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Derivation of Gauge and Gravitational Induced Chern-Simons Terms in Three Dimensions

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We study the theory of fermions coupled to external gauge and gravitational fields. In three dimensions, the existence of massive fermions is known to lead to the induced Chern-Simons term. We derive the induced Chern-Simons term by a path integral method.

§1. Introduction

In the investigation of quantum theory anomalies play an important role, since in modern theory the symmetries are fully utilized. Anomalies give restrictions on the symmetries from the quantum side. Especially, anomalies in even dimensions are well known, for example, chiral $U(1)$, gauge and gravitational anomalies, etc.

In odd dimensions, there are some different aspects[1)~15)] from even dimensional theory. One is the existence of the parity violating effective action. As is well known, the odd dimensional theory coupled to external gauge and gravitational fields has the invariance under infinitesimal gauge and infinitesimal general coordinate transformations and, when the fermion mass vanishes, under the parity transformation also. But the theory in which the number of fermions is odd is not invariant under the global transformation with a winding number. That is to say, if the winding number of the global transformation is n, the effective action changes by $\pi|n|$ under this transformation,[1)]

$$\det(-\not{D}) \to (-1)^n \det(-\not{D}) . \tag{1·1}$$

To cancel the global anomaly, one needs to add a term which changes in the same manner as above under the global transformation. This term is called a topological mass term.[2)] In gauge theory, it is given by

$$\mathcal{L}_{\mathrm{cs}} = \frac{\mu}{2e^2} \epsilon^{\mu\nu\alpha} \mathrm{tr}\left(F_{\mu\nu} A_\alpha - \frac{2}{3} A_\mu A_\nu A_\alpha \right), \tag{1·2}$$

where tr stands for a trace over the internal indices and e for the gauge coupling constant, and we use the following notations:

$$A_\mu = eT^a A_\mu{}^a , \quad F_{\mu\nu} = eT^a F^a_{\mu\nu} , \quad [T^a, T^b] = f^{abc} T^c . \tag{1·3}$$

Under the global gauge transformation,

$$A_\mu \to U^{-1} A_\mu U + U^{-1} \partial_\mu U , \tag{1·4}$$

(1·2) changes as

$$\int d^3x \mathcal{L}_{\mathrm{cs}} \to \int d^3x \mathcal{L}_{\mathrm{cs}} + \mu \frac{8\pi^2}{e^2} w(U),$$

$$w(U) = \frac{1}{24\pi^2} \int d^3x \epsilon^{\mu\nu\alpha} \mathrm{tr}(\partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\alpha U U^{-1}). \qquad (1\cdot5)$$

On the other hand, in gravity theory, the topological term is given by

$$\mathcal{L}_{\mathrm{cs}} = \frac{1}{4\kappa^2 \mu} \epsilon^{\mu\nu\alpha} \left[R_{\mu\nu ab} \omega_\alpha{}^{ab} + \frac{2}{3} \omega_{\mu b}{}^c \omega_{\nu c}{}^a \omega_{\alpha a}{}^b \right],$$

$$R_{\mu\nu ab} \equiv \partial_\mu \omega_{\nu ab} + \omega_{\mu a}{}^c \omega_{\nu cb} - (\mu \leftrightarrow \nu),$$

$$\omega_{\mu ab} = -\omega_{\mu ba} \qquad (1\cdot6)$$

with $\omega_{\mu ab}$ being the spin connection. The above topological mass term is added by hand. Meanwhile, this term is also introduced by the existence of a massive fermion. This is called the induced Chern-Simons term. This term is also generated by the regulator mass term in the case that the massless fermion theory is regularized by the Pauli-Villars regulator. But it has an opposite sign to the one induced by the fermion mass term. In this paper, we only consider the Chern-Simons term generated by the fermion mass and investigate a way how to compute it.

