[\dot{\omega}_\lambda] \in H^1(\mathrm{F}, G)$, where $\dot{\omega}_\lambda \in Z^1(\mathrm{F}, N_C)$ is any element whose image in W is the unique element ω_λ of $t_\lambda W^\circ \cap \Omega_C$.

2.5. **The dual group.** Corollary 2.4.3 is usually expressed in terms of the dual group $\hat{G}$ of $\mathbf{G}$. Let $Y := X^*(\mathbf{T})$ be the algebraic character group of $\mathbf{T}$, and let $\langle\ ,\ \rangle : X \times Y \to \mathbb{Z}$ be the natural pairing. The dual group of $\mathbf{T}$ is the complex torus $\hat{T} := Y \otimes \mathbb{C}^\times$; it is a maximal torus in $\hat{G}$. Let $\hat{Z}$ denote the center of $\hat{G}$.

For any $\sigma \in \mathrm{Aut}(X)$, let $\hat{\sigma} \in \mathrm{Aut}(Y)$ be defined by

$$\langle \sigma\lambda, \eta\rangle = \langle \lambda, \hat{\sigma}\eta\rangle, \qquad \lambda \in X,\ \eta \in Y.$$

The action of $\hat{\vartheta}$ on Y extends to the automorphism $\hat{\vartheta} \otimes 1$ of $\hat{T}$, thence by restriction to an automorphism of $\hat{Z}$.

We may identify

$$\bar{X} = \mathrm{Hom}(\hat{Z}, \mathbb{C}^\times),$$

via restriction of characters. Restricting further to $\hat{Z}^{\hat{\vartheta}}$, we may identify

$$\bar{X}/(1-\vartheta)\bar{X} = \mathrm{Hom}(\hat{Z}^{\hat{\vartheta}}, \mathbb{C}^\times).$$

The elements in $\bar{X}/(1-\vartheta)\bar{X}$ vanishing on the identity component of $\hat{Z}^{\hat{\vartheta}}$ are exactly the torsion elements in $\bar{X}/(1-\vartheta)\bar{X}$. Hence we may identify

$$\big[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} = \mathrm{Irr}[\pi_0(\hat{Z}^{\hat{\vartheta}})].$$

With these identifications, Corollary 2.4.3 becomes the usual expression of Kottwitz' isomorphism.

2.6. **A commutative diagram.** It is at this point that we first use seriously the assumption that F arises from a quasi-split k-structure on $\mathbf{G}$ which is K-split. Such an assumption ensures the existence of a F-fixed hyperspecial vertex $o \in j(\bar{C})$.

Let W_o be the image of $N_o = N \cap G_o$ in W. The latter has another semidirect product expression

$$W = X \rtimes W_o,$$

and both factors are preserved by ϑ.

Since $\mathbf{G}$ is K-split and k-quasi-split, there is a $\mathrm{Gal}(\bar{k}/k)$-invariant pinning in G. Applying Prop. 3 of [59] to this pinning, we see that there is an F-stable finite subgroup $\dot{W}_o \subset N_o$ projecting onto W_o.

Let $w \in W_o$, choose a lift $\dot{w} \in \dot{W}_o$ of w, and set $\mathrm{F}_w := \mathrm{Ad}(\dot{w}) \circ \mathrm{F}$. Applying Lemmas 2.1.2 and 2.3.1 to the groups $\dot{W}_o$ and G_o, respectively, there exists $p_0 \in G_o$ such that

$$\dot{w} = p_0^{-1}\,\mathrm{F}(p_0).$$

The map $\mathrm{Ad}(p_0) : T \longrightarrow G$ intertwines the pairs (T, F_w), (G, F). Let

$$r : H^1(\mathrm{F}_w, T) \longrightarrow H^1(\mathrm{F}, G), \tag{6}$$

be the map induced by $\mathrm{Ad}(p_0)$.

A version of the following result was proved by Kottwitz [35, Thm. 1.2].

Lemma 2.6.1. *We have a commutative diagram*

$$\begin{tikzcd} {[X/(1-w\vartheta)X]_{\mathrm{tor}}} \arrow[r] \arrow[d, "\simeq"'] & {[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}}} \arrow[d, "\simeq"] \\ H^1(\mathrm{F}_w, T) \arrow[r, "r"] & H^1(\mathrm{F}, G) \end{tikzcd}$$

where the vertical maps are from 2.4.3 applied to T and G, the top row is the natural projection and the map r is defined in Equation (6).

Proof. Starting at $[X/(1-w\vartheta)X]_{\mathrm{tor}}$ and going down the left side, then over on the bottom row, the class of $\lambda \in [X/(1-w\vartheta)X]_{\mathrm{tor}}$ goes to the class

$$[p_0 t_\lambda p_0^{-1}] = [t_\lambda p_0^{-1}\,\mathrm{F}(p_0)] = [t_\lambda \dot{w}] \in H^1(\mathrm{F}, G).$$

Equation (9) below shows that $[t_\lambda \dot{w}] = [\dot{\omega}_\lambda]$, which is the result of the other route, by Corollary 2.4.3. □

2.7. **Fixed points and cocycles.** We continue in the set-up of Section 2.4. In this section we show how cocycles in $Z^1(\mathrm{F}, G)$ arise from fixed-points in $\mathcal{A}$ of elements in the affine Weyl group W. This will be used to associate Frobenius endomorphisms on G to Langlands parameters in ${}^LG$.

Let $w \in W_o$, and let X_w be the preimage in X of $[X/(1-w\vartheta)X]_{\mathrm{tor}}$. For $\lambda \in X_w$, we define

$$\sigma_\lambda := t_\lambda w\vartheta \in W \rtimes \langle\vartheta\rangle.$$

Lemma 2.7.1. *The element $\sigma_\lambda \in W \rtimes \langle\vartheta\rangle$ has finite order.*

Proof. The element $w\vartheta$ has finite order, say n, since it belongs to the finite group $W_o \rtimes \langle\vartheta\rangle$. We let $N_{w\vartheta} = 1 + w\vartheta + \cdots + (w\vartheta)^{n-1} \in \mathrm{End}(X)$ be the associated norm mapping. Since $\lambda \in X_w$, there is $m \geq 1$ such that $m\lambda = (1-w\vartheta)\nu$, for some $\nu \in X$. Then

$$\sigma_\lambda^{nm} = N_{w\vartheta}(t_{m\lambda}) = N_{w\vartheta}(1-w\vartheta)(t_\nu) = 1.$$

□

By Lemma 2.7.1, σ_λ preserves a facet J_λ in $\mathcal{A}$. Choose an alcove C_λ in $\mathcal{A}$ containing J_λ in its closure. Let W_λ be the subgroup of W° generated by reflections in the hyperplanes containing J_λ. The group W_λ acts simply-transitively on alcoves in $\mathcal{A}$ containing J_λ in their closure. Hence there is a unique element $w_\lambda \in W_\lambda$ such that

$$\sigma_\lambda \cdot C_\lambda = w_\lambda \cdot C_\lambda.$$

Set $y_\lambda := w_\lambda^{-1} t_\lambda w$. Thus we have two expressions for σ_λ:

$$t_\lambda w\vartheta = \sigma_\lambda = w_\lambda y_\lambda \vartheta, \tag{7}$$

and the latter is characterized as the unique factorization of σ_λ such that $w_\lambda \in W_\lambda$ and $y_\lambda \in W$ satisfies $y_\lambda\vartheta \cdot C_\lambda = C_\lambda$. Since w_λ fixes J_λ pointwise, we also have $y_\lambda\vartheta \cdot J_\lambda = J_\lambda$; indeed, σ_λ and $y_\lambda\vartheta$ have the same action on J_λ.

To briefly look ahead: Equation (7) is the essence of our Langlands correspondence. The expression $t_\lambda w\vartheta$ will arise from a certain kind of Langlands parameter, that is, $t_\lambda w\vartheta$ is an object on the "geometric side". On the other hand, y_λ and w_λ will determine a twisted Frobenius F_λ and an unramified torus in G^{F_λ}, respectively, so y_λ and w_λ are objects on the "p-adic side". The next result leads us to F_λ.

Lemma 2.7.2. *There exists a lift $u_\lambda \in N$ of y_λ such that $u_\lambda \in Z^1(\mathrm{F}, N)$.*

Proof. If j is the order of σ_λ (see Lemma 2.7.1), then
$$1 = (w_\lambda y_\lambda\vartheta)^j = w'_\lambda(y_\lambda\vartheta)^j,$$
for some $w'_\lambda \in W^\circ$. Since W° acts simply-transitively on alcoves in $\mathcal{A}$, we can decompose
$$W \rtimes \langle\vartheta\rangle = W^\circ \rtimes \tilde{\Omega}_{C_\lambda},$$
where $\tilde{\Omega}_{C_\lambda}$ is the stabilizer of C_λ in $W \rtimes \langle\vartheta\rangle$. It follows that $(y_\lambda\vartheta)^j = 1$. Let k be the order of ϑ. Then
$$1 = (y_\lambda\vartheta)^{jk} = [y_\lambda\vartheta(y_\lambda)\cdots\vartheta^{jk-1}(y_\lambda)]\vartheta^{jk} = y_\lambda\vartheta(y_\lambda)\cdots\vartheta^{jk-1}(y_\lambda).$$
That is, $y_\lambda \in Z^1(\mathrm{F}, W)$. Hence, for all $x \in W$, we have $x^{-1}y_\lambda\vartheta(x) \in Z^1(\mathrm{F}, W)$.

Recall that $y_\lambda\vartheta \cdot C_\lambda = C_\lambda$. Let $x \in W^\circ$ be the element such that $C_\lambda = x \cdot C$. Then $y_\lambda\vartheta \cdot x \cdot C = x \cdot C$, so $x^{-1}y_\lambda\vartheta(x) \in \Omega_C$ (recall C is ϑ-stable). By the previous paragraph, we have in fact $x^{-1}y_\lambda\vartheta(x) \in Z^1(\mathrm{F}, \Omega_C)$.

By Lemma 2.3.4 there is a lift $n \in Z^1(\mathrm{F}, N)$ of $x^{-1}y_\lambda\vartheta(x)$ such that $n \cdot C = C$. Choose a lift $\dot{x} \in N$ of x. Then the element
$$u_\lambda := \dot{x}n\,\mathrm{F}(\dot{x})^{-1} \in Z^1(\mathrm{F}, N)$$
is a lift of y_λ as claimed. □

Lemma 2.7.3. *The class of u_λ in $H^1(\mathrm{F}, G)$ is equal to that of $\dot{\omega}_\lambda \in Z^1(\mathrm{F}, G)$. (See 2.4.3.)*

Proof. By the construction of u_λ in Lemma 2.7.2, we have $[u_\lambda] = [n]$, where $n \in Z^1(\mathrm{F}, N)$ is a lift of $x^{-1}y_\lambda\vartheta(x) \in \Omega_C$, and x is a certain element of W°. By Corollary 2.4.3, it suffices to show that
$$x^{-1}y_\lambda\vartheta(x) \in t_\lambda W^\circ.$$
First note that $t_\lambda W^\circ$ is preserved under conjugation by W°. The equation $t_\lambda w = w_\lambda y_\lambda$ then implies $y_\lambda \in t_\lambda W^\circ$. Since $x \in W^\circ$ as well, it follows that $x^{-1}y_\lambda\vartheta(x) \in t_\lambda W^\circ$. □

Fix once and for all a lift $\dot{w}$ of w in $\dot{W}_o$. Since $t_\lambda w y_\lambda^{-1} = w_\lambda \in W^\circ$, there exists a unique lift $\dot{w}_\lambda \in N$ of w_λ satisfying
$$t_\lambda\dot{w} = \dot{w}_\lambda u_\lambda.$$
Set
$$G_\lambda := G_{J_\lambda}, \qquad \mathrm{F}_\lambda := \mathrm{Ad}(u_\lambda) \circ \mathrm{F}\,.$$

Since $y_\lambda\vartheta \cdot C_\lambda = C_\lambda$, we have

$$\mathrm{F}_\lambda \cdot C_\lambda = C_\lambda.$$

We have

$$t_\lambda \in Z^1(\mathrm{F}_w, G), \qquad \dot{w} \in Z^1(\mathrm{F}, G),$$

the first by the definition of X_w and the second by Lemma 2.1.2 applied to the group $\dot{W}_o$. From Lemma 2.1.1 we conclude that

$$t_\lambda \dot{w} \in Z^1(\mathrm{F}, G).$$

But also $u_\lambda \in Z^1(\mathrm{F}, G)$, so using Lemma 2.1.1 again, we conclude that

$$\dot{w}_\lambda \in Z^1(\mathrm{F}_\lambda, G_\lambda).$$

By Lemma 2.3.1 there is an element $p_\lambda \in G_\lambda$ such that

$$p_\lambda^{-1}\,\mathrm{F}_\lambda(p_\lambda) = \dot{w}_\lambda.$$

This equation can be written as

$$t_\lambda \dot{w} = p_\lambda^{-1} u_\lambda\,\mathrm{F}(p_\lambda). \tag{8}$$

It follows that $t_\lambda \dot{w} \in Z^1(\mathrm{F}, G)$, and, in view of Lemma 2.7.3, we have

$$[t_\lambda \dot{w}] = [u_\lambda] = [\dot{\omega}_\lambda] \in H^1(\mathrm{F}, G), \tag{9}$$

as claimed in the proof of Lemma 2.6.1.

2.8. **A normal form for Frobenius endomorphisms.** Keep the set-up of Section 2.7. For each $\lambda \in X_w$ we have defined a Frobenius automorphism F_λ and an F_λ-stable alcove C_λ in $\mathcal{A}$. For certain $w \in W_o$, we will eventually associate to λ, and some additional data, a representation $\pi_\lambda \in \mathrm{Irr}(G^{\mathrm{F}_\lambda})$. This association will be quite natural, but it will leave us with infinitely many pairs $(G^{\mathrm{F}_\lambda}, \pi_\lambda)$, which are almost all conjugate to one another in some sense, and we will need to compare them. To do this, we seek a normal form for our Frobenius endomorphisms F_λ.

Fix a class $\omega \in H^1(\mathrm{F}, G)$, along with a representative $u \in \omega \cap N$, such that $u \cdot C = C$. This is possible by Lemma 2.3.4. In this section we will gather together all of the F_λ for which $u_\lambda \in \omega$. We will then use our explicit cohomology picture to keep track of conjugacy classes of tori and certain semisimple elements in a fixed group G^{F_u}. This, in turn, will be used in our stability calculations.

From Lemma 2.6.1, we have a map

$$r : X_w \to H^1(\mathrm{F}, G)$$

sending $\lambda \mapsto [\dot{\omega}_\lambda]$. For $\lambda \in r^{-1}(\omega)$, define $\sigma_\lambda = t_\lambda w \vartheta$, and choose $J_\lambda, C_\lambda, u_\lambda$ as in Section 2.7. Recall that the Frobenius $\mathrm{F}_\lambda = \mathrm{Ad}(u_\lambda) \circ \mathrm{F}$ stabilizes the alcove C_λ.

Lemma 2.8.1. *For each $\lambda \in r^{-1}(\omega)$, there exists $m_\lambda \in N$ such that*

$$m_\lambda * u_\lambda = u, \qquad m_\lambda \cdot C_\lambda = C.$$

Proof. Choose $k_\lambda \in N$ such that $k_\lambda \cdot C_\lambda = C$. Since $\mathrm{F}_\lambda \cdot C_\lambda = C_\lambda$, it follows that $k_\lambda * u_\lambda \in N_C$. In Lemma 2.7.3 we proved that $[u] = [u_\lambda]$ in $H^1(\mathrm{F}, G)$. Therefore $[u] = [k_\lambda * u_\lambda]$ in $H^1(\mathrm{F}, G)$. Since u and $k_\lambda * u_\lambda$ belong to N_C, and $H^1(\mathrm{F}, N_C) \to H^1(\mathrm{F}, G)$ is injective (see Lemma 2.3.4), we have $[u] = [k_\lambda * u_\lambda]$ in $H^1(\mathrm{F}, N_C)$. Hence there is $\ell_\lambda \in N_C$ such that $u = (\ell_\lambda k_\lambda) * u_\lambda$. Then $m_\lambda := \ell_\lambda k_\lambda$ has the required properties. □

As in Section 2.7, we have the alternative expression $\sigma_\lambda = w_\lambda y_\lambda \vartheta$, where $w_\lambda \in W_\lambda$ and y_λ is the image of u_λ in W. Recall that we have fixed a lift $\dot{w} \in \dot{W}_o$ of w, which determines a lift $\dot{w}_\lambda \in N \cap G_\lambda$ by the equation $t_\lambda \dot{w} = \dot{w}_\lambda u_\lambda$, and we have an element $p_\lambda \in G_\lambda$ such that $p_\lambda^{-1} \mathrm{F}_\lambda(p_\lambda) = \dot{w}_\lambda$. Choose m_λ as in Lemma 2.8.1 and set

$$q_\lambda := m_\lambda p_\lambda \in G, \quad S_\lambda := \mathrm{Ad}(q_\lambda)T.$$

Then in G we have, using equation (8),

$$q_\lambda^{-1} \mathrm{F}_u(q_\lambda) = p_\lambda^{-1} \cdot m_\lambda^{-1} u \, \mathrm{F}(m_\lambda \cdot p_\lambda) u^{-1} = p_\lambda^{-1} u_\lambda \, \mathrm{F}(p_\lambda) u^{-1} = t_\lambda \dot{w} u^{-1}.$$

Thus, we have the analogue of equation (8) for q_λ:

$$t_\lambda \dot{w} = q_\lambda^{-1} u \, \mathrm{F}(q_\lambda). \tag{10}$$

Equation (10) will be used repeatedly in future calculations. It implies that the map $\mathrm{Ad}(q_\lambda) : T \longrightarrow S_\lambda$ satisfies

$$\mathrm{F}_u \circ \mathrm{Ad}(q_\lambda) = \mathrm{Ad}(q_\lambda) \circ \mathrm{F}_w \,.$$

In particular, S_λ is an F_u-stable unramified maximal torus in G, whose underlying algebraic group $\mathbf{S}_\lambda$ is k-isomorphic to the twist of $\mathbf{T}$ by w.

In this section we have constructed an infinite family $\{S_\lambda : \lambda \in r^{-1}(\omega)\}$ of such tori, and our next task is to group these tori, and their strongly regular elements, into G^{F_u}-conjugacy classes.

2.9. **Conjugacy.** We will use several times another consequence of Steinberg's vanishing result, Theorem 2.2.1.

Lemma 2.9.1. *Let $\mathbf{G}_{ad}$ be the adjoint group of $\mathbf{G}$, and let $G_{ad} = \mathbf{G}_{ad}(K)$. Suppose $\mathbf{G}_{ad}$ acts on a k-variety $\mathbf{X}$, with connected stabilizers. For $x, y \in \mathbf{X}(K)$, the following are equivalent:*

1. *x and y are in the same G-orbit*
2. *x and y are in the same G_{ad}-orbit*
3. *x and y are in the same $\mathbf{G}$-orbit.*

Here $\mathbf{G}$ acts on $\mathbf{X}$ via the canonical map $j : \mathbf{G} \longrightarrow \mathbf{G}_{ad}$.

Proof. Implication $1 \Rightarrow 2$ is clear. Since $\mathbf{G} \longrightarrow \mathbf{G}_{ad}$ is surjective, $2 \Rightarrow 3$ is also clear. Assume 3 holds, so there is $g \in \mathbf{G}$ such that $g \cdot x = y$. Since $x, y \in \mathbf{X}(K) = \mathbf{X}^{\mathcal{I}}$, the map sending $\sigma \in \mathcal{I}$ to $g^{-1}\sigma(g) \in \mathbf{G}$ is a cocycle in $Z^1(K, \mathbf{G}_x)$. By hypothesis, $\mathbf{G}_x$ is the full stabilizer of x in $\mathbf{G}$. By Theorem 2.2.1 we have $H^1(K, \mathbf{G}_x) = 1$, so there is $h \in \mathbf{G}_x$ such that $(gh)^{-1}\sigma(gh) = 1$ for all $\sigma \in \mathcal{I}$. Hence $gh \in G$, and 1 follows. □

We say that $\gamma \in G^F$ is *strongly regular semisimple in* G if the centralizer of γ in $\mathbf{G}$ is a torus. By 2.9.1 we have for such γ the equalities

$$[\mathrm{Ad}(\mathbf{G})\gamma]^{\mathrm{Gal}(\bar{k}/k)} = [\mathrm{Ad}(G_{ad})\gamma]^F = [\mathrm{Ad}(G)\gamma]^F.$$

Such a set is called the *G-stable conjugacy-class of* γ. It is a finite union of $\mathrm{Ad}(G^F)$-orbits, which are called the *rational classes* in the stable class.

2.10. **Rational classes in a stable class.** We continue with the setup of Section 2.8. Our aim is to explicitly parametrize the rational classes in the stable classes of certain elements $\gamma \in G$.

Recall that the map $r : X_w \longrightarrow H^1(\mathrm{F}, G)$ is defined as a composition

$$r : X_w \longrightarrow [X/(1-w\vartheta)X]_{\mathrm{tor}} \longrightarrow H^1(\mathrm{F}, G).$$

We have fixed $\omega \in H^1(\mathrm{F}, G)$, and have considered the fiber $r^{-1}(\omega) \subset X_w$. Now let $[r^{-1}(\omega)]$ denote the image of $r^{-1}(\omega)$ in $[X/(1-w\vartheta)X]_{\mathrm{tor}}$. In other words, $[r^{-1}(\omega)]$ is the fiber over ω in the second map in the above composition. By Lemma 2.6.1, we may identify $[r^{-1}(\omega)]$ with the fiber over ω of the natural map $H^1(\mathrm{F}_w, T) \longrightarrow H^1(\mathrm{F}, G)$.

Let $\gamma \in T^{\mathrm{F}_w}$ be a strongly regular element of G. For $\lambda \in r^{-1}(\omega)$, we set

$$\gamma_\lambda := q_\lambda \gamma q_\lambda^{-1} \in S_\lambda^{\mathrm{F}_u}.$$

Lemma 2.10.1. *For $\lambda, \mu \in r^{-1}(\omega)$, the elements γ_λ and γ_μ are G^{F_u}-conjugate if and only if $\lambda \equiv \mu \mod (1-w\vartheta)X$. Thus, sending $\lambda \mapsto \gamma_\lambda$ defines a bijection*

$$[r^{-1}(\omega)] \xrightarrow{\sim} [\mathrm{Ad}(G)\gamma]^{\mathrm{F}_u}/G^{\mathrm{F}_u}.$$

Proof. Since $S_\lambda = G_{\gamma_\lambda}$, this is almost obvious from Lemma 2.6.1. However, we will give a direct proof which produces the conjugation from elements already in play.

By Equation (10) we have

$$q_\lambda^{-1} u\, \mathrm{F}(q_\lambda) = t_\lambda \dot{w}, \qquad q_\mu^{-1} u\, \mathrm{F}(q_\mu) = t_\mu \dot{w}.$$

Let $h = q_\mu q_\lambda^{-1}$, so that $\mathrm{Ad}(h)\gamma_\lambda = \gamma_\mu$. Then $h^{-1}\,\mathrm{F}_u(h) \in S_\lambda$ since γ is strongly regular. Moreover, $\gamma_\mu \in \mathrm{Ad}(G^{\mathrm{F}_u})\gamma_\lambda$ if and only if the class $[h^{-1}\,\mathrm{F}_u(h)]$ in $H^1(\mathrm{F}_u, S_\lambda)$ is the identity element. We have

$$\begin{aligned} h^{-1}\,\mathrm{F}_u(h) &= q_\lambda \cdot q_\mu^{-1} u\, \mathrm{F}(q_\mu) \cdot \mathrm{F}(q_\lambda)^{-1} u^{-1} \\ &= q_\lambda t_\mu \dot{w}\, \mathrm{F}(q_\lambda)^{-1} u^{-1} \\ &= q_\lambda t_\mu \dot{w} \dot{w}^{-1} t_{-\lambda} q_\lambda^{-1} \\ &= q_\lambda t_{\mu-\lambda} q_\lambda^{-1}, \end{aligned}$$

so

$$[h^{-1}\,\mathrm{F}_u(h)] = [q_\lambda t_{\mu-\lambda} q_\lambda^{-1}] \in H^1(\mathrm{F}_u, S_\lambda).$$

On the other hand, we have isomorphisms

$$\big[X/(1-w\vartheta)X\big]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}_w, T) \xrightarrow{\mathrm{Ad}(q_\lambda)} H^1(\mathrm{F}_u, S_\lambda).$$

The first isomorphism is Corollary 2.4.3 applied to T, F_w; for $\nu \in X_w$, it sends the class of $\nu \mod (1-w\vartheta)X$ to the class of t_ν in $H^1(\mathrm{F}_w, T)$. Thus, $[q_\lambda t_{\mu-\lambda} q_\lambda^{-1}]$ is trivial in $H^1(\mathrm{F}_u, S_\lambda)$ if and only if $\lambda - \mu \in (1-w\vartheta)X$. □

2.11. **A partition of the rational classes in a stable class.** We have seen in Lemma 2.10.1 that the fiber $[r^{-1}(\omega)]$ parametrizes the G^{F_u}-conjugacy classes in the stable class of γ_λ, for $\lambda \in r^{-1}(\omega)$. In this section we study an additional structure on this fiber. Namely, the group

$$W_o^{w\vartheta} := \{z_o \in W_o : \ w\vartheta(z_o)w^{-1} = z_o\}$$

acts naturally on X_w, $[X/(1-w\vartheta)X]_{\mathrm{tor}}$, and $[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}}$, and $W_o^{w\vartheta}$ acts trivially on the latter. Hence there is a natural $W_o^{w\vartheta}$-action on the fiber $[r^{-1}(\omega)]$. This action in fact corresponds to G^{F_u}-conjugacy among the family of tori $\{S_\lambda : \ \lambda \in r^{-1}(\omega)\}$, as follows.

Lemma 2.11.1. *For $\lambda, \mu \in r^{-1}(\omega)$ the following are equivalent.*

(1) *There is $z_o \in W_o^{w\vartheta}$ such that $z_o\mu \equiv \lambda \mod (1-w\vartheta)X$.*
(2) *There is $g \in G^{\mathrm{F}_u}$ such that ${}^g\gamma_\mu \in S_\lambda$.*
(3) *There is $g \in G^{\mathrm{F}_u}$ such that ${}^gS_\mu = S_\lambda$.*

Proof. Assertions 2 and 3 are equivalent because $S_\lambda = G_{\gamma_\lambda}$ for all $\lambda \in r^{-1}(\omega)$. (We have made them separate statements for later convenience.)

Assume 3 holds. Then $q_\lambda^{-1} g q_\mu \in N$. Applying Equation (10) for μ and λ, we find that

$$\begin{aligned} q_\lambda^{-1} g q_\mu \cdot t_\mu \dot{w} \cdot \mathrm{F}(q_\mu^{-1} g^{-1} q_\lambda) &= q_\lambda^{-1} g u \,\mathrm{F}(g)^{-1}\,\mathrm{F}(q_\lambda) \\ &= q_\lambda^{-1} g \cdot u\,\mathrm{F}(g)^{-1} u^{-1} \cdot q_\lambda t_\lambda \dot{w} \\ &= q_\lambda^{-1} g \cdot \mathrm{F}_u(g)^{-1} \cdot q_\lambda t_\lambda \dot{w} \\ &= t_\lambda \dot{w}. \end{aligned} \tag{11}$$

Let $z \in W$ be the image of $q_\lambda^{-1} g q_\mu$, and write $z = t_\nu z_o$ with $\nu \in X$, $z_o \in W_o$. Mapping the first and last terms of Equation (11) to W, we have

$$t_\nu z_o \cdot t_\mu w \cdot \vartheta(z_o^{-1} t_{-\nu}) = t_\lambda w.$$

This shows that $z_o \in W_o^{w\vartheta}$, and then projection onto W_o yields

$$\lambda = z_o\mu + (1-w\vartheta)\nu,$$

so 1 holds.

Conversely, if 1 holds, then $\lambda = z_o\mu + (1-w\vartheta)\nu$ for some $\nu \in X$, and we set $z = t_\nu z_o$. Since $H^1(\mathrm{F}_w, {}^0T) = 1$, there is a lift $\dot{z}_o \in N^{\mathrm{F}_w}$ of z_o, and we set $\dot{z} = t_\nu \dot{z}_o$. Then

$$\mathrm{F}(\dot{z}) = t_{\vartheta\nu} \dot{w}^{-1} \dot{z}_o \dot{w}.$$

Set $g = q_\lambda \dot z q_\mu^{-1}$. It is clear that ${}^g S_\mu = S_\lambda$. To prove 3, it remains to show that $g \in G^{\mathrm{F}_u}$. Using equation (10) again, we compute

$$\begin{aligned}\mathrm{F}_u(g) &= u\,\mathrm{F}(q_\lambda)\ \cdot\ \mathrm{F}(\dot z)\ \cdot\ \mathrm{F}(q_\mu)^{-1}u^{-1}\\ &= q_\lambda t_\lambda \dot w\ \cdot\ t_{\vartheta\nu}\dot w^{-1}\dot z_o \dot w\ \cdot \dot w^{-1} t_{-\mu} q_\mu^{-1}\\ &= q_\lambda\ \cdot\ t_{\lambda + w\vartheta\nu - z_o\mu}\ \cdot\ \dot z_o q_\mu^{-1}\\ &= q_\lambda t_\nu \dot z_o q_\mu^{-1}\\ &= g,\end{aligned}$$

as desired. □

Let

$$W_{o,\lambda}^{w\vartheta} := \{z \in W_o^{w\vartheta} :\ z\lambda \equiv \lambda \mod (1 - w\vartheta)X\}$$

be the stabilizer in $W_o^{w\vartheta}$ of the class of λ in $[r^{-1}(\omega)]$. The next result interprets $W_o^{w\vartheta}$ and $W_{o,\lambda}^{w\vartheta}$ as "large" and "small" Weyl groups of S_λ, respectively. This will be used to relate L-packets to stable conjugacy classes of tori.

Lemma 2.11.2. *For $\lambda \in r^{-1}(\omega)$, the map $\mathrm{Ad}(q_\lambda)$ induces isomorphisms*

$$W_o^{w\vartheta} \xrightarrow{\sim} N(G, S_\lambda^{\mathrm{F}_u})/S_\lambda, \qquad W_{o,\lambda}^{w\vartheta} \xrightarrow{\sim} N(G, S_\lambda)^{\mathrm{F}_u}/S_\lambda^{\mathrm{F}_u}.$$

Proof. First, a remark about normalizers of tori. Let F be a Frobenius on G arising from some k-structure, and let S be the group of K-points of a maximal k-torus $\mathbf{S} \subset \mathbf{G}$. We claim that

$$[N(G,S)/S]^F = N(G, S^F)/S. \tag{12}$$

For $\subseteq$: Let $n \in N(G,S)$ be such that $F(n) = ns$ for some $s \in S$. Then on S we have $\mathrm{Ad}(n) \circ F = F \circ \mathrm{Ad}(n)$, implying that $n \in N(G, S^F)$. For $\supseteq$: Choose $s_0 \in S^F \cap G^{\mathrm{srss}}$. For $n \in N(G, S^F)$, and $s \in S$, the element nsn^{-1} centralizes s_0, hence lies in S. This shows that $N(G,S^F) \subseteq N(G,S)$. Moreover, we have $\mathrm{Ad}(n)s_0 \in S^F$, implying that $\mathrm{Ad}(n^{-1}F(n))s_0 = s_0$, hence $F(n) \in nS$, as desired.

This remark shows that $W_o^{w\vartheta} = N(G, T^{\mathrm{F}_w})/T$, and the first isomorphism follows. The second isomorphism amounts to showing that the projections $N \to W \to W_o$ induces an isomorphism

$$N^{\mathrm{F}_{t_\lambda w}}/T^{\mathrm{F}_w} \longrightarrow W_{o,\lambda}^{w\vartheta}. \tag{13}$$

Let $n \in N^{\mathrm{F}_{t_\lambda w}}$, and let $t_\nu z$ be the image of n in W, where $\nu \in X$ and $z \in W_o$. We want to show that $z \in W_{o,\lambda}^{w\vartheta}$. From the equation $\mathrm{Ad}(t_\lambda)\,\mathrm{F}_w(n) = n$, we get

$$\mathrm{Ad}(t_\lambda w)\vartheta(t_\nu z) = t_\nu z,$$

which leads to

$$t_{\lambda + (w\vartheta - 1)\nu}\,\mathrm{Ad}(w)\vartheta(z) = t_{z\lambda} z,$$

hence $z \in W_o^{w\vartheta}$ and $z\lambda = \lambda + (w\vartheta - 1)\nu$, as desired. This shows also that (13) is injective.

To see that (13) is surjective, let $z \in W_{o,\lambda}^{w\vartheta}$, and choose a lift $\dot z \in N_o$. Since $\mathrm{Ad}(t_\lambda w)\vartheta(z) = z$ in $W_o = N_o/{}^0T$, we have

$$\mathrm{Ad}(t_\lambda)\,\mathrm{F}_w(\dot z) = \dot z t,$$

for some $t \in {}^0T$. Since $H^1(\mathrm{F}_w, {}^0T) = 1$ we can write $t = s\,\mathrm{F}_w(s^{-1})$ for some $s \in {}^0T$. Then $\dot{z}s$ is a lift of z in $N^{\mathrm{F}_{t_\lambda w}}$. $\square$

3. The conjectural local Langlands correspondence

Very roughly speaking, the conjectural local Langlands correspondence predicts a relationship between representations of a p-adic group and certain maps from the Weil group into the dual group. The latter maps are called "Langlands parameters"; they should partition the representations of the p-adic group into finite sets, called "L-packets", and it is conjectured that these L-packets have many nice properties. We now make these statements more precise.

3.1. **Frobenius endomorphisms and representations of p-adic groups.** Continue with the set-up of section 2.3: $\mathbf{G}$ is a connected reductive k-group which is k-quasi-split and K-split, with Frobenius automorphism F on the group $G = \mathbf{G}(K)$.

For each cocycle $u \in Z^1(\mathrm{F}, G)$, we have a twisted Frobenius

$$\mathrm{F}_u := \mathrm{Ad}(u) \circ \mathrm{F}$$

on G, and for $g \in G$, we have

$$\mathrm{Ad}(g) \circ \mathrm{F}_u \circ \mathrm{Ad}(g)^{-1} = \mathrm{F}_{g*u}\,.$$

Therefore $\mathrm{Ad}(g)$ is an isomorphism

$$\mathrm{Ad}(g) : G^{\mathrm{F}_u} \longrightarrow G^{\mathrm{F}_{g*u}},$$

which induces a bijection on irreducible representations, denoted by

$$\mathrm{Ad}(g)_* : \mathrm{Irr}(G^{\mathrm{F}_u}) \longrightarrow \mathrm{Irr}(G^{\mathrm{F}_{g*u}}).$$

This bijection preserves the sets $\mathrm{Irr}^2(\cdot)$ of square-integrable representations.

Thus we have a G-action on the set

$$\mathcal{R}^2(\mathrm{F}, G) := \{(u, \pi) :\ u \in Z^1(\mathrm{F}, G),\ \pi \in \mathrm{Irr}^2(G^{\mathrm{F}_u})\}.$$

Considering the u-coordinate, we can partition $\mathcal{R}^2(\mathrm{F}, G)$ into G-stable subsets

$$\mathcal{R}^2(\mathrm{F}, G) = \coprod_{\omega \in H^1(\mathrm{F}, G)} \mathcal{R}^2(\mathrm{F}, G, \omega),$$

where $\mathcal{R}^2(\mathrm{F}, G, \omega)$ consists of the pairs $(u, \pi) \in \mathcal{R}^2(\mathrm{F}, G)$ with $u \in \omega$.

3.2. **Dual group.** Let $\hat{G}$ be the dual group of $\mathbf{G}$. By definition, the dual torus

$$\hat{T} := Y \otimes \mathbb{C}^\times$$

is a maximal torus in $\hat{G}$. The operator $\hat{\vartheta} \in \mathrm{Aut}(Y)$ dual to ϑ extends to an automorphism of the torus $\hat{T}$, with trivial action on $\mathbb{C}^\times$.

We choose, once and for all, a pinning $(\hat{T}, \hat{B}, \{x_\alpha\})$ where $\hat{B}$ is a Borel subgroup of $\hat{G}$ containing $\hat{T}$ and the x_α are non-trivial elements in the simple root groups of $\hat{T}$ in $\hat{B}$. There is a unique extension of $\hat{\vartheta}$ to an automorphism of $\hat{G}$, satisfying $\hat{\vartheta}(x_\alpha) = x_{\vartheta\cdot\alpha}$ (see [7]). We can then form the semidirect product

$$^LG := \langle\hat{\vartheta}\rangle \ltimes \hat{G}.$$

3.3. **Weil group.** Recall that the inertia subgroup $\mathcal{I} \leq \mathrm{Gal}(\bar{k}/k)$ is the kernel of the natural map

$$\mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{Gal}(K/k).$$

The *Weil group* $\mathcal{W}$ is the subgroup of $\mathrm{Gal}(\bar{k}/k)$ generated by $\mathcal{I}$ and the Frobenius Frob. The *wild inertia* subgroup $\mathcal{I}^+ \lhd \mathcal{I}$ is the maximal pro-p subgroup of $\mathcal{I}$. The *tame inertia group* is the quotient $\mathcal{I}_t := \mathcal{I}/\mathcal{I}^+$, and the *tame Weil group* is the quotient $\mathcal{W}_t := \mathcal{W}/\mathcal{I}^+$. We will have more to say about these groups in Section 4.3.

3.4. **Elliptic Langlands parameters.** An *elliptic Langlands parameter* is a homomorphism

$$\varphi : \mathcal{W} \times SL_2(\mathbb{C}) \longrightarrow {}^LG$$

with the following properties:

- $\varphi(\mathcal{I})$ is a finite subgroup of $\hat{G}$,
- $\varphi(\mathrm{Frob}) = \hat{\vartheta}f$, where $f \in \hat{G}$ is semisimple,
- the restriction of φ to $SL_2(\mathbb{C})$ is algebraic,
- The identity component $C_{\hat{G}}(\varphi)^\circ$ of $C_{\hat{G}}(\varphi)$ is equal to the identity component $(\hat{Z}^{\hat{\vartheta}})^\circ$ of $\hat{Z}^{\hat{\vartheta}}$.

The last condition expresses the "ellipticity" of φ; it is equivalent to requiring that the image of φ is not contained in a proper Levi subgroup of ^LG, where the meaning of "Levi subgroup" is as in [7, 3.4].

