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Derivation of a stochastic cellular automaton model for the dynamics of bistable units with global and asymmetric local interactions

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By means of ultradiscretization and probabilistic treatment, we derive a stochastic cellular automaton (CA) model from a dynamical model for bistable units with global and asymmetric local interactions. The obtained CA model is composed of the elementary CA rule 254 with a probabilistic rule that regulates the average of the total cell states.

Subject Index A31, A34

1. Introduction

Spatiotemporal patterns in non-equilibrium and nonlinear systems have been widely studied from the viewpoint of statistical physics and nonlinear dynamics [1–3]. For their studies, cellular automaton (CA) models [4–6] and reaction–diffusion equations [7] have been developed. In addition, characterizing the relationship between CA models and nonlinear equations for describing the same phenomena has been an important problem [8]. For example, a CA model generating a Sierpinski gasket pattern can be obtained as a solution of a nonlinear reaction–diffusion equation for pulse propagation [9]. In general, ultradiscretization [10] has been used as a systematic mathematical method for deriving CA models from reaction–diffusion equations [11]. In particular, this method has been successfully applied to integrable systems [12].

Recently, one of the authors proposed the following dynamical model for spatiotemporal patterns of bistable units $\{\phi_j\}$ with global and asymmetric local interactions [13,14]:

$$\dot{\phi}_j = -(\phi_j - \alpha)(\phi_j - \beta)(\phi_j - \gamma) + D\{\theta(\phi_{j+1} - \phi_j) + \theta(\phi_{j-1} - \phi_j)\} - (\bar{\phi} - V) + \xi_j, \quad (1)$$

where $j = 1 \sim N$. The parameters α, β, and γ satisfy $0 < \alpha < \beta < \gamma$. V and D are positive constants. $\theta(x)$ is defined as $\theta(x) = x$ $(x \geq 0)$, and 0 $(x < 0)$. $\bar{\phi}$ represents the average of $\{\phi_j\}$: $\bar{\phi} = \frac{1}{N}\Sigma_{j=1}^{N}\phi_j$. ξ_j is a noise term. The first term of the right-hand side of Eq. (1) brings about the bistability of ϕ_j, where $\phi_j = \alpha$ and γ are stable points. The second and third terms show the asymmetric local interaction and the global interaction, respectively. There exist the following two characteristic dynamical properties in Eq. (1). (i) Due to the asymmetric local interaction, the state $\phi_j \approx \alpha$, which is one of the bistable states, changes to the other bistable state $\phi_j \approx \gamma$. Note that this property is independent of the existence of the global interaction and noise. It is clear that the number of ϕ_j taking γ increases and finally all ϕ_j become γ if only this asymmetric local property is considered. (ii) However, the system changes from bistable to monostable with $\phi_j \approx \alpha$ due to the global interaction when the number of ϕ_j having γ exceeds a threshold, which is given as a function of V. Furthermore, for the noise term, the change of ϕ_j from γ to α occurs stochastically. Accordingly, this global property regulates $\bar{\phi}$ to be a constant value.

The above dynamical properties have also been expressed by a CA model, which was constructed heuristically [15]. (In this paper, the previous CA model is referred as the "Y12" model hereafter.) The Y12 model gives the time evolution of a cell state and describes the competition between the elementary CA rule 254 and a probabilistic rule. The CA rule 254 and the probabilistic rule are considered to correspond to the asymmetric local interaction and the global interaction with noise in Eq. (1), respectively. Nevertheless, the mathematical derivation of the Y12 model from Eq. (1) is not yet clear.

In this paper, we show the derivation of a CA model from Eq. (1) by ultradiscretization and probabilistic treatments. In the next section, we review the ultradiscretization method briefly. In Sect. 3, we derive the elementary CA rule 254 from the asymmetric local interaction by means of ultradiscretization. In Sect. 4, a probabilistic function is introduced to consider the global interaction and noise terms. The discussion and conclusion are given in Sect. 5.