The induced Chern-Simons term due to the fermion mass term (not by the regulator mass term) is calculated by several methods. As one of simple methods, there is the perturbation method.[2),4)] One can obtain the term by computing one-loop amplitudes with two and three gauge fields. In addition, there is a topological method. The answer in this method is given as follows:

$$\mathrm{Im}\, \Gamma_{\mathrm{eff}} = \eta[e, A] = 2 \int_{M^{2n+1}} Q(A, \omega) + \text{constant},$$

$$dQ(A, \omega) = \hat{A}(R) \mathrm{ch}(F)|_{2n+2}, \qquad (1\cdot7)$$

where Γ_{eff} is the effective action; $\hat{A}(R)$ is the Dirac genus defined as

$$\hat{A}(R) = \det \frac{\sqrt{R}/4\pi}{\sinh\sqrt{R}/4\pi} = 1 + \frac{1}{24\pi^2} \mathrm{Tr}\, R^2 + \cdots, \qquad (1\cdot8)$$

and $\mathrm{ch}(F)$ is the Chern character defined as

$$\mathrm{Tr}\, e^{iF/2\pi} = 1 - \frac{\mathrm{Tr}\, F^2}{8\pi} + \cdots. \qquad (1\cdot9)$$

In these expressions, we use the differential form and the symbol $|_{2n+2}$ denotes that we only extract a $2n+2$ form. Im Γ_{eff} is given in arbitrary odd dimensions. In the path integral formalism,[2),16)] one may use the proper time method to evaluate the induced Chern-Simons term. In this case, the gauge field has been restricted to $A_\mu = \text{constant}$ to make the computation easy. In the topological method,[3)] the induced Chern-Simons terms were obtained in the theory of gauge fields and gravity. But in perturbation theory, the gravitational term was merely calculated to the second order within the weak field approximation.[5)] In the path integral method, the gravitational term has not been calculated before. In this paper, we would like to compute the induced Chern-Simons terms in both gauge and gravitational theories by the path integral method with no restrictions on field configurations and no approximations.

The organization of this paper is as follows. In the next section, we derive the induced Chern-Simons term from the one-loop effective action in the case of the theory of fermions coupled to gauge fields. In § 3, we derive the Chern-Simons term in gravity theory. In § 4, we present our conclusion.

§ 2. Chern-Simons term in gauge theory

Let us consider the induced Chern-Simons term in gauge theory. For convenience, we will work in the Euclidean space throughout this paper. Notations are as follows. The metric is taken as $g^{\mu\nu}=\mathrm{diag}(-1,-1,-1)$. The antihermitian Dirac matrices satisfy

$$\{\gamma^\mu,\gamma^\nu\}=-2\delta^{\mu\nu}\,, \tag{2·1}$$

and we choose Dirac matrices to be

$$\gamma_1=i\sigma_1\,,\quad \gamma_2=i\sigma_2\,,\quad \gamma_3=i\sigma_3\,, \tag{2·2}$$

where σ_i are the Pauli matrices. We will consider the theory of massive fermions coupled to external gauge fields. The Lagrangian is given by

$$\mathcal{L}=-\bar\psi(\not{D}+m)\psi=-\bar\psi[\gamma^\mu(i\partial_\mu+eA_\mu)+m]\psi\,. \tag{2·3}$$

Here A_μ stands for $A_\mu{}^a T^a$, where the T^a's are the antihermitian generators of a gauge group G. Since the induced Chern-Simons term is generated from the one-loop effective action Γ_{eff}, we need to define it. We define Γ_{eff} as

$$e^{-\Gamma_{\mathrm{eff}}}=\int d\bar\psi d\psi e^{\int\bar\psi(-\not{D}-m)\psi d^3x}\,. \tag{2·4}$$

This can be also written as

$$\Gamma_{\mathrm{eff}}=-\ln\det(-\not{D}-m)=-\mathrm{Tr}\ln(-\not{D}-m)\,. \tag{2·5}$$