We let C_φ denote the component group of $C_{\hat{G}}(\varphi)$. Since $\hat{Z}^{\hat{\vartheta}}$ is contained in the center of $C_{\hat{G}}(\varphi)$, each $\rho \in \mathrm{Irr}(C_\varphi)$ determines a central character on $\hat{Z}^{\hat{\vartheta}}$ hence, via Kottwitz' isomorphism (Corollary 2.4.3), a class $\omega_\rho \in H^1(\mathrm{F}, G)$. Thus we may partition

$$\mathrm{Irr}(C_\varphi) = \coprod_{\omega\in H^1(\mathrm{F},G)} \mathrm{Irr}(C_\varphi, \omega),$$

where $\mathrm{Irr}(C_\varphi, \omega)$ consists of the representations $\rho \in \mathrm{Irr}(C_\varphi)$ with $\omega_\rho = \omega$.

3.5. **The conjectures.** The version of the Langlands conjectures stated here is the product of many refinements, by Deligne, Lusztig, Vogan and others. The local Langlands correspondence for $\mathbf{G}$ is a conjectural bijection between the set of $\hat{G}$-orbits of pairs (φ, ρ), where φ is an elliptic Langlands parameter and $\rho \in \mathrm{Irr}(C_\varphi)$, and the set of G-orbits in $\mathcal{R}^2(\mathrm{F}, G)$. Among many other expected properties, the G-orbit corresponding to (φ, ρ) should lie in $\mathcal{R}^2(\mathrm{F}, G, \omega)$ precisely when $\omega_\rho = \omega$.

Thus, we expect to have, for each $\hat{G}$-conjugacy class of elliptic Langlands parameters φ, a finite set

$$\Pi(\varphi) = \coprod_{\omega \in H^1(\mathrm{F}, G)} \Pi(\varphi, \omega),$$

where

$$\Pi(\varphi, \omega) := \{[\pi(\varphi, \rho)] : \ \rho \in \mathrm{Irr}(C_\varphi, \omega)\}, \tag{14}$$

and $[\pi(\varphi, \rho)] = \{(u, \pi_u(\varphi, \rho)) : \ u \in \omega\}$ is a G-orbit in $\mathcal{R}^2(\mathrm{F}, G, \omega)$.

These putative sets $\Pi(\varphi)$ are known as "L-packets". These L-packets should form partitions

$$\mathcal{R}^2(\mathrm{F}, G)/G = \coprod_{\{\varphi\}/\hat{G}} \Pi(\varphi), \qquad \mathcal{R}^2(\mathrm{F}, G, \omega)/G = \coprod_{\{\varphi\}/\hat{G}} \Pi(\varphi, \omega).$$

To describe the properties we expect of an L-packet, we fix a representative $u \in Z^1(\mathrm{F}, N)$ of each class $\omega \in H^1(\mathrm{F}, G)$. We represent the trivial class by $u = 1$, recalling that $\mathrm{F}_1 = \mathrm{F}$. Then $\{\pi_u(\varphi, \rho) : \ \rho \in \mathrm{Irr}(C_\varphi, \omega)\}$ is a set of representatives for the G-orbits comprising $\Pi(\varphi, \omega)$.

We expect L-packets to have the following properties.

(i) The representation $\pi_u(\varphi, \rho)$ is unipotent [39] if and only if φ is unramified, (that is, if φ is trivial on the inertia subgroup $\mathcal{I}$ of $\mathcal{W}$). For $\mathbf{G}$ with connected center, Lusztig has constructed unipotent L-packets corresponding to unramified φ [39], [40]. See also [41] and [48] and for orthogonal and split adjoint exceptional groups, respectively.

(ii) $\pi_u(\varphi, \rho)$ has depth-zero (that is, has nonzero vectors fixed under the pro-unipotent radical of some parahoric subgroup in G^{F_u}) if and only if φ is tame (that is, φ is trivial on the wild inertia subgroup $\mathcal{I}^+$ of $\mathcal{I}$).

(iii) $\pi_1(\varphi, 1)$ should be generic (that is, has a Whittaker model). If $\mathbf{G}$ has connected center, then $\pi_1(\varphi, 1)$ should be the unique generic representation in $\Pi(\varphi)$.

(iv) Let ${}^LM$ be a minimal Levi subgroup of ${}^LG$ containing $\varphi(\mathcal{W})$. (It is unique up to conjugacy by the connected centralizer of $\varphi(\mathcal{W})$ [7].) If ${}^LM = {}^LG$, then every class in $\Pi(\varphi)$ should consist of supercuspidal representations. In this case, we say that $\Pi(\varphi)$ itself is "supercuspidal". The L-packets in this paper are all supercuspidal.

If ${}^LM \neq {}^LG$, then ${}^LM$ corresponds to an F-stable Levi subgroup $M \subset G$ contained in an F-stable proper parabolic subgroup $P \subset G$. The restriction $\varphi : \mathcal{W} \longrightarrow {}^LM$ inductively corresponds to a generic supercuspidal representation $\pi_1^M(\varphi, 1)$ of M^{F}, and $\pi_1(\varphi, 1)$ should be a generic constituent of the smoothly-induced representation $\mathrm{Ind}_{P^{\mathrm{F}}}^{G^{\mathrm{F}}} \pi_1^M(\varphi, 1)$. For $(u, \rho) \neq (1, 1)$, the representation $\pi_u(\varphi, \rho)$ should be supported on Levi subgroups of G^{F_u} whose center has k-rank no larger than that of M^{F}.

(v) For each u, normalize Haar measure on G^{F_u} so that the formal degree of the Steinberg representation of G^{F_u} is independent of u. (For example, one could make all Steinberg formal degrees equal to one, but we will choose a different normalization.) Let Deg denote formal degree with respect to these measures. Then we should have

$$\mathrm{Deg}[\pi_u(\varphi, \rho)] = \dim \rho \cdot \mathrm{Deg}[\pi_1(\varphi, 1)].$$

Recall that $\pi_u(\varphi, \rho)$ and $\pi_1(\varphi, 1)$ may be representations of non-isomorphic groups.

Properties (i-v) were verified in [48] for unipotent L-packets of split adjoint exceptional groups (see (i) above).

(vi) Fix u and φ, and let Θ_ρ be the character of $\pi_u(\varphi, \rho)$, viewed as a function on the set $(G^{\mathrm{rss}})^{\mathrm{F}_u}$ of regular semisimple elements of G^{F_u}. The function

$$\sum_{\rho \in \mathrm{Irr}(C_\varphi, \omega)} \dim \rho \cdot \Theta_\rho$$

should be stable. That is, if $\gamma, \gamma' \in (G^{\mathrm{srss}})^{\mathrm{F}_u}$ are G-conjugate[1] strongly regular elements (see Section 1), then we should have

$$\sum_{\rho \in \mathrm{Irr}(C_\varphi, \omega)} \dim \rho \cdot \Theta_\rho(\gamma) = \sum_{\rho \in \mathrm{Irr}(C_\varphi, \omega)} \dim \rho \cdot \Theta_\rho(\gamma').$$

This was verified in [41] for unipotent L-packets for inner forms of $SO(2n+1)$ (see (i) above).

4. FROM TAME REGULAR SEMISIMPLE PARAMETERS TO DEPTH-ZERO SUPERCUSPIDAL L-PACKETS

We shall construct L-packets satisfying (ii)-(vi) above, for tame parameters φ in "general position". We will first make this condition precise, and outline the construction.

Our construction relies on the tame Langlands correspondence for tori. A general Langlands correspondence for tori was proved by Langlands [37] but it seems more difficult to extract the depth-zero correspondence from [37] than to re-prove it from scratch, so we give a short self-contained account of the tame Langlands correspondence for tori. Then we construct our L-packets, using the material from Section 2.7.

4.1. **Tame regular semisimple parameters.** We say that a Langlands parameter φ is *tame regular semisimple* if it is trivial on the wild inertia subgroup $\mathcal{I}^+$ and the centralizer of $\varphi(\mathcal{I})$ in $\hat{G}$ is a torus. The latter condition is what we mean by "general position". This forces φ to be trivial on $SL_2(\mathbb{C})$. (There is a more general notion of "tame regular" parameter which we will consider elsewhere.)

Recall that $\mathcal{W}_t = \mathcal{W}/\mathcal{I}^+$ and $\mathcal{I}_t = \mathcal{I}/\mathcal{I}^+$. Our choice of inverse Frobenius determines a splitting

$$\mathcal{W}_t = \langle \mathrm{Frob} \rangle \ltimes \mathcal{I}_t,$$

where $\mathrm{Frob}^{-1} x \, \mathrm{Frob} = x^q$ for $x \in \mathcal{I}_t$.

Recall that the Weyl group N/T is identified with W_o, the image of N_o in W. We let $\hat{W}_o$ denote the Weyl group $\hat{N}/\hat{T}$ where $\hat{N}$ is the normalizer of $\hat{T}$ in $\hat{G}$. The restriction of the duality map

$$\mathrm{Aut}(X) \xrightarrow{\sigma \mapsto \hat{\sigma}} \mathrm{Aut}(Y)$$

defines an anti-isomorphism $w \mapsto \hat{w}$ from W_o to $\hat{W}_o$.

[1] It is customary to require the elements to be $\mathbf{G}$-conjugate, but we have seen in Lemma 2.9.1 that two strongly regular semisimple elements of G are $\mathbf{G}$-conjugate if and only if they are G-conjugate.

After conjugating by $\hat{G}$, we may assume that $\varphi(\mathcal{I}_t) \subset \hat{T}$ and $\varphi(\mathrm{Frob}) = \hat{\vartheta} f$, where $f \in \hat{N}$. Let $\hat{w}$ be the image of f in $\hat{W}_o$, corresponding to $w \in W_o$ via the above anti-isomorphism.

Then

$$C_{\hat{G}}(\varphi) = \hat{T}^{\widehat{w\vartheta}},$$

which implies that the restriction map $X \to \mathrm{Hom}(\hat{T}^{\widehat{w\vartheta}}, \mathbb{C}^\times)$ induces an isomorphism

$$\big[X/(1-w\vartheta)X\big]_{\mathrm{tor}} \xrightarrow{\sim} \mathrm{Irr}(C_\varphi), \qquad \lambda \mapsto \rho_\lambda. \tag{15}$$

Moreover, φ is elliptic if and only if

$$(\hat{T}^{\widehat{w\vartheta}})^\circ = (\hat{Z}^{\hat{\vartheta}})^\circ.$$

To summarize: A tame regular semisimple elliptic Langlands parameter (TRSELP) is given by two objects:

- a continuous homomorphism $s : \mathcal{I}_t \longrightarrow \hat{T}$, with $C_{\hat{G}}(s) = \hat{T}$, and
- an element $f \in \hat{N}$ satisfying the two conditions

$$\hat{\vartheta} \circ \mathrm{Ad}(f) \circ s^q = s, \qquad (\hat{T}^{\widehat{w\vartheta}})^\circ = (\hat{Z}^{\vartheta})^\circ,$$

  where $w \in W_o$ arises from f as above.

Remark 4.1.1. If $\hat{G}$ is semisimple, then the ellipticity condition on φ is that $\hat{T}^{\widehat{w\vartheta}}$ be finite. In this case, the map $\hat{T} \longrightarrow \hat{T}$ given by $t \mapsto t^{-1}\widehat{w\vartheta}(t)$ has finite fibers, hence is surjective. Hence, if we conjugate $\hat{\vartheta} f$ by elements of $\hat{T}$, we can change f to any other representative of $\hat{w}$. This means the $\hat{T}$-conjugacy class of $\hat{\vartheta} f$ is determined by the image $\hat{w}$ of f in $\hat{W}_o$, so the $\hat{G}$-conjugacy classes of TRSELPs are in bijection with $\hat{W}_o$-conjugacy classes of pairs $(s, \hat{w})$, where $s : \mathcal{I}_t \longrightarrow \hat{T}$ is continuous, with $C_{\hat{G}}(s) = \hat{T}$, and $\hat{w} \in \hat{W}_o$ satisfies

$$\widehat{w\vartheta} \circ s^q = s, \qquad \hat{T}^{\widehat{w\vartheta}} \text{ is finite.}$$

4.2. **Outline of the construction.** Suppose we have a TRSELP φ, with $s, f, \hat{w}$ as above. Recall from Section 2.7 that X_w denotes the preimage in X of $[X/(1-w\vartheta)X]_{\mathrm{tor}}$. For $\lambda \in X_w$, let ρ_λ be as in (15). In Section 2.7 we associated to λ a cocycle u_λ whose class in $H^1(\mathrm{F}, G)$ is ω_{ρ_λ}. The twisted Frobenius $\mathrm{F}_\lambda = \mathrm{F}_{u_\lambda}$ stabilizes a facet $J_\lambda \subset \mathcal{A}$ with corresponding parahoric subgroup G_λ. Ellipticity will imply that the facet J_λ is in fact a minimal F_λ-stable facet in $\mathcal{A}$, so $G_\lambda^{\mathrm{F}_\lambda}$ is a maximal parahoric subgroup of G^{F_λ}.

To (φ, λ) we will further associate an F_λ-minisotropic torus T_λ, a depth-zero character χ_λ of T_λ, whence an irreducible cuspidal representation κ_λ^0 of $\mathsf{G}_\lambda^{\mathrm{F}_\lambda} := (G_\lambda/G_\lambda^+)^{\mathrm{F}_\lambda}$ (viewed as a representation of $G_\lambda^{\mathrm{F}_\lambda}$), via the Deligne-Lusztig construction. In fact, χ_λ will define an extension κ_λ of κ_λ^0 to $Z^{\mathrm{F}} G_\lambda^{\mathrm{F}_\lambda}$ such that the smoothly-induced representation

$$\pi_\lambda := \mathrm{Ind}_{Z^{\mathrm{F}} G_\lambda^{\mathrm{F}_\lambda}}^{G^{\mathrm{F}_\lambda}} \kappa_\lambda$$

is irreducible. Here Z denotes the group of K-rational points of the maximal k-split torus in the center of $\mathbf{G}$. An exercise shows that the functions in π_λ necessarily have compact support modulo Z^{F}, so we could just as well define π_λ using the compact induction functor ind.

In our construction, u_λ and J_λ are not uniquely defined, but the G-orbit $[u_\lambda, \pi_\lambda] \in \mathcal{R}^2(\mathrm{F}, G)$ will be independent of the choices of u_λ and J_λ. Moreover, for $\lambda, \mu \in X_w$, we will have

$$[u_\lambda, \pi_\lambda] = [u_\mu, \pi_\mu] \quad \Leftrightarrow \quad \rho_\lambda = \rho_\mu.$$

Thus, to (φ, ρ) we will associate the G-orbit $[u_\lambda, \pi_\lambda] \in \mathcal{R}^2(\mathrm{F}, G, \omega_\rho)$, where $\lambda \in X_w$ is any character of $\hat{T}$ restricting to ρ.

The L-packets thus defined are the "natural" ones: All choices involved in the construction are rendered equivalent by taking G-orbits. However, to make the stability calculations, we need representations of a fixed group. Using Section 2.8, we will choose a representative (u, π) in each G-orbit $[u_\lambda, \pi_\lambda]$, so as to have all representations in the "unnatural" L-packet living on the single group G^{F_u}.

4.3. **Depth zero characters of unramified tori.** Recall that $X = X_*(\mathbf{T}),\ Y = X^*(\mathbf{T})$. Let $\sigma \in \mathrm{Aut}(X)$ be an automorphism of X of order n, and let $\mathrm{F}_\sigma = \sigma \otimes \mathrm{Frob}^{-1}$ be the corresponding twisted Frobenius of both $T = X \otimes K^\times$ and $\mathsf{T} = X \otimes \mathfrak{F}^\times$. (Recall that Frob^{-1} is the q-power map on $\mathfrak{F}$.) Let $\mathfrak{f}_n$ be the degree n extension of $\mathfrak{f}$ contained in $\mathfrak{F}$. Since σ has order n, the torus T with Frobenius F_σ splits over $\mathfrak{f}_n$, and $\mathsf{T}^{\mathrm{F}_\sigma^n} = X \otimes \mathfrak{f}_n^\times$.

Given automorphisms α, β of abelian groups A, B, respectively, let

$$\mathrm{Hom}_{\alpha,\beta}(A, B)$$

denote the set of homomorphisms $f : A \longrightarrow B$ such that $f \circ \alpha = \beta \circ f$.

We have an exact sequence

$$1 \longrightarrow \mathsf{T}^{\mathrm{F}_\sigma} \longrightarrow \mathsf{T}^{\mathrm{F}_\sigma^n} \xrightarrow{1-\mathrm{F}_\sigma} \mathsf{T}^{\mathrm{F}_\sigma^n} \xrightarrow{N_\sigma} \mathsf{T}^{\mathrm{F}_\sigma} \longrightarrow 1,$$

where $N_\sigma(t) = t\,\mathrm{F}_\sigma(t)\,\mathrm{F}_\sigma^2(t)\cdots\mathrm{F}_\sigma^{n-1}(t)$. So N_σ induces an isomorphism

$$\mathrm{Hom}(\mathsf{T}^{\mathrm{F}_\sigma}, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{F}_\sigma,\mathrm{Id}}(\mathsf{T}^{\mathrm{F}_\sigma^n}, \mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{F}_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_n^\times, \mathbb{C}^\times).$$

There is also an isomorphism

$$\mathrm{Hom}(\mathfrak{f}_n^\times, \hat{T}) \xrightarrow{\sim} \mathrm{Hom}(X \otimes \mathfrak{f}_n^\times, \mathbb{C}^\times), \quad s \mapsto \chi_s,$$

where $\chi_s(\lambda \otimes a) = \lambda(s(a))$, for $\lambda \in X,\ a \in \mathfrak{f}_n^\times$. One checks that

$$\chi_s \in \mathrm{Hom}_{\mathrm{F}_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_n^\times, \mathbb{C}^\times) \quad \Leftrightarrow \quad \hat{\sigma} \circ s = s \circ \mathrm{Frob},$$

where $\hat{\sigma} \in \mathrm{Aut}(Y)$ is dual to σ. (The action of $\hat{\sigma}$ on $\hat{T}$ is such that $\sigma \cdot \lambda = \lambda \circ \hat{\sigma}$ for all $\lambda \in X$.) Hence $s \mapsto \chi_s$ is an isomorphism

$$\mathrm{Hom}_{\mathrm{Frob},\hat{\sigma}}(\mathfrak{f}_n^\times, \hat{T}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{F}_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_n^\times, \mathbb{C}^\times).$$

The tame inertia group $\mathcal{I}_t$ is identified with the projective limit

$$\mathcal{I}_t = \varprojlim_m \mathfrak{f}_m^\times,$$

with respect to the norm mappings on the finite fields $\mathfrak{f}_m$. The canonical projection

$$\mathcal{I}_t \longrightarrow \mathfrak{f}_m^\times$$

induces an isomorphism as Frob-modules

$$\mathcal{I}_t/(1-\operatorname{Ad}\mathrm{Frob}^m)\mathcal{I}_t \xrightarrow{\sim} \mathfrak{f}_m^\times.$$

Since $\hat{\sigma}$ has order n, any $s \in \operatorname{Hom}_{\operatorname{Ad}\mathrm{Frob},\hat{\sigma}}(\mathcal{I}_t, \hat{T})$ is trivial on $(1-\operatorname{Ad}\mathrm{Frob}^n)\mathcal{I}_t$. It follows that

$$\operatorname{Hom}_{\mathrm{Frob},\hat{\sigma}}(\mathfrak{f}_n^\times, \hat{T}) \simeq \operatorname{Hom}_{\operatorname{Ad}\mathrm{Frob},\hat{\sigma}}(\mathcal{I}_t, \hat{T}).$$

Thus the map $s \mapsto \chi_s$ is a canonical bijection

$$\operatorname{Hom}_{\operatorname{Ad}\mathrm{Frob},\hat{\sigma}}(\mathcal{I}_t, \hat{T}) \xrightarrow{\sim} \operatorname{Hom}(\mathsf{T}^{\mathrm{F}_\sigma}, \mathbb{C}^\times).$$

Now $s \circ \operatorname{Ad}\mathrm{Frob} = \hat{\sigma} \circ s$ iff for some (equivalently, any) $\tau \in \hat{T}$, the assignment $\mathrm{Frob} \mapsto \hat{\sigma} \ltimes \tau$ extends s to a homomorphism

$$\varphi : \mathcal{W}_t \longrightarrow {}^L T_\sigma,$$

where ${}^L T_\sigma = \langle \hat{\sigma} \rangle \ltimes \hat{T}$ is the L-group of the torus T with Frobenius F_σ. The $\hat{T}$-conjugacy class of the extension φ is uniquely determined by the image of τ in $\hat{T}/(1-\hat{\sigma})\hat{T}$. The latter group is identified with the character group of X^σ, whereby τ corresponds to

$$\chi_\tau \in \operatorname{Hom}(X^\sigma, \mathbb{C}^\times), \qquad \chi_\tau(\lambda) = \lambda(\tau).$$

Our choice of uniformizer in k gives an isomorphism

$$T^{\mathrm{F}_\sigma} \simeq {}^0 T^{\mathrm{F}_\sigma} \times X^\sigma,$$

where ${}^0 T$ is the group of R_K-points of $\mathbf{T}$. Hence the above isomorphisms give a canonical bijection between $\hat{T}$-conjugacy classes of admissible homomorphisms $\varphi : \mathcal{W}_t \longrightarrow {}^L T_\sigma$ and depth-zero characters

$$\chi_\varphi := \chi_s \otimes \chi_\tau \in \operatorname{Irr}(T^{\mathrm{F}_\sigma}),$$

where $s = \varphi|_{\mathcal{I}_t}$ and $\varphi(\mathrm{Frob}) = \hat{\sigma} \ltimes \tau$. This bijection has the following naturality property.

Lemma 4.3.1. *Let α be an algebraic automorphism of T commuting with F_σ, so $\alpha \in \operatorname{Aut}(X)$ and $\hat{\alpha} \in \operatorname{Aut}(Y)$. Then $\chi_\varphi \circ \alpha = \chi_{\hat{\alpha}\circ\varphi}$.*

Proof. We check it first on X^σ. Since $\chi_\varphi(\mu) = \mu(\tau)$, for $\mu \in X^\sigma$, we have

$$\chi_\varphi(\alpha \cdot \mu) = (\alpha \cdot \mu)(\tau) = \mu(\hat{\alpha} \cdot \tau) = \chi_{\hat{\alpha}\circ\varphi}(\mu).$$

Now on $\mathsf{T}^{\mathrm{F}_\sigma}$ we have $\chi_\varphi = \chi_s$, where $\chi_s \in \operatorname{Hom}_{\mathrm{F}_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_n^\times, \mathbb{C}^\times)$. For $\lambda \in X$, $a \in \mathfrak{f}_n^\times$, we have

$$\chi_{\hat{\alpha}\circ\varphi}(\lambda \otimes a) = \chi_{\hat{\alpha}\circ s}(\lambda \otimes a) = \lambda\big(\hat{\alpha} \cdot (s(a))\big) = (\alpha \cdot \lambda)(s(a)) = (\chi_s \circ \alpha)(\lambda \otimes a) = (\chi_\varphi \circ \alpha)(\lambda \otimes a).$$

$\square$

Now let $\varphi : \mathcal{W}_t \to {}^LG$ be a TRSELP, with associated $w \in W_o$ and set $\sigma = w\vartheta$. We want to construct from φ a $\hat{T}$-conjugacy class of Langlands parameters

$$\varphi_T : \mathcal{W}_t \longrightarrow {}^LT_\sigma,$$

such that $\varphi_T = \varphi$ on $\mathcal{I}$, and such that $\varphi_T(\mathrm{Frob})$ and $\varphi(\mathrm{Frob})$ have the same action on $\hat{T}$. We will have

$$\varphi_T(\mathrm{Frob}) = \hat{\sigma} \ltimes \tau$$

for some $\tau \in \hat{T}$, which is only defined up to $\hat{\sigma}$-twisted conjugacy. That is, we need only define the coset of τ in $\hat{T}/(1-\hat{\sigma})\hat{T}$.

We define the coset of τ as follows. Let $\hat{G}'$ be the derived group of $\hat{G}$, and let $\hat{T}' = \hat{T} \cap \hat{G}'$. Ellipticity implies that the map $\tau \mapsto \tau\hat{\sigma}(\tau)^{-1}$ has finite kernel on $\hat{T}'$, which means that

$$(1-\hat{\sigma})\hat{T}' = \hat{T}',$$

so the inclusion $\hat{T} \hookrightarrow \hat{G}$ induces a bijection

$$\hat{T}/(1-\hat{\sigma})\hat{T}' \xrightarrow{\sim} \hat{G}/\hat{G}' =: \hat{G}_{ab}.$$

It follows that $\hat{T} \hookrightarrow \hat{G}$ induces a bijection

$$\hat{T}/(1-\hat{\sigma})\hat{T} \xrightarrow{\sim} \hat{G}_{ab}/(1-\hat{\vartheta})\hat{G}_{ab} \tag{16}$$

between the the set of $\hat{\sigma}$-twisted conjugacy classes in $\hat{T}$ and the set of $\hat{\vartheta}$-twisted conjugacy classes in the abelianization $\hat{G}_{ab}$. Now, if $\varphi(\mathrm{Frob}) = \hat{\vartheta} \ltimes f$, we take any $\tau \in \hat{T}$ whose class in $\hat{T}/(1-\hat{\sigma})\hat{T}$ corresponds under (16) to the image of f in $\hat{G}_{ab}/(1-\hat{\vartheta})\hat{G}_{ab}$.

Hence, from the TRSELP φ we get a character $\chi_{\varphi_T} \in \mathrm{Irr}(T^{\mathrm{F}_\sigma})$. We will abuse notation slightly and again denote this character by χ_φ.

4.4. **From tame parameters to depth-zero types.** Let $\varphi : \mathcal{W}_t \longrightarrow {}^LG$ be a TRSELP with $\varphi(\mathrm{Frob}) = \hat{\vartheta}f$ as in Section 4.1. Let $w \in W_o$ be the element such that $\hat{w}$ is the image of f in $\hat{W}_o$. Since φ is elliptic, we have

$$X^{w\vartheta} = X_*(\mathbf{Z}^\circ)^\vartheta, \qquad X_{ad}^{w\vartheta} = \{0\}, \tag{17}$$

where $\mathbf{Z}^\circ$ is the identity component of the center $\mathbf{Z}$ of $\mathbf{G}$.

Let $\lambda \in X_w$, and set

$$\sigma_\lambda = t_\lambda w\vartheta \in W \rtimes \langle\vartheta\rangle,$$

as in Section 2.7. By the second equation in (17), the operator $I - w\vartheta$ acts invertibly on $\mathcal{A}_{ad}$, so σ_λ has a unique fixed-point $x_\lambda \in \mathcal{A}_{ad}$, given by

$$x_\lambda = (I - w\vartheta)^{-1}t_{j\lambda} \cdot o.$$

Let $\tilde{x}_\lambda$ be the pre-image of x_λ in $\mathcal{A}^{\sigma_\lambda}$.

The facet J_λ from Section 2.7 is the unique facet in $\mathcal{A}$ containing $\tilde{x}_\lambda$. As in Section 2.7, we choose an alcove C_λ in $\mathcal{A}$ containing J_λ in its closure, and write

$$G_\lambda := G_{J_\lambda}, \qquad W_\lambda := [N \cap G_\lambda]/{}^0T, \qquad \mathsf{G}_\lambda := G_\lambda/G_\lambda^+.$$

We choose $u_\lambda \in Z^1(\mathrm{F}, N)$ as in Lemma 2.7.2, and define

$$\mathrm{F}_\lambda := \mathrm{Ad}(u_\lambda) \circ \mathrm{F}\,.$$

Then F_λ is a Frobenius endomorphism of G for some k-rational structure on $\mathbf{G}$ which is inner to the quasi-split structure on $\mathbf{G}$ given by F. Recall also that F_λ stabilizes the apartment $\mathcal{A}$, the alcove C_λ, and the facet J_λ.

Lemma 4.4.1. *We have*

$$\mathcal{A}^{\sigma_\lambda} = J_\lambda^{\sigma_\lambda} = J_\lambda^{\mathrm{F}_\lambda} = \tilde{x}_\lambda.$$

In particular, the point x_λ is a vertex in $\mathcal{B}(G_{ad})^{\mathrm{F}_\lambda}$.

Proof. From (7) of Section 2.7 we may decompose σ_λ in two ways:

$$\sigma_\lambda = t_\lambda w\vartheta = w_\lambda y_\lambda \vartheta.$$

Since w_λ fixes J_λ pointwise, we have

$$J_\lambda^{\mathrm{F}_\lambda} = J_\lambda^{y_\lambda\vartheta} = J_\lambda^{\sigma_\lambda} = \tilde{x}_\lambda.$$

Also, $\mathcal{A}^{\sigma_\lambda} = \tilde{x}_\lambda \subseteq J_\lambda$, implying that $\mathcal{A}^{\sigma_\lambda} = J_\lambda^{\sigma_\lambda}$. □

Since $\mathrm{F}_\lambda \cdot J_\lambda = J_\lambda$, F_λ induces a Frobenius endomorphism of G_λ, preserving T. Since $\mathrm{F}_\lambda \cdot C_\lambda = C_\lambda$, the Frobenius F_λ also preserves a Borel subgroup of G_λ containing T. It follows [6, 20.6] that T is a maximally $\mathfrak{f}$-split torus in G_λ with respect to F_λ.

From Section 2.7 we have the alternative expression

$$\sigma_\lambda = w_\lambda y_\lambda \vartheta,$$

where $w_\lambda \in W_\lambda$ and y_λ is the image of u_λ in W. Moreover, our fixed choice of lift $\dot{w}$ of w defines a lift $\dot{w}_\lambda \in N \cap G_\lambda$ of w_λ, via the equation

$$t_\lambda \dot{w} = \dot{w}_\lambda u_\lambda.$$

Recall we can then choose an element $p_\lambda \in G_\lambda$ such that

$$p_\lambda^{-1}\,\mathrm{F}_\lambda(p_\lambda) = \dot{w}_\lambda.$$

Note that

$$\mathrm{F}_\lambda \circ \mathrm{Ad}(p_\lambda) = \mathrm{Ad}(p_\lambda) \circ \mathrm{Ad}(\dot{w}_\lambda u_\lambda) \circ \mathrm{F}\,.$$

Define

$$T_\lambda := \mathrm{Ad}(p_\lambda)T.$$

Then T_λ is an F_λ-stable unramified torus in G. On T, we have $\mathrm{Ad}(\dot{w}_\lambda u_\lambda) = \mathrm{Ad}(w)$, so $\mathrm{Ad}(p_\lambda) : T \longrightarrow T_\lambda$ satisfies

$$\mathrm{F}_\lambda \circ \mathrm{Ad}(p_\lambda) = \mathrm{Ad}(p_\lambda) \circ \mathrm{F}_w,$$

where $\mathrm{F}_w = \mathrm{Ad}(\dot{w}) \circ \mathrm{F}$.

By ellipticity, we have

$$T^{\mathrm{F}_w} = X^{w\vartheta} \times {}^0T^{\mathrm{F}_w} = X_*(\mathbf{Z}^\circ)^\vartheta \times {}^0T^{\mathrm{F}_w} = Z^{\mathrm{F}} \cdot {}^0T^{\mathrm{F}_w}.$$

This implies that T_λ is F_λ-minisotropic. Moreover, we have ${}^0T_\lambda = T_\lambda \cap G_\lambda$, and ${}^0T_\lambda$ projects to an F_λ-minisotropic maximal torus T_λ in G_λ.

On $\mathcal{A}$ and $\mathcal{A}_{ad}$ we have $\mathrm{Ad}(\dot{w}_\lambda u_\lambda)\,\mathrm{F} = \sigma_\lambda$. By Lemma 4.4.1 the unique fixed-point of T_λ in $\mathcal{B}(G_{ad})^{\mathrm{F}_\lambda}$ is

$$[p_\lambda \cdot \mathcal{A}_{ad}]^{\mathrm{F}_\lambda} = p_\lambda \cdot \mathcal{A}_{ad}^{\sigma_\lambda} = p_\lambda \cdot x_\lambda = x_\lambda.$$

As in Section 4.3, we have a depth-zero character $\chi = \chi_\varphi$ of T^{F_w}. Since φ is in general position, Lemma 4.3.1 implies that χ is F_w-regular.

This character χ transports to a depth-zero F_λ-regular character

$$\chi_\lambda := \mathrm{Ad}(p_\lambda)_*\chi \in \mathrm{Irr}(T_\lambda^{\mathrm{F}_\lambda}).$$

The restriction of χ_λ to ${}^0T_\lambda^{\mathrm{F}_\lambda}$ factors through a character $\chi_\lambda^0 \in \mathrm{Irr}(\mathsf{T}_\lambda^{\mathrm{F}_\lambda})$, which is in "general position" with respect to F_λ, in the sense of [20, 5.16]. By [20, 8.3], Deligne-Lusztig induction then gives an irreducible cuspidal representation

$$\kappa_\lambda^0 := \epsilon(\mathsf{G}_\lambda, \mathsf{T}_\lambda) \cdot R^{\mathsf{G}_\lambda}_{\mathsf{T}_\lambda, \chi_\lambda^0} \in \mathrm{Irr}(\mathsf{G}_\lambda^{\mathrm{F}_\lambda}).$$

Inflate κ_λ^0 to a representation of $G_\lambda^{\mathrm{F}_\lambda}$ and define an extension to $Z^{\mathrm{F}}G_\lambda^{\mathrm{F}_\lambda}$ by

$$\kappa_\lambda := \chi_\lambda \otimes \kappa_\lambda^0.$$

This makes sense since $(Z \cap G_\lambda)^{\mathrm{F}_\lambda}$ acts on κ_λ^0 via the restriction of the scalar character χ_λ^0.

So far, to the TRSELP φ and $\lambda \in X_w$, we have associated a Frobenius F_λ, an F_λ-stable parahoric subgroup G_λ, and an irreducible representation κ_λ of $Z^{\mathrm{F}}G_\lambda^{\mathrm{F}_\lambda}$. In the process we made choices of $\dot{w}, C_\lambda,\ u_\lambda,\ p_\lambda$.

Lemma 4.4.2. *Given a TRSELP φ and $\lambda \in X_w$, both fixed, suppose we make two sets of choices $(\dot{w}, C_\lambda, u_\lambda, p_\lambda)$ and $(\dot{w}', C'_\lambda, u'_\lambda, p'_\lambda)$ as above, giving rise to $(\mathrm{F}_\lambda, T_\lambda, \chi_\lambda, \kappa_\lambda)$ and $(\mathrm{F}'_\lambda, T'_\lambda, \chi'_\lambda, \kappa'_\lambda)$ as above. Then there is $h \in G_\lambda$ such that*

(1) $h * u'_\lambda = u_\lambda$;
(2) $\mathrm{Ad}(h)_*(T'_\lambda, \chi'_\lambda, \kappa'_\lambda) = (T_\lambda, \chi_\lambda, \kappa_\lambda)$.

Proof. Note that (1) implies that $\mathrm{Ad}(h)\big(G_\lambda^{\mathrm{F}'_\lambda}\big) = G_\lambda^{\mathrm{F}_\lambda}$, so (2) makes sense. Since σ_λ is defined before the choices are made, we have

$$w_\lambda y_\lambda = t_\lambda w = w'_\lambda y'_\lambda,$$

so there is $t \in {}^0T$ such that

$$\dot{w}_\lambda u_\lambda = t\dot{w}'_\lambda u'_\lambda.$$

Here, both sides belong to $Z^1(\mathrm{F}, N)$ and act on T via w. Lemma 2.1.1 implies that $t \in Z^1(\mathrm{F}_w, {}^0T)$. By Lemma 2.3.1 for ${}^0T$, there is $s \in {}^0T$ such that

$$s\,\mathrm{F}_w(s)^{-1} = t. \tag{18}$$

Since $\mathrm{Ad}(w) = \mathrm{Ad}(\dot{w}'_\lambda u'_\lambda)$ on T, equation (18) can be written

$$s\dot{w}'_\lambda u'_\lambda = t\dot{w}'_\lambda u'_\lambda\,\mathrm{F}(s) = \dot{w}_\lambda u_\lambda\,\mathrm{F}(s). \tag{19}$$

Recall our equations characterizing p_λ and p'_λ:

(20) $$p_\lambda^{-1}\,\mathrm{F}_\lambda(p_\lambda) = \dot{w}_\lambda, \qquad {p'_\lambda}^{-1}\,\mathrm{F}'_\lambda(p'_\lambda) = \dot{w}'_\lambda.$$

These allow us to write (19) in the form

(21) $$s \cdot {p'_\lambda}^{-1} u'_\lambda\,\mathrm{F}(p'_\lambda) = p_\lambda^{-1} u_\lambda\,\mathrm{F}(p_\lambda) \cdot \mathrm{F}(s).$$

Equation (21) shows that the element $h := p_\lambda s {p'_\lambda}^{-1}$ satisfies $h * u'_\lambda = u_\lambda$. We have $h \in G_\lambda$ since p_λ, s, p'_λ are all in G_λ. It is clear that $\mathrm{Ad}(h)(T'_\lambda, \chi'_\lambda) = (T_\lambda, \chi_\lambda)$, which then implies that $\mathrm{Ad}(h)_*\kappa'_\lambda = \kappa_\lambda$. □

4.5. **Definition of the L-packets.** Given a TRSELP φ, an element $\lambda \in X_w$ and a set of choices $(C_\lambda, u_\lambda, p_\lambda)$, define

$$\pi_\lambda := \mathrm{Ind}_{Z^{\mathrm{F}} G_\lambda^{\mathrm{F}_\lambda}}^{G^{\mathrm{F}_\lambda}} \kappa_\lambda,$$

where Ind denotes smooth induction. (The functions in π_λ automatically have compact support modulo Z^F.) In this notation we have suppressed the choices $(C_\lambda, u_\lambda, p_\lambda)$, but by Lemma 4.4.2 the G-orbit (in fact the G_λ-orbit) of (u_λ, π_λ) is independent of these choices.

Lemma 4.5.1. *The representation π_λ of G^{F_λ} is irreducible supercuspidal.*

Proof. By [44, 6.6] it suffices to show that κ_λ induces irreducibly to the group

$$(G_\lambda^\star)^{\mathrm{F}_\lambda} = \{g \in G^{\mathrm{F}_\lambda} :\ g \cdot J_\lambda = J_\lambda\},$$

which is the normalizer of $G_\lambda^{\mathrm{F}_\lambda}$ in G^{F_λ}. For this, it is enough to show the stabilizer of κ_λ in $(G_\lambda^\star)^{\mathrm{F}_\lambda}$ is just $Z^{\mathrm{F}_\lambda} G_\lambda^{\mathrm{F}_\lambda}$.