2. Brief review of ultradiscretization with tropical discretization

Ultradiscretization is a limiting procedure transforming a difference equation into another type of difference equation subject to max-plus algebra [16]. Now we consider a difference equation of positive variables u_j^n, where n is discretized time and j is a discretized position. First, in ultradiscretization, u_j^n is replaced by using U_j^n as $u_j^n = \exp(U_j^n/\varepsilon)$, where ε is a positive parameter. After such a variable transformation from u_j^n to U_j^n, the formula called the ultradiscrete limit,

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon} + \cdots) = \max(A, B, \ldots),$$

is adopted. By this procedure, we can obtain a new time-dependent difference equation, namely, an ultradiscrete equation of U_j^n. If U_j^n takes a value included in a set $\{0, 1, 2, \ldots, N\}$, U_j^{n+1} can take one of the values in the same set [17]. This property implies that an ultradiscrete equation associates with CA. However, this ultradiscretization method is not applicable to all difference equations, since there is a "negativity" problem [10]. In general, ultradiscretization for a difference equation assumes that the equation does not have a minus sign because the ultradiscrete limit of $\varepsilon \log(e^{A/\varepsilon} - e^{B/\varepsilon})$ is not definable when $A \leq B$. To tackle this problem, Murata has developed the following method, which is called tropical discretization [18]. This method is applicable to ultradiscretization for a reaction–diffusion equation with a minus sign, such as

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u) - g(u), \tag{2}$$

where $u = u(x, t) > 0$, D is a diffusion coefficient, and $f(u),\ g(u) \geq 0$. In order to get an ultradiscrete equation for Eq. (2), it is necessary to derive a difference equation suitable for tropical discretization. For instance, the difference equation for Eq. (2) can be written as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + f(u_j^n) - g(u_j^n), \tag{3}$$

where $\Delta t = t/n$, $\Delta x = x/j$. In tropical discretization, we set $v_j^n = (u_{j+1}^n + u_{j-1}^n)/2$ and replace $f(u_j^n) - g(u_j^n)$ with $v_j^n(f(v_j^n) - g(v_j^n))/(v_j^n + \Delta t g(v_j^n))$. Then, we obtain

$$u_j^{n+1} = v_j^n \frac{v_j^n + \Delta t f(v_j^n)}{v_j^n + \Delta t g(v_j^n)}, \tag{4}$$

where we set $D\Delta t/(\Delta x)^2 = 1/2$ for simplicity. Equation (4) is the difference equation of Eq. (2) with tropical discretization. By adopting the variable transformations $\Delta t = (1/2)^{n+1}e^{T/\varepsilon}$, $u_j^n = (1/2)^n e^{U_j^n/\varepsilon}$, $f(u_j^n) = e^{F(U_j^n)/\varepsilon}$, and $g(u_j^n) = e^{G(U_j^n)/\varepsilon}$ into Eq. (4) and by the ultradiscrete limit, the ultradiscrete equation is obtained as

$$U_j^{n+1} = V_j^n + \max\{V_j^n, T + F(V_j^n)\} - \max\{V_j^n, T + G(V_j^n)\}. \tag{5}$$

Here $V_j^n = \max\{U_{j+1}^n, U_{j-1}^n\}$. Although we use the coefficients $(1/2)^n$ and $(1/2)^{n+1}$ for variable transformations of ultradiscretization, these coefficients depend on the properties of functions f and g in general. For instance, if f and g are positive homogeneous functions with respect to k, we should choose the coefficient for Δt as $2^{n(k-1)-1}$.

3. Ultradiscretization for the asymmetric local interaction

3.1. *Derivation of ultradiscrete equations*

Now we apply the ultradiscretization with tropical discretization to property (i) shown in Sect. 1. Focusing on the first and second terms on the right-hand side of Eq. (1), we consider the difference equation

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = f(\phi_j^n) - g(\phi_j^n) + D\{\theta(\phi_{j+1}^n - \phi_j^n) + \theta(\phi_{j-1}^n - \phi_j^n)\}. \tag{6}$$

Here f and g are defined as

$$f(\phi) = (\alpha + \beta + \gamma)\phi^2 + \alpha\beta\gamma,\ g(\phi) = \phi^3 + (\alpha\gamma + \alpha\beta + \beta\gamma)\phi. \tag{7}$$

For the asymmetric local interaction in Eq. (6), we first consider the case where $\phi_{j+1}^n > \phi_j^n$ and $\phi_{j-1}^n > \phi_j^n$. In this case, Eq. (6) becomes

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = f(\phi_j^n) - g(\phi_j^n) + D(\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n). \tag{8}$$