This Tr stands for the traces over Dirac matrices and internal indices as well as for the integration in coordinate space. We would like to compute the Chern-Simons term by starting from the above Lagrangian in the path integral formalism. We expect that the proper time method is not quite suitable and rather complicated; in this method one computes $\det(-\not{D}-m)$ essentially as $\det(-\not{D}-m)=\sqrt{\det(-\not{D}-m)\det(-\not{D}-m)^\dagger}=\sqrt{\det(-\not{D}-m)(-\not{D}-m)^\dagger}$, so the information on the phase factor may be missed. In order to obtain the Chern-Simons term easily, we use the method which is analogous to the one used to obtain the consistent anomalies from the path integral measure.[17] According to it, we may take the operator in the Lagrangian as

$$\Gamma_{\text{eff}}=-\mathrm{Tr}\ln\frac{(-\not{D}-m)}{(-i\not{\partial}-m)}$$
$$=-\mathrm{Tr}\ln\frac{(-\not{D}-m)(i\not{\partial}-m)}{(-i\not{\partial}-m)(i\not{\partial}-m)}$$
$$=-\mathrm{Tr}\ln[(-\not{D}-m)(i\not{\partial}-m)]+\mathrm{Tr}\ln[(-i\not{\partial}-m)(i\not{\partial}-m)]\,. \qquad (2\cdot6)$$

In this expression, in the first line, $(-i\not{\partial}-m)$ is added to be $\Gamma_{\text{eff}}=0$ for $A_\mu=0$. In the second line we add $(i\not{\partial}-m)/(i\not{\partial}-m)$ to obtain the Chern-Simons term easily and to convert the above effective action to an integration over the momentum. We can rewrite (2·6) as

$$\Gamma_{\text{eff}}=-\mathrm{Tr}\ln[(-\not{D}-m)(i\not{\partial}-m)]+\mathrm{Tr}\ln[(-i\not{\partial}-m)(i\not{\partial}-m)]$$
$$=-\mathrm{Tr}\ln(\not{\partial}\not{\partial}-e\not{A}m-ie\not{A}\not{\partial}+m^2)+\mathrm{Tr}\ln(\not{\partial}\not{\partial}+m^2)$$
$$=\int d^3x\int\frac{d^3k}{(2\pi)^3}\langle x|k\rangle\mathrm{tr}\int_0^\infty\frac{ds}{s}$$
$$\times[\exp\{-i(\partial_\mu\partial^\mu-e\not{A}m-ie\not{A}\not{\partial}+m^2)s\}-\exp\{-i(\partial_\mu\partial^\mu+m^2)s\}]\langle k|x\rangle$$
$$=\int d^3x\int\frac{d^3k}{(2\pi)^3}\mathrm{tr}\int_0^\infty\frac{ds}{s}$$
$$\times\exp\{-((\partial_\mu+ik_\mu)(\partial^\mu+ik^\mu)-e\not{A}m-ie\not{A}(\not{\partial}+i\not{k})+m^2)s\}$$
$$=\int d^3x\;\mathrm{tr}\int_0^\infty\frac{ds}{s^{5/2}}\int\frac{d^3k}{(2\pi)^3}\exp\{-m^2s\}\exp\{k_\mu k^\mu\}$$
$$\times\exp\{-\partial_\mu\partial^\mu s-2ik_\mu\partial^\mu\sqrt{s}+e\not{A}ms+ie\not{A}(\not{\partial}s+i\not{k}\sqrt{s})\}\,, \qquad (2\cdot7)$$

where in the fourth line of the above expression we have deformed the path of integration: $s\to is$, and in the fifth line we have rescaled the momentum: $\sqrt{s}k_\mu\to k_\mu$. The symbol tr stands for the trace over the internal indices and Dirac matrices. We omit $\exp\{-i(\partial_\mu\partial^\mu+m^2)s\}$ in the third line of (2·7), since it is easily shown that this term gives no contribution. In (2·7), we used the plane wave[18),19)]

$$\langle x|k\rangle=e^{-ikx}\,. \qquad (2\cdot8)$$

The next problem is how to extract the induced Chern-Simons term from this action in (2·7). One of the properties of the induced Chern-Simons term is to have the contraction with the Levi-Civita tensor and a proportionality factor of $m/|m|$. That is, it is enough to consider the contribution from the terms which have the above properties and are finite as $m\to0$. To do this computation, the following formula is very useful.