Suppose $h \in (G_\lambda^\star)^{\mathrm{F}_\lambda}$ and $\mathrm{Ad}(h)_*\kappa_\lambda = \kappa_\lambda$. By [20, Thm. 6.8], there is $g \in G_\lambda^{\mathrm{F}_\lambda}$ such that

$$\mathrm{Ad}(gh)_*(\mathsf{T}_\lambda, \chi_\lambda) = (\mathsf{T}_\lambda, \chi_\lambda).$$

Then by [14] there is $\ell \in (G_\lambda^+)^{\mathrm{F}_\lambda}$ such that

$$\mathrm{Ad}(\ell g h)_*(T_\lambda, \chi_\lambda) = (T_\lambda, \chi_\lambda).$$

That is, $\ell g h \in N(G, T_\lambda)^{\mathrm{F}_\lambda}$ and fixes χ_λ. Hence $p_\lambda^{-1}\ell g h p_\lambda \in N^{\mathrm{F}_w}$ and fixes χ. Let z be the projection of $p_\lambda^{-1}\ell g h p_\lambda$ to W_o. By Lemma 4.3.1 we have $\hat{z} \circ s = s$, but $C_{\hat{G}}(s) = \hat{T}$, so $z = 1$. It follows that $\ell g h \in T_\lambda^{\mathrm{F}_\lambda} \cap (G_\lambda^\star)^{\mathrm{F}_\lambda} \subset Z^{\mathrm{F}_\lambda} G_\lambda^{\mathrm{F}_\lambda}$. Since ℓ and g are in $G_\lambda^{\mathrm{F}_\lambda}$, this implies that $h \in Z^{\mathrm{F}_\lambda} G_\lambda^{\mathrm{F}_\lambda}$. □

At this point we have a supercuspidal representation $\pi_\lambda \in \mathrm{Irr}(G^{\mathrm{F}_\lambda})$ for every $\lambda \in X_w$. We now show that the G-orbit $[u_\lambda, \pi_\lambda] := \mathrm{Ad}(G) \cdot (u_\lambda, \pi_\lambda)$ depends only on the character $\rho_\lambda \in \mathrm{Irr}(C_\varphi)$ corresponding to the image of λ in $[X/(1 - w\vartheta)X]_{\mathrm{tor}} = \mathrm{Irr}(C_\varphi)$ (see Section 4.1).

Lemma 4.5.2. *Given φ, along with $\lambda, \mu \in X_w$, make choices $(C_\lambda, u_\lambda, p_\lambda)$, (C_μ, u_μ, p_μ) as above. Then $\rho_\lambda = \rho_\mu$ if and only if there exists $g \in G$ such that*

(1) $g * u_\lambda = u_\mu$;
(2) $g \cdot J_\lambda = J_\mu$;
(3) $\mathrm{Ad}(g)_*\kappa_\lambda \simeq \kappa_\mu$.

Proof. Suppose $\rho_\lambda = \rho_\mu$. This is equivalent to having $\mu = \lambda + (1 - w\vartheta)\nu$ for some $\nu \in X$, which amounts to the following equation in $W \rtimes \langle\vartheta\rangle$:

$$t_\nu \sigma_\lambda t_\nu^{-1} = t_\nu t_\lambda w\vartheta t_\nu^{-1} = t_\mu w\vartheta = \sigma_\mu.$$

Lifting to N, we have

$$t_\nu \dot{w}_\lambda u_\lambda \,\mathrm{F}(t_\nu)^{-1} = t\dot{w}_\mu u_\mu, \tag{22}$$

for some $t \in {}^0T$. Arguing as in the proof of Lemma 4.4.2, there is $s \in {}^0T_\mu$ such that

$$p_\mu t p_\mu^{-1} = s^{-1}\,\mathrm{F}_\mu(s).$$

Using Equations (20) we then find that

$$g * u_\lambda = u_\mu,$$

where $g = sp_\mu t_\nu p_\lambda^{-1}$.

Since σ_λ and σ_μ have unique fixed-points x_λ and x_μ in $\mathcal{A}_{ad}$, we must have $t_\nu \cdot x_\lambda = x_\mu$, hence $t_\nu \cdot J_\lambda = J_\mu$, from which 2 is immediate.

Finally, we have

$$\mathrm{Ad}(g)_*(T_\lambda, \chi_\lambda) = \mathrm{Ad}(sp_\mu t_\nu)_*(T, \chi) = \mathrm{Ad}(s)_*(T_\mu, \chi_\mu),$$

so $\mathrm{Ad}(g)_*\kappa_\lambda \simeq \kappa_\mu$.

Turning to the converse, suppose we have $g \in G$ satisfying items 1-3 above. By 2 and 3 and [20, Thm. 6.8], the pairs

$$(\mathrm{Ad}(g)T_\lambda, \mathrm{Ad}(g)_*\chi_\lambda), \qquad (T_\mu, \chi_\mu)$$

are conjugate in $G_\mu^{\mathrm{F}_\mu}$, so without loss of generality, we may assume these two pairs are equal. Then, the element $n := p_\mu^{-1} g p_\lambda$ belongs to N. By 1, $\mathrm{Ad}(n)$ preserves T^{F_w}, and it preserves the F_w-regular character χ, since $\mathrm{Ad}(g)_*\chi_\lambda = \chi_\mu$. It follows that $n \in T$. Let t_ν be the image of n in W. As in the first paragraph of the proof, it suffices to prove that $t_\nu\sigma_\lambda t_\nu^{-1} = \sigma_\mu$. But this follows from the equation

$$\mathrm{Ad}(n) \circ \mathrm{Ad}(\dot{w}_\lambda u_\lambda) \circ \mathrm{F} \circ \mathrm{Ad}(n)^{-1} = \mathrm{Ad}(\dot{w}_\mu u_\mu) \circ \mathrm{F},$$

which is proved using Equations (20) as before. □

Now we have our first main result.

Theorem 4.5.3. *Given a TRSELP φ with associated $w \in W_o$, let $r : X_w \to H^1(\mathrm{F}, G)$ be as in section 2.8. For each $\omega \in H^1(\mathrm{F}, G)$ define*

$$\Pi(\varphi, \omega) := \{[u_\lambda, \pi_\lambda] : \ \lambda \in r^{-1}(\omega)\}.$$

Then we have a well-defined bijection $\mathrm{Irr}(C_\varphi, \omega) \xrightarrow{\sim} \Pi(\varphi, \omega)$, as follows. Given $\rho \in \mathrm{Irr}(C_\varphi, \omega)$, choose any $\lambda \in r^{-1}(\omega)$ such that $\rho_\lambda = \rho$, and associate to ρ the G-orbit $[u_\lambda, \pi_\lambda] \in \Pi(\varphi, \omega)$.

Proof. Recall that $r(\lambda) = \omega$ if and only if $\rho_\lambda \in \mathrm{Irr}(\varphi, \omega)$. Suppose we have $\lambda, \mu \in X_w$ such that $\rho_\lambda, \rho_\mu \in \mathrm{Irr}(\varphi, \omega)$. From [44, 6.2] it follows that conditions 1-3 of Lemma 4.5.2 are equivalent to having $g \in G$ such that

$$\mathrm{Ad}(g) \cdot (u_\lambda, \pi_\lambda) = (u_\mu, \pi_\mu).$$

So we have proved that

$$[u_\lambda, \pi_\lambda] = [u_\mu, \pi_\mu] \quad \Leftrightarrow \quad \rho_\lambda = \rho_\mu,$$

as desired. □

Remark 4.5.4. Recall that $\mathrm{Irr}(C_\varphi, \omega)$ is equal to the fiber over ω under the composition

$$\mathrm{Irr}(C_\varphi) \xrightarrow{\sim} \left[X/(1 - w\vartheta)X\right]_{\mathrm{tor}} \to \left[\bar{X}/(1 - \vartheta)\bar{X}\right]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}, G),$$

whereby $\rho = \rho_\lambda \mapsto r(\lambda)$. By Lemma 2.7.3 we have $u_\lambda \in \omega_\lambda = r(\lambda) = \omega$. Hence our representation π_λ lives on an inner twist of G belonging to the class $\omega \in H^1(\mathrm{F}, G)$, in accordance with the conjectures in Section 3.

4.6. **Choosing representatives in an L-packet.** We now use Section 2.8 to choose representatives, living on a single group, of each G-orbit in an L-packet $\Pi(\varphi, \omega)$. We fix $u \in \omega \cap N$, and for each $\lambda \in r^{-1}(\omega)$ we choose m_λ as in Section 2.8. For each $\rho \in \mathrm{Irr}(C_\varphi, \omega)$, define

$$\pi_u(\varphi, \rho) := \mathrm{Ad}(m_\lambda)_* \pi_\lambda \in \mathrm{Irr}(G^{\mathrm{F}_u}),$$

for any $\lambda \in r^{-1}(\omega)$ such that $\rho_\lambda = \rho$. We have seen that the isomorphism class of π_λ is independent of the choice of λ. Two choices of m_λ differ by an element of G^{F_u}, so the isomorphism class of $\pi_u(\varphi, \rho_\lambda)$ likewise does not depend on the choice of m_λ. The normalized L-packet is then defined as

$$\Pi_u(\varphi) := \{\pi_u(\varphi, \rho) : \ \rho \in \mathrm{Irr}(C_\varphi, \omega)\}.$$

More explicitly, the representation $\pi_u(\varphi, \rho)$ is given as follows. Recall that $m_\lambda \cdot C_\lambda$ is our fixed F_u-stable alcove C. The facet $I_\lambda := m_\lambda \cdot J_\lambda$ is contained in $\bar{C}$, and is likewise F_u-stable. The F_u-minisotropic torus $S_\lambda = \mathrm{Ad}(m_\lambda) T_\lambda = \mathrm{Ad}(q_\lambda) T$ (see Section 2.8) has the property that $S_\lambda \cap G_{I_\lambda}$ projects to an F_u-minisotropic torus S_λ in G_{I_λ}. The character $\theta_\lambda := \mathrm{Ad}(m_\lambda)_* \chi_\lambda = \mathrm{Ad}(q_\lambda)_* \chi$ is F_u-regular, and gives a(n inflated) Deligne-Lusztig representation

$$\varkappa_\lambda^0 := \varepsilon(\mathsf{G}_{I_\lambda}, \mathsf{S}_\lambda) \cdot R_{\mathsf{S}_\lambda, \theta_\lambda}^{\mathsf{G}_{I_\lambda}} \in \mathrm{Irr}(G_{I_\lambda}^{\mathrm{F}_u}),$$

and an extension of $\varkappa_\lambda^0$ to a representation $\varkappa_\lambda$ of $Z^F G_{I_\lambda}^{\mathrm{F}_u}$. Finally, we have

$$\pi_u(\varphi, \rho) = \mathrm{Ind}_{Z^F G_{I_\lambda}^{\mathrm{F}_u}}^{G^{\mathrm{F}_u}} \varkappa_\lambda.$$

5. Normalizations of measures and formal degrees

We now move toward Harmonic Analysis. The first step is a uniform normalization of Haar measures on groups of the form G^F, where $G = \mathbf{G}(K)$ and $\mathbf{G}$ is a connected reductive k-group, split over K. We then verify the equality of formal degrees in an L-packet, according to the conjectures in Section 3. (Note that the group C_φ is abelian for these L-packets.) Except where noted, our Frobenius on G is now unspecified, and is denoted by F, according to our conventions.

5.1. **Haar measure.** We denote the Lie algebra of G by $\mathfrak{g}$, and again let F denote the induced Frobenius action on $\mathfrak{g}$.

Suppose $x \in \mathcal{B}(G)$ or $\mathcal{B}(G_{ad})$. Just as we could attach a parahoric G_x and its pro-unipotent radical G_x^+ to x, so we can define lattices $\mathfrak{g}_x$ and $\mathfrak{g}_x^+$ in $\mathfrak{g}$ (see [43, §3.2], [3, §2.2], where the corresponding objects are called $\mathfrak{g}_{x,0}$ and $\mathfrak{g}_{x,0^+}$). As before, the lattices $\mathfrak{g}_x$ and $\mathfrak{g}_x^+$ are independent of the facet to which x belongs. If J is any subset of a facet and $x \in J$, then we set $\mathfrak{g}_J = \mathfrak{g}_x$ and $\mathfrak{g}_J^+ = \mathfrak{g}_x^+$. If J is an F-stable subset of a facet, then

$$\mathsf{L}_J := \mathfrak{g}_J/\mathfrak{g}_J^+$$

is the Lie algebra of G_J, and we have

$$\mathsf{L}_J^F = \mathfrak{g}_J^F/(\mathfrak{g}_J^{+F}).$$

Let dg denote the Haar measure on G^F, normalized so that

$$\operatorname{meas}_{dg}(G_J^F) = \frac{|\mathsf{G}_J^F|}{|\mathsf{L}_J^F|^{1/2}}$$

for one, in fact every, F-stable facet J in $\mathcal{B}(G)$.

Let dX denote the Haar measure on $\mathfrak{g}^F$, normalized so that

$$\operatorname{meas}_{dX}(\mathfrak{g}_J^F) = |\mathsf{L}_J^F|^{1/2}$$

for one (in fact, every) F-stable facet J in $\mathcal{B}(G)$.

To show that these normalizations are independent of the choice of J as claimed, it is enough to show that if J and J' are F-stable facets in $\mathcal{B}(G)$ with $J' \subset \bar{J}$, then

$$\operatorname{meas}_{dg}(G_J^F) = \frac{|\mathsf{G}_J^F|}{|\mathsf{L}_J^F|^{1/2}}$$

implies

$$\operatorname{meas}_{dg}(G_{J'}^F) = \frac{|\mathsf{G}_{J'}^F|}{|\mathsf{L}_{J'}^F|^{1/2}}$$

(and similarly for the measure dX on $\mathfrak{g}$). Since $J' \subset \bar{J}$, we have

$$G_{J'}^+ \subset G_J^+ \subset G_J \subset G_{J'}.$$

Moreover, the image of G_J in $\mathsf{G}_{J'}$ is a parabolic $\mathfrak{f}$-subgroup with unipotent radical $G_J^+/G_{J'}^+$ and Levi component isomorphic to G_J. A short calculation gives the desired result.

Remark 5.1.1. The above expression for $\operatorname{meas}_{dg}(G_J^F)$ can be simplified a bit. Let G be a connected reductive group over $\mathfrak{f}$ with Frobenius F. Let $\mathsf{T} \subset \mathsf{B}$ be an F-stable maximal torus and an F-stable Borel subgroup in G. Then

$$|\mathsf{G}^F| = [\mathsf{G}^F : \mathsf{B}^F] \cdot |\mathsf{B}^F| = q^\nu [\mathsf{G}^F : \mathsf{B}^F] \cdot |\mathsf{T}^F|,$$

where ν is the number of (absolute) roots of T in B. The latter two factors are prime to p, so

$$|\mathsf{G}^F|_{p'} = [\mathsf{G}^F : \mathsf{B}^F] \cdot |\mathsf{T}^F|,$$

where $|\cdot|_{p'}$ is the largest factor of $|\cdot|$ which is prime to p. We have $\dim \mathsf{G} = \dim \mathsf{T} + 2\nu$. It follows that

$$\mathrm{meas}_{dg}(G_J^F) = q^{-\operatorname{rk}(\mathbf{G})/2}|\mathsf{G}_J^F|_{p'}.$$

where $\operatorname{rk}(\mathbf{G})$ is the absolute rank of $\mathbf{G}$.

This normalization applies as well to the largest k-split torus $\mathbf{Z}$ of the center of $\mathbf{G}$, and gives

$$\mathrm{meas}_{dz}({}^0Z^F) = q^{-\operatorname{rk}(\mathbf{Z})/2}|\mathsf{Z}^F| = (q^{1/2} - q^{-1/2})^{\operatorname{rk}(\mathbf{Z})},$$

where $\mathsf{Z} = {}^0Z/{}^0Z^+$.

For any irreducible admissible representation π of G^F which is square-integrable modulo Z^F, let $\mathrm{Deg}(\pi)$ denote the formal degree of π with respect to the quotient measure dg/dz on G^F/Z^F (c.f. [26]).

5.2. **Formal degree of the Steinberg representations.** The formal degree conjectures in Section 3 require Haar measures for which the formal degree of the Steinberg representation of G^F is unchanged by inner twists of F, for $F = \mathrm{F}_u$. In this section we show that the measures dg defined above have this property. First we consider some constants arising in this formal degree.

Recall that the quasi-split Frobenius F acts on $X = X_*(\mathbf{T})$ by the automorphism ϑ, and that $\mathbf{Z}$ denotes the largest k-split torus in the center of $\mathbf{G}$. Note that $G/Z = (\mathbf{G}/\mathbf{Z})^{\mathcal{I}}$.

Let $X_1 = X_*(\mathbf{T}/\mathbf{Z})$ and let C_1 be the projection to the apartment of T/Z in $\mathcal{B}(G/Z)$ of the ϑ-stable alcove C in $\mathcal{A}$. Let Ω_1 be the stabilizer of C_1 in the affine Weyl group of T/Z in G/Z. The inclusion $X_*(\mathbf{Z}) \hookrightarrow X$ projects to an embedding

$$X_*(\mathbf{Z}) \hookrightarrow (X/X^\circ)^\vartheta \simeq \Omega_C^\vartheta,$$

where X° is the co-root lattice of $\mathbf{T}$. Identifying, as we may, X° with the co-root lattice of $\mathbf{T}/\mathbf{Z}$, we have

$$\Omega_1 \simeq X_1/X^\circ \simeq \Omega_C/X_*(\mathbf{Z}),$$

and a finite subgroup $\Omega_2 := \Omega_C^\vartheta/X_*(\mathbf{Z}) \hookrightarrow \Omega_1$ fitting into the exact sequence

$$1 \longrightarrow \Omega_2 \longrightarrow \Omega_1 \xrightarrow{1-\vartheta} \Omega_1 \longrightarrow \Omega_1/(1-\vartheta)\Omega_1 \longrightarrow 1,$$

showing that

$$|\Omega_2| = |\Omega_1/(1-\vartheta)\Omega_1| = |H^1(\mathrm{F}, G/Z)|. \tag{23}$$

Now take a cocycle $u \in Z^1(\mathrm{F}, N_C)$, with corresponding twist $\mathrm{F}_u = \mathrm{Ad}(u)\circ\mathrm{F}$ as before. Since $u \in N_C$ and Ω_C is abelian, we have $\Omega_C^{u\vartheta} = \Omega_C^\vartheta$. It follows that Ω_2 is unchanged if we replace ϑ by an inner twist $u\vartheta$. Of course this also follows from (23).

Next, let $V_1 = X_1 \otimes \mathbb{C}$, let R be the graded $\mathbb{C}$-algebra of W_o-invariant polynomial functions on the $\mathbb{C}$-vector space V_1, and let $\mathfrak{m}$ be the maximal ideal in R of functions vanishing at $0 \in V_1$. Then $V := \mathfrak{m}/\mathfrak{m}^2$ is a vector space of dimension $\ell := \dim V_1$. The space V inherits a grading from R, written $V = \oplus V(d)$. Moreover, ϑ acts naturally on R and V, preserving the grading. Choose a basis of eigenvectors for ϑ in each $V(d)$ and let $f_1, \ldots, f_\ell$ be the collection of eigenvectors obtained. Let $d_j = \deg(f_j)$ and let ϵ_j be the eigenvalue of ϑ on f_j.

Define the constant

$$c(\mathbf{G}/\mathbf{Z}) := |\Omega_2| \cdot \prod_{i=1}^{\ell} \frac{q^{d_i} - \epsilon_i}{q^{d_i-1} - \epsilon_i}.$$

The denominators in $c(\mathbf{G}/\mathbf{Z})$ are nonzero because each ϵ_i is a root of unity and $V(1)^{\vartheta} = \{0\}$. Since u acts trivially on R and $|\Omega_2|$ is invariant under inner-twists, it follows that $c(\mathbf{G}/\mathbf{Z})$ is invariant under inner-twists.

Let G_{C_1} be the Iwahori subgroup of G/Z at the alcove C_1. From [5, 5.3] and [55, 3.10], (see also [22, 5.5]) it follows that the formal degree of the Steinberg representation St_u of G^{F_u} is given, using our normalizations in section 5.1, by

$$\begin{aligned}
\mathrm{Deg}(St_u) &= \frac{|\mathsf{T}^{\mathrm{F}_u}/\mathsf{Z}^{\mathrm{F}_u}|}{c(\mathbf{G}/\mathbf{Z})} \cdot \frac{1}{\mathrm{meas}_{dg/dz}(G_{C_1}^{\mathrm{F}_u})} \\
&= \frac{|\mathsf{G}_{C_1}^{\mathrm{F}_u}|_{p'}}{c(\mathbf{G}/\mathbf{Z})} \cdot \frac{q^{\mathrm{rk}(\mathbf{G}/\mathbf{Z})/2}}{|\mathsf{G}_{C_1}^{\mathrm{F}_u}|_{p'}} \\
&= \frac{q^{\mathrm{rk}(\mathbf{G}/\mathbf{Z})/2}}{c(\mathbf{G}/\mathbf{Z})}.
\end{aligned}$$

This last expression is independent of u, as claimed.

5.3. **Formal degrees in our L-packets.** Now suppose π is an irreducible cuspidal representation of G^F of the sort considered in 4.5, namely $\pi = \mathrm{Ind}_{G_J^F Z^F}^{G^F} \kappa$, for some minimal F-stable facet $J \subset \mathcal{B}(G)$ and $\kappa \in \mathrm{Irr}(G_J^F Z^F)$. The formal degree of π is given by

$$\mathrm{Deg}(\pi) = \dim \kappa \cdot \frac{\mathrm{meas}_{dz}({}^0 Z^F)}{\mathrm{meas}_{dg}(G_J^F)}.$$

Recall also that κ is of the following form. We have an F-minisotropic torus $S < G$ such that $\mathcal{A}(S)^F = J^F$, a regular character $\theta \in \mathrm{Irr}(S^F)$ whose restriction to $S \cap G_J^F$ factors through $\mathsf{S}^F = S \cap G^F / S \cap G_J^{+F}$, and on G_J^F we have

$$\kappa = \varepsilon(\mathsf{G}_J, \mathsf{S}) \cdot R_{\mathsf{S},\theta}^{\mathsf{G}_J}.$$

By [20, Thm. 7.1] we have

$$\dim \kappa = \frac{|\mathsf{G}_J^F|_{p'}}{|\mathsf{S}^F|}.$$

Using also Remark 5.1.1, we find that

$$\mathrm{Deg}(\pi) = \frac{|\mathsf{Z}^F|}{|\mathsf{S}^F|} q^{\mathrm{rk}(\mathbf{G}/\mathbf{Z})/2}.$$

Now if $F = \mathrm{F}_u$ and $\pi = \pi_u(\varphi, \rho)$ as in 4.6, then the torus S is k-isomorphic to the platonic torus T with twisted Frobenius F_w (see 2.8). Therefore, we have

$$\mathrm{Deg}\big(\pi_u(\varphi, \rho)\big) = \frac{q^{\mathrm{rk}(\mathbf{G}/\mathbf{Z})/2}}{|\mathsf{T}^{\mathrm{F}_w}/\mathsf{Z}^{\mathrm{F}}|}.$$

The right side of this equation is independent of u and ρ, so all representations in an L-packet $\Pi(\varphi)$ (see Section 4.5) have the same formal degree.

6. GENERIC REPRESENTATIONS

In this section we determine the generic representations in our L-packets $\Pi(\varphi)$. Only quasi-split groups have generic representations, so these can only occur in packets $\Pi(\varphi, \omega)$ for ω belonging to the kernel of the map $j_G : H^1(\mathrm{F}, G) \to H^1(\mathrm{F}, G_{ad})$ induced by the adjoint map $j : G \to G_{ad}$.

Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ defined over k, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. We may and shall assume that $\mathbf{B}$ contains our fixed maximal torus $\mathbf{T}$, which is the centralizer of a maximal k-split torus $\mathbf{S}$.

A character $\psi : U^{\mathrm{F}} \longrightarrow \mathbb{C}^\times$ is *generic* if ψ is nontrivial on each simple root group of S^{F} in U^{F}. A representation $\pi \in \mathrm{Irr}(G^{\mathrm{F}})$ is *generic* if $\mathrm{Hom}_{U^{\mathrm{F}}}(\pi, \psi) \neq 0$, for some generic character ψ of U^{F}. We say that π is *ψ-generic* if we want to specify ψ.

If $\omega \in \ker j_G$ and $\rho \in \mathrm{Irr}(C_\varphi, \omega)$, we say the class $\pi(\varphi, \rho) \in \Pi(\varphi, \omega)$ is *generic* if some (equivalently, every) representation in $\pi(\varphi, \rho)$ is generic.

Generic characters and representations for finite reductive groups are defined similarly.

6.1. **Depth-zero generic characters and representations.** This first section of this chapter concerns all generic depth-zero supercuspidal representations, not just those arising in our L-packets.

Given a hyperspecial vertex $x \in \mathcal{A}_{ad}^{\vartheta}$, set $U_x := U \cap G_x$, $U_x^+ := U \cap G_x^+$. The quotient $\mathsf{U}_x := U_x/U_x^+$ is the unipotent radical of an F-stable Borel subgroup of G_x. We say that a character $\psi : U^{\mathrm{F}} \longrightarrow \mathbb{C}^\times$ has *depth-zero at x* if the restriction of ψ to U_x^{F} factors through a generic character ψ_x of $\mathsf{U}_x^{\mathrm{F}}$. Note that a depth-zero character at x is automatically generic for U^{F}, since x is hyperspecial. Moreover, any generic character ψ_x of $\mathsf{U}_x^{\mathrm{F}}$ arises from some ψ having depth-zero at x (using, for example, [27, 24.12]).

Let $\kappa^\circ \in \mathrm{Irr}(G_x^{\mathrm{F}})$ be the inflation of an irreducible cuspidal representation of $\mathsf{G}_x^{\mathrm{F}}$, and let κ be an extension of κ° to $Z^{\mathrm{F}} G_x^{\mathrm{F}}$. In this chapter, it is convenient to use the notation

$$\pi(x, \kappa) := \mathrm{ind}_{Z^{\mathrm{F}} G_x^{\mathrm{F}}}^{G^{\mathrm{F}}} \kappa \tag{24}$$

for the compactly induced representation of G^{F}. Since x is hyperspecial, the normalizer of G_x^{F} in G^{F} is $Z^{\mathrm{F}} G_x^{\mathrm{F}}$, so [44, 6.6] implies that $\pi(x, \kappa)$ is an irreducible depth-zero supercuspidal representation of G^{F}.

Lemma 6.1.1. *Let $x \in \mathcal{A}_{ad}^{\vartheta}$ be a hyperspecial vertex, let ψ be a character of U^{F} having depth-zero at x, and let ψ_x be the corresponding generic character of $\mathsf{U}_x^{\mathrm{F}}$ as above. Assume that κ° is ψ_x-generic. Then $\pi(x, \kappa)$ is ψ-generic.*

Proof. This follows from Frobenius reciprocity: Let $V \subset \pi(x, \kappa)$ be the space of functions supported on $Z^{\mathrm{F}} G_x^{\mathrm{F}} U^{\mathrm{F}}$. Then $V \simeq \mathrm{ind}_{U_x^{\mathrm{F}}}^{U^{\mathrm{F}}} \kappa$ as representations of U^{F}, and V is a U^{F}-stable direct summand of $\pi(x,\kappa)$. We have

$$\begin{aligned} 0 \neq \mathrm{Hom}_{U_x^{\mathrm{F}}}(\kappa, \psi_x) &= \mathrm{Hom}_{U^{\mathrm{F}}}(\mathrm{ind}_{U_x^{\mathrm{F}}}^{U^{\mathrm{F}}} \kappa, \psi) \\ &= \mathrm{Hom}_{U^{\mathrm{F}}}(V, \psi) \\ &\subset \mathrm{Hom}_{U^{\mathrm{F}}}(\pi(x,\kappa), \psi). \end{aligned}$$

□

The next result shows that all generic depth-zero supercuspidals are of the form $\pi(x,\kappa)$ as constructed in (24) above.

Lemma 6.1.2. *Let ψ be a generic character of U^{F}, and let π be an irreducible supercuspidal depth-zero ψ-generic representation of G^{F}. Then there is a hyperspecial vertex $x \in \mathcal{A}_{ad}^{\vartheta}$ and a cuspidal representation κ° of $\mathsf{G}_x^{\mathrm{F}}$ (which we inflate to a representation of G_x^{F}), such that the following hold.*

1. *ψ has depth-zero at x, and κ° is a ψ_x-generic representation of $\mathsf{G}_x^{\mathrm{F}}$.*
2. *There is an an extension of κ° to a representation κ of $Z^{\mathrm{F}} G_x^{\mathrm{F}}$, such that $\pi \simeq \pi(x,\kappa)$.*

Proof. From [44, 6.8] there is a vertex $z \in \mathcal{A}_{ad}^{\vartheta}$, a cuspidal representation κ_z of $\mathsf{G}_z^{\mathrm{F}}$, and a representation $\dot\kappa_z$ of the normalizer $\dot G_z^{\mathrm{F}}$ of G_z^{F} in G^{F} such that κ_z appears in $\dot\kappa_z|_{G_z^{\mathrm{F}}}$ and $\pi \simeq \mathrm{ind}_{\dot G_z^{\mathrm{F}}}^{G^{\mathrm{F}}} \dot\kappa_z$.

We may assume that z is contained in the closure of our fixed alcove $jC^{\vartheta} \subset \mathcal{A}_{ad}^{\vartheta}$. Let $\tilde\Phi$ be the set of affine roots of $\mathbf{S}$ in $\mathbf{G}$. For any point y in the closure of jC^{ϑ}, we set

$$\begin{aligned} \tilde\Phi_y &:= \{\tilde\alpha \in \tilde\Phi :\ \tilde\alpha(y) = 0\}, \\ \tilde\Phi_y^+ &:= \{\tilde\alpha \in \tilde\Phi_y :\ \tilde\alpha|_C > 0\}. \end{aligned}$$

Then $\tilde\Phi_y$ is a spherical root system, and $\tilde\Phi_y^+$ is a set of positive roots in $\tilde\Phi_y$. We let $\tilde\Pi_y$ be the unique base of $\tilde\Phi_y$ contained in $\tilde\Phi_y^+$.

Let $\Phi_y,\ \Phi_y^+,\ \Pi_y$ be the respective sets of gradients of the affine roots in $\tilde\Phi_y,\ \tilde\Phi_y^+,\ \tilde\Pi_y$. Each of these sets lies in Φ_o, a set upon which W_o^{ϑ} acts. The roots in Π_y are non-divisible in Φ_y, and form a base of the reduced root system consisting of non-divisible roots in Φ_y.

Let

$${}_zW_o^{\vartheta} := \{w \in W_o^{\vartheta} :\ w^{-1}\Pi_z \subset \Phi_o^+\}.$$

Since π is ψ-generic and is a quotient of $\mathrm{ind}_{G_z^{\mathrm{F}}}^{G^{\mathrm{F}}} \kappa_z$, we have

$$\mathrm{Hom}_{G^{\mathrm{F}}}(\mathrm{ind}_{G_z^{\mathrm{F}}}^{G^{\mathrm{F}}} \kappa_z, \mathrm{Ind}_{U^{\mathrm{F}}}^{G^{\mathrm{F}}} \psi) \neq 0.$$

As in the proof of [47, Lemma 4], this implies that there exists $n \in N^{\mathrm{F}}$ whose image $v \in W_o^{\vartheta}$ belongs to ${}_zW_o^{\vartheta}$ and such that $n_*\psi|_{G_z^+ \cap {}^nU^{\mathrm{F}}}$ is trivial, while $n_*\psi|_{G_z \cap {}^nU^{\mathrm{F}}}$ appears in $\kappa_z|_{G_z \cap {}^nU^{\mathrm{F}}}$.

By [47, Lemma 2], the image $\mathsf{V}_z^{\mathrm{F}}$ of $G_z \cap {}^nU^{\mathrm{F}}$ in $\mathsf{G}_z^{\mathrm{F}}$ is the maximal unipotent subgroup of $\mathsf{G}_z^{\mathrm{F}}$ generated by root groups $\mathsf{U}_\beta^{\mathrm{F}}$ for $\beta \in \Pi_z$. Let θ be the character of $\mathsf{V}_z^{\mathrm{F}}$ obtained from the restriction of ${}^n\psi$ to $G_z \cap {}^nU^{\mathrm{F}}$. We have seen that θ appears in $\kappa_z|_{\mathsf{V}_z^{\mathrm{F}}}$.

We claim that $v^{-1}\Pi_z \subset \Pi_o$. Suppose not, and choose $\beta \in \Pi_z$ such that $v^{-1}\beta \in \Phi_o^+ \smallsetminus \Pi_o$. Then the root group $U^{\mathrm{F}}_{v^{-1}\beta}$ is contained in the kernel of ψ, so θ is trivial on the simple root group $\mathsf{U}^{\mathrm{F}}_\beta$ in $\mathsf{V}^{\mathrm{F}}_z$. This contradicts the cuspidality of κ_z. So $v^{-1}\Pi_z \subset \Pi_o$, hence in fact $v^{-1}\Pi_z = \Pi_o$, since $|\Pi_z| = |\Pi_o| = \dim \mathcal{A}^\vartheta_{ad}$.

We have shown, moreover, that for each $\alpha \in \Pi_o$, the character ${}^n\psi$ is trivial on $G_z^+ \cap U^{\mathrm{F}}_{v\alpha}$, and nontrivial on $G_z \cap U^{\mathrm{F}}_{v\alpha}$. Hence ψ is trivial on $G^+_{n^{-1}\cdot z} \cap U^{\mathrm{F}}_\alpha$ and nontrivial on $G_{n^{-1}\cdot z} \cap U^{\mathrm{F}}_\alpha$.

It now suffices to prove that the vertex z is hyperspecial. For then the previous paragraph shows that ψ has depth-zero at $x := n^{-1}\cdot z$, and taking $\kappa^\circ := \mathrm{Ad}(n^{-1})_*\kappa_z$, $\kappa = \mathrm{Ad}(n^{-1})_*\dot\kappa_z$ will satisfy the conclusions of the lemma.

Since $v\Pi_o = \Pi_z$, it is clear that z is special, but not immediately clear that it is hyperspecial. Let $\alpha \in \Pi_o$. Since $v\alpha \in \Pi_z$, there is $k_\alpha \in \mathbb{Z}$ such that $v\alpha - k_\alpha \in \tilde\Pi_z$. It follows that

$$z = \prod_{\alpha\in\Pi_o} t_{k_\alpha v\lambda_\alpha}\cdot o,$$

where $\{\lambda_\alpha :\ \alpha \in \Pi_o\} \subset X_{ad}$ is the dual basis of Π_o. Hence $z = t\cdot o$, for an element $t \in T^{\mathrm{F}}_{ad}$. Since $\mathrm{Ad}(t)$ is a k-rational automorphism of G^{F}, it follows that z is hyperspecial. □

6.2. **Generic representations in our L-packets.** Fix a TRSELP φ with corresponding $w \in W_o$. We identify

$$\mathrm{Irr}(C_\varphi) = H^1(\mathrm{F}_w, T) = [X/(1-w\vartheta)X]_{\mathrm{tor}}.$$

We likewise identify

$$H^1(\mathrm{F}_w, T_{ad}) = X_{ad}/(1-w\vartheta)X_{ad}.$$

(Note that the latter group is finite.) For $\lambda \in X_w$, let ρ_λ denote the image of λ in $H^1(\mathrm{F}_w, T)$, and $\rho_{j\lambda}$ the image of $j\lambda$ in $H^1(\mathrm{F}_w, T_{ad})$. Then $\rho_{j\lambda} = j_w(\rho_\lambda)$, where

$$j_w : H^1(\mathrm{F}_w, T) \longrightarrow H^1(\mathrm{F}_w, T_{ad})$$

is the map induced by the map $j : G \longrightarrow G_{ad}$. Recall that x_λ is the unique fixed-point of $t_\lambda w\vartheta$ in $\mathcal{A}_{ad}$.

Lemma 6.2.1. *For $\lambda \in X_w$, the following are equivalent.*

1. $\rho_{j\lambda} = 1$*;*
2. *the vertex x_λ is hyperspecial;*
3. *the representation π_λ of Section 4.5 is generic.*

Proof. The representations κ_λ are generic, by [21, 3.10]. The equivalence of 2 and 3 now follows from Lemmas 6.1.1 and 6.1.2.

To prove the equivalence of 1 and 2, recall that x_λ is defined by the relation

$$(1-w\vartheta)x_\lambda = t_{j\lambda}\cdot o.$$

Now x_λ is hyperspecial iff $x_\lambda \in X_{ad}\cdot o$, iff $j\lambda \in (1-w\vartheta)X_{ad}$, iff $\rho_{j\lambda} = 1$. □

For $\omega \in \ker j_G$, we set

$$\mathrm{Irr}(C_\varphi,\omega)_{gen} := \{\rho \in \mathrm{Irr}(C_\varphi,\omega) :\ \pi(\varphi,\rho) \text{ is generic}\}.$$

Lemma 6.2.2. *For $\omega \in \ker j_G$, we have*

$$|\operatorname{Irr}(C_\varphi, \omega)_{gen}| = [X_{ad}^\vartheta : j(X^\vartheta)].$$

In particular, the number of generic representations in $\Pi(\varphi, \omega)$ is independent of the TRSELP φ and the class $\omega \in \ker j_G$.