The tropical discretization of Eq. (8) is done in the same way as that of Eq. (3), and we obtain

$$\phi_j^{n+1} = w_j^n \frac{w_j^n + \Delta t f(w_j^n)}{w_j^n + \Delta t g(w_j^n)}, \tag{9}$$

where $w_j^n = \mu(\phi_{j+1}^n + \phi_{j-1}^n) + (1 - 2\mu)\phi_j^n$ and $\mu = D\Delta t$. Similarly, we obtain the following difference equations in other cases:

$$\phi_j^{n+1} = \phi_{j+1}^n \frac{\phi_{j+1}^n + \Delta t f(\phi_j^n)}{\phi_{j+1}^n + \Delta t g(\phi_j^n)},\ (\phi_{j+1} > \phi_j, \phi_{j-1} \le \phi_j) \tag{10}$$

$$\phi_j^{n+1} = \phi_{j-1}^n \frac{\phi_{j-1}^n + \Delta t f(\phi_j^n)}{\phi_{j-1}^n + \Delta t g(\phi_j^n)},\ (\phi_{j+1} \le \phi_j, \phi_{j-1} > \phi_j) \tag{11}$$

$$\phi_j^{n+1} = \phi_j^n \frac{\phi_j^n + \Delta t f(\phi_j^n)}{\phi_j^n + \Delta t g(\phi_j^n)},\ (\phi_{j+1} \le \phi_j,\ \phi_{j-1} \le \phi_j). \tag{12}$$

The difference equations (9–12) with Eq. (7) have no subtraction. Then the ultradiscrete equations in each case can be derived. First, the variables in these difference equations are transformed as follows:

$$\begin{cases} \Delta t = e^{T/\varepsilon}/(\mu^{n+1})^2, \quad \phi_j^n = \mu^n e^{U_j^n/\varepsilon}, \\ \alpha = \mu^{n+1} e^{A/\varepsilon}, \quad \beta = \mu^{n+1} e^{B/\varepsilon}, \quad \gamma = \mu^{n+1} e^{\Gamma/\varepsilon}, \\ (1-2\mu)/\mu = e^{M/\varepsilon}, \quad (1-\mu)/\mu = e^{M'/\varepsilon}. \end{cases} \tag{13}$$

After these transformations, we adopt the ultradiscrete limit ($\varepsilon \to 0$) in the four cases.
(i) When $\phi_{j+1}^n > \phi_j^n$ and $\phi_{j-1}^n > \phi_j^n$, namely, $U_{j+1}^n > U_j^n$, and $U_{j-1}^n > U_j^n$, we obtain

$$\phi_j^{n+1} = w_j^n \frac{w_j^n + \Delta t\{(\alpha+\beta+\gamma)(w_j^n)^2 + \alpha\beta\gamma\}}{w_j^n + \Delta t\{(w_j^n)^3 + (\alpha\gamma+\alpha\beta+\beta\gamma)(w_j^n)\}} \tag{14}$$

from Eq. (9) with Eq. (7). Equation (14) is rewritten by Eq. (13) as

$$\begin{aligned} U_j^{n+1} &= \varepsilon \log\Big[\Big(e^{U_{j+1}^n/\varepsilon} + e^{U_{j-1}^n/\varepsilon} + e^{(M+U_j^n)/\varepsilon}\Big) \\ &\quad + e^{T/\varepsilon}\Big\{\Big(e^{A/\varepsilon} + e^{B/\varepsilon} + e^{\Gamma/\varepsilon}\Big)\Big(e^{U_{j+1}^n/\varepsilon} + e^{U_{j-1}^n/\varepsilon} + e^{(M+U_j^n)/\varepsilon}\Big)^2 + e^{(A+B+\Gamma)/\varepsilon}\Big\}\Big] \\ &\quad - \varepsilon \log\Big\{e^{0/\varepsilon} + e^{T/\varepsilon}\Big(e^{U_{j+1}^n/\varepsilon} + e^{U_{j-1}^n/\varepsilon} + e^{(M+U_j^n)/\varepsilon}\Big)^2 \\ &\quad + e^{(A+B)/\varepsilon} + e^{(A+\Gamma)/\varepsilon} + e^{(\Gamma+B)/\varepsilon}\Big\}. \end{aligned}$$

Using the ultradiscrete limit, we obtain

$$\begin{aligned} U_j^{n+1} &= \max\{W_j^n, T + \max(A + 2W_j^n, B + 2W_j^n, \Gamma + 2W_j^n, A + B + \Gamma)\} \\ &\quad - \max\{0, T + \max(2W_j^n, A + B, A + \Gamma, B + \Gamma)\}, \end{aligned} \tag{15}$$

where $W_j^n = \max(U_{j+1}^n, M + U_j^n, U_{j-1}^n)$. This equation is equivalent to Eq. (5) if we set $F(U) = \max(A + 2U, B + 2U, \Gamma + 2U, A + B + \Gamma)$, $G(U) = \max(3U, A + B + U, A + \Gamma + U, B + \Gamma + U)$. Additionally, when we replace A with $-\infty$ and Γ with 0, Eq. (15) is the same as the ultradiscrete Allen–Cahn equation reported by Murata [18]. Moreover, by the relation $A < B < \Gamma$, Eq. (15) becomes simpler in form:

$$U_j^{n+1} = \max\{W_j^n, T + \max(\Gamma + 2W_j^n, A + B + \Gamma)\} - \max\{0, T + \max(2W_j^n, B + \Gamma)\}. \tag{16}$$