$$\int_0^\infty e^{-m^2s}s^{z-1}ds=\frac{\Gamma(z)}{|m^2|^z}\,. \qquad (2\cdot9)$$

By this formula, we notice that if we would like to obtain the contribution of finite terms as $m\to0$, it is sufficient to extract the terms which are proportional to s^2 with a factor m or proportional to s^3 with a factor m^3 in the expansion of $\exp\{-\partial_\mu\partial^\mu s$

$-2ik_\mu\partial^\mu\sqrt{s}+e\not{A}ms+ie\not{A}(\not{\partial}s+i\not{k}\sqrt{s})\}$ in the last line of (2·7). From the above expression, we thus obtain the terms in Γ_{eff} which contain the induced Chern-Simons term: in the second order of the expansion of $\exp\{-\partial_\mu\partial^\mu s-2ik_\mu\partial^\mu\sqrt{s}+e\not{A}ms+ie\not{A}(\not{\partial}s+i\not{k}\sqrt{s})\}$,

$$\frac{1}{2!}(ie)(em)s^2\not{A}\not{\partial}\not{A}\,, \tag{2·10}$$

and in the third order,

$$\frac{1}{3!}[(e)^3\not{A}\not{A}\not{A}m^3s^2+3(e^3)ms^2\not{A}\not{A}\not{k}\not{A}\not{k}]\,. \tag{2·11}$$

In these expressions, we omit the terms which do not satisfy the expected properties of the Chern-Simons term. In three dimensions, the integration over the momentum and Dirac matrices are given by

$$\frac{1}{(2\pi)^3}\int e^{k^2}d^3k=\frac{1}{8\pi^{3/2}}\,,\quad \frac{1}{(2\pi)^3}\int k^\mu k^\nu e^{k^2}d^3k=\frac{1}{2}\frac{-g^{\mu\nu}}{8\pi^{3/2}}\,,$$

$$\gamma^\mu\gamma^\nu=-\delta^{\mu\nu}-\epsilon^{\mu\nu\rho}\gamma_\rho\,,\qquad \gamma^\mu\gamma^\rho\gamma_\mu=-\gamma^\rho\,,$$

$$\mathrm{tr}\,\gamma^\mu\gamma^\nu=2g^{\mu\nu}\,,\qquad \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho=-2\epsilon^{\mu\nu\rho}\,. \tag{2·12}$$

The integration over s gives rise to

$$\int_0^\infty\frac{1}{s^{1/2}}e^{-m^2s}=\frac{\Gamma\left(\frac{1}{2}\right)}{|m|}=\frac{\sqrt{\pi}}{|m|}\,,$$

$$\int_0^\infty s^{1/2}e^{-m^2s}=\frac{\Gamma\left(\frac{3}{2}\right)}{|m|^3}=\frac{\sqrt{\pi}}{2|m|^3}\,. \tag{2·13}$$

Using these formulas, we obtain

$$\int d^3x\frac{1}{16\pi}ie^2\frac{m}{|m|}(-2\epsilon^{\mu\nu\rho}\mathrm{tr}A_\mu\partial_\nu A_\rho)$$
$$+\frac{1}{6}\left[\frac{e^3m^3}{16\pi|m|^3}(-2\epsilon^{\mu\nu\rho}\mathrm{tr}A_\mu A_\nu A_\rho)+\frac{3e^3m}{16\pi|m|}(-2\epsilon^{\mu\nu\rho}\mathrm{tr}A_\mu A_\nu A_\rho)\right], \tag{2·14}$$

where tr stands for the trace over the internal indices. As a final result, we get

$$\int d^3x\frac{1}{8\pi}\frac{m}{|m|}\epsilon^{\mu\nu\rho}\mathrm{tr}\left(-ie^2A_\mu\partial_\nu A_\rho-\frac{2e^3}{3}A_\mu A_\nu A_\rho\right). \tag{2·15}$$

At a glance, it may appear that (1·2) is different from (2·15). This difference is due to that of notational convention, and if we change $A_\mu\to iA_\mu$ in (2·15), (2·15) agrees with (1·2). In (2·15), the overall factor i comes from the imaginary part of Γ_{eff}. The result obtained by the above method agrees with the one obtained by the topological method with a topological mass term corresponding to $\mu=e^2m/8\pi|m|$.