Proof. We give the proof assuming that $p \nmid [X_{ad} : jX]$. The argument for general p is more complicated (see [19]). In this proof only, we change notation and let $\mathbf{Z}$ denote the full center of $\mathbf{G}$, and set $Z = G \cap \mathbf{Z}$. We have a diagram of group homomorphisms

$$\begin{array}{ccccc}
H^1(k, \mathbf{Z}) & \xrightarrow{\iota} & H^1(\mathrm{F}, G) & \xrightarrow{j_G} & H^1(\mathrm{F}, G_{ad}) \\
\shortparallel & & r \uparrow & & \\
H^1(k, \mathbf{Z}) & \xrightarrow{\iota_w} & H^1(\mathrm{F}_w, T) & \xrightarrow{j_w} & H^1(\mathrm{F}_w, T_{ad}) \\
& & \shortparallel & & \\
& & \operatorname{Irr}(C_\varphi) & &
\end{array}$$

induced by the inclusions $\mathbf{Z} \hookrightarrow \mathbf{T} \hookrightarrow \mathbf{G}$, the adjoint map $j : G \longrightarrow G_{ad}$, and $\operatorname{Ad}(p_0) : T \longrightarrow G$, where $p_0^{-1}\,\mathrm{F}(p_0) = \dot{w}$ (see Section 2.7). The rows are exact at the middle term [51, Prop.38], and $\iota = r \circ \iota_w$. Recall that $r^{-1}(\omega) = \operatorname{Irr}(C_\varphi, \omega)$. We prove the result by computing $|\ker \iota|$ in two ways.

We have $\ker \iota_w \subseteq \ker \iota$, so the ι-fibers are unions of ι_w-fibers. From Lemma 6.2.1 it follows that

$$\iota_w(\iota^{-1}(\omega)) = \operatorname{Irr}(C_\varphi, \omega)_{gen}.$$

This implies that

$$|\ker \iota| = |\iota^{-1}(\omega)| = |\operatorname{Irr}(C_\varphi, \omega)_{gen}| \cdot |\ker \iota_w|.$$

Now

$$\ker \iota_w \simeq T_{ad}^{\mathrm{F}_w}/j(T^{\mathrm{F}_w}),$$

and we have

$$T_{ad}^{\mathrm{F}_w} = X_{ad}^{w\vartheta} \times ({}^0T_{ad})^{\mathrm{F}_w}, \quad j(T^{\mathrm{F}_w}) = j(X^{w\vartheta}) \times j({}^0T^{\mathrm{F}_w}).$$

Since $X_{ad}^{w\vartheta} = \{0\}$, it follows that

$$|\ker \iota| = |\operatorname{Irr}(C_\varphi, \omega)_{gen}| \cdot |({}^0T_{ad})^{\mathrm{F}_w}/j({}^0T^{\mathrm{F}_w})|. \tag{25}$$

On the other hand, we have

$$|\ker \iota| = |G_{ad}^{\mathrm{F}}/j(G^{\mathrm{F}})|.$$

Since $\mathbf{G}$ is quasi-split, [8, 5.6] implies that the inclusion $T_{ad} \hookrightarrow G_{ad}$ induces a bijection

$$T_{ad}^{\mathrm{F}}/j(T^{\mathrm{F}}) \xrightarrow{\sim} G_{ad}^{\mathrm{F}}/j(G^{\mathrm{F}}).$$

Since $T_{ad}^{\mathrm{F}} = X_{ad}^\vartheta \times ({}^0T_{ad})^{\mathrm{F}}$, we have

$$T_{ad}^{\mathrm{F}}/j(T^{\mathrm{F}}) = [X_{ad}^\vartheta/j(X^\vartheta)] \times ({}^0T_{ad})^{\mathrm{F}}/j({}^0T^{\mathrm{F}}),$$

so

$$|\ker \iota| = |X_{ad}^\vartheta/j(X^\vartheta)| \cdot |({}^0T_{ad})^{\mathrm{F}}/j({}^0T^{\mathrm{F}})|. \tag{26}$$

Comparing Equations (25) and (26), the proof boils down to showing that

$$|({}^0T_{ad})^{\mathrm{F}_w}/j({}^0T^{\mathrm{F}_w})| = |({}^0T_{ad})^{\mathrm{F}}/j({}^0T^{\mathrm{F}})|. \tag{27}$$

If $p \nmid [X_{ad} : jX]$, then $jX \otimes R_K^\times = X_{ad} \otimes R_K^\times$, so we have an exact sequence

$$1 \longrightarrow {}^0T \cap Z \longrightarrow {}^0T \xrightarrow{\ j\ } {}^0T_{ad} \longrightarrow 1.$$

Since $H^1(\mathrm{F}_w, {}^0T) = H^1(\mathrm{F}, {}^0T) = 1$ and w acts trivially on Z, it follows that both sides of Equation (27) are equal to $|H^1(\mathrm{F}, {}^0T \cap Z)|$. □

It follows from [60, 2.5] that $[X_{ad}^\vartheta : j(X^\vartheta)]$ is the number of G^{F_u}-orbits of hyperspecial vertices in $\mathcal{B}(G^{\mathrm{F}_u})$. Lemma 6.2.2 leads one to expect that each of these orbits supports a unique generic representation in $\Pi_u(\varphi)$. We will prove this in a few steps, as follows.

Lemma 6.2.3. *Let F_u be a quasi-split Frobenius, and let S be an F_u-minisotropic torus in G. Assume that the unique fixed-point x of S^{F_u} in $\mathcal{B}(G_{ad})^{\mathrm{F}_u}$ is hyperspecial. Then*

$$N(G^{\mathrm{F}_u}, S)/S^{\mathrm{F}_u} = N(G, S^{\mathrm{F}_u})/S.$$

Proof. Let $n \in N(G, S^{\mathrm{F}_u}) \subset N(G, S)$. Since x is hyperspecial and is contained in the apartment of S in $\mathcal{B}(G_{ad})$, we have $N(G,S) = N(G_x, S)S$, so we may assume $n \in N(G_x, S^{\mathrm{F}_u})$. Then $\mathrm{F}_u(n) = nt$ for some $t \in S \cap G_x = {}^0S$. Choose $d \geq 1$ such that $\mathrm{F}_u^d(n) = n$. If $d = 1$ there is nothing to prove, so assume $d > 1$. This implies that $t\,\mathrm{F}_u(t)\cdots\mathrm{F}_u^{d-1}(t) = 1$. By Lemma 2.3.1 there is $s \in {}^0T$ such that $t = s\,\mathrm{F}_u(s)^{-1}$, so that $\mathrm{F}_u(ns) = ns$. □

Returning to the notation of Section 4.6, let $u \in \omega \in \ker j_G$, and suppose $\lambda, \mu \in r^{-1}(\omega)$ are such that $\rho_\lambda, \rho_\mu \in \ker j_w$. It follows from Lemma 6.2.1 that $v_\lambda := m_\lambda \cdot x_\lambda$ and $v_\mu := m_\mu \cdot x_\mu$ are hyperspecial vertices in $\mathcal{A}_{ad}^{\mathrm{F}_u}$. The representations $\pi_u(\varphi, \rho_\lambda)$ and $\pi_u(\varphi, \rho_\mu)$ are induced from the stabilizers in G^{F_u} of v_λ and v_μ, respectively.

Lemma 6.2.4. *Assume that v_λ and v_μ are G^{F_u}-conjugate hyperspecial vertices. Then $\rho_\lambda = \rho_\mu$.*

Proof. We first prove that S_λ and S_μ are G^{F_u}-conjugate. Since $G^{\mathrm{F}_u} = G_{v_\lambda}^{\mathrm{F}_u} N^{\mathrm{F}_u} G_{v_\mu}^{\mathrm{F}_u}$, there is $n \in N^{\mathrm{F}_u}$ such that $n \cdot v_\mu = v_\lambda$. The F_u-minisotropic tori

$$S_1 := S_\lambda, \qquad S_2 := {}^nS_\mu$$

both have v_λ as their unique fixed-point in $\mathcal{B}(G_{ad})^{\mathrm{F}_u}$. Let T and S_i be the images of $T \cap G_{v_\lambda}$ and $S_i \cap G_{v_\lambda}$, respectively, in G_{v_λ}.

Set

$$k_1 := q_\lambda m_\lambda^{-1}, \qquad k_2 := nq_\mu m_\mu^{-1} n^{-1}.$$

Then $k_i \in G_{v_\lambda}$ and $S_i = \mathrm{Ad}(k_i)T$ for $i = 1, 2$. Let $\bar{k}_i$ be the image of k_i in G_{v_λ}, so that $\mathsf{S}_i = \mathrm{Ad}(\bar{k}_i)\mathsf{T}$.

Using Equation (10) we find that

$$\begin{aligned} k_1^{-1}\,\mathrm{F}_u(k_1) &\equiv m_\lambda \cdot wu^{-1} \cdot \mathrm{F}_u(m_\lambda)^{-1} \mod T,\\ k_2^{-1}\,\mathrm{F}_u(k_2) &\equiv nm_\mu \cdot wu^{-1} \cdot \mathrm{F}_u(nm_\mu)^{-1} \mod T. \end{aligned}$$

Since v_λ is hyperspecial, every class in N/T has a representative in $N \cap G_{v_\lambda}$. Applying this to $m_\lambda T$, $nm_\mu T$ and $wu^{-1}T$, it follows that $\bar{k}_1^{-1}\,\mathrm{F}_u(\bar{k}_1)$ and $\bar{k}_2^{-1}\,\mathrm{F}_u(\bar{k}_2)$ are F_u-conjugate in the Weyl group of T in G_{v_λ}. This means (c.f. [12, 3.3.3]) that S_1 and S_2 are $\mathsf{G}_{v_\lambda}^{\mathrm{F}_u}$-conjugate. The uniqueness part of Lemma 8.0.10 then implies that S_1 and S_2 are G^{F_u}-conjugate. Hence S_λ and S_μ are G^{F_u}-conjugate, as claimed.

By Lemma 2.11.1 there is $z_o \in W_o^{w\vartheta}$ such that

$$\lambda \equiv z_o\mu \mod (1 - w\vartheta)X.$$

But Lemmas 6.2.3 and 2.11.2 imply that

$$W_o^{w\vartheta} = W_{o,\mu}^{w\vartheta}.$$

Hence $\lambda \equiv \mu \mod (1 - w\vartheta)X$, so $\rho_\lambda = \rho_\mu$. $\square$

Remark 6.2.5. Lemma 6.2.3 and the last step in the above proof can be seen in another way, as follows. Since v_λ is hyperspecial, Lemma 6.2.2 implies that $\rho_\lambda \in \ker j_w = \operatorname{im} i_w : [H^1(k, \mathbf{Z}) \to H^1(\mathrm{F}_w, T)]$. Since $W_o^{w\vartheta}$ acts trivially on $H^1(k, \mathbf{Z})$, it follows that ρ_λ is a $W_o^{w\vartheta}$-fixed point in $H^1(\mathrm{F}_w, T)$.

Combining Lemmas 6.2.2 and 6.2.4 yields the promised result:

Corollary 6.2.6. *There is a bijection between the set of generic representations in $\Pi_u(\varphi)$ and the set of G^{F_u}-orbits of hyperspecial vertices in $\mathcal{B}(G_{ad})^{\mathrm{F}_u}$, such that a generic representation is induced from the stabilizer of any hyperspecial vertex in the corresponding orbit.*

Remark 6.2.7. If $\mathbf{G}$ has connected center, then G_x has connected center for any hyperspecial vertex $x \in \mathcal{B}(G)$. Assume x is F_u-stable. It follows from Proposition 5.26 and Theorems 6.8 and 10.7 of [20] that every cuspidal generic representation of G_x is of the form $\pm R_{\mathsf{S},\theta}^{\mathsf{G}_x}$ for some F_u-minisotropic maximal torus $\mathsf{S} \subset \mathsf{G}_x$ and $\theta \in \mathrm{Irr}(\mathsf{S}^{\mathrm{F}_u})$ in general position. Using Lemma 6.1.2, this implies that every depth-zero generic supercuspidal representation of G^{F_u} appears in $\Pi_u(\varphi)$ for some TRSELP φ.

7. Topological Jordan decomposition

We define the set of *compact elements* in G by

$$G_0 := \bigcup_{x \in \mathcal{B}(G)} G_x,$$

and the set of *topologically unipotent elements* in G by

$$G_{0^+} = \bigcup_{x \in \mathcal{B}(G)} G_x^+.$$

We define $\mathfrak{g}_0$ and $\mathfrak{g}_{0^+}$ similarly. These $G \rtimes \langle F \rangle$-stable subsets of G will play an important role in this paper.

Remark 7.0.8. From [16] we have that if $x \in \mathcal{B}(G)$, then

$$G_0 \cap \operatorname{stab}_G(x) = G_x.$$

Let p denote the characteristic of $\mathfrak{f}$. Choose m such that for all F-stable facets J in $\mathcal{B}(G)$ and all elements $g \in \mathsf{G}_J^F$ we have $g^{(p^m)} = s$ where s denotes the semisimple component in the Jordan decomposition of g.

Suppose $\gamma \in G_0^F$. Let $J \subset \mathcal{B}(G)$ be any F-stable facet such that $\gamma \in G_J$. Since $\gamma \in G_J^F$, it follows that we can define

$$\gamma_s := \lim_{n \to \infty} \gamma^{(p^{mn})}.$$

This limit does not depend on m, and the element γ_s has finite order prime to p. We set

$$\gamma_u = \gamma \cdot \gamma_s^{-1}.$$

The *topological Jordan decomposition* is the commuting factorization

$$\gamma = \gamma_s \gamma_u = \gamma_u \gamma_s.$$

We have $\gamma_s,\ \gamma_u \in G_J^F$. Moreover, γ_s is semisimple and has semisimple image in G_J while γ_u has unipotent image in G_J. In particular, γ_u is topologically unipotent. We say that γ is *topologically semisimple* if $\gamma = \gamma_s$, that is, if $\gamma = \gamma^{p^m}$.

The topological Jordan decomposition $\gamma = \gamma_s \gamma_u$ is the unique commuting factorization of γ as a product of a topologically semisimple element and a topologically unipotent element. This implies that if $g \in \mathbf{G}$ is chosen so that ${}^g\gamma \in G^F$, then ${}^g(\gamma_s) = ({}^g\gamma)_s$ and ${}^g(\gamma_u) = ({}^g\gamma)_u$.

Lemma 7.0.9. *Suppose $\gamma \in G_0^F$ has topological Jordan decomposition $\gamma = \gamma_s \gamma_u$. Then γ, γ_s, and γ_u all belong to G_{γ_s}. Moreover, if $\gamma \in G^{rss}$, then $\gamma_u \in G_{\gamma_s}^{rss}$.*

Proof. Choose a Borel subgroup $\mathbf{B} < \mathbf{G}$ containing γ. Since $\mathbf{B} \cap G$ is a closed subgroup of G, both γ_s and γ_u belong to $\mathbf{B} \cap G$. Since γ_s is semisimple, it follows from [6, Theorem 10.6 (5ii)] that the centralizer in $\mathbf{B}$ of γ_s is connected. Thus, γ, γ_s, and γ_u belong to $\mathbf{B}_{\gamma_s} \cap G \subset G_{\gamma_s}$.

The centralizer of γ_u in G_{γ_s} has finite index in the centralizer of γ in G. This implies the last assertion. $\square$

Since γ_s is compact and has finite order prime to p, the results of [46] combined with Remark 7.0.8 allow us to identify

$$\mathcal{B}(G_{\gamma_s}) = \mathcal{B}(G)^{\gamma_s}. \tag{28}$$

More precisely, there is an unramified maximal torus S of G containing γ_s, and a bijection from the apartment of S in $\mathcal{B}(G_{\gamma_s})$ to the apartment of S in $\mathcal{B}(G)$ which extends to a G_{γ_s}-equivariant bijection $\mathcal{B}(G_{\gamma_s}) \xrightarrow{\sim} \mathcal{B}(G)^{\gamma_s}$. In particular, $\mathbf{G}_{\gamma_s}$ and $\mathbf{G}$ have the same K-rank.

For an exhaustive treatment of the topological Jordan decomposition, see [52].

8. UNRAMIFIED AND MINISOTROPIC MAXIMAL TORI

Recall that we are assuming $\mathbf{G}$ is K-split, and that we say a subgroup $S < G$ is a *maximal unramified torus in* G if $S = \mathbf{S}(K)$, where $\mathbf{S}$ is a K-split maximal torus in $\mathbf{G}$ such that $\mathbf{S}$ is defined over k.

All maximal unramified tori in G can be found as follows.

Lemma 8.0.10. *Suppose we are given a nonempty F-stable subset J of a facet in $\mathcal{B}(G)$ or $\mathcal{B}(G_{ad})$ and an F-stable maximal torus $\mathsf{S} < \mathsf{G}_J$. Then there exists a maximal unramified torus S in G such that*

(1) $J \subset \mathcal{A}(S)$;
(2) *the image of* $S \cap G_J$ *in* G_J *is exactly* S.

Moreover, S is unique up to conjugacy by G_J^{+F}.

Proof. The existence of S is shown in the proof of [10, 5.1.10]. The uniqueness is proved in [14, Lemma 2.2.2]. □

Such an S is called a *lift* of (J, S).

A maximal unramified torus S in G is called F*-minisotropic in* G if $X_*(\mathbf{S})^F = X_*(\mathbf{Z})$, where $\mathbf{Z}$ is the identity component of the maximal k-split torus in the center of $\mathbf{G}$.

Likewise, a maximal $\mathfrak{f}$-torus S in a reductive $\mathfrak{f}$-group G with Frobenius F is called F*-minisotropic in* G if $X_*(\mathsf{S})^F = X_*(\mathsf{Z})$, where Z is the maximal $\mathfrak{f}$-split torus the center of G.

Let $\mathfrak{T}(G)$ be the set of F-minisotropic maximal tori in G. If $S \in \mathfrak{T}(G)$, then there exists a unique F-stable facet $J \subset \mathcal{B}(G)$ such that

$$\mathcal{A}(S)^F = J^F.$$

The unique parahoric subgroup ${}^0S$ of S is given by

$${}^0S = S \cap G_J.$$

Note that $N(G,S)^F$ preserves $\mathcal{A}(S)^F$, hence normalizes G_J^F and G_J^{+F}. In particular, $G_J^{+F} N(G,S)^F$ is a subgroup of G^F.

Let S be the image of $S \cap G_J$ in G_J. Then S is an F-minisotropic torus in G_J, and S is a lift of (J, S).

Fix now $S \in \mathfrak{T}(G)$ and a topologically semisimple element $\gamma \in G_0^F$. For our later integral calculations we must consider the two sets

$$E(\gamma, S) := \{g \in G^F : {}^g\gamma \in G_J, \quad \overline{{}^g\gamma} \in \mathsf{S}\},$$
$$\tilde{D}(\gamma, S) := \{d \in G^F : {}^d\gamma \in S\}.$$

In other terms, $\tilde{D}(\gamma, S)$ is the set of elements of G^F which conjugate S into G_γ, and $E(\gamma, S)$ is the set of elements of G^F which send some G_J^{+F}-conjugate of S into G_γ and whose inverse sends J into $\mathcal{B}(G_\gamma)$. Since $\gamma \in G_0^F$, we have $\tilde{D}(\gamma, S) \subset E(\gamma, S)$.

We have obvious actions, by multiplication, of $G_J^{+F} N(G,S)^F \times G_\gamma^F$ on $E(\gamma, S)$, and of $N(G,S)^F \times G_\gamma^F$ on $\tilde{D}(\gamma, S)$.

Lemma 8.0.11. *The inclusion $\tilde{D}(\gamma, S) \hookrightarrow E(\gamma, S)$ induces a bijection*

$$N(G,S)^F \backslash \tilde{D}(\gamma, S)/G_\gamma^F \xrightarrow{\sim} G_J^{+F} N(G,S)^F \backslash E(\gamma, S)/G_\gamma^F.$$

Both sets of double cosets are finite.

Proof. The set $N(G,S)^F \backslash \tilde{D}(\gamma, S)/G_\gamma^F$ parametrizes G_γ^F-conjugacy classes of F-minisotropic tori in G_γ which lie in the G^F-conjugacy class of S. Since G_γ^F has only finitely many conjugacy classes of unramified maximal tori, the set $N(G,S)^F \backslash \tilde{D}(\gamma, S)/G_\gamma^F$ is finite.

We now prove injectivity. Suppose we have $d, d' \in \tilde{D}(\gamma, S)$, and $h \in G_J^{+F}$, $n \in N(G,S)^F$, $g \in G_\gamma^F$, such that $d' = nhdg$. Replacing d' by $n^{-1}d'g^{-1}$, we may assume without loss of generality that $d' = hd$. This means that ${}^d\gamma$ and ${}^{hd}\gamma$ both belong to S, and being compact, ${}^d\gamma$ and ${}^{hd}\gamma$ in fact belong to $S \cap G_J$. Since $h \in G_J^{+F}$, both ${}^d\gamma$ and ${}^{hd}\gamma$ have the same image in S. Hence we can write ${}^{hd}\gamma = {}^d\gamma\gamma_1$, where $\gamma_1 \in G_J^{+F} \cap S$ is topologically unipotent. But then ${}^d\gamma\gamma_1 = \gamma_1{}^d\gamma$, and since ${}^{hd}\gamma$ is topologically semisimple, we must have $\gamma_1 = 1$, by uniqueness of the topological Jordan decomposition. It follows that S and $h^{-1}Sh$ are two lifts of (S, J) in ${}^dG_{\gamma_s}$. By Lemma 8.0.10, there is $k \in ({}^dG_{\gamma_s})_J^{+F}$ such that $kSk^{-1} = h^{-1}Sh$. This implies that $h \in N(G,S)^F \cdot {}^d(G_{\gamma_s}^F)$, proving injectivity.

For surjectivity, suppose $g \in E(\gamma, S)$, and let $H = {}^gG_\gamma$. Then ${}^g\gamma$ fixes J pointwise, so by Equation (28), J is contained in a facet in the building $\mathcal{B}(H)$ of H. We let H_J denote the corresponding parahoric subgroup of H. Then ${}^g\gamma \in H_J$.

Considering root data, we find a $\mathfrak{f}$-isomorphism $\iota : (\mathsf{G}_J)_{\overline{{}^g\gamma}} \xrightarrow{\sim} \mathsf{H}_J$ making the following diagram commutative.

$$\begin{array}{ccc} H \cap G_J & = & H_J \\ \downarrow & & \downarrow \\ (\mathsf{G}_J)_{\overline{{}^g\gamma}} & \xrightarrow{\iota} & \mathsf{H}_J \end{array}.$$

We have $\mathsf{S} < (\mathsf{G}_J)_{\overline{{}^g\gamma}}$ by hypothesis, hence $\iota\mathsf{S}$ is an F-stable maximal torus in H_J. Choose a lift S' in H of $(J, \iota\mathsf{S})$. Then $S' \cap H_J = S' \cap G_J$ so S' is a lift of (J, S) in G. But S is also a lift of (J, S) in G, so by 8.0.10 there is $k \in G_J^{+F}$ such that ${}^kS' = S$. Since ${}^g\gamma \in S'$, we have ${}^{kg}\gamma \in S$. This means $kg \in \tilde{D}(\gamma, S)$, proving surjectivity. □

9. Some character computations

In this chapter we give an integral formula for the characters of the representations constructed in Section 4.4. In fact, we define a set of integrals on G^F which include these characters as a subset. Our eventual goal is to express these integrals as combinations of similar integrals on the set of topologically unipotent elements, in the same way that a Deligne-Lusztig character is expressed as a combination of Green functions.

9.1. **Harish-Chandra's character formula.** Recall that Z denotes the group of K-rational points of the maximal k-split torus in the center of $\mathbf{G}$.

Suppose that Q is an open subgroup of G^F containing Z^F such that Q is compact modulo Z^F. Suppose also that κ is a representation of Q for which the compactly-induced representation $\pi := \operatorname{ind}_Q^{G^F} \kappa$ of G^F is irreducible. Let $\dot{\chi}_\kappa$ denote the extension by zero of the character of κ to a function on G^F. In [26] Harish-Chandra showed that the value of the character of π at $\gamma \in (G^{\mathrm{rss}})^F$ is given by the formula

$$\frac{\operatorname{Deg}(\pi)}{\chi_\kappa(1)} \int_{G^F/Z^F} dg^* \int_L \dot{\chi}_\kappa({}^{gl}\gamma)\, dl.$$

Here dg^* denotes the quotient measure on G^F/Z^F with respect to Haar measures dg and dz on G^F and Z^F, respectively, $\operatorname{Deg}(\pi)$ denotes the formal degree of π with respect to $dg^* = dg/dz$ (see Section 5), and L is an arbitrary compact open subgroup of G^F with Haar measure dl normalized so that $\operatorname{meas}_{dl}(L) = 1$.

9.2. **The character integral.** Let S be an F-minisotropic maximal torus in G, and let J be the unique minimal F-stable facet in $\mathcal{B}(G)$ such that $\mathcal{A}(S)^F = J^F$. Recall that ${}^0S^F = S^F \cap G_J$, and $S \cap G_J$ projects onto an F-minisotropic torus S in G_J.

Let $\operatorname{Irr}_0(S^F)$ denote the set of depth-zero characters of S^F. For $\theta \in \operatorname{Irr}_0(S^F)$, the restriction of θ to ${}^0S^F$ factors through S^F, and thus defines a Deligne-Lusztig virtual character $R^{\mathsf{G}_J}_{\mathsf{S},\theta}$. Let $\dot{R}^{\mathsf{G}_J}_{\mathsf{S},\theta}$ denote the natural inflation of $R^{\mathsf{G}_J}_{\mathsf{S},\theta}$ to a function on G_J^F, extended by zero to the rest of G^F.

Define a function $R(G,S,\theta)$ on $(G^{\mathrm{rss}})^F$ by the integral

$$R(G,S,\theta)(\gamma) := \frac{\operatorname{meas}_{dz}(Z_J^F)}{\operatorname{meas}_{dg}(G_J^F)} \cdot \int_{G^F/Z^F} dg^* \int_L \dot{R}^{\mathsf{G}_J}_{\mathsf{S},\theta}({}^{gl}\gamma)\, dl.$$

Here L and the measures dg^*, dl are as in Section 9.1. (The integral converges; see, for example, Lemma 10.0.7.)

Remark 9.2.1. For $h \in G^F$, a change of variables shows that

$$R(G, {}^hS, h_*\theta) = R(G,S,\theta),$$

where $h_*\theta = \theta \circ \operatorname{Ad}(h)^{-1}$. If $\mathcal{T}$ is a G^F-orbit of pairs (S,θ) with $S \in \mathfrak{T}(G)$ and $\theta \in \operatorname{Irr}_0(S^F)$, we sometimes write

$$R(G,\mathcal{T}) := R(G,S,\theta),$$

for any $(S,\theta) \in \mathcal{T}$.

9.3. **Relation to characters.** Suppose $\theta \in \operatorname{Irr}_0(S^F)$ is *regular*, in the sense that θ has trivial stabilizer in $N(G,S^F)/S$. There is a unique representation κ of $Z^F G_J^F$ such that

(1) the restriction to G_J^F of κ has the character $\varepsilon(\mathsf{G}_J,\mathsf{S}) \cdot \dot{R}^{\mathsf{G}_J}_{\mathsf{S},\theta}$, and
(2) the restriction of κ to Z^F is given by the scalar character $\theta|_{Z^F}$, times the identity.

We have seen that the induced representation $\pi := \operatorname{Ind}_{Z^F G_J^F}^{G^F} \kappa$ is irreducible and supercuspidal.

Lemma 9.3.1. *Let Θ_π be the character of the representation π just defined. Then Θ_π vanishes off the set $Z^F G_0^F$. For $z \in Z^F$ and regular semisimple $\gamma \in G_0^F$ we have*

$$\Theta_\pi(z\gamma) = \varepsilon(\mathsf{G}_J, \mathsf{S}) \cdot \theta(z) \cdot R(G, S, \theta)(\gamma).$$

Proof. Harish-Chandra's integral formula (see Section 9.1) makes the vanishing assertion obvious and gives, for $z \in Z^F$ and regular semisimple $\gamma \in G_0^F$, the formula

$$\Theta_\pi(z\gamma) = \theta(z) \cdot \frac{\mathrm{meas}_{dg}(G_J^F)}{\mathrm{meas}_{dz}(Z_J^F)} \cdot \frac{\mathrm{Deg}(\pi)}{\dot{R}^{\mathsf{G}_J}_{\mathsf{S},\theta}(1)} \cdot R(G, S, \theta)(\gamma).$$

Consequently, we need to show that

$$\frac{\mathrm{meas}_{dg}(G_J^F)}{\mathrm{meas}_{dz}(Z_J^F)} \cdot \frac{\mathrm{Deg}(\pi)}{\dot{R}^{\mathsf{G}_J}_{\mathsf{S},\theta}(1)} = \varepsilon(\mathsf{G}_J, \mathsf{S}).$$

But, from Remark 5.3 we have

$$\mathrm{Deg}(\pi) \cdot \mathrm{meas}_{dg^*}(Z^F G_J^F / Z^F) = \dim(\kappa),$$

and the claim follows. □

9.4. **Stable conjugacy of tori and their characters.** We want to produce a sum of character integrals that will be be stable. In the situation of Section 9.3, these sums will specialize to the sum of characters over an L-packet, as defined in Section 4.6. Our integral sums are based on the notion of stable conjugacy of unramified tori and their characters.

Recall that $\mathfrak{T}(G)$ denotes the set of F-minisotropic maximal tori in G. We say that two tori $S_1, S_2 \in \mathfrak{T}(G)$ are *G-stably conjugate* if there is $g \in G$ such that ${}^g(S_1^F) = S_2^F$. This defines an equivalence relation on $\mathfrak{T}(G)$, whose equivalence classes are called *G-stable classes* . The set of G-stable classes injects into $H^1(F, N/T)$ as follows. Any two maximal unramified tori in G are conjugate by an element of G. For $S \in \mathfrak{T}(G)$, write $S = {}^gT$, for $g \in G$. Since $F(S) = S$, we have an element $n := g^{-1}F(g) \in Z^1(F, N)$. Projecting to N/T gives an element $\bar{n} := g^{-1}F(g)T \in Z^1(F, N/T)$. One checks that the class $[\bar{n}]$ of $\bar{n}$ in $H^1(F, N/T)$ is independent of g. Note that $S^F = {}^g(T^{F_n})$, where, as usual, $F_n = \mathrm{Ad}(n) \circ F$.

Lemma 9.4.1. *Suppose $h \in G$, and $n, m \in N$. Then*

$$h^{-1}mF(h) \in nT \Leftrightarrow {}^h(T^{F_n}) = ({}^hT)^{F_m}.$$

Proof. Implication $\Rightarrow$ is straightforward. For the converse, choose a strongly regular element $t \in T^{F_n}$. From the equation $F_m({}^ht) = {}^ht$, we find that the element $h^{-1}mF(h)n^{-1}$ centralizes t, hence lies in T. □

For $v = nT \in N/T$, set $F_v = F_n$, and define

$$\mathcal{T}_v := \{S \in \mathfrak{T}(G) : \ S^F = {}^g(T^{F_v}) \quad \text{for some } g \in G\}.$$

Lemma 9.4.2. *The sets $\mathcal{T}_v$ have the following properties.*

(1) *If $\mathcal{T}_v$ is nonempty, then $\mathcal{T}_v$ is a G-stable class in $\mathfrak{T}(G)$.*

(2) *Every G-stable class in $\mathfrak{T}(G)$ is of the form $\mathcal{T}_v$ for some $v \in N/T$.*
(3) *For $v, v' \in N/T$, we have $\mathcal{T}_v = \mathcal{T}_{v'}$ if and only if $[v] = [v']$ in $H^1(F, N/T)$.*
(4) *If $\mathbf{G}$ is k-quasi-split, then $\mathcal{T}_v$ is nonempty.*

Proof. See [14]. □

For each $S \in \mathcal{T}_v$, Lemma 9.4.1 implies that there is $g \in G$ such that $S = {}^gT$ and $g^{-1}F(g) \in v$. Note that the choice of g is not uniquely determined by S; two choices of g differ by an element of $N(G, S^F)$. The map $\mathrm{Ad}(g) : T \longrightarrow S$ intertwines (T, F_v) and (S, F). For each depth-zero character $\chi \in \mathrm{Irr}_0(T^{F_v})$, we have a corresponding character $g_*\chi \in \mathrm{Irr}_0(S^F)$, which depends on the choice of g.

This dependence on g is eliminated by passing to a "covering" of $\mathcal{T}_v$, as follows. Consider the set of pairs

$$\hat{\mathfrak{T}}(G) := \{(S, \theta) : \ S \in \mathfrak{T}(G) \quad \text{and} \quad \theta \in \mathrm{Irr}_0(S^F)\}.$$

We say that two pairs $(S_1, \theta_1),\ (S_2, \theta_2) \in \hat{\mathfrak{T}}(G)$ are *G-stably conjugate* if there is $g \in G$ such that

(1) ${}^g(S_1^F) = S_2^F$, and
(2) $g_*\theta_1 = \theta_2$.

The G-stable classes of pairs $(S, \theta) \in \hat{\mathfrak{T}}(G)$ are parametrized as follows. Fix $v \in N/T$, $\chi \in \mathrm{Irr}_0(T^{vF})$, and define

$$\hat{\mathcal{T}}_{v,\chi} := \{(S, \theta) \in \hat{\mathfrak{T}}(G) : \text{ there exists } g \in G \text{ such that } S^F = {}^g(T^{F_v}), \text{ and } \theta = g_*\chi\}.$$

Lemma 9.4.3. (1) *If $\mathcal{T}_v$ is nonempty, then $\hat{\mathcal{T}}_{v,\chi}$ is a nonempty G-stable class in $\hat{\mathfrak{T}}(G)$.*
(2) *Every G-stable class in $\hat{\mathfrak{T}}(G)$ is of the form $\hat{\mathcal{T}}_{v,\chi}$ for some $v \in N/T$, $\chi \in \mathrm{Irr}_0(T^F)$.*
(3) *For $v \in N/T$, $\chi, \chi' \in \mathrm{Irr}_0(T^{F_v})$, we have $\hat{\mathcal{T}}_{v,\chi} = \hat{\mathcal{T}}_{v,\chi'}$ if and only if there is $n \in N(G, T^{F_v})$ such that $n_*\chi = \chi'$.*

Proof. This follows easily from Lemma 9.4.2. □

Thus, we have a partition

$$\hat{\mathfrak{T}}(G) = \coprod_{\substack{v \in N/T \\ \mathcal{T}_v \neq \varnothing}} \ \coprod_{\chi \in \mathrm{Irr}_0(T^{F_v})/N(G, T^{F_v})} \hat{\mathcal{T}}_{v,\chi}$$

of $\hat{\mathfrak{T}}(G)$ into nonempty G-stable classes.

Projection onto the first factor is a surjection $p_1 : \hat{\mathcal{T}}_{v,\chi} \longrightarrow \mathcal{T}_v$. Given $S \in \mathcal{T}_v$ we can project the fiber $p_1^{-1}(S)$ onto the second factor. This gives a map

$$p_2 : p_1^{-1}(S) \longrightarrow \mathrm{Irr}_0(S^F).$$

We define

$$\theta_S^\chi = \sum_{\theta \,\in\, p_2 p_1^{-1}(S)} \theta.$$

To see the dependence on χ, choose g as in the definition of $\hat{\mathcal{T}}_{v,\chi}$ above. Then

$$\theta_S^\chi = \sum_{\bar{n}\in N(G,S^F)/S} (ng)_*\chi,$$

and the sum is independent of the choice of g.

The character sums θ_S^χ have the following stability property.

Lemma 9.4.4. *Suppose $S_1, S_2 \in \mathcal{T}_v$, $\gamma \in S_1^F$, and $\chi \in \mathrm{Irr}_0(T^{F_v})$. Then for any $h \in G$ such that ${}^h(S_1^F) = S_2^F$, we have*

$$\theta_{S_1}^\chi(\gamma) = \theta_{S_2}^\chi({}^h\gamma).$$

Proof. This is immediate from the observation that $h_*[p_2p_1^{-1}(S_1)] = p_2p_1^{-1}(S_2)$. □

9.5. **The stable character integral.** Fix a G-stable class $\hat{\mathcal{T}}_{\mathrm{st}} \subset \hat{\mathfrak{T}}(G)$. The group G^F acts on $\hat{\mathfrak{T}}(G)$ via $g\cdot(S,\theta) = ({}^gS, g_*\theta)$, and $\hat{\mathcal{T}}_{\mathrm{st}}$ is the union of finitely many G^F-orbits in $\hat{\mathfrak{T}}(G)$. By Remark 9.2.1 the function $R(G,S,\theta)$ depends only on the G^F-orbit of (S,θ). We can therefore define a function $R(G,\hat{\mathcal{T}}_{\mathrm{st}})$ on $(G^{\mathrm{rss}})^F$ by

$$R(G,\hat{\mathcal{T}}_{\mathrm{st}}) := \sum_{(S,\theta)\in\hat{\mathcal{T}}_{\mathrm{st}}/G^F} R(G,S,\theta),$$

where $R(G,S,\theta)$ was defined in Section 9.2. Our eventual goal is to show that the function $R(G,\hat{\mathcal{T}}_{\mathrm{st}})$ is stable. But first, we relate $R(G,\hat{\mathcal{T}}_{\mathrm{st}})$ to the sum of characters in an L-packet.

9.6. **Relation to L-packets.** In this section we show that the sum of characters in an L-packet, as defined in Section 4.6, can be expressed, up to a sign, as one of the functions $R(G,\hat{\mathcal{T}}_{\mathrm{st}})$ as defined in Section 9.5. We return to the notation used in Section 4.6 and previously, so that $F = \mathrm{F}_u$. Set $v = \dot{w}u^{-1}T \in N/T$, and let $\chi \in \mathrm{Irr}_0(T^{\mathrm{F}_w})$ be regular. Note that $F_v = \mathrm{F}_w$, and by the proof of Lemma 2.11.2, we may identify

$$N(G,T^{\mathrm{F}_w})/T = W_o^{w\vartheta}.$$

For each $\lambda \in r^{-1}(\omega)$ we have the pair $(S_\lambda,\theta_\lambda) = q_\lambda\cdot(T,\chi) \in \hat{\mathcal{T}}_{v,\chi}$. Recall from Lemma 2.6.1 the commutative diagram

$$\begin{array}{ccc} [X/(1-w\vartheta)X]_{\mathrm{tor}} & \longrightarrow & [\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} \\ \simeq\downarrow & & \downarrow\simeq \\ H^1(\mathrm{F}_w,T) & \xrightarrow{\mathrm{Ad}(p_0)} & H^1(F,G) \end{array}$$

where the vertical maps are bijections. Recall that $[r^{-1}(\omega)]$ denotes the fiber of the map in the top row, and this fiber carries a natural action of $W_o^{w\vartheta}$.