The ultradiscrete equations in the other cases are obtained from Eqs. (10–12) in a similar way.
(ii) When $U_{j+1}^n > U_j^n$, and $U_{j-1}^n \le U_j^n$, from Eq. (10),

$$U_j^{n+1} = \max\{L_j^n,\ T + \max(\Gamma + 2L_j^n, A + B + \Gamma)\} - \max\{0, T + \max(2L_j^n, B + \Gamma)\}. \tag{17}$$

(iii) When $U_{j+1}^n \le U_j^n$, and $U_{j-1}^n > U_j^n$, from Eq. (11),

$$U_j^{n+1} = \max\{R_j^n,\ T + \max(\Gamma + 2R_j^n, A + B + \Gamma)\} - \max\{0, T + \max(2R_j^n, B + \Gamma)\}. \tag{18}$$

(iv) When $U_{j+1}^n \le U_j^n$, and $U_{j-1}^n \le U_j^n$, from Eq. (12),

$$U_j^{n+1} = \max\{U_j^n, T + \max(\Gamma + 2U_j^n, A + B + \Gamma)\} - \max\{0, T + \max(2U_j^n, B + \Gamma)\}. \tag{19}$$

Here $L_j^n = \max(U_{j+1}^n, M' + U_j^n)$ and $R_j^n = \max(M' + U_j^n, U_{j-1}^n)$. The four equations (16–19) are the ultradiscrete equations of the bistable and asymmetric local interaction parts of Eq. (1).

3.2. The bistability and correspondence to the CA rule 254

We consider the bistability of the ultradiscrete equations (16–19). Now we assume $\phi_j^0 \in [\alpha, \gamma]$. This assumption is followed by $A \leq U_j^0 \leq \Gamma$. Additionally, we set the parameter $T > \max\{0, -(A + B + \Gamma)\}$. For this setting, it is found that the solutions of these ultradiscrete equations take only two states after a number of time steps from the initial condition; the state eventually reaches either A or Γ from any $\phi_j^0 \in [\alpha, \gamma]$. The proof is as follows. In the case of Eq. (19), if U_j^0 has an arbitrary value with $A < U_j^0 < \frac{A+B}{2}$, U_j^1 becomes A, owing to $T > -(A + B + \Gamma)$. If $\frac{A+B}{2} < U_j^0 < B$, then U_j^1 becomes $2U_j^0 - B(< U_j^0)$. Also, there is a time n at which U_j^{n-1} satisfies $A < U_j^{n-1} < \frac{A+B}{2}$. Then, the next state U_j^n takes A. Furthermore, if a state U_j^n is smaller than A, it is proved that U_j^n converges with A. In the same way, it is also proved that, if U_j^0 has an arbitrary value satisfying $B < U_j^0 < \Gamma$, the value of U_j^n becomes Γ at a finite time n. Hence, the bistability of Eq. (19) is proved.

For Eqs. (16–18), the above proof of bistability will be changed by the existence of W_j^n, L_j^n, and R_j^n. However, if the parameters M, M' satisfy the inequalities $M, M' < A - \Gamma$, it is found that the effect of the local interaction can be ignored and that the bistability holds in the time evolution of U_j^n, even for Eqs. (16–18). In fact, when $M, M' < A - \Gamma$ are satisfied, we obtain $W_j^n = \max(U_{j+1}^n, U_{j-1}^n)$, $L_j^n = U_{j+1}^n$, $R_j^n = U_{j-1}^n$. Then, from the above proof, the bistability for Eqs. (16–18) is confirmed.

Regarding the inequalities for M and M', we focus on a solution of Eq. (2) with Eq. (7). Due to the effect of the local interaction, which is represented by D, a solution of Eq. (2) forms a monotonically increasing function connected with $u = \alpha$ and $u = \gamma$. When D is small, the rise of the solution from α to γ is so steep that u is supposed to take the two states (α and γ) approximately. Since M and M' involve the diffusion coefficient D in Eq. (1) through μ, $A - \Gamma$ gives the threshold for M and M', which determines whether the state can be regarded as bistable or not.