§ 3. Chern-Simons term in gravitational theory

Now we would like to compute the induced Chern-Simons term in gravity theory. The method of computation in this section is similar to that in gauge theory. We extract the terms which contain the Levi-Civita tensor and are proportional to $m/|m|$. We take the starting effective action as

$$e^{-\Gamma_{\rm eff}}=\int[d(h^{1/2}\bar{\psi})][d(h^{1/2}\psi)]e^{\int h\bar{\psi}(-i\not{D}-m)\psi d^3x}\,, \tag{3·1}$$

where

$$\not{D}=\gamma^\mu(\partial_\mu-\frac{i}{2}\omega_{\mu mn}\sigma^{mn})\,,\qquad \sigma^{mn}=\frac{1}{4}[\gamma^m,\gamma^n]\,,$$
$$\gamma^\mu(x)=h_a{}^\mu(x)\gamma^a\,,\qquad h(x)\equiv\det(h_\mu{}^a)\,. \tag{3·2}$$

The connection ω does not contain the torsion and $h_\mu{}^a$ is the vielbein. When we compute the above $\Gamma_{\rm eff}$, we have to take the change into more careful account from x space to k space since there exists the geodesic bi-scalar in gravity theory.[19)~21)] This is a major difference from the case of gauge theory. We thus compute $\Gamma_{\rm eff}$ by taking into account the possible contribution of the geodesic bi-scalar. We first rewrite Eq. (3·1).

$$\begin{aligned}\Gamma_{\rm eff}&=-\ln\det\frac{(-i\not{D}-m)(i\not{D}-\omega-m)}{(-i\not{D}+\omega-m)(i\not{D}-\omega-m)}\\&=-\mathrm{Tr}\ln[(-i\not{D}-m)(i\not{D}-\omega-m)]+\mathrm{Tr}\ln[(-i\not{D}+\omega-m)(i\not{D}-\omega-m)]\\&=-\mathrm{Tr}\ln[\not{D}\not{D}+i\not{D}\omega+m\omega+m^2]+\mathrm{Tr}\ln[(\not{D}+i\omega)(\not{D}+i\omega)+m^2]\\&=\int d^3x\lim_{x\to x'}\langle x|\int_0^\infty\frac{ds}{s}\mathrm{tr}[e^{-i(D_\mu D^\mu+R/4+i\not{D}\omega+m\omega+m^2)s}-e^{-i(D_\mu+i\omega_\mu)(D^\mu+i\omega^\mu)+m^2)s}]|x'\rangle\,,\end{aligned} \tag{3·3}$$

where $\omega=\frac{1}{2}\gamma^\mu\omega_{\mu mn}\sigma^{mn}$ and $\omega_\mu=\frac{1}{2}\omega_{\mu mn}\sigma^{mn}$. The symbol tr stands for the trace over Dirac matrices. It can be confirmed that the last term in (3·3) gives no contribution to the induced Chern-Simons term. We thus omit the last term. Similar to the calculation in the previous section, we calculate the above expression in the momentum space. In the present case, $\langle x|x'\rangle$ is given by

$$\langle x|x'\rangle=\delta(x-x')\,. \tag{3·4}$$

In gravity theory, δ function is given by a formula including the geodesic bi-scalar. Here we introduce the geodesic bi-scalar $\sigma(x,x')$ as follows:[19)]

$$\delta(x-x')=\int\frac{d^3k}{(2\pi)^3}e^{ik_\mu D^\mu\sigma(x,x')}\,. \tag{3·5}$$