Lemma 9.6.1. *Recall that $v = \dot{w}u^{-1}T$, and $\chi \in \mathrm{Irr}_0(T^{\mathrm{F}_w})$ is regular. The mappings $\lambda \mapsto (S_\lambda, \theta_\lambda)$, $\lambda \mapsto S_\lambda$, respectively, induce bijections*

$$\alpha : [r^{-1}(\omega)] \xrightarrow{\sim} \hat{\mathcal{T}}_{v,\chi}/G^{\mathrm{F}_u}, \qquad \beta : [r^{-1}(\omega)]/W_o^{w\vartheta} \xrightarrow{\sim} \mathcal{T}_v/G^{\mathrm{F}_u},$$

which make the following diagram commute.

$$\begin{array}{ccc} [r^{-1}(\omega)] & \xrightarrow{\alpha} & \hat{\mathcal{T}}_{v,\chi}/G^{\mathrm{F}_u} \\ p\downarrow & & \downarrow \bar{p}_1 \\ \left[r^{-1}(\omega)\right]/W_o^{w\vartheta} & \xrightarrow{\beta} & \mathcal{T}_v/G^{\mathrm{F}_u} \end{array}$$

Here p is the quotient map and $\bar{p}_1$ is induced by the projection p_1 onto the first factor.

Proof. The map β is well-defined and bijective, by Lemma 2.11.1.

If $\lambda, \mu \in r^{-1}(\omega)$ are congruent modulo $(1 - w\vartheta)X$, then from the proof of Lemma 2.10.1 there exists $s \in S_\lambda$ such that $q_\mu q_\lambda^{-1} s \in G^{\mathrm{F}_u}$. Since

$$q_\mu q_\lambda^{-1} s \cdot (S_\lambda, \theta_\lambda) = (S_\mu, \theta_\mu),$$

this shows that the map α is well-defined.

The fiber of $\bar{p}_1$ over the G^{F_u}-orbit of S_λ in $\mathcal{T}_v$ is in evident bijection with $N(G, S_\lambda^{\mathrm{F}_u})/N(G^{\mathrm{F}_u}, S_\lambda)$. By Lemma 2.11.2, the latter is in bijection with the fiber of p over the class of λ in $\left[r^{-1}(\omega)\right]/W_o^{w\vartheta}$.

It therefore suffices to prove that α is injective. Suppose $g \in G^{\mathrm{F}_u}$, $\lambda, \mu \in r^{-1}(\omega)$ and $g \cdot (S_\mu, \theta_\mu) = (S_\lambda, \theta_\lambda)$. As in the proof of Lemma 2.11.1, the element $q_\lambda^{-1} g q_\mu$ belongs to $N(T^{\mathrm{F}_w})$, and projects to an element $z_o \in W_o^{w\vartheta}$ such that $z_o\mu \equiv \lambda \mod (1 - w\vartheta)X$. But also $g_*\theta_\mu = \theta_\lambda$, which means that z_0 fixes χ. Since χ is regular, we have $z_o = 1$, hence $\mu \equiv \lambda \mod (1 - w\vartheta)X$. □

Recall that for $\lambda \in r^{-1}(\omega)$, $u \in \omega$, and a TRSELP φ we defined in Section 4.6 the representation

$$\pi_u(\varphi, \rho_\lambda) = \mathrm{Ad}(m_\lambda)_* \pi_\lambda \in \mathrm{Irr}(G^{\mathrm{F}_u}),$$

where m_λ is as in Lemma 2.8.1. This construction involved the character $\chi = \chi_\varphi \in \mathrm{Irr}_0(T^{\mathrm{F}_w})$ corresponding to φ as in Section 4.3.

Lemma 9.6.2. *Let $\mathbf{G}_u$ be the inner twist of $\mathbf{G}$ given by the cocycle $u \in \omega$, and let $\mathbf{T}_w$ be the twist of $\mathbf{T}$ determined by w. Then for $\lambda \in r^{-1}(\omega)$ we have*

$$\varepsilon(\mathsf{G}_\lambda, \mathsf{T}_\lambda) = \varepsilon(\mathbf{G}_u, \mathbf{T}_w).$$

Hence, this sign is independent of $\lambda \in r^{-1}(\omega)$.

Proof. The $\mathfrak{f}$-rank of G_λ equals the k-rank of $\mathbf{G}_{u_\lambda}$, and $\mathbf{G}_{u_\lambda} \simeq \mathbf{G}_u$ over k. Likewise, we have seen that $\mathbf{T}_\lambda \simeq \mathbf{T}_w$ over k. □

For $\lambda \in r^{-1}(\omega)$, let Θ_{ρ_λ} be the character of $\pi_u(\varphi, \rho_\lambda)$. By construction, the function Θ_{ρ_λ} depends only on the class of λ in $[r^{-1}(\omega)]$. We can now prove the desired result of this section.

Lemma 9.6.3. *Let $v = wu^{-1}$ and let $\chi = \chi_\varphi$ be as in Section 4.3. Then*

$$\sum_{\lambda \in [r^{-1}(\omega)]} \Theta_{\rho_\lambda} = \varepsilon(\mathbf{G}_u, \mathbf{T}_w) \cdot R(G, \hat{\mathcal{T}}_{v,\chi}).$$

Proof. By Lemma 9.3.1, we have

$$\Theta_{\rho_\lambda} = \varepsilon(\mathsf{G}_\lambda, \mathsf{T}_\lambda) \cdot R(G, S_\lambda, \theta_\lambda),$$

so the claim follows from Lemmas 9.6.1 and 9.6.2. □

10. Reduction formulae for character integrals

If G is a connected reductive $\mathfrak{f}$-group with Frobenius F, S is a maximal $\mathfrak{f}$-torus in G, and $\theta \in \operatorname{Irr}(\mathsf{S}^F)$, then from [20, Thm 4.2] we have the reduction formula

$$R^{\mathsf{G}}_{\mathsf{S},\theta}(x) = \sum_{\substack{g \in \mathsf{G}^F \\ {}^g\mathsf{S} \subset \mathsf{G}_s}} g_*\theta(s) \cdot Q^{\mathsf{G}_s}_{g\mathsf{S}g^{-1}}(u), \tag{29}$$

where $x = su \in \mathsf{G}^F$ is the Jordan decomposition, and for any maximal $\mathfrak{f}$-torus $\mathsf{S}_1 \subset \mathsf{G}_s$, the normalized Green function $Q^{\mathsf{G}_s}_{\mathsf{S}_1}$ is defined on all of G^F by

$$Q^{\mathsf{G}_s}_{\mathsf{S}_1}(h) := \begin{cases} \dfrac{1}{|\mathsf{G}_s^F|} R^{\mathsf{G}_s}_{\mathsf{S}_1,\theta_1}(h) & \text{if } h \in \mathsf{G}_s^F \text{ and } h \text{ is unipotent} \\ 0 & \text{otherwise ,} \end{cases} \tag{30}$$

the right side being independent of $\theta_1 \in \operatorname{Irr}(\mathsf{S}_1^F)$.

In this section we prove an analogue of Equation (29) for our functions $R(G, S, \theta)$, using now the topological Jordan decomposition.

Fix a pair $(S, \theta) \in \hat{\mathfrak{T}}(G)$, and let $\hat{\mathcal{T}}$ denote the G^F-orbit of (S, θ). For $\gamma \in G_0^F \cap G^{\mathrm{rss}}$ with topological Jordan decomposition $\gamma = \gamma_s\gamma_u$, we define

$$\hat{\mathcal{T}}(\gamma_s) := \{(S', \theta') \in \hat{\mathcal{T}} :\ \gamma_s \in S'\}.$$

Then $G^F_{\gamma_s}$ preserves $\hat{\mathcal{T}}(\gamma_s)$, and acts on $\hat{\mathcal{T}}(\gamma_s)$ with finitely many orbits.

Our reduction formula for $R(G, S, \theta)$ is as follows.

Lemma 10.0.4. *For $\gamma = \gamma_s\gamma_u$ as above, we have*

$$R(G, S, \theta)(\gamma) = \sum_{(S',\theta') \in \hat{\mathcal{T}}(\gamma_s)/G^F_{\gamma_s}} \theta'(\gamma_s) \cdot R(G_{\gamma_s}, S', 1)(\gamma_u).$$

The proof of Lemma 10.0.4 will require some preliminary steps. Let J be the facet in $\mathcal{A}(S)$ such that $J^F = \mathcal{A}(S)^F$, and let S be the image of $S \cap G_J$ in G_J. Any compact element $\delta \in S^F$ belongs to $S \cap G_J$, and we let $\overline{\delta} \in \mathsf{S}$ denote the image of δ.

Applying Equation (30) with $\mathsf{G} = \mathsf{G}_J$, $s = \overline{\gamma}_s$, and $\mathsf{S}_1 = \mathsf{S}$, we have the normalized Green function $Q^{(\mathsf{G}_J)_{\overline{\gamma}_s}}_{\mathsf{S}}$ defined on all of G_J^F. We let $\dot{Q}^{(\mathsf{G}_J)_{\overline{\gamma}_s}}_{\mathsf{S}}$ denote the natural inflation of $Q^{(\mathsf{G}_J)_{\overline{\gamma}_s}}_{\mathsf{S}}$ to a function on G_J^F, extended by zero to the rest of G^F.

Lemma 10.0.5. *Let $\gamma \in G_0^F$ be regular semisimple with $\gamma_s \in S$, and let L_{γ_s} be a compact open subgroup of G_{γ_s} with Haar measure di. Then the support of the function on $G_{\gamma_s}^F$ given by*

$$h \mapsto \int_{L_{\gamma_s}} \dot{Q}_{\mathsf{S}}^{(\mathsf{G}_J)_{\overline{\gamma}_s}}({}^{hi}\gamma_u)\, di,$$

is compact modulo the center of $G_{\gamma_s}^F$.

Proof. The function $Q_{\mathsf{S}}^{(\mathsf{G}_J)_{\overline{\gamma}_s}}$ on the unipotent set in $(\mathsf{G}_J)_{\overline{\gamma}_s}$ is the restriction of $R_{\mathsf{S},\theta}^{(\mathsf{G}_J)_{\overline{\gamma}_s}}$, for any $\theta \in \mathrm{Irr}_0(S^F)$. Take θ to be regular. Since S is F-minisotropic in $(\mathsf{G}_J)_{\overline{\gamma}_s}$, the function $\dot{R}_{\mathsf{S},\theta}^{(\mathsf{G}_J)_{\overline{\gamma}_s}}$ is a matrix coefficient of a supercuspidal representation of $G_{\gamma_s}^F$, constructed as in Section 4.4 with G^F there replaced by $G_{\gamma_s}^F$. Hence the function $\dot{Q}_{\mathsf{S}}^{(\mathsf{G}_J)_{\overline{\gamma}_s}}$ is the restriction of a supercuspidal matrix coefficient to the compact topological unipotent set in $G_{\gamma_s}^F$. The result now follows from [26, Lemma 23, p. 59]. □

The restriction of θ to $S^F \cap G_J$ is the inflation of a character $\theta_0 \in \mathrm{Irr}(\mathsf{S}^F)$. Let $\dot{\theta}$ denote the function on G^F defined by

$$\dot{\theta}(\delta) = \begin{cases} \theta_0(\bar{\delta}) & \text{if } \delta \in G_J^F \text{ and } \bar{\delta} \in \mathsf{S}, \\ 0 & \text{otherwise.} \end{cases}$$

For each regular semisimple element $\gamma \in G_0^F$, define a locally constant function f_γ on G^F by

$$f_\gamma(g) := \dot{\theta}({}^g\gamma_s) \cdot \dot{Q}_{\mathsf{S}}^{(\mathsf{G}_J)_{\overline{{}^g\gamma_s}}}({}^g\gamma_u).$$

Note that f_γ is supported on the set $E(\gamma_s, S)$ defined in Section 8, and is left-invariant under G_J^{+F}.

Lemma 10.0.6. *Let $\gamma \in G_0^F$ be regular semisimple, and let L_{γ_s} be a compact open subgroup of G_{γ_s} with Haar measure di. Then the function $\tau_\gamma \colon G^F \to \mathbb{C}$ defined by*

$$\tau_\gamma(g) := \int_{L_{\gamma_s}} f_\gamma(gi)\, di$$

is locally constant and compactly supported modulo Z^F.

Proof. Since $\tau_\gamma(jg) = \tau_\gamma(g)$ for all $j \in (G_J^+)^F$ and $g \in G^F$, it is clear that τ_γ is locally constant.

Without loss of generality, we assume that $\gamma \in G_J^F$ and $\overline{\gamma}_s \in \mathsf{S}$. By Lemma 8.0.10, there is a lift of (J, S) in G_{γ_s}. Any such lift is F-minisotropic in G. It follows that the center of $G_{\gamma_s}^F$ is compact modulo Z^F.

Choose a set $D(\gamma_s, S)$ of representatives for the double cosets in

$$N(G, S)^F \backslash \tilde{D}(\gamma_s, S) / G_{\gamma_s}^F.$$

By Lemma 8.0.11 the set $D(\gamma_s, S)$ is finite, and the support of τ_γ is contained in

$$E(\gamma_s, S) = \coprod_{d \in D(\gamma_s, S)} G_J^{+F} N(G, S)^F d G_{\gamma_s}^F.$$

Since S is F-minisotropic, the group $N(G,S)^F$ is compact modulo Z^F. It suffices therefore to show, for fixed $d \in D(\gamma_s, S)$, that the function $h \mapsto \tau_\gamma(dh)$ on $G^F_{\gamma_s}$ has compact support modulo the center of $G^F_{\gamma_s}$. This is Lemma 10.0.5 with γ there replaced by ${}^d\gamma$. $\square$

The key to the reduction formula is the following "localization" result.

Lemma 10.0.7. *Suppose $\gamma \in G^F_0$ is regular semisimple and L is a compact open subgroup of G^F, and let $L_{\gamma_s} = L \cap G_{\gamma_s}$. Normalize Haar measures so that $\mathrm{meas}_{d\ell}(L) = \mathrm{meas}_{di}(L_{\gamma_s}) = 1$. Then the integrals*

$$\int_{G^F/Z^F} dg^* \int_{L_{\gamma_s}} f_\gamma(gi)\, di$$

and

$$\int_{G^F/Z^F} dg^* \int_L f_\gamma(gl)\, dl$$

both converge and are equal. Moreover, these integrals are independent of L.

Proof. The first integral is

$$\int_{G^F/Z^F} \tau_\gamma(g)\, dg^*.$$

Lemma 10.0.6 shows that this integral converges and allows us to rewrite it as

$$\begin{aligned}\int_{G^F/Z^F} \tau_\gamma(g)\, dg^* &= \int_{G^F/Z^F} dg^* \int_L \tau_\gamma(gl)\, dl \\ &= \int_{G^F/Z^F} dg^* \int_L dl \int_{L_{\gamma_s}} f_\gamma(gli)\, di \\ &= \int_{G^F/Z^F} dg^* \int_L f_\gamma(gl)\, dl,\end{aligned}$$

absorbing i into the integral over L.

To see that the integrals are independent of L, it suffices to show they are unchanged if we replace L by a compact open subgroup $L' < L$. We have

$$\int_{G^F/Z^F} dg^* \int_{L_{\gamma_s}} f_\gamma(gl)\, dl = \int_{G^F/Z^F} dg^* \int_{L_{\gamma_s}} dl \int_{(L')_{\gamma_s}} f_\gamma(gll')\, dl'.$$

By Lemma 10.0.6 again, the integral over $(L')_{\gamma_s}$ has compact support as a function on $G^F/Z^F \times L_{\gamma_s}$. Hence we may switch the integrals over G^F/Z^F and L_{γ_s}. The claim follows. $\square$

Now we can prove Lemma 10.0.4. From Equation (29), we have

$$\begin{aligned}\frac{\mathrm{meas}_{dg}(G^F_J)}{\mathrm{meas}_{dz}({}^0Z^F)} R(G,S,\theta)(\gamma) &= \int_{G^F/Z^F} dg^* \int_L \dot{R}^{\mathsf{G}_J}_{\mathsf{S},\theta}({}^{gl}\gamma)\, dl \\ &= \sum_{x \in G^F_J/G^{+F}_J} \int_{G^F/Z^F} dg^* \int_L f_\gamma(xgl)\, dl.\end{aligned}$$

Absorbing x into the integral over G^F/Z^F and using Lemma 10.0.7 , we get

$$\frac{\mathrm{meas}_{dg}(G_J^F)}{\mathrm{meas}_{dz}({}^0Z^F)} R(G,S,\theta)(\gamma) = |\mathsf{G}_J^F| \int_{G^F/Z^F} dg^* \int_{L_{\gamma_s}} f_\gamma(gi)\, di.$$

During the rest of this calculation only, we use the abbreviations

$$N := N(G,S)^F, \quad U := G_J^{+F}, \quad H = G_{\gamma_s}.$$

Let $D(\gamma_s, S)$ be as in the proof of 10.0.6. The integral over L_{γ_s} is supported on

$$E(\gamma_s, S) = \coprod_{d \in D(\gamma_s,S)} \coprod_{\bar{n} \in N/N_d} UndH^F/Z^F,$$

where $N_d = {}^dH \cap N$. Consequently, we have

$$\frac{\mathrm{meas}_{dg}(G_J^F)}{\mathrm{meas}_{dz}({}^0Z^F)} R(G,S,\theta)(\gamma) = |\mathsf{G}_J^F| \sum_{d \in D(\gamma_s,S)} \sum_{\bar{n} \in N/N_d} \int_{UndH^F/Z^F} dg^* \int_{L_{\gamma_s}} f_\gamma(gi)\, di. \tag{31}$$

Note that the map $(d, \bar{n}) \mapsto (nd)^{-1} \cdot (S, \theta)$ induces a bijection

$$D(\gamma_s, S) \times N/N_d \xrightarrow{\sim} \hat{\mathcal{T}}(\gamma_s)/H.$$

Hence the sum in Equation (31) matches the sum in Lemma 10.0.4.

Fix $d \in D(\gamma_s, S)$ and $\bar{n} \in N/N_d$, and set

$$J' = (nd)^{-1}J, \quad U' = G_{J'}^{+F} = \mathrm{Ad}(nd)^{-1}U, \quad (S', \theta') = (nd)^{-1} \cdot (S, \theta), \quad \gamma_s' = {}^{nd}\gamma_s.$$

We then have

$$\int_{UndH^F/Z^F} dg^* \int_{L_{\gamma_s}} f_\gamma(gi)\, di = \frac{\mathrm{meas}_{dg}(U)}{\mathrm{meas}_{dh}(H \cap U')} \int_{H^F/Z^F} dh^* \int_{L_{\gamma_s}} f_\gamma(ndhi)\, di.$$

From the definitions, we have

$$f_\gamma(ndhi) = \dot{\theta}(\gamma_s') \cdot \dot{Q}_{\mathsf{S}}^{(\mathsf{G}_J)_{\overline{\gamma'}_s}}({}^{ndhi}\gamma_u) = \dot{\theta}'(\gamma_s) \cdot \dot{Q}_{\mathsf{S}'}^{(\mathsf{G}_{J'})_{\overline{\gamma}_s}}({}^{hi}\gamma_u).$$

As in the proof of Lemma 8.0.11, the projection $H \cap G_{J'} \longrightarrow \mathsf{G}_{J'}$ allows us to identify

$$(\mathsf{G}_{J'})_{\overline{\gamma}_s} = \mathsf{H}_{J'},$$

so that

$$f_\gamma(ndhi) = \dot{\theta}'(\gamma_s) \cdot \dot{Q}_{\mathsf{S}'}^{\mathsf{H}_{J'}}({}^{hi}\gamma_u).$$

Since $U = G_J^{+F}$ and $H_{J'}^{+F} = H \cap U'$, we have

$$\int_{UndH^F/Z^F} dg^* \int_{L_{\gamma_s}} f_\gamma(gi)\, di = \frac{\mathrm{meas}_{dg}(G_J^{+F})}{\mathrm{meas}_{dh}(H_{J'}^{+F})} \int_{H^F/Z^F} dh^* \int_{L_{\gamma_s}} \dot{\theta}'(\gamma_s) \cdot \dot{Q}_{\mathsf{S}'}^{\mathsf{H}_{J'}}({}^{hi}\gamma_u)\, di.$$

Since the center of H is contained in the F-minisotropic torus S', we conclude that Z is the group of K-rational points of the maximal k-split torus in the center of $\mathbf{G}_{\gamma_s}$. Hence, from the definition of $R(H, S', 1)$, we have

$$|\mathsf{G}_J^F| \int_{UndH^F/Z^F} dg^* \int_{L_{\gamma_s}} f_\gamma(gi)\, di = \frac{\mathrm{meas}_{dg}(G_J^F)}{\mathrm{meas}_{dz}({}^0Z^F)} \theta'(\gamma_s) \cdot R(H, S', 1)(\gamma_u).$$

Inserting this into Equation (31) completes the proof of Lemma 10.0.4.

10.1. **Characters in a simple case.** We illustrate Lemma 10.0.4 in the simple case where $\gamma \in G_0^F$ is strongly regular and topologically semisimple. We have $\gamma_s = \gamma$ and $\gamma_u = 1$.

Let $\hat{\mathcal{T}} \subset \hat{\mathfrak{T}}(G)$ be a G^F-orbit. We write

$$R(G, \hat{\mathcal{T}}) := R(G, S', \theta')$$

for any $(S', \theta') \in \hat{\mathcal{T}}$. Then $\hat{\mathcal{T}}(\gamma)$ is nonempty if and only if $(S, \theta) \in \hat{\mathcal{T}}$, where $S = G_\gamma$ and $\theta \in \mathrm{Irr}_0(S^F)$, in which case we have

$$\hat{\mathcal{T}}(\gamma) = \{(S, n_*\theta) :\ n \in N(G, S)^F/S^F\}.$$

Since $R(G_\gamma, S, 1)(1) = 1$, Lemma 10.0.4 gives the formula

$$R(G, \hat{\mathcal{T}})(\gamma) = \sum_{n \in N(G,S)^F/S^F} n_*\theta(\gamma)$$

if $\hat{\mathcal{T}}(\gamma)$ is nonempty, and $R(G, \hat{\mathcal{T}})(\gamma) = 0$ otherwise.

Return now to the situation of Section 9.6, with $F = \mathrm{F}_u$ etc. By Lemma 9.6.1, $\hat{\mathcal{T}}$ contains

$$(S_\lambda, \theta_\lambda) = \mathrm{Ad}(q_\lambda) \cdot (T, \chi),$$

for some $\lambda \in r^{-1}(\omega)$. If S is not G^{F_u}-conjugate to S_λ, then $R(G, \hat{\mathcal{T}})(\gamma) = 0$. Suppose $S = {}^hS_\lambda$ for some $h \in G^{\mathrm{F}_u}$. Let $\theta = h_*\theta_\lambda$, so that

$$\hat{\mathcal{T}}(\gamma) = \{(S, n_*\theta) :\ n \in N(G, S)^{\mathrm{F}_u}/S\}.$$

From Lemmas 2.11.2 and 10.0.4 it follows that

$$R(G, \hat{\mathcal{T}})(\gamma) = \sum_{y \in W_{o,\lambda}^{w\vartheta}} (hq_\lambda y)_*\chi(\gamma).$$

From Lemma 9.3.1 we get the following character values.

Proposition 10.1.1. *Suppose $\gamma \in G_0^{\mathrm{F}_u}$ is strongly regular and topologically semisimple. Then $\Theta_{\rho_\lambda}(\gamma) = 0$ unless γ lies in a G^{F_u}-conjugate of S_λ, and if $\gamma \in {}^hS_\lambda$ for $h \in G^{\mathrm{F}_u}$, we have*

$$\Theta_{\rho_\lambda}(\gamma) = \varepsilon(\mathsf{G}_\lambda, \mathsf{T}_\lambda) \sum_{y \in W_{o,\lambda}^{w\vartheta}} (hq_\lambda y)_*\chi(\gamma).$$

11. Reduction formula for stable character integrals

In this section we prove the analogue of Lemma 10.0.4 for stable character integrals. Fix a G-stable class $\hat{\mathcal{T}}_{\mathrm{st}} \subset \hat{\mathfrak{T}}(G)$. Recall from Lemma 9.4.3 that there is $v \in N/T$ and $\chi \in \mathrm{Irr}_0(T^{F_v})$ such that every pair $(S,\theta) \in \hat{\mathcal{T}}_{\mathrm{st}}$ is of the form

$$(S,\theta) = ({}^gT, g_*\chi)$$

for some $g \in G$ with $g^{-1}F(g) \in v$.

Given $\gamma \in G_0^F$ regular semisimple, with topological Jordan decomposition $\gamma = \gamma_s\gamma_u$, we define

$$\hat{\mathcal{T}}_{\mathrm{st}}(\gamma_s) := \{(S,\theta) \in \hat{\mathcal{T}}_{\mathrm{st}} :\ \gamma_s \in S\}.$$

This set is a finite disjoint union

$$\hat{\mathcal{T}}_{\mathrm{st}}(\gamma_s) = \coprod_{\hat{i} \in \hat{I}(\gamma_s)} \hat{\mathcal{T}}_{\mathrm{st}}(\gamma_s, \hat{i}),$$

where each $\hat{\mathcal{T}}_{\mathrm{st}}(\gamma_s,\hat{i})$ is a G_{γ_s}-stable class in $\hat{\mathfrak{T}}(G_{\gamma_s})$, and $\hat{I}(\gamma_s)$ is an index set for these G_{γ_s}-stable classes.

Applying p_1, we have

$$\mathcal{T}_{\mathrm{st}}(\gamma_s) := p_1[\hat{\mathcal{T}}_{\mathrm{st}}(\gamma_s)] = \coprod_{i \in I(\gamma_s)} \mathcal{T}_{\mathrm{st}}(\gamma_s, i),$$

where each $\mathcal{T}_{\mathrm{st}}(\gamma_s, i)$ is a G_{γ_s}-stable class in $\mathfrak{T}(G_{\gamma_s})$, and $I(\gamma_s)$ is an index set for these G_{γ_s}-stable classes. There is a surjective map $\hat{i} \mapsto i$ from $\hat{I}(\gamma_s)$ to $I(\gamma_s)$, such that

$$p_1[\hat{\mathcal{T}}_{\mathrm{st}}(\gamma_s,\hat{i})] = \mathcal{T}_{\mathrm{st}}(\gamma_s, i).$$

The fiber of this map over $i \in I(\gamma_s)$ has cardinality

$$N(i) := |N(G_{\gamma_s}, S^F)/S|,$$

where S is any element of $\mathcal{T}_{\mathrm{st}}(\gamma_s, i)$.

For any G_{γ_s}-stable class $\mathcal{T}^1_{\mathrm{st}} \subset \mathfrak{T}(G_{\gamma_s})$, we set

$$Q(G_{\gamma_s}, \mathcal{T}^1_{\mathrm{st}}) := \sum_{S \in \mathcal{T}^1_{\mathrm{st}}/G^F_{\gamma_s}} |N(G_{\gamma_s}, S^F)/N(G^F_{\gamma_s}, S)| \cdot R(G_{\gamma_s}, S, 1).$$

This will turn out to be a stable p-adic analogue of a Green function. We will consider the sums $Q(G_{\gamma_s}, \mathcal{T}_{\mathrm{st}}(\gamma_s, i))$, for $i \in I(\gamma_s)$. But first we need more notation.

For each $\hat{i} \in \hat{I}(\gamma_s)$ and $S \in \mathcal{T}_{\mathrm{st}}(\gamma_s, \hat{i})$, we have a character sum

$$\theta^{\hat{i}}_S = \sum_{\theta \in p_2^{\hat{i}}(p_1^{\hat{i}})^{-1}(S)} \theta,$$

where $p_1^{\hat{i}}$ (resp. $p_2^{\hat{i}}$) is the restriction of p_1 (resp. p_2) to $\hat{\mathcal{T}}_{\mathrm{st}}(\gamma_s, \hat{i})$.

In fact, this sum is independent of S: Given two tori $S, S' \in \mathcal{T}_{\mathrm{st}}(\gamma_s, i)$, we have

$$\theta_S^{\hat{i}}(\gamma_s) = \theta_{S'}^{\hat{i}}(\gamma_s),$$

as a special case of Lemma 9.4.4. We therefore define

$$\theta_{\hat{i}}^{\chi}(\gamma_s) := \theta_S^{\hat{i}}(\gamma_s),$$

for any $S \in \mathcal{T}_{\mathrm{st}}(\gamma_s, i)$. Note that the sum

$$\theta_i^{\chi}(\gamma_s) := \sum_{\hat{i} \mapsto i} \theta_{\hat{i}}^{\chi}(\gamma_s)$$

is none other than the character sum $\theta_S^{\chi}(\gamma_s)$, for any $S \in \mathcal{T}_{st}(\gamma_s, i)$, as defined in Section 9.4.

Finally, recall (Section 10) that for each G^F-orbit $\hat{\mathcal{T}} \subset \hat{\mathcal{T}}_{\mathrm{st}}$, we have defined

$$\hat{\mathcal{T}}(\gamma_s) = \{(S, \theta) \in \hat{\mathcal{T}} : \ \gamma_s \in S\}.$$

Now we are ready to state the reduction formula for stable character integrals.

Lemma 11.0.2. *For $\gamma \in G_0^F$ regular semisimple, with topological Jordan decomposition $\gamma = \gamma_s \gamma_u$, we have*

$$R(G, \hat{\mathcal{T}}_{\mathrm{st}})(\gamma) = \sum_{i \in I(\gamma_s)} \frac{\theta_i^{\chi}(\gamma_s)}{N(i)} \cdot Q(G_{\gamma_s}, \mathcal{T}_{\mathrm{st}}(\gamma_s, i))(\gamma_u).$$

Proof. Using Lemma 10.0.4, we compute

$$\begin{aligned}
R(G, \hat{\mathcal{T}}_{\mathrm{st}})(\gamma) &= \sum_{\hat{\mathcal{T}} \in \hat{\mathcal{T}}_{\mathrm{st}}/G^F} R(G, \hat{\mathcal{T}})(\gamma) \\
&= \sum_{\hat{\mathcal{T}} \in \hat{\mathcal{T}}_{\mathrm{st}}/G^F} \ \sum_{(S,\theta) \in \hat{\mathcal{T}}(\gamma_s)/G^F_{\gamma_s}} \theta(\gamma_s) \cdot R(G_{\gamma_s}, S, 1)(\gamma_u) \\
&= \sum_{\hat{i} \in \hat{I}(\gamma_s)} \ \sum_{(S,\theta) \in \hat{\mathcal{T}}_{\mathrm{st}}(\gamma_s, \hat{i})/G^F_{\gamma_s}} \theta(\gamma_s) \cdot R(G_{\gamma_s}, S, 1)(\gamma_u) \\
&= \sum_{\hat{i} \in \hat{I}(\gamma_s)} \ \sum_{S \in \mathcal{T}_{\mathrm{st}}(\gamma_s, i)/G^F_{\gamma_s}} \frac{\theta_S^{\hat{i}}(\gamma_s)}{|N(G^F_{\gamma_s}, S)/S^F|} \cdot R(G_{\gamma_s}, S, 1)(\gamma_u) \\
&= \sum_{i \in I(\gamma_s)} \theta_i^{\chi}(\gamma_s) \sum_{S \in \mathcal{T}_{\mathrm{st}}(\gamma_s, i)/G^F_{\gamma_s}} \frac{1}{|N(G^F_{\gamma_s}, S)/S^F|} \cdot R(G_{\gamma_s}, S, 1)(\gamma_u) \\
&= \sum_{i \in I(\gamma_s)} \frac{\theta_i^{\chi}(\gamma_s)}{N(i)} \cdot Q(G_{\gamma_s}, \mathcal{T}_{\mathrm{st}}(\gamma_s, i))(\gamma_u).
\end{aligned}$$

$\square$

11.1. **A bijection between stable classes of unramified tori.** Lemma 11.0.2 reduces the proof of stability to the topologically unipotent set, as follows. Let $\mathcal{T}_{\rm st}(\gamma_s)/G_{\gamma_s}$ denote the set of G_{γ_s}-stable classes in $\mathcal{T}_{\rm st}(\gamma_s)$. So $\mathcal{T}_{\rm st}(\gamma_s)/G_{\gamma_s}$ is indexed by $I(\gamma_s)$.

We now assume that $\gamma \in G_0^F$ is in fact strongly regular semisimple, that is, the centralizer of γ in $\mathbf{G}$ is a torus. Then if $g \in G$ and ${}^g\gamma$ is again in G^F, we can construct a bijection

$$\iota_g : \mathcal{T}_{\rm st}(\gamma_s)/G_{\gamma_s} \xrightarrow{\sim} \mathcal{T}_{\rm st}({}^g\gamma_s)/G_{{}^g\gamma_s}$$

as follows. Let $S \in \mathcal{T}_{st}(\gamma_s)$. Since $\gamma \in G^F$ and has connected centralizer, we have $g^{-1}F(g) \in Z^1(F, G_{\gamma_s})$.

Let $\mathbf{Z}_{\gamma_s}$ be the maximal k-split torus in the center of $\mathbf{G}_{\gamma_s}$. Since S is F-minisotropic in G_{γ_s}, the group of co-invariants of F in $X_*(\mathbf{S})$ has the same rank as $X_*(\mathbf{Z}_{\gamma_s})$. It then follows from [35, Thm.1.2] (see also Lemma 2.6.1) that the map $H^1(F,S) \longrightarrow H^1(F, G_{\gamma_s})$ is surjective.

This means there is $h \in G_{\gamma_s}$ such that $(gh)^{-1}F(gh) \in S$. Hence $\mathrm{Ad}(gh) : S \longrightarrow {}^{gh}S$ commutes with F, so ${}^{gh}S \in \mathcal{T}_{st}({}^g\gamma_s)$.

Suppose also $S' \in \mathcal{T}_{\rm st}(\gamma_s)$, and $(S')^F = {}^k(S^F)$ for some $k \in G_{\gamma_s}$. This implies that $k^{-1}F(k) \in S$. As above, there exists $h' \in G_{\gamma_s}$ such that $(gh')^{-1}F(gh') \in S'$. Then the element $j := gh'kh^{-1}g^{-1} \in G_{{}^g\gamma_s}$ satisfies $j^{-1}F(j) \in {}^{gh}S$, ${}^{jgh}S = {}^{gh'}S'$, which means that ${}^{gh}S$ is $G_{{}^g\gamma_s}$-stably conjugate to ${}^{gh'}S'$. Therefore, sending the G_{γ_s}-stable class of S to the $G_{{}^g\gamma_s}$-stable class of ${}^{gh}S$ gives a well-defined injection ι_g, as above. It is straightforward to check that $\iota_{(g^{-1})}$ is the inverse of ι_g, so ι_g is actually a bijection.

We may view ι_g as a bijection on index sets:

$$\iota_g : I(\gamma_s) \longrightarrow I({}^g\gamma_s).$$

This map has the property that

$$N(i) = N(\iota_g(i)),$$

for each $i \in I(\gamma_s)$.

Lemma 11.1.1. *Let $\gamma \in G_0^F$ be strongly regular semisimple, with topological Jordan decomposition $\gamma = \gamma_s\gamma_u$, and let $g \in G$ be such that ${}^g\gamma \in G^F$. Let $\hat{\mathcal{T}}_{\rm st}$ be a G-stable class in $\hat{\mathfrak{T}}(G)$, and assume that for all $i \in I(\gamma_s)$ we have*

$$Q(G_{\gamma_s}, \mathcal{T}_{\rm st}(\gamma_s, i))(\gamma_u) = Q(G_{{}^g\gamma_s}, \mathcal{T}_{\rm st}({}^g\gamma_s, \iota_g(i)))({}^g\gamma_u).$$

Then we have

$$R(G, \hat{\mathcal{T}}_{\rm st})(\gamma) = R(G, \hat{\mathcal{T}}_{\rm st})({}^g\gamma).$$

Proof. From Lemma 11.0.2 we have

$$R(G, \hat{\mathcal{T}}_{\rm st})(\gamma) = \sum_{i\in I(\gamma_s)} \frac{\theta_i^\chi(\gamma_s)}{N(i)} \cdot Q(G_{\gamma_s}, \mathcal{T}_{\rm st}(\gamma_s, i))(\gamma_u). \tag{32}$$

On the other hand, by Lemma 9.4.4 again (this time in full force) we have

$$\theta_i^\chi(\gamma_s) = \theta_{\iota_g(i)}^\chi({}^g\gamma_s).$$

It follows that

$$R(G,\hat{\mathcal{T}}_{\mathrm{st}})({}^g\gamma)=\sum_{i\in I(\gamma_s)}\frac{\theta_i^{\chi}(\gamma_s)}{N(i)}\cdot Q(G_{{}^g\gamma_s},\mathcal{T}_{\mathrm{st}}({}^g\gamma_s,\iota_g(i)))({}^g\gamma_u), \tag{33}$$

whence the result. □

11.2. **Stable characters in a simple case.** We illustrate Section 11.1 by considering the stable version of Section 10.1. As in the latter section, we suppose $\gamma\in G^F$ is strongly regular and topologically semisimple, and let $S=G_\gamma$. Let $\hat{\mathcal{T}}_{\mathrm{st}}\subset\hat{\mathfrak{T}}(G)$ be a G-stable class.

Let us describe the objects in 11.0.2 in this case. If $\hat{\mathcal{T}}_{\mathrm{st}}(\gamma)$ is empty, then $R(G,\hat{\mathcal{T}}_{\mathrm{st}})(\gamma)=0$. Assume $\hat{\mathcal{T}}_{\mathrm{st}}(\gamma)$ is nonempty. Then there is $\theta\in\mathrm{Irr}_0(S^F)$ such that

$$\hat{\mathcal{T}}_{\mathrm{st}}(\gamma)=\{(S,n_*\theta):\ n\in N(G,S^F)/S\}.$$

Thus, we may identify $\hat{I}(\gamma)=N(G,S^F)/S$, and for each $n\in\hat{I}(\gamma)$, we have

$$\hat{\mathcal{T}}_{\mathrm{st}}(\gamma,n)=\{(S,n_*\theta)\}.$$

The index set $I(\gamma)$ consists of a single element, i, and

$$Q(G_\gamma,\mathcal{T}_{\mathrm{st}}(\gamma,i))(\gamma_u)=Q(S,\{S\})(1)=1.$$

In terms of tori, the map ι_g simply sends S to ${}^gS$. Hence the conditions of Lemma 11.1.1 hold trivially, so that $R(G,\hat{\mathcal{T}}_{\mathrm{st}})$ is constant on the G-stable class of γ.

Lemma 11.0.2 gives the formula

$$R(G,\hat{\mathcal{T}}_{\mathrm{st}})(\gamma)=\sum_{n\in N(G,S^F)/S}n_*\theta(\gamma).$$

From Lemma 9.6.3 it follows that the sum of characters in the L-packet $\Pi(\varphi,\omega)$ is constant on the G-stable class of γ.