We consider the correspondence of the ultradiscrete equations to an elementary CA rule. In general, A and Γ are not necessarily integer values, and U_j^n takes continuous values. However, it is clear that U_j^n is integer for all j and n if U_j^0, A, and Γ take integer values. From Eqs. (16–19), if is found that U_j^{n+1} is determined by a set $[U_{j+1}^n, U_j^n, U_{j-1}^n]$. For example, if $[U_{j+1}^n, U_j^n, U_{j-1}^n]$ is given as $[\Gamma, A, A]$ at time n, U_j^{n+1} becomes Γ by Eq. (17). In a similar way, U_j^{n+1} is given from Eqs. (16–19) as follows:

$U_{j+1}^n\ U_j^n\ U_{j-1}^n$	1 1 1	1 1 0	1 0 1	1 0 0	0 1 1	0 1 0	0 0 1	0 0 0
U_j^{n+1}	1	1	1	1	1	1	1	0

Here, we use “1” and “0” instead of Γ and A, respectively. This is exactly the elementary CA rule 254. By formally denoting the CA rule 254 as f_{254}, we show the time evolution of U_j^n as

$$U_j^{n+1} = f_{254}(U_{j+1}^n, U_j^n, U_{j-1}^n). \tag{20}$$

4. Interpretation of the global interaction as a probabilistic rule

In Sect. 3, we applied ultradiscretization to Eq. (6) and derived the CA rule 254. In this derivation, we did not consider the global interaction and noise terms in Eq. (1). This treatment is considered to be valid if $\bar{\phi}$ does not exceed the threshold, as described in Sect. 1. But owing to the effect of the asymmetric local interaction, ϕ_j tends to change from α to γ, and $\bar{\phi}$ can exceed the threshold, and the effect of global interaction and noise terms cannot be ignored for the dynamics of ϕ_j. The global interaction brings about a change from a bistable state to a monostable state with one stable point

$\phi_j = \alpha$. In addition, ϕ_j changes from γ to α stochastically because of the existence of noise. These effects can be represented by the following probabilistic function $\theta_j^n(x)$ [19]. If $0 \leq x \leq 1$, then

$$\theta_j^n(x) = \begin{cases} U_j^n, \text{ with the probability } x \\ 0 \text{ , with the probability } 1 - x \end{cases}$$

and $\theta_j^n(x) \equiv 0 \ (x < 0, \ 1 < x)$. By using this function θ_j^n, the probabilistic rule for U_j^n is given as follows. The global interaction $-(\bar{\phi} - V)$ in Eq. (1) is replaced by $-(\bar{U}^n - V')$, where $\bar{U}^n$ is $\frac{1}{N}\Sigma_{j=1}^N U_j^n$, V' is some constant (<1). Furthermore, in order to take the noise term ξ_j into account, we modify $-(\bar{U}^n - V')$ as $-\theta_j^n(\bar{U}^n - V')$. Here, we can regard the parameter V' as a threshold of $\bar{U}^n$. Then the time evolution of U_j^n is given by a combination of the CA rule 254 (derived from the asymmetric local interaction) and the probabilistic rule (reflecting the global interaction and noise):

$$U_j^{n+1} = f_{254}(U_{j+1}^n, U_j^n, U_{j-1}^n) - \theta_j^n(\bar{U}^n - V'). \tag{21}$$

5. Discussion and conclusion

We comment on the difference of the present CA model (Eq. (21)) from the Y12 model. In the Y12 model, the following probabilistic rules are adopted for $U_j^n \in (0, 1)$. If $\bar{U}^n \geq V'$, each cell having "1" is converted to "0" with the probability $\frac{\bar{U}^n - V'}{\bar{U}^n}$ at each time step. On the other hand, if $\bar{U}^n < V'$, each cell having "0" is converted to "1" with the probability $\frac{V' - \bar{U}^n}{1 - \bar{U}^n}$. In the present model, each state U_j^n is converted from "1" to "0" with the probability $\bar{U} - V'$. However, U_j^n is not converted from "0" to "1" stochastically. In fact, considering the dynamics around the threshold, there is no effect if U_j^n converts its state from "0" to "1" stochastically in Eq. (1). Hence, Eq. (21) expresses the dynamical behavior of Eq. (1) more adequately than the Y12 model.

In conclusion, we show the derivation of the stochastic CA model (Eq. (21)) from the nonlinear difference equation (Eq. (1)) by means of ultradiscretization and probabilistic treatment. The ultradiscrete equations (16–19) possess bistability, and bring about the CA rule 254. The global interaction and the noise in Eq. (1) can be interpreted as a probabilistic rule for changing from the "1" state to the "0" state, and regulate the average of the total cell states.

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