The scalar $\sigma(x,x')$ is equal to one half of the square of the distance along the geodesic between x and x'. The geodesic interval $\sigma(x,x')$, which is a bi-scalar, satisfies

$$\frac{1}{2}D_\mu\sigma(x,x')D^\mu\sigma(x,x')=\sigma(x,x')\,,$$
$$\frac{1}{2}D_\mu'\sigma(x,x')D'^\mu\sigma(x,x')=\sigma(x,x')\,,\qquad(3\cdot6)$$

where D_μ' represents the covariant derivative with respect to x'. The coincidence limit is given by[20),21)]

$$\lim_{x\to x'}\sigma(x,x')=0\,,\quad \lim_{x\to x'}D_\mu\sigma(x,x')=0\,,$$
$$\lim_{x\to x'}D_\mu D_\nu\sigma(x,x')=g_{\mu\nu}\,,\quad \lim_{x\to x'}D_\mu D_\nu D_\alpha\sigma(x,x')=0\,,$$
$$\lim_{x\to x'}D_\mu D_\nu D_\alpha D_\beta\sigma(x,x')=\frac{1}{3}R_{\beta\nu\alpha\mu}+\frac{1}{3}R_{\beta\mu\alpha\nu}\,.\qquad(3\cdot7)$$

By using Eqs. (3·4) and (3·5), the formula (3·3) can be written as

$$\begin{aligned}(3\cdot3)=&\int d^3x\lim_{x\to x'}\int\frac{d^3k}{(2\pi)^3}\int\frac{ds}{s}\mathrm{tr}\exp\{-i[(D_\mu+i\varDelta_\mu)(D^\mu+i\varDelta^\mu)\\&+\frac{R}{4}+(i\not{D}-\not{\varDelta})\omega+m\omega+m^2]s\}\\=&\int d^3x\lim_{x\to x'}\int\frac{ds}{s^{5/2}}\int\frac{d^3k}{(2\pi)^3}\mathrm{tr}\exp\{-m^2s\}\exp\{\varDelta_\mu\varDelta^\mu\}\exp\{-\varDelta_\mu\varDelta^\mu\}\\&\times\exp\Big\{\varDelta_\mu\varDelta^\mu-\Big(D_\mu D^\mu+\frac{R}{4}+i\not{D}\omega\Big)s\\&-i(\varDelta_\mu D^\mu+D_\mu\varDelta^\mu+i\not{\varDelta}\omega)\sqrt{s}-m\omega s\Big\}\end{aligned}\qquad(3\cdot8)$$

with $\varDelta^\mu\equiv k_\alpha D^\mu D^\alpha\sigma(x,x')$. In (3·8), we have deformed the path of integration: $s\to is$, and we have rescaled the momentum: $\sqrt{s}k_\mu\to k_\mu$. In these evaluations, there is one important technical point. If we define

$$A=\varDelta_\mu\varDelta^\mu\,,$$
$$B=-\Big(D_\mu D^\mu+\frac{R}{4}+i\not{D}\omega\Big)s-i(\varDelta_\mu D^\mu+D_\mu\varDelta^\mu+i\not{\varDelta}\omega)\sqrt{s}-m\omega s\,,\qquad(3\cdot9)$$

then,

$$e^{-A}e^{A+B}=e^{B+(1/2)[-A,A+B]\cdots}\,,\qquad(3\cdot10)$$

where we must consider commutators except for the first term B in the above expansion. This is a difficult point in the calculation in gravity theory. Moreover, being different from the chiral anomaly, the trace over Dirac matrices does not contain the chirality operator γ_5. A priori, we cannot drop these commutator terms in (3·10). In the following, however, we explain that these commutator terms give no contribution to the induced Chern-Simons term.

We expect that very higher commutator terms give no contribution, since they give the terms higher order in ω and R. In fact, we can easily confirm that it is sufficient to analyse the terms up to the fourth order,

$$[A, B], \quad [[A, B], A], \quad [[A, B], B],$$
$$[[[A, B], A], B], \quad [[[A, B], A], A],$$
$$[[[B, A], B], B], \quad [[[B, A], B], A]. \tag{3·11}$$

Since the terms which contain m are primarily important, we first consider them. In the expression B in (3·9), there is a term with ms which does not include the derivative. The induced Chern-Simons term is given by the terms with ms^2 or m^3s^3, as was explained in § 2. Since the order of the terms with a factor m, which survive after the commutator, is inevitably not less than $O(s^{3/2})$, the m^3s^3 term is generated only by the expansion of B.