12. Transfer to the Lie algebra

Lemma 11.1.1 reduces the proof of stability to the following.

Lemma 12.0.1. *Assume as above that $\gamma\in G_0^F$ is strongly regular semisimple, and $g\in G$ is such that ${}^g\gamma\in G^F$. Let $\mathcal{T}_{\mathrm{st}}$ be a G_{γ_s}-stable class in $\mathfrak{T}(G_{\gamma_s})$. Then*

$$Q(G_{\gamma_s},\mathcal{T}_{\mathrm{st}})(\gamma_u)=Q(G_{{}^g\gamma_s},\iota_g\mathcal{T}_{\mathrm{st}})({}^g\gamma_u).$$

We will prove Lemma 12.0.1 under some restrictions on k, to be installed as they are needed. The first step in the proof of Lemma 12.0.1 is to transfer the calculation to the Lie algebras $\mathfrak{g}_{\gamma_s}$ and $\mathfrak{g}_{{}^g\gamma_s}$ of G_{γ_s} and $G_{{}^g\gamma_s}$ respectively. We then invoke a deep result of Waldspurger [63], which states that, for groups which are inner forms of each other, the fundamental lemma for the Lie algebra is true.

12.1. **Orbital Integrals.** Fix γ and g as in the statement of Lemma 12.0.1. Since the calculation takes place mostly in the groups G_{γ_s} and $G_{{}^g\gamma_s}$, we adjust the notation slightly for clarity. Let $H = G_{\gamma_s}$, and let $\mathfrak{h} = \mathrm{Lie}(H)$ be the Lie algebra of H. We fix an additive character $\Lambda : k \longrightarrow \mathbb{C}^\times$ which is trivial on the prime ideal of R but non-trivial on R. Suppose B is a nondegenerate, symmetric, $\langle F\rangle \ltimes H$-invariant bilinear form on $\mathfrak{h}$. For $f \in C_c^\infty(\mathfrak{h}^F)$, the space of locally constant, compactly supported functions on $\mathfrak{h}^F$, we define the Fourier transform (with respect to B) of f by

$$\hat{f}(X) = \int_{\mathfrak{h}^F} f(Y) \cdot \Lambda(B(X,Y))\, dY,$$

where dY is Haar measure on $\mathfrak{h}$, normalized as in Section 5.

Suppose X is a regular semisimple element of $\mathfrak{h}^F$. For $f \in C_c^\infty(\mathfrak{h}^F)$ we define $\mu_X^{H^F}(f)$, the orbital integral of f with respect to X, by

$$\mu_X^{H^F}(f) := \int_{H^F/(C'_H(X))^F} f({}^hX)\, \frac{dh}{dt}$$

where $C'_H(X)$ is the maximal unramified torus in the torus $C_H(X)$ and dh, dt are Haar measures on H^F, $C'_H(X)^F$, respectively, normalized as in Section 5.

Remark 12.1.1. If $X' \in \mathfrak{h}^F$ is H-conjugate to X, then the tori $C'_H(X)$ and $C'_H(X')$ are H-conjugate. Consequently, if dt' denotes the Haar measure on $C'_H(X')$, it follows that the measures $\frac{dh}{dt}$ and $\frac{dh}{dt'}$ determine the same multiple of the top degree form on the orbit ${}^{\mathbf{H}}X = {}^{\mathbf{H}}X'$.

We define $\hat{\mu}_X^{H^F}(f) := \mu_X^{H^F}(\hat{f})$ for $f \in C_c^\infty(\mathfrak{h}^F)$. In this way, we have a distribution $\hat{\mu}_X^{H^F}$ on $C_c^\infty(\mathfrak{h}^F)$. Thanks to Harish-Chandra [25, Theorem 4.4], we know that $\hat{\mu}_X^{H^F}$ is represented on $\mathfrak{h}^F$ by a function, which we also denote by $\hat{\mu}_X^{H^F}$. (The same result is true for the Fourier transform of any orbital integral.)

12.2. **A result of Waldspurger.** In this section, $\mathbf{H}$ is any connected reductive k-group splitting over K. As usual, F is the Frobenius action on both $H := \mathbf{H}(K)$, and $\mathfrak{h} := \mathrm{Lie}(H)$. For $X \in \mathfrak{h}^F$ regular semisimple, write

$$[\mathrm{Ad}(H)X]^F = \coprod_i \mathrm{Ad}(H^F)X_i$$

where the X_i run over a (finite) set of representatives for the $\mathrm{Ad}(H^F)$-orbits in $[\mathrm{Ad}(H)X]^F$ (see Section 2.9.1). We set

$$\hat{S}_X^{\mathfrak{h}} = \sum_i \hat{\mu}_{X_i}^{H^F}.$$

The measures used for each orbital integral are compatible, in the sense of Remark 12.1.1.

Let $\mathbf{H}^*$ denote a k-quasi-split inner form of $\mathbf{H}$, and let $\mathbf{H}^*_{ad}$ be the adjoint group of $\mathbf{H}^*$. Let H^* and H^*_{ad} denote the groups of K-rational points of $\mathbf{H}^*$ and $\mathbf{H}^*_{ad}$, respectively, and let F^* denote the action of Frobenius on H^*, H^*_{ad} and $\mathfrak{h}^* = \mathrm{Lie}(H^*)$. Choose an inner twist

$$\phi\colon H \to H^*.$$

That is, ϕ is a K-isomorphism, and there is $h_\phi^* \in H_{ad}^*$ such that

$$\mathrm{Ad}(h_\phi^*) = F^* \circ \phi \circ F^{-1} \circ \phi^{-1} \in \mathrm{Aut}_K(H^*).$$

Here we implicitly use the isomorphism $H^1(F, H_{ad}) = H^1(k, \mathbf{H}_{ad})$, see Section 2.2. The choice of ϕ defines an injective map S_ϕ from the set of stable regular semisimple orbits in $\mathfrak{h}^F$ to the set of stable regular semisimple orbits in $(\mathfrak{h}^*)^{F^*}$, as follows. If $X \in \mathfrak{h}^F$, then $F^*(d\phi(X)) = \mathrm{Ad}(h_\phi^*)d\phi(X)$, so the $\mathrm{Ad}(H^*)$-orbit of $d\phi(X)$ is F^*-stable. If X is regular semisimple, then so too is $d\phi(X)$. The existence of an F^*-stable Kostant section shows that the $\mathrm{Ad}(H^*)$-orbit of $d\phi(X)$ contains an F^*-fixed point X^* (see, for example, [56, 9.5] or [36]). Finally, S_ϕ sends $[\mathrm{Ad}(H)X]^F$ to $[\mathrm{Ad}(H^*)X^*]^{F^*}$.

Suppose now that $\mathbf{H}'$ is any inner form of $\mathbf{H}$. Let H' denote the group of K-rational points of $\mathbf{H}'$ and let F' denote the action of Frobenius on H' and $\mathfrak{h}' = \mathrm{Lie}(H')$. Suppose $X \in \mathfrak{h}^F$ and $X' \in (\mathfrak{h}')^{F'}$ are regular semisimple elements, and $\phi\colon H \to H^*$ and $\phi'\colon H' \to H^*$ are inner twists. We say that X and X' are *(ϕ, ϕ')-comparable* provided that

$$S_\phi\big([\mathrm{Ad}(H)X]^F\big) = S_{\phi'}\big([\mathrm{Ad}(H')X']^{F'}\big)$$

as stable regular semisimple orbits in $(\mathfrak{h}^*)^{F^*}$.

Example 12.2.1. Take $\mathbf{H} = \mathbf{G}_{\gamma_s}$ as in the situation of Section 12.1. Let $\log\colon G_{0+} \to \mathfrak{g}$ be any injective $\langle F\rangle \ltimes G$-equivariant map which takes regular semisimple elements to regular semisimple elements. (The existence of such a map with just these properties follows from [9, p. 333, §7.6, Proposition 10].) Then $C_{\mathbf{G}}(\gamma)$ is a torus in $\mathbf{H}$, and $F(g) = gs$ for some $s \in C_H(\gamma)$. Moreover, $\mathrm{Ad}(g) : H \longrightarrow H' := {}^gH$ is an inner twist, with $F' = F$. Let $X := \log(\gamma_u)$. From [29, Theorem 13.4(a)] it follows that $X \in \mathfrak{h}$ and ${}^gX \in {}^g\mathfrak{h}$ are regular semisimple elements. Since F and $\mathrm{Ad}(s)$ fix γ_u, it follows that $F(X) = X$, and $F({}^gX) = {}^gX$.

Suppose $\phi\colon H \to H^*$ is an inner twist, and let $X^* \in [\mathrm{Ad}(H^*)d\phi(X)]^{F^*}$. One checks that the map $\phi' := \phi \circ \mathrm{Ad}(g^{-1})\colon {}^gH \to H^*$ is also an inner twist, and that $X^* \in [\mathrm{Ad}(H^*)d\phi'({}^gX)]^{F^*}$. It follows that X and ${}^gX$ are (ϕ, ϕ')-comparable.

Example 12.2.2. Continue with the notation of Example 12.2.1 and also Section 11.1. Let $\mathcal{T}_{st}$ be an H-stable class in $\mathfrak{T}(H)$, and let $\mathcal{T}'_{st} = \iota_g\mathcal{T}_{st}$, an H'-stable class in $\mathfrak{T}(H')$, be as in Section 11.1. Suppose $S_0 \in \mathcal{T}_{st}$ and X_0 is a $\mathfrak{g}$-regular element of $\mathrm{Lie}(S_0)^F$. Let $X \in [\mathrm{Ad}(H)X_0]^F$. Note that X is regular in $\mathfrak{g}$. As in the definition of ι_g, there is $h \in H$ such that $(gh)^{-1}F(gh) \in C_G(X)$, and the elements X and $X' := {}^{gh}X \in [\mathrm{Ad}(H')({}^gX_0)]^F$ are $(\phi, \phi \circ \mathrm{Ad}(g)^{-1})$-comparable.

Lemma 12.2.3. *Let $\phi\colon H \to H^*$ and $\phi'\colon H' \to H^*$ denote inner twists. Suppose X, Y (resp. X', Y') are regular semisimple elements in $\mathfrak{h}^F$ (resp. $(\mathfrak{h}')^{F'}$). If X and X' are (ϕ, ϕ')-comparable elliptic elements and Y and Y' are (ϕ, ϕ')-comparable elements, then we have*

$$\hat{S}_X^{\mathfrak{h}}(Y) = \varepsilon(\mathbf{H}, \mathbf{H}') \cdot \hat{S}_{X'}^{\mathfrak{h}'}(Y').$$

Remark 12.2.4. The above lemma may be viewed as more evidence for Kottwitz' sign conjecture [33].

Proof. Without loss of generality, $\mathbf{H}'$ is k-quasi-split, and we may replace $\mathbf{H}'$ by $\mathbf{H}^*$. Waldspurger has already shown [63, Théorème 1.5] that for X, Y, X', Y' as in the statement of the lemma, we have

$$\hat{S}_X^{\mathfrak{h}}(Y) = c \cdot \hat{S}_{X'}^{\mathfrak{h}^*}(Y')$$

where c is an eighth root of unity. (In the notation of [63], this is actually the special case $s = 1, \xi = I$ of [63, Théorème 1.5], and $c = \gamma_\Lambda(\mathfrak{h}^*)/\gamma_\Lambda(\mathfrak{h})$.) We will give two proofs that $c = \varepsilon(\mathbf{H}, \mathbf{H}^*)$.

The first proof uses Shalika germs. For all $n \in \mathbb{Z}$ we have

$$\hat{S}_X^{\mathfrak{h}}(\varpi^n Y) = c \cdot \hat{S}_{X'}^{\mathfrak{h}^*}(\varpi^n Y').$$

From Harish-Chandra [25, Theorem 5.1.1], for all $n \in \mathbb{Z}$ sufficiently large we have

$$\begin{aligned}\hat{S}_X^{\mathfrak{h}}(\varpi^{2n} Y) &= \sum_{\mathcal{O} \in \mathcal{O}_{\mathfrak{h}}(0)} c_{\mathcal{O}}^{\mathfrak{h}}(X) \cdot \hat{\mu}_{\mathcal{O}}(\varpi^{2n} Y) \\ &= c_0^{\mathfrak{h}}(X) + \sum_{\mathcal{O} \in \mathcal{O}_{\mathfrak{h}}(0) \setminus \{0\}} c_{\mathcal{O}}^{\mathfrak{h}}(X) \cdot q^{-n \cdot \dim \mathcal{O}} \cdot \hat{\mu}_{\mathcal{O}}(Y)\end{aligned}$$

where $\mathcal{O}_{\mathfrak{h}}(0)$ denotes the set of nilpotent H^F-orbits in $\mathfrak{h}^F$, the $c_{\mathcal{O}}^{\mathfrak{h}}(X)$ are complex constants, and 0 denotes the zero orbit $\{0\}$. A similar statement is true for $\hat{S}_{X'}^{\mathfrak{h}^*}$. Thus,

$$\begin{aligned}c_0^{\mathfrak{h}}(X) &= \lim_{n \to \infty} \hat{S}_X^{\mathfrak{h}}(\varpi^{2n} Y) \\ &= \lim_{n \to \infty} c \cdot \hat{S}_{X'}^{\mathfrak{h}^*}(\varpi^{2n} Y') \\ &= c \cdot c_0^{\mathfrak{h}^*}(X').\end{aligned}$$

Let $X_1, X_2, \ldots, X_m$ be representatives for the H^F-orbits in $[\mathrm{Ad}(H)X]^F$. From [25, Theorem 8.1] we have

$$c_0^{\mathfrak{h}}(X) = \sum_{j=1}^{m} \Gamma_0^{C_{\mathfrak{h}}(X_j)}(X_j),$$

where $\Gamma_0^{C_{\mathfrak{h}}(X_j)}(X_j)$ denotes the evaluation of the (unnormalized) Shalika germ corresponding to the zero orbit at X_j.

If the center of H^F is compact, then so too is the center of $(H^*)^{F^*}$. Thanks to Rogawski [49] we have

$$\Gamma_0^{C_{\mathfrak{h}}(X_j)}(X_j) = \frac{\varepsilon(\mathbf{H}, \mathbf{Z})}{\mathrm{Deg}(\mathrm{St}^H)}.$$

Thus, if the center of H^F is compact, we conclude from the above, the fact that $\mathrm{Deg}(\mathrm{St}^H) > 0$, and the fact that c is an eighth root of unity that $c = \varepsilon(\mathbf{H}, \mathbf{H}^*)$.

Suppose that the center Z^F of H^F is not compact. Let $\mathbf{H}_d$ denote the derived group of $\mathbf{H}$ and let $\mathfrak{h}_d$ denote the Lie algebra of $H_d = \mathbf{H}_d(K)$. The center of H_d is finite and $H^F/(H_d^F)Z^F$ is a finite group. Without loss of generality, we assume $X \in \mathfrak{h}_d^F$. From Lemma 2.9.1, we have that two regular semisimple elements of $\mathfrak{h}_d^F$ are H-stably conjugate if and only if they are H_d-stably conjugate. However, since two regular semisimple elements of $\mathfrak{h}_d^F$ may be H^F-conjugate without being H_d^F-conjugate, for $1 \leq i \leq m$ we introduce the group

$$H_i^F := \{h \in H^F : \text{ there is an } h' \in H_d^F \text{ such that } {}^{h'h}X_i = X_i\}.$$

We have $H_d^F Z^F \trianglelefteq H_i^F \trianglelefteq H^F$. Thus, we can write

$$\hat{S}_X^{\mathfrak{h}_d} = \sum_{i=1}^m \sum_{\bar{h} \in H^F/H_i^F} \hat{\mu}^{H_d^F}_{h^{-1}X_i}.$$

Suppose we can show that the restriction of $\hat{S}_X^{\mathfrak{h}}$ to $\mathfrak{h}_d^F$ equals $e \cdot \hat{S}_X^{\mathfrak{h}_d}$ for some constant $e > 0$. We would then have $c_0^{\mathfrak{h}}(X) = e \cdot c_0^{\mathfrak{h}_d}(X)$. Arguing as in the previous paragraph, we would again conclude that $c = \varepsilon(\mathbf{H}, \mathbf{H}^*)$.

To complete the proof, we now show that such a constant e exists. We will use Harish-Chandra's integral formula for the Fourier transform of a regular semisimple orbital integral [25, Lemma 7.9]. Since we only wish to establish the positivity of e, in what follows we are not careful about specifying our invariant measures nor about accounting for the (positive) constants that occur. Let L be a compact open subgroup of H^F which lies in $H_d^F Z^F$. There is a positive constant const so that for regular semisimple $Y \in \mathfrak{h}_d^F$

$$\begin{aligned}
\hat{S}_X^{\mathfrak{h}}(Y) &= \mathrm{const} \cdot \sum_i \int_{H^F/Z^F} dg^* \int_L \Lambda(B({}^{g\ell}Y, X_i))\, d\ell \\
&= \mathrm{const} \cdot \sum_i \sum_{\bar{h} \in H^F/H_i^F} \sum_{\bar{g}_1 \in H_i^F/H_d^F Z^F} \int_{H_d^F Z^F/Z^F} dg_2^* \int_L \Lambda(B({}^{g_1 g_2 \ell}Y, {}^{h^{-1}}X_i))\, d\ell \\
&\qquad \text{which, from the definition of } H_i^F \text{, becomes} \\
&= \mathrm{const} \cdot \sum_i \sum_{\bar{h} \in H^F/H_i^F} \left|H_i^F/H_d^F Z^F\right| \cdot \int_{H_d^F Z^F/Z^F} dg_2^* \int_L \Lambda(B({}^{g_2 \ell}Y, {}^{h^{-1}}X_i))\, d\ell.
\end{aligned}$$

We claim that for $1 \leq i, j \leq m$ we have

$$\left|H_i^F/H_d^F Z^F\right| = \left|H_j^F/H_d^F Z^F\right|.$$

In fact, we will show that the group H_i^F is independent of i. Note that $H_i^F/H_d^F Z^F$ can be characterized as the set of cosets in $H^F/H_d^F Z^F$ which intersect $(C_H(X_i))^F$ nontrivially. Thus, it is enough to show that for $h \in H^F$ we have

$$h(H_d^F Z^F) \cap (C_H(X_i))^F \neq \varnothing \iff h(H_d^F Z^F) \cap (C_H(X_j))^F \neq \varnothing.$$

Suppose $h \in H^F$ and $g \in H_d^F Z^F$ so that $hg \in (C_H(X_i))^F$. It is enough to produce a $g' \in H_d^F Z^F$ such that $hg' \in (C_H(X_j))^F$. Since X_i and X_j are H_d-stably conjugate, there is an $h' \in H_d$ so that ${}^{h'}X_i = X_j$. Since $C_H(X_i)$ is abelian, this implies that ${}^{h'}((C_H(X_i))^F) = (C_H(X_j))^F$. Consequently, ${}^{h'}(hg) \in (C_H(X_j))^F$ and ${}^{h'}(hg) = h(h^{-1}h'hg(h')^{-1}) \in hH_d Z^F$. Set $g' := (h^{-1}h'hg(h')^{-1})$. Note that $g \in H_d^F Z^F$ implies $g' \in H_d(Z^F)$. But also $g' \in h^{-1}(C_H(X_j))^F \leq H^F$, so in fact $g' \in H_d^F Z^F$ and $hg' \in (C_H(X_j))^F$, as desired.

Therefore,

$$\begin{aligned}\hat{S}_X^{\mathfrak{h}}(Y) &= \mathrm{const}' \cdot \sum_i \sum_{\bar{h}\in H^F/H_i^F} \int_{H_d^F Z^F/Z^F} dg_2^* \int_L \Lambda(B({}^{g_2\ell}Y, {}^{h^{-1}}X_i))\, d\ell \\ &= \mathrm{const}' \cdot \sum_i \sum_{\bar{h}\in H^F/H_i^F} \int_{H_d^F/(H_d^F\cap Z^F)} dg_2^* \int_L \Lambda(B({}^{g_2\ell}Y, {}^{h^{-1}}X_i))\, d\ell.\end{aligned}$$

□

12.3. **Another calculation of Waldspurger's sign.** In this section we give a second proof of Lemma 12.2.3 in terms of our pure inner forms G_λ. This proof continues in the vein of [63].

For $\lambda \in X_w$, we have a pure inner form $\mathbf{G}_\lambda$ of $\mathbf{G}$ with Frobenius $\mathrm{F}_\lambda = \mathrm{Ad}(u_\lambda) \circ \mathrm{F}$. In particular $\mathbf{G} = \mathbf{G}_0$ is k-quasi-split. Note that $\mathbf{G}_\lambda = \mathbf{G}$ as groups; the subscript indicates the variation in k-structure. To simplify the notation, we write $\sigma := w\vartheta$, and set $\Sigma := \langle\sigma\rangle$.

We define an inner twisting $\phi_\lambda : \mathbf{G}_\lambda \longrightarrow \mathbf{G}_0$ by $\phi_\lambda = \mathrm{Ad}(h_\lambda)$, where $h_\lambda = p_0 p_\lambda^{-1}$ and $p_\lambda, p_0 \in G$ satisfy the equations

$$p_0^{-1}\,\mathrm{F}(p_0) = \dot{w}, \qquad p_\lambda^{-1} u_\lambda\, \mathrm{F}(p_\lambda) = t_\lambda \dot{w} \tag{34}$$

of Chapter 2.7. Let $\Phi(\mathbf{T})$ denote the set of roots of $\mathbf{T}$ in $\mathbf{G}$. Likewise, let $\mathbf{T}_\lambda = \mathrm{Ad}(p_\lambda)\mathbf{T}$, and let $\Phi(\mathbf{T}_\lambda)$ denote the set of roots of $\mathbf{T}_\lambda$ in $\mathbf{G}_\lambda$.

The map $\mathrm{Ad}(p_\lambda) : \mathbf{T} \longrightarrow \mathbf{T}_\lambda$ intertwines F_w on $\mathbf{T}$ with F_λ on $\mathbf{T}_\lambda$. It induces a map $\Phi(\mathbf{T}) \longrightarrow \Phi(\mathbf{T}_\lambda)$ given by

$$\alpha \mapsto \alpha_\lambda := \alpha \circ \mathrm{Ad}(p_\lambda)^{-1},$$

satisfying

$$\mathrm{F}_\lambda \cdot \alpha_\lambda = (\sigma \cdot \alpha)_\lambda.$$

(Recall that F_w acts on $\Phi(\mathbf{T})$ via σ.)

Fix $\lambda \in X_w$. By Hilbert's Theorem 90, there exists a set $\{E_{\alpha_\lambda} :\ \alpha \in \Phi(\mathbf{T})\}$ of $\mathbf{T}_\lambda$-root vectors in $\mathfrak{g}$ having the property that

$$\mathrm{F}_\lambda \cdot E_{\alpha_\lambda} = E_{(\sigma\cdot\alpha)_\lambda}.$$

The transformed root vectors

$$E_\alpha^* := \phi_\lambda(E_{\alpha_\lambda})$$

are only preserved by F up to scalar multiples. That is, for each $\alpha \in \Phi(\mathbf{T})$ there is $c_{\alpha_\lambda} \in \bar{k}$ such that

$$\mathrm{F}(c_{\alpha_\lambda} E_\alpha^*) = c_{(\sigma\cdot\alpha)_\lambda} E_{\sigma\cdot\alpha}^*.$$

A straightforward computation shows, for each $\alpha \in \Phi(\mathbf{T})$, that

$$\mathrm{Frob}(c_{\alpha_\lambda}) = \varpi^{\langle\lambda,\sigma\cdot\alpha\rangle} \cdot c_{(\sigma\cdot\alpha)_\lambda}. \tag{35}$$

Following [63], a Σ-orbit in $\Phi(\mathbf{T})$ is called *symmetric* if it is closed under $\alpha \mapsto -\alpha$ and *anti-symmetric* otherwise. Let $\dot{\mathrm{Sym}}(\mathbf{T})$ be a set of representatives for the symmetric Σ-orbits in $\Phi(\mathbf{T})$.

For any $\alpha \in \Phi(\mathbf{T})$, define
$$\Sigma_\alpha = \{\tau \in \Sigma : \ \tau \cdot \alpha = \alpha\},$$
and let $k_\alpha \subset \bar{k}$ be the fixed field of the pre-image of Σ_α in $\mathrm{Gal}(\bar{k}/k)$.

For each $\alpha \in \dot{\mathrm{Sym}}(\mathbf{T})$, define
$$\Sigma_{\pm\alpha} = \{\tau \in \Sigma : \ \tau \cdot \alpha = \pm\alpha\},$$
and let $k_{\pm\alpha} \subset \bar{k}$ be the fixed field of the pre-image of $\Sigma_{\pm\alpha}$ in $\mathrm{Gal}(\bar{k}/k)$. There is an integer $m = m(\alpha)$ such that
$$\Sigma_{\pm\alpha} = \langle \sigma^m \rangle, \qquad \Sigma_\alpha = \langle \sigma^{2m} \rangle.$$
We have $\sigma^m \alpha = -\alpha$. The extension k_α/k is unramified of degree $2m$.

Moreover, $k_\alpha/k_{\pm\alpha}$ is an unramified quadratic extension, hence corresponds via class-field theory to the character $\chi_\alpha : k_{\pm\alpha}^\times \longrightarrow \{\pm 1\}$ given by
$$\chi_\alpha(x) = (-1)^{v(x)},$$
where v is the valuation on K. Using [63], Lemma 12.2.3 is equivalent to the following formula.

Lemma 12.3.1.
$$\prod_{\alpha \in \dot{\mathrm{Sym}}(\mathbf{T})} \chi_\alpha(c_{\alpha_\lambda} \cdot c_{-\alpha_\lambda}) = \varepsilon(\mathbf{G}_\lambda, \mathbf{G}_0).$$

The proof requires a few steps.

Lemma 12.3.2. *We have*
$$c_{\alpha_\lambda} \cdot c_{-\alpha_\lambda} = n \cdot \varpi^{-\langle \lambda, \alpha + \sigma\cdot\alpha + \cdots + \sigma^{(m-1)}\cdot\alpha \rangle},$$
where $m = m(\alpha)$ and $n \in k_{\pm\alpha}^\times$ is a norm from k_α.

.

Proof. Applying Equation (35) repeatedly, we have
$$c_{(\sigma^k\cdot\alpha)_\lambda} = \mathrm{Frob}^k(c_{\alpha_\lambda}) \cdot \varpi^{-\langle \lambda, \sigma\cdot\alpha + \cdots + \sigma^k\cdot\alpha \rangle},$$
for $k \geq 1$. Since $\sigma^m \cdot \alpha = -\alpha$, we have
$$c_{-\alpha_\lambda} = \mathrm{Frob}^m(c_{\alpha_\lambda}) \cdot \varpi^{-\langle \lambda, \sigma\cdot\alpha + \cdots + \sigma^{(m-1)}\cdot\alpha - \alpha \rangle}.$$
Hence
$$c_{\alpha_\lambda} \cdot c_{-\alpha_\lambda} = c_{\alpha_\lambda} \cdot \mathrm{Frob}^m(c_{\alpha_\lambda}) \cdot \varpi^{2\langle \lambda, \alpha \rangle} \cdot \varpi^{-\langle \lambda, \alpha + \sigma\cdot\alpha + \cdots + \sigma^{m-1}\cdot\alpha \rangle}.$$
Since $c_\alpha \in k_\alpha$, this proves the claim, with
$$n = c_{\alpha_\lambda} \cdot \mathrm{Frob}^m(c_{\alpha_\lambda}) \cdot \varpi^{2\langle \lambda, \alpha \rangle}.$$
□

Choose a set of positive roots $\Phi^+(\mathbf{T}) \subset \Phi(\mathbf{T})$, and set
$$2\rho = \sum_{\beta \in \Phi^+(\mathbf{T})} \beta.$$

Lemma 12.3.3.

$$\sum_{\alpha\in\dot{\mathrm{Sym}}(\mathbf{T})} \langle\lambda, \alpha+\sigma\cdot\alpha+\cdots+\sigma^{m(\alpha)-1}\cdot\alpha\rangle \equiv \langle\lambda, 2\rho\rangle \mod 2.$$

Proof. Let $\mathcal{O}'_1, \ldots, \mathcal{O}'_p$ be a choice of one from each pair $\{\mathcal{O}'_i, -\mathcal{O}'_i\}$ of anti-symmetric Σ-orbits in $\Phi(\mathbf{T})$, and let $\mathcal{O}_1, \ldots, \mathcal{O}_q$ be the symmetric Σ-orbits. For $\alpha \in \Phi(\mathbf{T})$, define $|\alpha| = \alpha$ if $\alpha \in \Phi^+(\mathbf{T})$, and $|\alpha| = -\alpha$ if $-\alpha \in \Phi^+(\mathbf{T})$. Set

$$\|\mathcal{O}'_i\| = \{|\alpha| :\ \alpha \in \mathcal{O}'_i\}, \qquad \mathcal{O}_j^+ = \mathcal{O}_j \cap \Phi^+(\mathbf{T}).$$

Then we have a disjoint union

$$\Phi^+(\mathbf{T}) = \coprod_{i=1}^{p} \|\mathcal{O}'_i\| \,\sqcup\, \coprod_{j=1}^{q} \mathcal{O}_j^+.$$

For any $1 \leq i \leq p$, we have

$$\sum_{\beta\in\|\mathcal{O}'_i\|} \beta \equiv \sum_{\alpha\in\mathcal{O}'_i} \alpha \mod 2\mathbb{Z}\Phi(\mathbf{T}).$$

The latter sum is Σ-invariant, hence it vanishes, since σ is elliptic. It follows that

$$\langle\lambda, 2\rho\rangle \equiv \sum_{j=1}^{q} \sum_{\beta\in\mathcal{O}_j^+} \langle\lambda, \beta\rangle \mod 2.$$

Working modulo two, we can replace each sum over $\mathcal{O}_j^+$ by

$$\sum_{k=0}^{m(\alpha)-1} \langle\lambda, \sigma^k\cdot\alpha\rangle,$$

for any $\alpha \in \mathcal{O}_j$. This proves the lemma. □

Combining Lemmas 12.3.2 and 12.3.3, we get

Corollary 12.3.4.

$$\prod_{\alpha\in\dot{\mathrm{Sym}}(\mathbf{T})} \chi_\alpha(c_{\alpha_\lambda}\cdot c_{-\alpha_\lambda}) = (-1)^{\langle\lambda,2\rho\rangle}.$$

We next give another expression for $\varepsilon(\mathbf{G}_\lambda, \mathbf{G}_0)$. Let $z_\lambda \in W_o$ be the projection of u_λ. Then z_λ and ϑ act linearly on the $\mathbb{Q}$-vector space $V := X \otimes \mathbb{Q}$ (recall that $X = X_*(\mathbf{T})$), and the k-rank of $\mathbf{G}_\lambda$ is given by

$$\mathrm{rk}(\mathbf{G}_\lambda) = \dim V^{z_\lambda\vartheta}.$$

Let $\det(A)$ denote the determinant of an operator $A \in GL(V)$.

Lemma 12.3.5.

$$\varepsilon(\mathbf{G}_\lambda, \mathbf{G}_0) = \det(z_\lambda).$$

Proof. Since $z_\lambda\vartheta$ has finite order and preserves the lattice $X \subset V$, we have

$$\det(z_\lambda\vartheta) = (-1)^{\dim V - \dim V^{z_\lambda\vartheta}}.$$

Likewise,

$$\det(\vartheta) = (-1)^{\dim V - \dim V^{\vartheta}}.$$

Together, these give

$$\det(z_\lambda) = (-1)^{\dim V^{z_\lambda\vartheta} - \dim V^{\vartheta}} = \varepsilon(\mathbf{G}_\lambda, \mathbf{G}_0).$$

□

To prove Lemma 12.3.1 it remains to prove

Lemma 12.3.6.

$$\det(z_\lambda) = (-1)^{\langle\lambda, 2\rho\rangle}.$$

Proof. From the definitions we see that $z_\lambda = z_{\lambda+\nu}$ for any $\nu \in X^\circ + X^W$, where X° is the co-root lattice of $\mathbf{T}$. Likewise, the parity of $\langle\lambda, 2\rho\rangle$ depends only on the class of λ in $X/(X^\circ + X^W)$. We have $\det(z_\lambda) = (-1)^{\langle\lambda,2\rho\rangle} = +1$ if $\lambda \in X^\circ + X^W$.

Assume now that $\lambda \notin X^\circ + X^W$. Recall that $\mathbf{T}_{ad}$ is the image of $\mathbf{T}$ in the adjoint group $\mathbf{G}_{ad}$ of $\mathbf{G}$, and that $X_{ad} = X_*(\mathbf{T}_{ad})$. We may view X° as a subgroup of X_{ad}. The natural map $X \to X_{ad}$ induces an injection

$$X/(X^\circ + X^W) \hookrightarrow X_{ad}/X^\circ.$$

The nontrivial elements in the group X_{ad}/X° are represented by the minuscule co-weights of $\mathbf{T}_{ad}$ [9, p. 240]. Hence the class of λ in $X/(X^\circ + X^W)$ determines a simple root $\alpha \in \Phi^+(\mathbf{T})$ such that $\langle\lambda, \beta\rangle = 0$ for all simple roots $\beta \neq \alpha$, and $\langle\lambda, \alpha\rangle = 1$. Moreover, we have a disjoint union

$$\Phi(\mathbf{T}) = \Phi_{-1} \sqcup \Phi_0 \sqcup \Phi_1,$$

where

$$\Phi_i = \{\beta \in \Phi(\mathbf{T}) : \ \langle\lambda, \beta\rangle = i\}$$

(see [9, p. 239]).

Iwahori-Matsumoto [30, 1.18] show that z_λ is W_o-conjugate to the unique element of W_o whose set of positive roots made negative is exactly Φ_1. This implies that

$$\det(z_\lambda) = (-1)^{|\Phi_1|}.$$

On the other hand, since

$$\Phi^+(\mathbf{T}) = [\Phi_0 \cap \Phi^+(\mathbf{T})] \sqcup \Phi_1,$$

it follows that

$$\langle\lambda, 2\rho\rangle = \sum_{\beta\in\Phi_1} \langle\lambda, \beta\rangle = |\Phi_1|.$$

This proves the present lemma, as well as Lemma 12.3.1.

□

12.4. **Murnaghan-Kirillov theory.** In this section, $\mathbf{H}$ is any connected reductive k-group, split over K, with Frobenius F on $H := \mathbf{H}(K)$. Let $H_{0^+},\ \mathfrak{h}_{0^+}$ denote respectively the sets of topologically unipotent elements in H, and topologically nilpotent elements in $\mathfrak{h} = \mathrm{Lie}(H)$.

We make the following restrictions on k and $\mathbf{H}$. Recall that q, a power of a prime p, is the cardinality of the residue field $\mathfrak{f}$. Let e denote the ramification degree of k over $\mathbb{Q}_p$, and let $\nu(\mathbf{H})$ be the number of positive roots in $\mathbf{H}$.

Restrictions 12.4.1. (1) $q \geq \nu(\mathbf{H})$.
(2) *There is a faithful k-embedding $\varphi : \mathbf{H} \hookrightarrow \mathrm{GL}_n$ such that $p \geq (2+e)n$.*

Note that if $\mathbf{G}$ is as in the previous part of the paper, $\mathbf{H}$ is the identity component of the centralizer of a topological semisimple element in G_0, and Restrictions 12.4.1 hold for $\mathbf{G}$ and some n, then they hold for $\mathbf{H}$, with the same n.

In Appendices A and B we will prove:

Lemma 12.4.2. *Assume Restrictions 12.4.1 hold. Then we have:*

(1) *For every F-stable facet $J \subseteq \mathcal{B}(H)$, and maximal F-stable torus $\mathsf{S} \subset \mathsf{H}_J$ with Lie algebra L_S, there is an element $\bar{X}_\mathsf{S} \in \mathsf{L}_\mathsf{S}^F$ whose centralizer in H_J is exactly S.*
(2) *There is an $\langle F\rangle \ltimes H$-equivariant bijection $\log : H_{0^+} \longrightarrow \mathfrak{h}_{0^+}$, which induces, for every minimal F-stable facet $J \subseteq \mathcal{B}(H)$, an $\langle F\rangle \ltimes \mathsf{H}_J$-equivariant bijection from the set of unipotent elements of H_J to the set of nilpotent elements of the Lie algebra of H_J.*

Recall that for $S \in \mathfrak{T}(H)$ there is a unique F-stable facet $J \subset \mathcal{B}(H)$ such that $J^F = \mathcal{B}(S)^F$, and that S denotes the image of S in H_J. Let $\mathbf{Z}$ denote the maximal k-split torus in the center of $\mathbf{H}$. The following lemma is a special case of a result in [18].

Lemma 12.4.3. *Assume Restrictions 12.4.1 hold. For each $S \in \mathfrak{T}(H)$, with (S, J) as above, and any $X_\mathsf{S} \in \mathrm{Lie}(S) \cap \mathfrak{h}_J^F$ whose projection to L_S^F is an element $\bar{X}_\mathsf{S}$ as in 12.4.2, we have the equality*

$$R(H,S,1)(\gamma) = \varepsilon(\mathbf{H},\mathbf{Z}) \cdot \hat{\mu}^{H^F}_{X_\mathsf{S}}(\log(\gamma)),$$

for every regular semisimple $\gamma \in H^F_{0^+}$, where $\log$ is as in Lemma 12.4.2

Proof. Fix a regular semisimple $\gamma \in H^F_{0^+}$. Let $d\ell$ denote the Haar measure on H^F_J with $\mathrm{meas}_{d\ell}(H^F_J) = 1$. We have

$$R(H,S,1)(\gamma) = \frac{\mathrm{meas}_{dz}(Z^F_J)}{\mathrm{meas}_{dh}(H^F_J)} \cdot \int_{H^F/Z^F} dh^* \int_{H^F_J} \dot{R}^{\mathsf{H}_J}_{\mathsf{S},1}({}^{h\ell}\gamma)\, d\ell.$$

On the other hand, from [2, Proposition 3.3.1], we can write

$$\hat{\mu}^{H^F}_{X_\mathsf{S}}(X) = \frac{\mathrm{meas}_{dz}(Z^F_J)}{\mathrm{meas}_{ds}((C_H(X_\mathsf{S}))^F_J)} \cdot \int_{H^F/Z^F} dh^* \int_{H^F_J} d\ell \int_{H^F_J} \Lambda(B({}^{\ell' h\ell}X, X_\mathsf{S}))\, d\ell' \tag{36}$$

where $X = \log\gamma$ and ds is the Haar measure on $(C_H(X_\mathsf{S}))^F$, normalized as in Section 5. (Note that in [2] the quotient measure is normalized slightly differently.)

From [2, Lemma 6.1.1] we see that the inner integral in Equation (36) is zero unless ${}^{h\ell}X \in \mathfrak{h}^F_J$. Consequently, it is enough to show that if ${}^{h\ell}X \in \mathfrak{h}^F_J$, then

$$\dot{R}^{\mathsf{H}_J}_{\mathsf{S},1}({}^{h\ell}\gamma) = \frac{\varepsilon(\mathbf{H},\mathbf{Z}) \cdot \mathrm{meas}_{dh}(H^F_J)}{\mathrm{meas}_{ds}((C_H(X_{\mathsf{S}}))^F_J)} \cdot \int_{H^F_J} \Lambda(B({}^{\ell' h\ell}X, X_{\mathsf{S}}))\, d\ell'. \tag{37}$$

But since

$$\mathrm{meas}_{ds}((C_H(X_{\mathsf{S}}))^F_J) = \frac{|\mathsf{S}^F|}{|\mathsf{L}^F_{\mathsf{S}}|^{1/2}},$$

and

$$\varepsilon(\mathbf{H},\mathbf{Z}) = \varepsilon(\mathsf{H}_J, \mathsf{S})$$

because S is minisotropic, Equation (37) follows immediately from [31, Theorem 3] and the properties of the map $\log$ in Lemma 12.4.2. $\square$

12.5. **Completion of the proof of stability.** In this section, we prove 12.0.1, assuming that Restrictions 12.4.1 are in place.

Let $\mathcal{T}_{\mathrm{st}}$ be an H-stable class in $\mathfrak{T}(H)$. We fix $S_0 \in \mathcal{T}_{\mathrm{st}}$ and $X_0 := X_{S_0} \in \mathfrak{h}^F$ as in 12.4.3.

Lemma 12.5.1. *The map $X \mapsto C_G(X)$ induces a surjective map*

$$c : [\mathrm{Ad}(H)\cdot X_0]^F/H^F \longrightarrow \mathcal{T}_{\mathrm{st}}/H^F,$$

whose fiber over the H^F-orbit of $S \in \mathcal{T}_{\mathrm{st}}$ is in bijection with $N(H, S^F)/N(H^F, S)$.

Proof. Note that

$$[\mathrm{Ad}(H)\cdot X_0]^F = \{{}^hX_0 :\ h \in H,\ \text{ and }\ h^{-1}F(h) \in S_0\},$$

and recall that

$$\mathcal{T}_{\mathrm{st}} = \{{}^hS_0 :\ h \in H,\ \text{ and }\ {}^h(S_0^F) = ({}^hS_0)^F\}.$$

Since $h^{-1}F(h) \in S_0$ if and only if ${}^h(S_0^F) = ({}^hS_0)^F$, it follows that

$$\{C_G(X) :\ X \in [\mathrm{Ad}(H)\cdot X_0]^F\} = \mathcal{T}_{\mathrm{st}}.$$

One checks that for $k, h \in H$ with ${}^kX, {}^hX \in \mathfrak{h}^F$, we have that $C_G({}^kX)$ is H^F-conjugate to $C_G({}^hX)$ if and only if there is $\ell \in H^F$ such that $k^{-1}\ell h \in N(H, S_0^F)$. It follows that the fiber of c over the H^F-orbit of ${}^kS_0$ consists of the distinct H^F-orbits $\mathrm{Ad}(H^F)({}^{gn}X)$, as n ranges over $N(H, ({}^kS_0)^F)/N(H^F, {}^kS_0)$. $\square$

Lemma 12.5.2. *If Restrictions 12.4.1 hold, then Lemma 12.0.1 holds.*

Proof. Using Lemmas 12.4.3 and 12.5.1, we have

$$\begin{aligned}Q(H,\mathcal{T}_{\mathrm{st}})(\gamma_u) &= \sum_{S\in\mathcal{T}_{\mathrm{st}}/H^F} |N(H,S^F)/N(H^F,S)|\cdot R(H,S,1)(\gamma_u)\\ &= \varepsilon(\mathbf{H},\mathbf{Z})\cdot \sum_{S\in\mathcal{T}_{\mathrm{st}}/H^F} |N(H,S^F)/N(H^F,S)|\cdot \hat{\mu}^{H^F}_{X_S}(\log(\gamma_u))\\ &= \varepsilon(\mathbf{H},\mathbf{Z})\cdot \sum_{X\in[\mathrm{Ad}(H)\cdot X_0]^F/H^F} \hat{\mu}^{H^F}_{X}(\log(\gamma_u))\\ &= \varepsilon(\mathbf{H},\mathbf{Z})\cdot \hat{S}^{\mathfrak{h}}_{X_0}(\log(\gamma_u)).\end{aligned}$$

A similar result holds for $Q({}^gH,\iota_g\mathcal{T}_{\mathrm{st}})({}^g\gamma_u)$. The result now follows from Examples 12.2.1, 12.2.2 and Lemma 12.2.3. ☐

13. L-PACKETS ARISING FROM THE OPPOSITION INVOLUTION

We illustrate our L-packets with a canonical example. For simplicity, take $\mathbf{G}$ to be absolutely quasi-simple and simply-connected, and let w_o be the unique element of W_o such that $w_o\cdot C = -C$. Up to isomorphism, there is a unique K-split k-structure on $\mathbf{G}$ for which the Frobenius F acts on X by $\vartheta = -w_o$. This k-structure is quasi-split, and we have $H^1(\mathrm{F},G)=1$.

We tabulate the groups $\mathbf{G}$ below, using their names from the tables of [60], and give the number $r := [X^{\vartheta}_{ad} : j(X^{\vartheta})]$ of generic representations in an L-packet $\Pi(\varphi)$ (see Lemma 6.2.2).

$\mathbf{G}$	${}^2A'_{2m}$	${}^2A'_{2m-1}$	B_n	C_n	D_{2m}	${}^2D_{2m+1}$	G_2	F_4	${}^2E_6$	E_7	E_8
r	1	2	2	2	4	2	1	1	1	2	1

Now let φ be a TRSELP whose associated w is w_o. Since $w_o\vartheta = -\mathrm{Id}$, the L-packet $\Pi(\varphi)$ is parametrized by

$$\mathrm{Irr}(C_\varphi) = X/2X,$$

where $X = X^\circ$ is the co-root lattice of $\mathbf{T}$ in $\mathbf{G}$. In particular, $|\Pi(\varphi)| = 2^n$, where n is the absolute rank of $\mathbf{G}$. With Haar measure normalized as in Section 5.3, each representation $\pi\in\Pi(\varphi)$ has formal degree

$$\mathrm{Deg}(\pi) = (q^{1/2}+q^{-1/2})^{-n}.$$

Since $W_o^{w_o\vartheta} = W_o$, the full Weyl group W_o acts on $\mathrm{Irr}(C_\varphi)$. This action has several interpretations.

First, by Lemma 9.6.1, the W_o-orbits on $X/2X$ are in bijection with the G^{F}-orbits in the G-stable class $\mathcal{T}_{w_o}$. The tori in this stable class are k-isomorphic to $\mathbf{U}_1^n$.

Second, the W_o-orbits in $X/2X$ are in bijection, via evaluation at -1, with conjugacy-classes of 2-torsion elements in $\mathbf{G}$ (or G, since $\mathbf{G}$ is simply-connected, and Lemma 2.9.1 applies). For each $\lambda\in X$, we have

$$x_\lambda = \frac{1}{2}t_\lambda\cdot o,$$

and the root datum, with F_λ-action, of G_{x_λ} is that of the centralizer in $\mathbf{G}$ of $\lambda(-1)$. The generic representations in $\Pi(\varphi)$ correspond to the 2-torsion elements in the center of $\mathbf{G}$.

For exceptional groups, the 2-torsion picture places a strong limitation on the type of inducing parahorics that appear in $\Pi(\varphi)$. For example, in E_8 there are three W_o-orbits in $X/2X$. The L-packet $\Pi(\varphi)$ has $256 = 1 + 120 + 135$ representations, induced from parahoric subgroups of type $E_8,\ A_1E_7,\ D_8$, respectively.

Third and finally, the generic representations in $\Pi(\varphi)$ are parametrized by the W_o-invariants:

$$\mathrm{Irr}(C_\varphi)_{gen} = \mathrm{Irr}(C_\varphi)^{W_o}.$$

Containment "$\subseteq$" is shown in Remark 6.2.5. For the other containment, note that a W_o-invariant element in $X_{ad}/2X_{ad}$ corresponds to a central 2-torsion element in $\mathbf{G}_{ad}$, hence must be trivial. Containment "$\supseteq$" now follows from Lemma 6.2.1.

APPENDIX A. GOOD BILINEAR FORMS AND REGULAR ELEMENTS

In the appendices, we prove various results used in the proof of stability. Here $\mathbf{G}$ is any connected reductive k-group, not necessarily split over K, and F is the corresponding Frobenius automorphism of G.

A.1. **Good bilinear forms.** We say that a symmetric bilinear form B on $\mathfrak{g}$ is "good" if B is $\langle F\rangle \ltimes G$-equivariant, nondegenerate, and restricts to the Killing form, B', on the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$.

Let $\mathfrak{g}_{x,t}$, $\mathfrak{g}_{x,t^+}$ be the Moy-Prasad filtration subalgebras of $\mathfrak{g}$ attached to $x \in \mathcal{B}(G)$ and $t \in \mathbb{R}$. (See Section B.5 below for a brief introduction to Moy-Prasad filtrations.)

Lemma A.1.1. *If $p > n+1$, where $n \geq 2$ is the dimension of a faithful k-representation of $\mathbf{G}$, then there exists a good bilinear form B on $\mathfrak{g}$ which induces, for all $x \in \mathcal{B}(G)$ and for all $t \in \mathbb{R}$, a nondegenerate pairing*

$$\mathfrak{g}_{x,t}/\mathfrak{g}_{x,t^+} \times \mathfrak{g}_{x,(-t)}/\mathfrak{g}_{x,(-t)^+} \to \mathfrak{F}.$$

Remark A.1.2. If B satisfies Lemma A.1.1 and x is F-fixed, then the induced pairing,

$$\mathfrak{g}_{x,t}/\mathfrak{g}_{x,t^+} \times \mathfrak{g}_{x,(-t)}/\mathfrak{g}_{x,(-t)^+} \to \mathfrak{F},$$

is $\langle F\rangle \ltimes G_x$-equivariant.

Proof. The existence of such a form B follows from the proof of [4, Proposition 4.1] under the condition that $p \nmid B'(H_\alpha, H_\alpha)$ for any root α of a maximal torus $\mathbf{T} \subset \mathbf{G}$, where H_α is the corresponding Chevalley basis vector in the Lie algebra of $\mathbf{T}$.

Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ be the simple factors of $\mathfrak{g}'$. Let m_i^* be the sum of the coefficients in the expression of the highest co-root of $\mathfrak{g}_i$ in terms of simple co-roots. From [54, I.4.8], any prime dividing $B'(H_\alpha, H_\alpha)$ must divide $6(m_i^*+1)$, where $\mathfrak{g}_i$ is the factor containing α.

Let $m^* = \max\{m_i : 1 \leq i \leq r\}$. We have $n \geq m^*$. To prove this, one may assume $\mathfrak{g}$ simple, and check the result case-by-case (recall that k has characteristic zero). The result follows. $\square$

A.2. **Regular elements.** Suppose J is an F-stable facet in $\mathcal{B}(G)$ and S is a maximal $\mathfrak{f}$-torus in G_J. We wish to establish conditions on p and q which will guarantee that the Lie algebra L_S^F contains a regular semisimple element of L_J.

Let F_0 be the q-power Frobenius of $\mathfrak{F}/\mathfrak{f}$. Let Φ_J be the set of $\mathfrak{F}$-roots of G_J with respect to S, and let $\ell = \dim \mathsf{L}_\mathsf{S}$. There is a permutation τ of Φ_J such that

$$\alpha \circ F = F_0 \circ \tau(\alpha)$$

for all $\alpha \in \Phi_J$. Let d be the order of τ. Let $\bar{\Phi}_J$ be the set of orbits in Φ_J under the group generated by τ and $\alpha \mapsto -\alpha$.

Lemma A.2.1. *If $p \neq 2$ and $q > \left|\bar{\Phi}_J\right|$, then L_S^F contains a regular element of L_J.*

Proof. Set $\mathfrak{f}_d := \mathfrak{f}^{F_0^d}$, $\mathsf{L}_\mathsf{S}^d := \mathsf{L}_\mathsf{S}^{F^d}$. The $\mathfrak{f}$-linear map

$$\mathrm{tr} : \mathsf{L}_\mathsf{S}^d \longrightarrow \mathsf{L}_\mathsf{S}^F,$$

given by

$$\mathrm{tr}\, X := \sum_{j=0}^{d-1} F^j(X)$$

has the property that for all $\alpha \in \Phi_J$, the composition $\alpha \circ \mathrm{tr}$ is not identically zero on L_S^d. Indeed, suppose there exists $\alpha \in \Phi_J$ for which $\alpha \circ \mathrm{tr}$ is zero. Since S is $\mathfrak{f}_d$-split, we can assume that the Chevalley basis vector H_α belongs to L_S^d. For all $t \in \mathfrak{f}_d$, we have

$$\begin{aligned} 0 &= \alpha(\mathrm{tr}(tH_\alpha)) \\ &= \alpha(tH_\alpha + t^q H_{\tau\alpha} + \cdots + t^{q^{d-1}} H_{\tau^{d-1}\alpha}) \\ &= t\langle \alpha, \check{\alpha}\rangle + t^q \langle \alpha, \check{\tau\alpha}\rangle + \cdots + t^{q^{d-1}} \langle \alpha, \check{\tau^{d-1}\alpha}\rangle. \end{aligned}$$

Since $p \neq 2$, we have $\langle \alpha, \check{\alpha}\rangle \neq 0$. Hence we have a nonzero polynomial of degree at most q^{d-1} but with q^d zeros in $\mathfrak{F}$, a contradiction.

Thus, for each $\alpha \in \Phi_J$ we have a nonzero $\mathfrak{f}$-linear map

$$\alpha \circ \mathrm{tr} : \mathsf{L}_\mathsf{S}^d \longrightarrow \mathfrak{f}_d.$$

Let Z_α be the kernel of this linear map. Since

$$F_0\big(\tau(\alpha)(\mathrm{tr}\, X)\big) = \alpha(\mathrm{tr}\, X),$$

we have $Z_{\tau(\alpha)} = Z_\alpha$. Also, we have $Z_\alpha = Z_{-\alpha}$. Hence the subspace Z_α depends only on the image of α in $\bar{\Phi}_J$.

It suffices to show that the set

$$\mathsf{L}_\mathsf{S}^\circ := \mathsf{L}_\mathsf{S}^d \setminus \bigcup_{\alpha \in \bar{\Phi}_J} Z_\alpha$$

is nonempty. We have

$$\left|\mathsf{L}_\mathsf{S}^\circ\right| = \left|\mathsf{L}_\mathsf{S}^d\right| - \left|\bigcup_{\bar{\alpha} \in \bar{\Phi}_J} Z_\alpha\right| \geq q^{\ell d} - \left|\bar{\Phi}_J\right| \left|Z_\beta\right|,$$

where β is chosen so that $|Z_\beta| = \max\{|Z_\alpha| : \alpha \in \bar{\Phi}_J\}$. Since $\dim_{\mathfrak{f}} Z_\beta \leq \ell d - 1$, we have $|Z_\beta| \leq q^{\ell d-1}$. Consequently,

$$|\mathsf{L}_{\mathsf{S}}^\circ| \geq q^{\ell d} - \left|\bar{\Phi}_J\right| q^{\ell d-1}.$$

Therefore $q > \left|\bar{\Phi}_J\right|$ ensures that $\mathsf{L}_{\mathsf{S}}^\circ$ is nonempty. □

Note that

$$|\bar{\Phi}_J| \leq \nu(\mathfrak{g}),$$

where $\nu(\mathfrak{g})$ is the number of positive (absolute) roots in $\mathfrak{g}$.

If p is not a torsion prime for G_J, then the centralizer in G_J of any semisimple element in L_J is connected [58, Theorem 3.14]. The torsion primes of G_J are also torsion primes of $\mathbf{G}$. Consequently, if p is not a torsion prime for $\mathbf{G}$, then any regular element of L_{S} has centralizer equal to S. The torsion primes of $\mathbf{G}$ are less than the number m^* defined in the proof of Lemma A.1.1. Putting all this together with Lemma A.2.1 gives the following result. Let n be as in Lemma A.1.1.

Lemma A.2.2. *If $p > n+1$ and $q > \nu(\mathfrak{g})$, then for every F-stable facet J in $\mathcal{B}(G)$, and every maximal F-stable maximal torus $\mathsf{S} \subset \mathsf{G}_J$, the Lie algebra $\mathsf{L}_{\mathsf{S}}^F$ contains an element whose centralizer in G_J is exactly S.*

APPENDIX B. A LOGARITHM MAPPING FOR G

Let e denote the ramification degree of k over $\mathbb{Q}_p$, and let $\varphi : \mathbf{G} \longrightarrow \mathbf{GL}_n$ be a faithful k-representation. We suppose that $\nu(K^\times) = \mathbb{Z}$ where ν is the valuation on K. For notational convenience we sometimes write $(G^F)_{0^+}$ instead of $(G_{0^+})^F$.

The purpose of this appendix is to prove the following Lemma.

Lemma B.0.3. *If $p \geq (2+e)n$, then there exists a $\langle F\rangle \ltimes G$-equivariant bijective map*

$$\log\colon G_{0^+} \to \mathfrak{g}_{0^+}$$

which, for each F-stable facet J in $\mathcal{B}(G)$, induces a $\langle F\rangle \ltimes \mathsf{G}_J$-equivariant bijective map from the set of unipotent elements in G_J to the set of nilpotent elements in L_J.

B.1. **The exponential map for the general linear group.** Recall that q is the order of the residue field of k. For each $X \in \mathfrak{gl}_n(k)$ we have $X \in \mathfrak{gl}_n(k)_{0^+}$ if and only if $|\mu| \leq q^{-1/n}$ for each eigenvalue μ of X. For each $g \in \mathrm{GL}_n(k)$, we have $g \in \mathrm{GL}_n(k)_{0^+}$ if and only if $|\mu - 1| \leq q^{-1/n}$ for each eigenvalue μ of g.

We begin with a technical result.

Lemma B.1.1. *If $p > en+1$, then*

(1) $\dfrac{q^{-j/n}}{|j!|} \leq q^{-1/n}$ *for $j \geq 2$ and*

(2) $\dfrac{q^{-j/n}}{|j!|} \to 0.$

Proof. Set

$$A(j) := \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{j}{p^2} \right\rfloor + \left\lfloor \frac{j}{p^3} \right\rfloor + \cdots .$$

Note that

$$\frac{q^{-j/n}}{|j!|} = \frac{q^{-j/n}}{q^{-eA(j)}} = q^{(neA(j)-j)/n}.$$

To establish item (1) it is enough to show

$$neA(j) - j \leq -1, \tag{38}$$

and to establish item (2) it is enough to show

$$neA(j) - j \leq \frac{-j}{(p-1)}. \tag{39}$$

Write

$$j = \sum_{i=0}^{\ell} b_i p^i$$

with $b_i \in \{0, 1, 2, \ldots, (p-1)\}$ and $b_\ell \neq 0$. We have

$$\left\lfloor \frac{j}{p^t} \right\rfloor = \sum_{t}^{\ell} b_i p^{i-t}$$

for $1 \leq t \leq \ell$. Consequently, $(p-1)A(j) = \sum_{i=0}^{\ell} b_i(p^i - 1) = j - \sum b_i$. Thus,

$$enA(j) < (p-1)A(j) \leq (j-1),$$

establishing (38), and

$$\begin{aligned}
&(p-1)(neA(j) - j) \\
&\qquad = nej - ne\sum b_i - (p-1)j \\
&\qquad < (ne - p + 1)j \\
&\qquad \leq -j
\end{aligned}$$

establishing (39). □

Our assumption $p \geq (2+e)n$ ensures that $p > en + 1$. Thus, thanks to Lemma B.1.1 and [25, §10.1], the map $\exp$ defined by

$$X \mapsto \sum_{\ell=0}^{\infty} \frac{X^\ell}{\ell!}$$

converges to a $\mathrm{GL}_n(k)$-equivariant bijective analytic map from $\mathfrak{gl}_n(k)_{0+}$ to $\mathrm{GL}_n(k)_{0+}$. We extend $\exp$ to a $\langle F \rangle \ltimes \mathrm{GL}_n(K)$-equivariant bijective analytic map from $\mathfrak{gl}_n(K)_{0+}$ to $\mathrm{GL}_n(K)_{0+}$ as follows. For each $m \in \mathbb{Z}_{\geq 1}$, by replacing k by K^{F^m} in the discussion above, we obtain an analytic map $\exp_m\colon (\mathfrak{gl}_n(K)_{0+})^{F^m} \to (\mathrm{GL}_n(K)_{0+})^{F^m}$. Thus, if $X \in \mathfrak{gl}_n(K)_{0+}$, we may choose $m \in \mathbb{Z}_{\geq 1}$ so that $X \in (\mathfrak{gl}_n(K)_{0+})^{F^m}$ and define $\exp(X) := \exp_m(X) \in (\mathrm{GL}_n(K)_{0+})^{F^m} \subset$

$\mathrm{GL}_n(K)_{0^+}$. This gives a well-defined $\langle F\rangle \ltimes \mathrm{GL}_n(K)$-equivariant bijective analytic map from $\mathfrak{gl}_n(K)_{0^+}$ to $\mathrm{GL}_n(K)_{0^+}$

For each facet J in $\mathcal{B}(\mathrm{GL}_n(K))$, the map $\exp$ takes $\mathfrak{gl}_n(K)_J \cap \mathfrak{gl}_n(K)_{0^+}$ to $\mathrm{GL}_n(K)_J \cap \mathrm{GL}_n(K)_{0^+}$. Finally, the map $\exp$ also takes the Haar measure on $\mathfrak{gl}_n(k)$ into the Haar measure on $\mathrm{GL}_n(k)$.

B.2. **The logarithmic mapping ψ.** From [9, III, §7.3, 2, Proposition 3], there is a neighborhood V of 0 in $\mathfrak{g}^F$ and a map $\phi\colon V \to G^F$ such that $\phi(V)$ is an open subgroup of G^F and $\phi\colon V \to \phi(V)$ is a k-analytic isomorphism of analytic manifolds with the property that $\phi(mX) = \phi(X)^m$ for all $m \in \mathbb{Z}$ and for all $X \in V$. From [9, III, §7.6, 6, Proposition 10], there is a neighborhood U of the identity in $(G^F)_{0^+}$ and a unique k-analytic map $\psi\colon (G^F)_{0^+} \to \mathfrak{g}^F$ such that $\psi(U) = V$, $\phi \circ \psi = 1_U$, and $\psi(g^m) = m\psi(g)$ for all $g \in (G^F)_{0^+}$ and all $m \in \mathbb{Z}$. Note that ψ is locally injective, hence injective.

Recall that the exponential map, $\exp$, for the general linear group was defined in section B.1. The unique map from $\mathrm{GL}_n(k)_{0^+}$ to $\mathfrak{gl}_n(k)$ determined (in the sense of the previous paragraph) by $\exp$ is called $\log$. It has the usual power series expansion. Since $p \geq (2+e)n > en+1$, the map $\log\colon \mathfrak{gl}_n(k)_{0+} \to \mathrm{GL}_n(k)_{0^+}$ is the inverse of $\exp\colon\ \mathrm{GL}_n(k)_{0^+} \to \mathfrak{gl}_n(k)_{0^+}$ (see, for example, [25, Lemma 10.1]).

From [9, III, §4.4, Corollary 2] there is a neighborhood $V' \subset V$ in $\mathfrak{g}^F$ such that

$$\varphi(\phi(X)) = \exp(d\varphi(X)) \tag{40}$$

for all $X \in V'$ and

$$d\varphi(\psi(g)) = \log(\varphi(g))$$

for all $g \in \phi(V')$. Suppose $g \in (G)^F_{0^+}$. Choose $m \in \mathbb{Z}_{\geq 1}$ so that $g^{p^m} \in \phi(V')$. We have

$$d\varphi(\psi(g)) = p^{-m} \cdot d\varphi(\psi(g^{p^m})) = p^{-m} \cdot \log(\varphi(g^{p^m})) = \log(\varphi(g)).$$

Thus

$$d\varphi(\psi(g)) = \log(\varphi(g)) \tag{41}$$

for all $g \in (G_{0^+})^F$.

B.3. **An extension of ψ.** The map ψ has a unique extension, which we shall also call ψ, to a $\langle F\rangle \ltimes G$-equivariant map from G_{0^+} to $\mathfrak{g}$. Indeed, for each $m \in \mathbb{Z}_{\geq 1}$, by replacing k by K^{F^m} in the discussion above, we obtain a (unique) K^{F^m}-analytic map $\psi_m\colon (G_{0^+})^{F^m} \to \mathfrak{g}^{F^m}$ for which

$$d\varphi(\psi_m(g)) = \log(\varphi(g))$$

for all $g \in (G_{0^+})^{F^m}$. Thus, since $d\varphi$ is injective, for $m' \geq m \geq 1$ we have $\psi_{m'}(g) = \psi_m(g)$ whenever $g \in (G_{0^+})^{F^m}$. In particular, $\psi_m(g) = \psi_1(g)$ whenever $g \in (G_{0^+})^F$. Thus, we may define $\psi\colon G_{0^+} \to \mathfrak{g}$ by setting $\psi(g) = \psi_m(g)$ whenever $g \in (G_{0^+})^{F^m}$. To see that ψ is $\langle F\rangle \ltimes G$ - equivariant, it is enough to check that it is F-equivariant. Since $d\varphi$ is injective, it is enough to check that $d\varphi(\psi(Fg)) = d\varphi(F(\psi(g)))$ for all $g \in G_{0^+}$. However,

$$\begin{aligned} d\varphi(\psi(Fg)) &= \log(\varphi(Fg)) = F\log(\varphi(g)) \\ &= Fd\varphi(\psi(g)) = d\varphi(F\psi(g)). \end{aligned}$$

B.4. **The adjoint representation and ψ.** Suppose $Y \in \mathfrak{gl}_n(K)_{0^+}$. Since the valuations of the eigenvalues of $\mathrm{ad}(Y)$ are bounded (below) by those of Y, and $p \geq (2+e)n$, the power series for $\exp(\mathrm{ad}(Y))$ converges in $\mathrm{GL}(\mathfrak{gl}_n)(K)$, and we have

$$\exp(\mathrm{ad}(Y)) = \mathrm{Ad}(\exp(Y)). \tag{42}$$

Similarly, for all $g \in \mathrm{GL}_n(K)_{0^+}$ we have

$$\log(\mathrm{Ad}(g)) = \mathrm{ad}(\log(g)). \tag{43}$$

For $h \in G_{0^+}$, we define $\log(\mathrm{Ad}(h)) \in \mathfrak{gl}(\mathfrak{g})(K)$ by

$$\log(\mathrm{Ad}(h)) := -\sum_{m\geq 1} \frac{(1-\mathrm{Ad}(h))^m}{m}.$$

Thus, for all $h \in G_{0^+}$ and $X \in \mathfrak{g}$,

$$\begin{aligned} d\varphi[\mathrm{ad}(\psi(h))X] &= [\mathrm{ad}(d\varphi(\psi(h)))]d\varphi(X) \\ &\qquad \text{(from Equation (41))} \\ &= [\mathrm{ad}(\log(\varphi(h)))]d\varphi(X) \\ &\qquad \text{(from Equation (43))} \\ &= [\log(\mathrm{Ad}(\varphi(h)))]d\varphi(X) \\ &= d\varphi(\log(\mathrm{Ad}(h))(X)). \end{aligned}$$

Since $d\varphi$ is injective, we conclude that

$$\log(\mathrm{Ad}(h))X = \mathrm{ad}(\psi(h))X \tag{44}$$

for all $h \in G_{0^+}$ and $X \in \mathfrak{g}$.

B.5. **A brief introduction to the filtrations of Moy and Prasad.** We recall here what we need from the theory of Moy-Prasad filtration lattices ([44, 43]).

Let T denote the group of K-rational points of a maximally K-split torus in $\mathbf{G}$. Let $\mathcal{A}$ denote the apartment in $\mathcal{B}(G)$ corresponding to T, let Φ denote the set of roots of G with respect to T, and let Δ denote the set of affine roots of G with respect to T and our valuation on K. The elements of Δ are affine functions on $\mathcal{A}$. For $\delta \in \Delta$, we let $\dot{\delta} \in \Phi$ denote the gradient of δ.

For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ denote the corresponding root space in $\mathfrak{g}$. For $\delta \in \Delta$, define the lattices $\mathfrak{g}_\delta^+$ and $\mathfrak{g}_\delta$ in $\mathfrak{g}_{\dot{\delta}}$ as follows: Choose a facet J in $\mathcal{A}$ on which δ is zero. Set

$$\mathfrak{g}_\delta := \mathfrak{g}_J \cap \mathfrak{g}_{\dot{\delta}} \text{ and } \mathfrak{g}_\delta^+ := \mathfrak{g}_J^+ \cap \mathfrak{g}_{\dot{\delta}}.$$

These definitions are independent of the choice of J.

Since $\mathbf{G}$ is K-quasi-split, the centralizer $M := C_G(T)$ is the group of K-rational points of a maximal K-torus $\mathbf{M}$ of $\mathbf{G}$. Let $\mathfrak{m}$ denote the Lie algebra of M. For $s \in \mathbb{R}$, we define

$$\mathfrak{m}_s := \{X \in \mathfrak{m} \mid \nu(d\chi(X)) \geq s \text{ for all } \chi \in \mathbf{X}^*(\mathbf{M})\}.$$

For $x \in \mathcal{A}$ and $s \in \mathbb{R}$, we define the lattice

$$\mathfrak{g}_{x,s} := \mathfrak{m}_s \oplus \sum_{\delta \in \Delta;\, \delta(x) \geq s} \mathfrak{g}_\delta.$$

For $t \geq s$ we have $\mathfrak{g}_{x,t} \subset \mathfrak{g}_{x,s}$; in fact,

$$\bigcup \mathfrak{g}_{x,s} = \mathfrak{g}, \text{ and } \bigcap_s \mathfrak{g}_{x,s} = \{0\}.$$

We set

$$\mathfrak{g}_{x,s^+} := \bigcup_{t>s} \mathfrak{g}_{x,t}.$$

If y is in $\mathcal{B}(G)$, then there is a g in G so that $gy \in \mathcal{A}$. For $s \in \mathbb{R}$, we define

$$\mathfrak{g}_{y,s} := {}^g\mathfrak{g}_{x,s} \text{ and } \mathfrak{g}_{y,s^+} := {}^g\mathfrak{g}_{x,s^+}.$$

This is independent of the choice of g.

Recall [3, §3] that, for $s \in \mathbb{R}$, we have the closed, open, G-invariant subsets

$$\mathfrak{g}_s := \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,s} \text{ and } \mathfrak{g}_{s^+} := \bigcup_{t>s} \mathfrak{g}_t.$$

For each $s \in \mathbb{R}_{\geq 0}$ and each $x \in \mathcal{B}(G)$, we also define, in a completely analogous manner, Moy-Prasad filtration subgroups $G_{x,s} \leq G_{x,0} = G_x$ (see [43]).

The Moy-Prasad filtration lattices and subgroups have the following properties (which we shall use without further comment). The first two properties are proved in [45, §2], the third is a formal consequence of the definitions, and the final is [1, Proposition 1.4.3].

(1) For $s, t \in \mathbb{R}$ and $x \in \mathcal{B}(G)$, we have $[\mathfrak{g}_{x,t}, \mathfrak{g}_{x,s}] \subset \mathfrak{g}_{x,(t+s)}$.
(2) For $s, t \in \mathbb{R}_{\geq 0}$ and $x \in \mathcal{B}(G)$, we have $(G_{x,s}, G_{x,t}) \subset G_{x,(t+s)}$.
(3) For $s \in \mathbb{R}$ and $x \in \mathcal{B}(G)$ we have

$$\varpi \cdot \mathfrak{g}_{x,s} = \mathfrak{g}_{x,(s+1)}.$$

(4) For $t \in \mathbb{R}_{\geq 0}$, $s \in \mathbb{R}$, and $x \in \mathcal{B}(G)$, we have $(\mathrm{Ad}(g) - 1)\mathfrak{g}_{x,s} \subset \mathfrak{g}_{x,s+t}$ for all $g \in G_{x,t}$.

B.5.1. *A technical result.* The purpose of this section is to establish a (weak) connection between the Moy-Prasad filtrations for $\mathfrak{g}$ and those for $\mathfrak{gl}_n(K)$. We do this so as to avoid introducing another constant (r_G below) into our hypotheses.

Fix a facet $J \subset \mathcal{B}(G)$. Define a continuous, piecewise-linear function $r\colon J \to \mathbb{R}_{>0}$ by sending $x \in J$ to the unique real number $r(x)$ for which

$$\mathfrak{g}_J^+ = \mathfrak{g}_{x,r(x)} \neq \mathfrak{g}_{x,r(x)^+}$$

After extending by zero, the function r becomes a continuous function on the closure of J. Hence, we may choose $x_J \in J$ so that

$$r(x_J) \geq r(x)$$

for all x in the closure of J. Define $r_J := r(x_J)$. The (rational) number r_J depends only on the G-conjugacy class of J. We set

$$r_G := \min_J r_J.$$

Note that, if J is F-stable, then, from the concavity of r and the Bruhat-Tits fixed-point theorem (see, for example, [60, §2.3.1]), we may assume that x_J is F-fixed.

Lemma B.5.1. *If C is an alcove in $\mathcal{B}(G)$, then $r_G = r_C$.*

Proof. Without loss of generality, $\mathbf{G}$ is semisimple. We can write

$$\mathcal{B}(G) = \prod_{i=1}^{m} \mathcal{B}(G_i)$$

where the $\mathbf{G}_i$ are the simple factors of $\mathbf{G}$. This decomposition respects the polysimplicial structure of $\mathcal{B}(G)$. If, with respect to this decomposition, $x \in J$ is written as

$$(x_1, \ldots, x_m),$$

then, from the way in which the Moy-Prasad filtration lattices are defined,

$$r(x) = \min\{r_1(x_1), \ldots, r_m(x_m)\}.$$

Here $r_1, \ldots, r_m$ are the analogues of $r \colon J \to \mathbb{R}$. Hence, we may in fact assume that $\mathbf{G}$ is simple.

Let J be a facet in $\mathcal{B}(G)$ and let C be an alcove in $\mathcal{B}(G)$. We shall show that $r_J \geq r_C$. After conjugating, we may assume that J is contained in the boundary of C and that $C \subset \mathcal{A}$. Let Δ_C denote the set of simple affine roots in Δ determined by C. Let Δ_J be the set

$$\{\delta \in \Delta_C \mid \operatorname{res}_J \delta \neq 0\} = \{\delta \in \Delta_C \mid \operatorname{res}_J \delta > 0\}.$$

We set

$$r'_J := \max_{x \in J} \min_{\delta \in \Delta_J} \delta(x),$$

and we let s denote the smallest positive number for which $\mathfrak{m}_s \neq \mathfrak{m}_{s+}$. From the way in which the Moy-Prasad filtration lattices are defined, we have

$$r_J = \min\{s, r'_J\} \text{ and } r_C = \min\{s, r'_C\}.$$

Thus, it is enough to show that $r'_C \leq r'_J$.

One can show that

$$r'_J = \Big[\prod_{\delta \in \Delta_J} r_\delta\Big] / \Big[\sum_{\delta' \in \Delta_J} \Big(\prod_{\delta \in \Delta_J \smallsetminus \{\delta'\}} r_\delta\Big)\Big]$$

where r_δ denotes the maximum value that δ obtains on the closure of J (and hence, on the closure of C).

Suppose J' is a facet in the closure of C such that J is contained in the closure of J' and $\dim(J') = \dim(J) + 1$. Let $\tilde{\delta} \in \Delta_C$ denote the affine root for which $\Delta_{J'} = \Delta_J \cup \{\tilde{\delta}\}$. Algebraic manipulation yields

$$r'_{J'} = \frac{r_{\tilde{\delta}}}{r_{\tilde{\delta}} + r'_J} \cdot r'_J < r'_J.$$

By iterating the above process, we conclude that $r'_C \leq r'_J$.

□

Remark B.5.2. If G is simple modulo its center and K-split, then

$$r_G = (1 + \sum m_\alpha)^{-1},$$

where m_α runs over the coefficients of the simple roots in the expression for the highest root. In particular, for $G = GL_n(K)$, we have $r_G = n^{-1}$.

Lemma B.5.3.

$$\mathfrak{g}_{0^+} = \mathfrak{g}_{r_G} \neq \mathfrak{g}_{r_G^+}.$$

Proof. Let C be an alcove in $\mathcal{B}(G)$. For all J in the closure of C we have $\mathfrak{g}_J^+ \subset \mathfrak{g}_C^+$. Consequently

$$\mathfrak{g}_{0^+} = \bigcup_{g \in G} {}^g\mathfrak{g}_C^+.$$

Thus, the equality follows from the fact that $\mathfrak{g}_C^+ = \mathfrak{g}_{x_C, r_G}$.

As in the proof of Lemma B.5.1, we may assume that $\mathbf{G}$ is simple; we use the notation of that proof.

Suppose $\mathfrak{g}_{r_G} = \mathfrak{g}_{r_G^+}$. Under this assumption, from [3, Corollary 3.2.2] we have $\mathfrak{g}_{x_C, r_G} \subset \mathfrak{g}_{x_C, r_G^+} + \mathcal{N}$. Thus, from, for example, [17, §4.1.2] or [43, Proposition 4.3], every coset Ξ in $\mathfrak{g}_{x_C, r_G}/\mathfrak{g}_{x_C, r_G^+}$ is killed by a one-parameter subgroup of $\mathsf{M} := M_0/M_0^+$; that is, for each Ξ there exists a one parameter subgroup $\mu = \mu_\Xi$ of the $\mathfrak{f}$-group M so that $\lim_{t \to 0} {}^{\mu(t)}\Xi = 0$. Consequently, in order to show that $\mathfrak{g}_{r_G} \neq \mathfrak{g}_{r_G^+}$, it is enough to find an $X \in \mathfrak{g}_{x_C, r_G}$ for which the coset $\Xi_X := X + \mathfrak{g}_{x_C, r_G^+}$ is not killed by any one-parameter subgroup of M.