Hence we have to carefully analyse the commutator terms which contribute to the ms^2 terms. For this purpose, we first note that the terms with $ms^{3/2}$ are not generated from the commutators: In $[A, B]$ the $O(s^{1/2})$ term is $2i\Delta_\nu D^\nu(\Delta_\mu\Delta^\mu)$ and it commutes with $m\omega s$, so in $[[A, B], B]$ the $O(ms^{3/2})$ terms are not generated. Moreover, the $O(ms^{3/2})$ term which is generated in the product of B and $[A, B]$ also vanishes in the coincidence limit.

In the commutator terms, there can exist a term with ms^2, but this has the form,

$$[[A, B], B] \sim [2D_\nu(\Delta_\mu\Delta^\mu)D^\nu + 2(D_\nu D^\nu\Delta_\mu)\Delta^\mu$$
$$+2(D^\nu\Delta_\mu)(D_\nu\Delta^\mu) + i\gamma_\nu\omega D^\nu(\Delta_\mu\Delta^\mu), m\omega]s^2, \tag{3·12}$$

and it can be easily shown that these terms vanish in the coincidence limit or by taking the γ matrix trace. There exists no term with a factor m which we must take into account except for (3·12); this will be shown in the Appendix. Moreover, in the Appendix, we will show that the cross terms between B and the commutator terms of order of $O(s)$ or $O(s^{1/2})$ give no contribution to the Chern-Simons term.

It has thus been established that the commutator terms $[A, B] + \cdots$ in (3·10) give no contribution to the induced Chern-Simons term.

From now on, let us try to compute the induced Chern-Simons term coming back to Eq. (3·8). To compute the Chern-Simons term, it is enough to expand e^B to the third order. We write them down explicitly as

$$\int d^3x \lim_{x\to x'} \int \frac{ds}{s^{5/2}} \int \frac{d^3k}{(2\pi)^3} \mathrm{tr}\Big[e^{-m^2 s} e^{\Delta_\mu\Delta^\mu}$$
$$\times \frac{1}{2!}\Big[\omega\Big(D_\mu D^\mu + \frac{R}{4} + i\not{D}\omega\Big) + \Big(D_\mu D^\mu + \frac{R}{4} + i\not{D}\omega\Big)\omega\Big] ms^2$$
$$-\frac{1}{3!}\{[(i\Delta_\mu D^\mu + iD^\mu\Delta_\mu - \not{\Delta}\omega)(i\Delta_\mu D^\mu + iD^\mu\Delta_\mu - \not{\Delta}\omega)\,\omega$$
$$+(i\Delta_\mu D^\mu + iD^\mu\Delta_\mu - \not{\Delta}\omega)\,\omega(i\Delta_\mu D^\mu + iD^\mu\Delta_\mu - \not{\Delta}\omega)$$
$$+\omega(i\Delta_\mu D^\mu + iD^\mu\Delta_\mu - \not{\Delta}\omega)(i\Delta_\mu D^\mu + iD^\mu\Delta_\mu - \not{\Delta}\omega)]ms^2 + (m\omega)^3\}\Big]. \tag{3·13}$$

We discard the terms which do not possess the expected properties of the Chern-Simons term, such as the Levi-Civita tensor structure; the expansion in ω to the third order gives

$$\int d^3x \lim_{x\to x'} \int \frac{ds}{s^{5/2}} \int \frac{dk}{(2\pi)^3} \mathrm{tr}\Big[e^{-m^2 s} e^{\Delta_\mu \Delta^\mu} \frac{1}{2!} (i\not{D}\omega\omega + \omega i\not{D}\omega) ms^2$$

$$- \frac{1}{3!} (3\omega \not{\Delta} \omega \not{\Delta} \omega ms^2 + m^3 \omega^3 s^3) \Big] . \qquad (3\cdot 14)$$