If $s < r'_C$, then choose $X \in \mathfrak{m}_s \smallsetminus \mathfrak{m}_{s^+}$. Since M is abelian, no one parameter subgroup of M can kill Ξ_X. If $s \geq r'_C$, then for each $\delta \in \Delta_C$, we may choose X_δ in the root space corresponding to the gradient of δ so that $X_\delta \in \mathfrak{g}_{x_C, r_G}$ yet $X_\delta \notin \mathfrak{g}_{x_C, r_G^+}$. From, for example, [13, Proposition 1.2], the coset Ξ_X for

$$X := \sum_\delta X_\delta$$

cannot be killed by a one-parameter subgroup of M.

□

Lemma B.5.4. *We have $r_G \geq n^{-1}$. In particular, $G_C^+ = G_{x_C, 1/n}$ and $\mathfrak{g}_C^+ = \mathfrak{g}_{x_C, 1/n}$.*

Proof. Since we are assuming that $p \geq (2+e)n$, it follows that every K-torus in $\mathbf{G}$ or GL_n splits over a tame extension of K. Hence, from the discussion in [3, §3.6] we have

$$\mathfrak{g} \cap \mathfrak{gl}_n(K)_{s^+} = \mathfrak{g}_{s^+}$$

and

$$\mathfrak{g} \cap \mathfrak{gl}_n(K)_s = \mathfrak{g}_s$$

for all s. From Remark B.5.2 and Lemma B.5.3, we have $\mathfrak{gl}_n(K)_{0^+} = \mathfrak{gl}_n(K)_{1/n} \neq \mathfrak{gl}_n(K)_{1/n^+}$. We conclude that $1/n \leq r_G$. For the last assertion, note that $G_{x_C, r} = G_{x_C, 0^+}$ for $0 < r \leq r_G$, and likewise for $\mathfrak{g}_{x_C, r}$. □

B.6. **A logarithmic map for semisimple groups.** Suppose that $\mathbf{G}$ is semisimple.

Moy-Prasad filtrations and the adjoint representation.

Lemma B.6.1. *Suppose $x \in \mathcal{B}(G)$, $t \in \mathbb{R}$, and $X \in \mathfrak{g}_{x,t}$. We have*

$$X \notin \mathfrak{g}_{x,t^+}$$

if and only if there exist $q \in \mathbb{R}$ and $Q \in \mathfrak{g}_{x,q} \smallsetminus \mathfrak{g}_{x,q^+}$ such that

$$\mathrm{ad}(X)Q \in \mathfrak{g}_{x,(t+q)} \smallsetminus \mathfrak{g}_{x,(t+q)^+}.$$

Proof. "$\Leftarrow$": Suppose $X \in \mathfrak{g}_{x,t^+}$. Then for all $q \in \mathbb{R}$ and for all $Q \in \mathfrak{g}_{x,q}$ we have

$$\mathrm{ad}(X)Q \in \mathfrak{g}_{x,(t+q)^+},$$

a contradiction.

"$\Rightarrow$": From Lemma A.1.1, there exists $Y \in \mathfrak{g}_{x,-t} \smallsetminus \mathfrak{g}_{x,(-t)^+}$ such that

$$B(X,Y) \in R^\times. \tag{45}$$

For all $s \in \mathbb{R}$ we have

$$(\mathrm{ad}(X)\,\mathrm{ad}(Y))\mathfrak{g}_{x,s} \subset \mathfrak{g}_{x,s}.$$

Since $\mathbf{G}$ is semisimple, we have that B is the Killing form. We conclude from Equation (45) that there exist a $q \in \mathbb{R}$ and a $Z \in \mathfrak{g}_{x,(t+q)} \smallsetminus \mathfrak{g}_{x,(t+q)^+}$ such that

$$\mathrm{ad}(X)(\mathrm{ad}(Y)Z) \in \mathfrak{g}_{x,(t+q)} \smallsetminus \mathfrak{g}_{x,(t+q)^+}.$$

Let $Q := \mathrm{ad}(Y)Z \in \mathfrak{g}_{x,q}$. Since $\mathrm{ad}(X)Q \in \mathfrak{g}_{x,(t+q)} \smallsetminus \mathfrak{g}_{x,(t+q)^+}$, we conclude that $Q \in \mathfrak{g}_{x,q} \smallsetminus \mathfrak{g}_{x,q^+}$. $\square$

Corollary B.6.2. *Suppose $x \in \mathcal{B}(G)$, $s \in \mathbb{R}$, and $X \in \mathfrak{g}$. We have*

$$X \in \mathfrak{g}_{x,s}$$

if and only if for all $q \in \mathbb{R}$ and for all $Q \in \mathfrak{g}_{x,q}$ we have

$$\mathrm{ad}(X)Q \in \mathfrak{g}_{x,(s+q)}.$$

Proof. "$\Rightarrow$": There is nothing to prove.

"$\Leftarrow$": If $X \notin \mathfrak{g}_{x,s}$, then there exists $t < s$ such that $X \in \mathfrak{g}_{x,t} \smallsetminus \mathfrak{g}_{x,t^+}$. From Lemma B.6.1, as $X \notin \mathfrak{g}_{x,t^+}$, there exist $q \in \mathbb{R}$ and $Q \in \mathfrak{g}_{x,q}$ such that $\mathrm{ad}(X)Q \notin \mathfrak{g}_{x,(t+q)^+}$. But $\mathfrak{g}_{x,(s+q)} \subset \mathfrak{g}_{x,(t+q)^+}$, so $\mathrm{ad}(X)Q \notin \mathfrak{g}_{x,(s+q)}$. $\square$

Moy-Prasad filtrations and ψ, I.

Remark B.6.3. Since $p \geq (2+e)\cdot n$, we have $m \geq n\cdot\nu(m)+2$ for $m \geq 2$. If we assume that $m \geq (2n-1)$, then we have $m \geq n\cdot(2+\nu(m))-1$.

Lemma B.6.4. *Suppose $x \in \mathcal{B}(G)$ and $t \in \mathbb{R}_{\geq 1/n}$. If $g \in G_{x,t}$, then for all $q \in \mathbb{R}$ and for all $Q \in \mathfrak{g}_{x,q}$ we have*

$$\log(\mathrm{Ad}(g))Q \equiv (\mathrm{Ad}(g)-1)Q \quad \textit{modulo } \mathfrak{g}_{x,(2t+q)}.$$

Proof. Fix $q \in \mathbb{R}$ and $Q \in \mathfrak{g}_{x,q}$. For $m > 1$ we have
$$\frac{(1-\mathrm{Ad}(g))^m}{m} Q$$
belongs to
$$\begin{aligned}\frac{1}{m} \cdot \mathfrak{g}_{x,(q+tm)} &\subset \mathfrak{g}_{x,(q+tm-\nu(m))} \\ &\subset \mathfrak{g}_{x,(q+2t+t(m-2)-\nu(m))}.\end{aligned}$$
From Remark B.6.3 we have $m \geq n \cdot \nu(m) + 2$. Thus
$$\mathfrak{g}_{x,(q+2t+t(m-2)-\nu(m))} \subset \mathfrak{g}_{x,(q+2t)}.$$
Consequently,
$$\log(\mathrm{Ad}(g))Q \equiv (\mathrm{Ad}(g)-1)Q \text{ modulo } \mathfrak{g}_{x,(q+2t)}.$$
□

Corollary B.6.5. *For all $x \in \mathcal{B}(G)$ and for all $s \geq 1/n$, we have*
$$\psi(G_{x,s}) \subset \mathfrak{g}_{x,s}.$$

Proof. From Corollary B.6.2, it is enough to show that for all $q \in \mathbb{R}$, for all $Q \in \mathfrak{g}_{x,q}$, and for all $g \in G_{x,s}$, we have
$$\mathrm{ad}(\psi(g))Q \in \mathfrak{g}_{x,(s+q)}.$$
However, from Equation (44) we have
$$\mathrm{ad}(\psi(g))Q = \log(\mathrm{Ad}(g))Q,$$
and $\log(\mathrm{Ad}(g))Q \in \mathfrak{g}_{x,(s+q)}$ from Lemma B.6.4. □

Logarithmic behavior of ψ.

Lemma B.6.6. *Suppose $x \in \mathcal{B}(G)$ and $s, t \in \mathbb{R}_{>0}$ with $s \leq t$. If $g \in G_{x,s}$ and $h \in G_{x,t}$, then for all $q \in \mathbb{R}$ and for all $Q \in \mathfrak{g}_{x,q}$ we have*
$$(1-\mathrm{Ad}(gh))^m Q \equiv (1-\mathrm{Ad}(g))^m Q \ \textit{ modulo } \ \mathfrak{g}_{x,(t+q+(m-1)s)}$$
for all $m \in \mathbb{Z}_{\geq 1}$.

Proof. We will argue by induction on m. Suppose $x \in \mathcal{B}(G)$, $s, t \in \mathbb{R}_{>0}$ with $s \leq t$, $g \in G_{x,s}$, and $h \in G_{x,t}$. For $q \in \mathbb{R}$ and $Q \in \mathfrak{g}_{x,q}$, we define $T = T(Q,h) \in \mathfrak{g}_{x,(q+t)}$ by $T := {}^hQ - Q$.

When $m = 1$, we have that for all $q \in \mathbb{R}$ and for all $Q \in \mathfrak{g}_{x,q}$
$$\begin{aligned}(1-\mathrm{Ad}(gh))Q &= Q - {}^{gh}Q = Q - {}^gQ - {}^gT \\ &\equiv Q - {}^gQ \text{ modulo } \mathfrak{g}_{x,(q+t)} \\ &= (1-\mathrm{Ad}(g))Q.\end{aligned}$$

If
$$(1-\mathrm{Ad}(gh))^m Q' \equiv (1-\mathrm{Ad}(g))^m Q' \text{ modulo } \mathfrak{g}_{x,(q'+t+(m-1)s)}$$

for all $q' \in \mathbb{R}$ and for all $Q' \in \mathfrak{g}_{x,q'}$, then for all $q \in \mathbb{R}$ and for all $Q \in \mathfrak{g}_{x,q}$

$$\begin{aligned}
(1-\mathrm{Ad}(gh))^{(m+1)}Q &= (1-\mathrm{Ad}(gh))^m[(1-\mathrm{Ad}(gh))Q] \\
&\qquad \text{(since } s \leq t\text{, we have } (1-\mathrm{Ad}(gh))Q \in \mathfrak{g}_{x,(q+s)}) \\
&\equiv (1-\mathrm{Ad}(g))^m[(1-\mathrm{Ad}(gh))Q] \ \text{ modulo } \mathfrak{g}_{x,(q+s+t+(m-1)s)} \\
&\big(\equiv (1-\mathrm{Ad}(g))^m[(1-\mathrm{Ad}(gh))Q] \ \text{ modulo } \mathfrak{g}_{x,(q+t+ms)}\big) \\
&= [(1-\mathrm{Ad}(g))^{(m+1)}Q] - [(1-\mathrm{Ad}(g))^m({}^gT)] \\
&\qquad \text{(since } {}^gT \in \mathfrak{g}_{x,(t+q)} \text{ and } s \leq t) \\
&\equiv (1-\mathrm{Ad}(g))^{(m+1)}Q \ \text{ modulo } \mathfrak{g}_{x,(t+q+ms)}.
\end{aligned}$$

□

Lemma B.6.7. *Suppose $x \in \mathcal{B}(G)$ and $s, t \in \mathbb{R}$ with $t \geq s \geq 1/n$. For all $g \in G_{x,s}$ and for all $h \in G_{x,t}$, we have*

$$\psi(gh) \equiv \psi(g) + \psi(h) \ \textit{modulo } \mathfrak{g}_{x,(s+t)}.$$

Proof. Suppose x, s, t, g, and h are as in the statement of the lemma. From Corollary B.6.2, it will be enough to show that if $q \in \mathbb{R}$ and $Q \in \mathfrak{g}_{x,q}$, then

$$\mathrm{ad}[\psi(gh) - \psi(g) - \psi(h)]Q \in \mathfrak{g}_{x,(q+s+t)}.$$

Thus, from Equation (44), it will be enough to show that

$$[\log(\mathrm{Ad}(gh)) - \log(\mathrm{Ad}(g)) - \log(\mathrm{Ad}(h))]Q \in \mathfrak{g}_{x,(q+s+t)}.$$

Since

$$\frac{(1-\mathrm{Ad}(gh))^m}{m}, \ \frac{(1-\mathrm{Ad}(g))^m}{m}, \ \text{and } \frac{(1-\mathrm{Ad}(h))^m}{m}$$

all tend to zero in $\mathfrak{gl}(\mathfrak{g})(K)$, there exists $N \in \mathbb{Z}_{>1}$, independent of q and Q, so that

$$[\log(\mathrm{Ad}(gh)) - \log(\mathrm{Ad}(g)) - \log(\mathrm{Ad}(h))]Q$$

is equivalent to

$$-\sum_1^N \frac{1}{m} \cdot [(1-\mathrm{Ad}(gh))^m - (1-\mathrm{Ad}(g))^m - (1-\mathrm{Ad}(h))^m]Q$$

modulo $\mathfrak{g}_{x,(s+t+q)}$. Fix $2 \leq m \leq N$. From Lemma B.6.6 we have

$$\begin{aligned}
[(1-\mathrm{Ad}(gh))^m - &(1-\mathrm{Ad}(g))^m - (1-\mathrm{Ad}(h))^m]Q \\
&\equiv -(1-\mathrm{Ad}(h))^m Q \ \text{ modulo } \mathfrak{g}_{x,(t+q+s(m-1))} \\
&\qquad \text{(since } t \geq s) \\
&\equiv 0 \ \text{ modulo } \mathfrak{g}_{x,(t+q+s(m-1))}.
\end{aligned}$$

Thanks to Remark B.6.3, for $m \geq 2$ we have

$$s(m-2) - \nu(m) \geq \frac{1}{n}(m-2) - \nu(m) \geq 0,$$

We conclude that for $m \geq 2$
$$\frac{1}{m}[(1-\mathrm{Ad}(gh))^m - (1-\mathrm{Ad}(g))^m - (1-\mathrm{Ad}(h))^m]Q$$
belongs to $\mathfrak{g}_{x,(q+t+s)}$. Consequently,
$$[\log(\mathrm{Ad}(gh)) - \log(\mathrm{Ad}(g)) - \log(\mathrm{Ad}(h))]Q$$
is equivalent to
$$-[(1-\mathrm{Ad}(gh)) - (1-\mathrm{Ad}(g)) - (1-\mathrm{Ad}(h))]Q$$
modulo $\mathfrak{g}_{x,(q+t+s)}$. But the latter is $(\mathrm{Ad}(g)-1)(\mathrm{Ad}(h)-1)Q$, which belongs to $\mathfrak{g}_{x,(q+t+s)}$. □

Remark B.6.8. The condition $t \geq s$ in Lemma B.6.7 is not required. Suppose x, g, h are as in the statement of the lemma and $1/n \leq t < s$. Choose $u \in G_{x,(s+t)}$ so that ${}^g h = hu$. Then
$$\begin{aligned}
\psi(gh) = \psi(({}^g h)g) &\equiv \psi({}^g h) + \psi(g) \text{ modulo } \mathfrak{g}_{x,(s+t)} \\
&= \psi(hu) + \psi(g) \equiv \psi(h) + \psi(u) + \psi(g) \text{ modulo } \mathfrak{g}_{x,(2t+s)} \\
&\quad \text{(from Corollary B.6.5)} \\
&\equiv \psi(h) + \psi(g) \text{ modulo } \mathfrak{g}_{x,(s+t)} \\
&= \psi(g) + \psi(h).
\end{aligned}$$

We can now reformulate Lemma B.6.7 as follows.

Corollary B.6.9. *Suppose $x \in \mathcal{B}(G)$ and $s,t \in \mathbb{R}_{\geq 1/n}$. For all $g \in G_{x,s}$ and for all $h \in G_{x,t}$, we have*
$$\psi(gh) \equiv \psi(g) + \psi(h) \textit{ modulo } \mathfrak{g}_{x,(s+t)}.$$

Filtration quotients and ψ.

Remark B.6.10. Since every torus of $\mathbf{G}$ splits over a tamely ramified extension of K, for all $t \in \mathbb{R}_{>0}$ and for all $x \in \mathcal{B}(G)$ we have an isomorphism of abelian groups
$$G_{x,t}/G_{x,t^+} \cong \mathfrak{g}_{x,t}/\mathfrak{g}_{x,t^+}.$$
This isomorphism has the property that for each coset Ξ_G in $G_{x,t}/G_{x,t^+}$ the isomorphism identifies a coset $\Xi_{\mathfrak{g}}$ in $\mathfrak{g}_{x,t}/\mathfrak{g}_{x,t^+}$ so that for all $q \in \mathbb{R}$ and for each $Q \in \mathfrak{g}_{x,q}$ we have
$$\mathrm{ad}(X)Q \equiv (\mathrm{Ad}(g)-1)Q \text{ modulo } \mathfrak{g}_{x,(t+q)^+}$$
for all $X \in \Xi_{\mathfrak{g}}$ and for all $g \in \Xi_G$. See [64, Corollary 2.4] or [65] for details.

Lemma B.6.11. *Suppose $x \in \mathcal{B}(G)$ and $t \in \mathbb{R}_{>0}$. If $g \in G_{x,t} \smallsetminus G_{x,t^+}$, then there exist $q \in \mathbb{R}$ and $Q \in \mathfrak{g}_{x,q} \smallsetminus \mathfrak{g}_{x,q^+}$ such that*
$$(\mathrm{Ad}(g)-1)Q \in \mathfrak{g}_{x,(t+q)} \smallsetminus \mathfrak{g}_{x,(t+q)^+}.$$

Proof. Choose g as in the statement of the lemma. If $X \in \Xi_{\mathfrak{g}}$, where $\Xi_{\mathfrak{g}} \in \mathfrak{g}_{x,t}/\mathfrak{g}_{x,t^+}$ corresponds to the coset gG_{x,t^+} in $G_{x,t}/G_{x,t^+}$, then $X \in \mathfrak{g}_{x,t} \smallsetminus \mathfrak{g}_{x,t^+}$. From Lemma B.6.1 we can choose $q \in \mathbb{R}$ and $Q \in \mathfrak{g}_{x,q} \smallsetminus \mathfrak{g}_{x,q^+}$ so that

$$\mathrm{ad}(X)Q \in \mathfrak{g}_{x,(t+q)} \smallsetminus \mathfrak{g}_{x,(t+q)^+}.$$

Since, from Remark B.6.10,

$$(\mathrm{Ad}(g) - 1)Q \equiv \mathrm{ad}(X)Q \text{ modulo } \mathfrak{g}_{x,(t+q)^+},$$

the lemma follows. □

Lemma B.6.12. *Suppose $t \geq 1/n$ and $x \in \mathcal{B}(G)$. The restriction of ψ to $G_{x,t}$ induces an isomorphism*

$$G_{x,t}/G_{x,t^+} \cong \mathfrak{g}_{x,t}/\mathfrak{g}_{x,t^+}$$

of abelian groups.

Proof. Fix $t \geq 1/n$ and $x \in \mathcal{B}(G)$. Since $\psi \colon G_{0^+} \to \mathfrak{g}$ is injective and from Corollary B.6.5

$$\psi(G_{x,t^+}) \subset \mathfrak{g}_{x,t^+}$$

while

$$\psi(G_{x,t}) \subset \mathfrak{g}_{x,t},$$

from Lemma B.6.7 we have that ψ induces a group homomorphism

$$G_{x,t}/G_{x,t^+} \to \mathfrak{g}_{x,t}/\mathfrak{g}_{x,t^+}.$$

We will show that this map is surjective. Since $G_{x,t}/G_{x,t^+}$ and $\mathfrak{g}_{x,t}/\mathfrak{g}_{x,t^+}$ are finite-dimensional $\mathfrak{F}$-vector spaces of the same dimension, injectivity will follow.

To show that the induced map is surjective, we must show that for each $X \in \mathfrak{g}_{x,t}$ there is a $g \in G_{x,t}$ for which

$$X - \psi(g) \in \mathfrak{g}_{x,t^+}.$$

Equivalently, from Corollary B.6.2, we need that for all $q \in \mathbb{R}$ and for all $Q \in \mathfrak{g}_{x,q}$

$$[\mathrm{ad}(X) - \mathrm{ad}(\psi(g))]Q \in \mathfrak{g}_{x,(q+t)^+}.$$

Suppose $X \in \mathfrak{g}_{x,t}$. From Remark B.6.10, there is a $g \in G_{x,t}$ so that for all $q \in \mathbb{R}$ and each $Q \in \mathfrak{g}_{x,q}$ we have

$$(\mathrm{Ad}(g) - 1)Q \equiv \mathrm{ad}(X)Q \text{ modulo } \mathfrak{g}_{x,(q+t)^+}. \tag{46}$$

Now, for all $q \in \mathbb{R}$ and for all $Q \in \mathfrak{g}_{x,q}$, we have

$$\begin{aligned}
[\mathrm{ad}(X) - \mathrm{ad}(\psi(g))]Q &= [\mathrm{ad}(X) - \log(\mathrm{Ad}(g))]Q \\
&\qquad \text{(from Corollary B.6.4)} \\
&\equiv [\mathrm{ad}(X) - (\mathrm{Ad}(g) - 1)]Q \text{ modulo } \mathfrak{g}_{x,(q+2t)} \\
&\qquad \text{(from Equation (46))} \\
&\equiv 0 \text{ modulo } \mathfrak{g}_{x,(q+t)^+}.
\end{aligned}$$

Thus $[\mathrm{ad}(X) - \mathrm{ad}(\psi(g))]Q \in \mathfrak{g}_{x,(q+t)^+}$. □

Moy-Prasad filtrations and ψ, II. We begin with an abstract result about maps between complete topological groups.

Lemma B.6.13. *Suppose H and L are complete topological groups. Let $f\colon H \to L$ be a map for which there exist neighborhood bases at the identity*

$$\{H_i \leq H \mid H_1 := H \geq H_2 \geq H_3 \geq \cdots \geq \{1\}\}$$

and

$$\{L_i \leq L \mid L_1 := L \geq L_2 \geq L_3 \geq \cdots \geq \{1\}\}$$

for H and L so that

(1) *$f(H_i) \subset L_i$ for all i and*
(2) *if $h \in H_i$ and $h' \in H_j$, then $f(hh') \equiv f(h)f(h')$ modulo L_{i+j}.*

If the induced map

$$H_i/H_{(i+1)} \to L_i/L_{(i+1)},$$

is surjective for all i, then f is surjective.

Remark B.6.14. Note that the first condition on f implies that it is continuous at the identity, while the second implies that f is continuous everywhere.

Proof. Suppose $\ell \in L$. Fix $j_0 \in \mathbb{Z}_{\geq 1}$ so that $\ell \in L_{j_0} \smallsetminus L_{(j_0+1)}$. By hypothesis, there is an $h_0 \in H_{j_0}$ such that $f(h_0) \equiv \ell$ modulo $L_{(j_0+1)}$. Fix $j_1 > j_0$ so that $f(h_0)^{-1}\ell \in L_{j_1} \smallsetminus L_{(j_1+1)}$. Since the induced map

$$H_{j_1}/H_{(j_1+1)} \to L_{j_1}/L_{(j_1+1)}$$

is surjective, there is an $h_1' \in H_{j_1}$ so that $f(h_1') \equiv f(h_0)^{-1}\ell$ modulo $L_{(j_1+1)}$. Set $h_1 := h_0h_1'$. We have

$$f(h_1) \equiv f(h_0)f(h_1') \text{ modulo } L_{j_1+j_0}.$$

Thus, $f(h_1) \equiv \ell$ modulo L_{j_1+1}.

Choose $j_2 > j_1$ so that $f(h_1)^{-1}\ell \in L_{j_2} \smallsetminus L_{(j_2+1)}$. Since the induced map

$$H_{j_2}/H_{(j_2+1)} \to L_{j_2}/L_{(j_2+1)}$$

is surjective, there is an $h_2' \in H_{j_2}$ so that $f(h_2') \equiv f(h_1)^{-1}\ell$ modulo $L_{(j_2+1)}$. Set $h_2 := h_1h_2'$. We have

$$f(h_2) \equiv f(h_1)f(h_2') \text{ modulo } L_{j_2+j_1}.$$

Thus, $f(h_2) \equiv \ell$ modulo L_{j_2+1}.

Continuing in this fashion, we produce a convergent sequence (h_i) in H. If $h = \lim h_i$, then $f(h) = \ell$. □

Lemma B.6.15. *For all facets $J \subset \mathcal{B}(G)$ and for all $s > 0$ we have $\psi(G_{x_J,s}) = \mathfrak{g}_{x_J,s}$.*

Proof. From Lemma B.5.4 we may assume that $s \geq 1/n$. Thanks to Corollary B.6.5 it suffices to prove surjectivity.

Choose $m' \in \mathbb{Z}_{\geq 1}$ so that J is $F^{m'}$-stable. We let $x = x_J$. It will be enough to show that for all $m \in \mathbb{Z}_{\geq m'}$ we have

$$\psi(G_{x,s}^{F^m}) = \mathfrak{g}_{x,s}^{F^m}.$$

Note that $G_{x,s}^{F^m}$ and $\mathfrak{g}_{x,s}^{F^m}$ are complete topological groups. Thanks to Corollary B.6.5, Lemma B.6.7, and Lemma B.6.12, the result follows from Lemma B.6.13. $\square$

Corollary B.6.16.

$$\psi(G_{0^+}) = \mathfrak{g}_{0^+}.$$

Proof. From Lemma B.6.15, for all facets J in $\mathcal{B}(G)$ we have $\psi(G_J^+) = \mathfrak{g}_J^+$. Since

$$G_{0^+} = \bigcup_J G_J^+$$

and

$$\mathfrak{g}_{0^+} = \bigcup_J \mathfrak{g}_J^+,$$

the result follows. $\square$

The map over the residue field induced by ψ.

Lemma B.6.17. *Suppose $x \in \mathcal{B}(G)$. If $t \geq 1/n$ and $g \in G_{x,t}$, then for all $q \in \mathbb{R}$ and for each $Q \in \mathfrak{g}_{x,q}$, we have*

$$\log(\mathrm{Ad}(g))Q \equiv -\sum_{m=1}^{2(n-1)} \frac{(1-\mathrm{Ad}(g))^m}{m} Q$$

modulo $\mathfrak{g}_{x,(q+2-1/n)}$.

Proof. Fix $t \geq 1/n$ and $g \in G_{x,t}$. Suppose $q \in \mathbb{R}$ and $Q \in \mathfrak{g}_{x,q}$. For all $m \in \mathbb{Z}_{\geq 1}$ we have

$$\frac{(1-\mathrm{Ad}(g))^m}{m} Q \in \mathfrak{g}_{x,(q+mt-\nu(m))}.$$

Since $p \geq (2+e)n$, we conclude that for $1 \leq m \leq (3n-2)$,

$$\frac{(1-\mathrm{Ad}(g))^m}{m} Q \in \mathfrak{g}_{x,(q+mt)}$$

(since m is a unit). In particular, as $t \geq 1/n$, we conclude that

$$\sum_{m=1}^{2(n-1)} \frac{(1-\mathrm{Ad}(g))^m}{m} Q \equiv \sum_{m=1}^{(3n-2)} \frac{(1-\mathrm{Ad}(g))^m}{m} Q \text{ modulo } \mathfrak{g}_{x,(q+2-1/n)}.$$

To finish the proof, it is enough to show that if $m \geq (3n-1)$, then $mt - \nu(m) \geq 2 - 1/n$. This follows from Remark B.6.3.

$\square$

Lemma B.6.18. *Suppose $J \subset \mathcal{B}(G)$ is a facet and $C \subset \mathcal{B}(G)$ is an alcove which contains J in its closure. If $g \in G_C^+$ and $h \in G_J^+$, then*

$$\psi(gh) \in \psi(g) + \mathfrak{g}_J^+.$$

Proof. Since $G_J^+ \leq G_C^+$, both g and gh belong to $G_C^+ = G_{x_C,1/n}$ (see Lemma B.5.4). Consequently, from Lemma B.6.15, both $\psi(g)$ and $\psi(gh)$ belong to $\mathfrak{g}_C^+ \leq \mathfrak{g}_J$. Hence, the images of $\psi(g)$ and $\psi(gh)$ in L_J belong to the nilradical of the Borel subgroup of G_J corresponding to C. Hence, they both belong to the derived Lie algebra of L_J. Since the restriction of B to $\mathfrak{g}_J$ induces the Killing form on the derived Lie algebra of L_J, it will be enough to show that for all $Q \in \mathfrak{g}_J$ we have

$$[\mathrm{ad}(\psi(gh)) - \mathrm{ad}(\psi(g))]Q \in \mathfrak{g}_J^+.$$

Fix $Q \in \mathfrak{g}_J$. Since $\varpi Q \in \mathfrak{g}_J^+$, we have

$$Q \in \varpi^{-1}\mathfrak{g}_J^+ \leq \varpi^{-1}\mathfrak{g}_C^+ = \mathfrak{g}_{x_C,1/n-1}.$$

From Lemma B.6.17 and Equation (44) we have

$$\mathrm{ad}(\psi(gh))Q = \log(\mathrm{Ad}(gh))Q \equiv -\sum_{m=1}^{2(n-1)} \frac{(1-\mathrm{Ad}(gh))^m}{m} Q$$

modulo $\mathfrak{g}_{x_C,1} = \varpi\mathfrak{g}_C \leq \varpi\mathfrak{g}_J \leq \mathfrak{g}_J^+$. (Note: m is a unit for $1 \leq m \leq 2(n-1)$.) Similarly,

$$\mathrm{ad}(\psi(g))Q \equiv -\sum_{m=1}^{2(n-1)} \frac{(1-\mathrm{Ad}(g))^m}{m} Q$$

modulo $\mathfrak{g}_J^+$. Consequently,

$$[\mathrm{ad}(\psi(gh) - \psi(g))]Q \equiv \sum_{m=1}^{2(n-1)} \left[\frac{(1-\mathrm{Ad}(g))^m - (1-\mathrm{Ad}(gh))^m}{m}\right] Q$$

modulo $\mathfrak{g}_J^+$. Since h acts trivially on $\mathfrak{g}_J/\mathfrak{g}_J^+$, we have

$$(1-\mathrm{Ad}(gh))^m Q \equiv (1-\mathrm{Ad}(g))^m Q$$

modulo $\mathfrak{g}_J^+$. The result follows. □

Corollary B.6.19. *Suppose J is an F-stable facet in $\mathcal{B}(G)$. The restriction of ψ to $G_{0^+} \cap G_J$ induces a $\langle F \rangle \ltimes \mathsf{G}_J$-equivariant bijective map from $\mathcal{U}_J$, the $\mathfrak{f}$-variety of unipotent elements in G_J, to $\mathcal{N}_J$, the $\mathfrak{f}$-variety of nilpotent elements in L_J.*

Proof. If $\bar{g} \in \mathcal{U}_J$, then there exist an alcove C and a $g \in G_C^+$ such that $J \subset \bar{C}$ and g is a lift of $\bar{g}$. From Lemma B.6.15 we have $\psi(g) \in \mathfrak{g}_C^+ \subset \mathfrak{g}_J$. Thus, the image of $\psi(g)$ in L_J belongs to $\mathcal{N}_J$. From Lemma B.6.18, the image of $\psi(g)$ in L_J is independent of the choice of g. Hence ψ induces a map $\bar{\psi}\colon \mathcal{U}_J \to \mathcal{N}_J$. As ψ is $\langle F \rangle \ltimes G_J$-equivariant, it follows that $\bar{\psi}$ is $\langle F \rangle \ltimes \mathsf{G}_J$-equivariant.

To see that $\bar{\psi}\colon \mathcal{U}_J \to \mathcal{N}_J$ is bijective, we note that $p \geq (2+e)n$ implies (see for example [12, §1.15]) that there is a (non-unique) bijective, G_J-equivariant $\mathfrak{f}$-morphism identifying $\mathcal{U}_J$ with $\mathcal{N}_J$. Thus, for all $m \in \mathbb{Z}_{\geq 1}$ the sets $\mathcal{U}_J^{F^m}$ and $\mathcal{N}_J^{F^m}$ have the same cardinality. Consequently, it is enough to show that the restriction to $\mathcal{U}_J^{F^m}$ of $\bar{\psi}$ surjects onto $\mathcal{N}_J^{F^m}$. If $\bar{X} \in \mathcal{N}_J^{F^m}$, then there exist an F^m-stable alcove C and $X \in (\mathfrak{g}_C^+)^{F^m}$ such that $J \subset \bar{C}$ and X is a lift of $\bar{X}$. From the proof of Lemma B.6.15 there exists a $g \in (G_C^+)^{F^m}$ such that $\psi(g) = X$.

Since $\mathcal{U}_J$ is the image of $G_{0+} \cap G_J$ in G_J, the corollary follows. $\square$

B.7. **An extension to reductive groups.** Drop the assumption that $\mathbf{G}$ is semisimple. In this section, we prove that the $\langle F \rangle \ltimes G$- equivariant map $\psi\colon G_{0+} \to \mathfrak{g}$ has the properties described in Lemma B.0.3.

Let G' denote the group of K-rational points of the derived group of $\mathbf{G}$. Let Z denote the group of K-rational points of $\mathbf{Z}$, the identity component of the center of $\mathbf{G}$. We recall that $Z \cap G'$ is finite. We let $\mathfrak{g}'$ (resp. $\mathfrak{z}$, resp. $\boldsymbol{\mathfrak{z}}$) denote the Lie algebra of G' (resp. Z, resp. $\mathbf{Z}$).

In Section B.6 we proved that the map

$$\operatorname{res}_{G'} \psi\colon G'_{0+} \to \mathfrak{g}'$$

has the properties required by Lemma B.0.3. From [9, III, §7, Proposition 11] we have

$$\psi(zh) = \psi(z) + \psi(h) \tag{47}$$

for all $z \in Z_{0+}$ and all $h \in G'_{0+}$.

Suppose $\mathbf{S}$ is any torus in $\mathbf{G}$. Let S denote the group of K-rational points of $\mathbf{S}$. Let $\mathfrak{s}$ (resp. $\boldsymbol{\mathfrak{s}}$) denote the Lie algebra of S (resp. $\mathbf{S}$).

Lemma B.7.1. *With our assumptions on p, we have*

$$\psi(S_{0+}) = \mathfrak{s}_{0+}.$$

Proof. Let E denote the splitting field of $\mathbf{S}$ over K. Since $\varphi\colon \mathbf{G} \to \mathrm{GL}_n$ is faithful and $\varphi(\mathbf{S})$ is a torus in GL_n, the field E is a tame Galois extension of K and $\nu_E(p) \leq n\nu(p)$.

Since E is a tame Galois extension of K, from [2, Lemma 2.2.3], we have

$$S_{0+} = \mathbf{S}(E)_{0+} \cap S.$$

By an argument similar to that given in Section B.3, there is a unique $\mathrm{Gal}(E/K) \ltimes \mathbf{G}(E)$-equivariant extension of ψ to a map $\psi\colon \mathbf{G}(E)_{0+} \to \boldsymbol{\mathfrak{g}}(E)$. From Equation (44) the image of the restriction to $\mathbf{S}(E)_{0+}$ of this map lies in $\boldsymbol{\mathfrak{s}}(E)$. It will be enough to show that $\psi(\mathbf{S}(E)_{0+}) = \boldsymbol{\mathfrak{s}}(E)_{0+}$.

Since $\mathbf{S}$ is E-split, there is an E-isomorphism φ_S from $\mathbf{S}$ to $(\mathrm{GL}_1)^j$ for some j. Since

$$p \geq (2 + \nu(p))n \geq 2n + \nu_E(p),$$

we have $p \geq 2 + \nu_E(p)$. We conclude (see the discussion concerning GL_n in §B.1) that

$$\log((\mathrm{GL}_1(E))^j_{0+}) = (M_1(E))^j_{0+}.$$

Since φ_S and $d\varphi_S$ are E-isomorphisms, the result follows from the fact that $d\varphi_S(\psi(s)) = \log(\varphi_S(s))$ for $s \in \mathbf{S}(E)_{0+}$ (see Equation (41)). $\square$

Lemma B.7.2. *Under our assumptions on p, the map $(z, h) \mapsto zh$ from $Z_{0+} \times G'_{0+}$ to G_{0+} is bijective.*

The proof below is due to Loren Spice; it is shorter than our original proof.

Proof (Spice). Since for each $x \in \mathcal{B}(G)$ we have $Z_{0^+} \subseteq G_{x,0^+}$ and $G'_{x,0^+} \subseteq G_{x,0^+}$, it suffices to check that the map $i_x \colon Z_{0^+} \times G'_{x,0^+} \to G_{x,0^+}$ which sends (z,h) to zh is bijective for all $x \in \mathcal{B}(G)$.

Fix $x \in \mathcal{B}(G)$. To show i_x is bijective, it is enough to check that the induced map on successive quotients of Moy-Prasad filtration subgroups is bijective. Fix $r \in \mathbb{R}_{>0}$. From [64, Corollary 2.4], it is enough to check that the induced map

$$\mathfrak{z}_r/\mathfrak{z}_{r^+} \times \mathfrak{g}'_{x,r}/\mathfrak{g}'_{x,r^+} \to \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$$

is bijective. From [4, Proposition 3.2], it is surjective. If $(\bar{Z}, \bar{X})$ is in its kernel, then there exist $Z \in \mathfrak{z}_r$ (resp., $X \in \mathfrak{g}'_{x,r}$) lifting $\bar{Z}$ (resp., $\bar{X}$) so that $Z + X \in \mathfrak{g}_{x,r^+}$. From [4, Proposition 3.2], we conclude that $Z \in \mathfrak{z}_{r^+}$ and $X \in \mathfrak{g}'_{x,r^+}$. Thus, the map is injective as well. □

Thanks to Equation (47), from Lemma B.7.1 and Lemma B.7.2, the map ψ is a bijective $\langle F \rangle \ltimes G$-equivariant map from G_{0^+} to $\mathfrak{g}_{0^+} = \mathfrak{z}_{0^+} + \mathfrak{g}'_{0^+}$. Moreover, since, for all $x \in \mathcal{B}(G)$, the image of $\mathfrak{z}_{0^+}$ in L_x is trivial, it follows from Lemma B.7.1 (with $S = Z$) that ψ has the properties required by Lemma B.0.3.

Index of selected notation and terms

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