By using Eq. (3·7) and after integrating over k and s variables, we take the trace over γ matrices and obtain the final result,

$$- \int d^3x \frac{1}{64\pi} \frac{m}{|m|} \epsilon^{\mu\nu\rho} [i\partial_\mu \omega_{\nu mn} \omega_\rho{}^{nm} - \frac{2}{3} \omega_{\mu l m} \omega_\nu{}^{mn} \omega_{\rho n}{}^l] . \qquad (3\cdot 15)$$

As in the case of gauge theory, because of notational difference, if we change $\omega \to -i\omega$ then (3·15) is in agreement with (1·6). The reason for the appearance of the overall factor i in (3·15) is the same as that in gauge theory. This result, which has been obtained with no approximations, agrees with the one in Ref. 5) and corresponds to the Chern-Simons term with $\mu = 32\pi |m| / \kappa^2 m$ in the notation of (1·6).

§ 4. Conclusion

In this paper, we calculated the induced Chern-Simons term in the theory with fermions coupled to external gauge and gravitational fields by the path integral method. The induced Chern-Simons term which we obtained in this paper is in agreement with the standard results.[1),2)]

In the present method, we can get the answer including the precise numerical coefficients and with no restrictions on the background field configurations. Moreover, we could also treat the gravity theory as well as gauge theory. In the past, at least in the gravity case, except for the topological method, we have not been able to obtain the exact formula without the weak field approximation.

In this paper we took the basic operator associated with the Lagrangian as the form like $(-\not{D} - m)(i\not{\partial} - m)$, but we do not understand how this choice is related to the three dimensional topology. We would like to understand this point still better.

As the future research of this method, there are applications to supersymmtric theory and also higher dimensional extensions. These are under consideration.

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Appendix

From the considerations in the text, the commutator terms in (3·11) which we must examine are of the order of $O(s^{1/2})$ or $O(s)$. We investigate these terms in sequence:

$$[A, B]\,. \tag{A·1}$$

The term of $O(s^{1/2})$ exists only in $[A, B]$. This term vanishes in the coincidence limit and does not contribute to the Chern-Simons term. The term of $O(s)$ gives no contribution because the commutator with $D_\mu D^\mu$ has different characteristics from that of the Chern-Simons term and the commutator with $\not{D}\omega$ vanishes in the coincidence limit. The terms of $O(ms^2)$, which are generated by the products of some powers B and some powers of $[A, B]$, vanish in the coincidence limit or have the different tensor structure from the Chern-Simons term.

$$[[A, B], A]\,. \tag{A·2}$$

There are only $O(s)$ terms, which give no contribution by the same reason as the terms of $O(s)$ in $[A, B]$.

$$[[A, B], B]\,. \tag{A·3}$$

There are $O(s)+O(s^{3/2})+\cdots$ terms which do not contribute to the Chern-Simons term because of the vanishing or different tensorial structures of these terms in the coincidence limit similar to the cross terms in (A·1).

In the next order, we encounter

$$[[[A, B], A], A]\sim 0\,, \tag{A·4}$$

and these is no series which contains this commutator. A series of $[[[A, B], A], A]$ here means the terms which contain the repeated commutator between $[[[A, B], A], A]$ and A or B.

$$[[[A, B], A], B]\,. \tag{A·5}$$

Since $[[A, B], A]\sim O(s)$, the leading order is $O(s^{3/2})$. Moreover, $[[A, B], A]$ commutes with $m\omega s$ in the last factor B so the term with a factor m is of order higher than $O(s^2)$. This term and its series have no effect on the Chern-Simons term.

$$[[[B, A], B], A]\,. \tag{A·6}$$

Using the Jacobi identity, we can show that this is equal to (A·5).

$$[[[B, A], B], B]\,. \tag{A·7}$$

The order of the term with a factor m is higher than $O(s^2)$ because the terms of $O(s)$ generated in $[[A, B], B]$ commute with $m\omega$ in the last factor B. Other terms without m are the order of $O(s^{3/2})$, at least. Therefore this series gives no contribution.

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