Derivation of the amplitude equation at the Rayleigh–Bènard instability

M. C. Cross

Bell Laboratories, Murray Hill, New Jersey 07974
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The full amplitude equation (as introduced by Newell and Whitehead and Segel) is derived directly from the hydrodynamic equations for both "free" and "rigid" upper and lower boundaries. The coefficients involved, including the interaction parameter of nonparallel rolls, are explicitly calculated for both boundary conditions and all Prandtl numbers.

I. INTRODUCTION

In their pioneering work, Newell and Whitehead[1] and Segel[2] demonstrated the advantages of using a single differential equation for an order parameter $\psi(\mathbf{r})$ describing the slow time and (transverse) spatial variation of the convective roll pattern close to the onset of convection. The equation turns out to have the form of a "time-dependent Ginzburg–Landau equation"

$$\dot{\psi}=\tau_0^{-1}(\delta F/\delta\psi), \tag{1}$$

with F a functional of ψ reminiscent of a Ginzburg–Landau free energy at a second-order phase transition (specifically for a two-dimensional smectic).[3] Explicitly, the equation is

$$\tau_0\dot{\psi}_{\mathbf{q}}=[\varepsilon-\xi_0^2(q-q_0)^2]\psi_q-\sum_{\mathbf{q}'\mathbf{q}''\mathbf{q}'''}g(\mathbf{q},\mathbf{q}',\mathbf{q}'',\mathbf{q}''')\times\psi^*_{\mathbf{q}'}\psi_{\mathbf{q}''}\psi_{\mathbf{q}'''}\delta_{\mathbf{q}+\mathbf{q}',\mathbf{q}''+\mathbf{q}'''}, \tag{2}$$

where $\psi_{\mathbf{q}}$ is the Fourier transform of the order parameter, q_0 the onset wave vector, and τ_0, ξ_0, and g are parameters to be calculated. It is convenient to normalize ψ so that the convective heat flow, defining the Nusselt number N, is given by

$$(N-1)\frac{R}{R_c}=\sum_{\mathbf{q}}|\psi_{\mathbf{q}}|^2, \tag{3}$$

where R is the Rayleigh number and R_c is its value at the onset of convection in a laterally infinite system. The small parameter of the expansion is $\varepsilon=R/R_c-1$. Note that as defined here $\psi(\mathbf{r})$ is proportional to the hydrodynamic fields, including the sinusoidal dependence of the roll pattern. Equation (2) is very general, containing both effects of a continuous spread of wave vectors $\mathbf{q}$ about a particular onset wave vector $q_0\hat{\mathbf{q}}_1$ (with $\hat{q}_1$ a unit vector), and the interaction of order parameters around different onset wavevectors $q_0\hat{\mathbf{q}}_1$, $q_0\hat{\mathbf{q}}_2$, etc.

The parameters τ_0, ξ_0, and g for the case of only one wave vector (and its negative) have all been derived for the case of stress-free upper and lower boundaries, and the full expression for g has been calculated for this case in the infinite Prandtl number limit.[1,2,4] On the other hand, most experiments are performed with rigid upper and lower boundaries (all fluid velocities are zero here), and with finite Prandtl number fluids. The main purpose of this paper is to derive the full amplitude equation for this experimentally useful case. Wesfried *et al.*[5] evaluated τ_0 and ξ_0, and for the special case of single rolls in a rectangular geometry g, for these boundary conditions, by assuming the form of the amplitude equation and then calculating each coefficient in turn. The coefficients can also be derived from the work of Kelly and Pal.[6] Here, we first complement this approach by formally deriving the amplitude equation from the hydrodynamic equations for this case. This is done by a very compact method valid for both rigid and free boundaries. This formal derivation leads to expressions for τ_0 and ξ_0 and the full general expression for g that is explicitly evaluated. In particular, the calculation of τ_0 seems particularly straightforward. Rather than following the "method of multiple scales" commonly used, we follow a more physical approach, somewhat like that of Swift and Hohenberg.[4] This approach, I believe, makes the content of the amplitude equation more evident. However, much of the detailed working relies heavily on results proved by Schlüter *et al.*[7]

II. FORMAL DERIVATION OF THE AMPLITUDE EQUATION

The equations of motion describing the evolution of the velocity field $(u,v,w)=(\mathbf{u},w)$ and the deviation of the temperature θ from the linear conducting profile between boundaries at $z=\pm\frac{1}{2}$ are taken to be the Boussinesq equations. If distance, time, and temperature are scaled by d, d^2/κ, and $\kappa\nu/\alpha g d^3$, respectively, where d is the cell height, κ and ν the thermal and viscous diffusivities, and α the thermal expansion coefficient, the equations take the form[7]

$$\begin{aligned}\dot{\mathbf{u}}&=\sigma\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right)\mathbf{u}-\left(w\frac{\partial}{\partial z}+\mathbf{u}\cdot\nabla\right)\mathbf{u}-\nabla P,\\ \dot{w}&=\sigma\theta+\sigma\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right)w-\left(w\frac{\partial}{\partial z}+\mathbf{u}\cdot\nabla\right)w-\frac{\partial}{\partial z}P,\\ \dot{\theta}&=Rw+\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right)\theta-\left(w\frac{\partial}{\partial z}+\mathbf{u}\cdot\nabla\right)\theta,\end{aligned} \tag{4}$$

together with the continuity equation

$$\nabla\cdot\mathbf{u}+\frac{\partial}{\partial z}w=0, \tag{5}$$

by which the pressure P may be eliminated. In these and all later equations ∇ is the two-dimensional transverse differential operator. Fluid properties are contained in the Prandtl number $\sigma=\nu/\kappa$. The Rayleigh

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number R is the conducting temperature gradient, scaled as prescribed here.

Defining a four-dimensional vector $\mathbf{V}=(\theta,\mathbf{u},w)$ and a gradient vector $\boldsymbol{\partial}=(0,\nabla,\partial/\partial z)$, these equations can be summarized:

$$\dot{\mathbf{V}}=\mathbf{D}\mathbf{V}-\boldsymbol{\partial}P+\mathbf{N}(\mathbf{V},\mathbf{V}), \quad (6)$$

$$\boldsymbol{\partial}\cdot\mathbf{V}=0, \quad (7)$$

where $\mathbf{D}=\mathbf{D}^0+\delta\mathbf{D}$ is the matrix operator with

$$\mathbf{D}^0=\begin{bmatrix}\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right) & 0 & R_c \\ 0 & \sigma\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right) & 0 \\ \sigma & 0 & \sigma\left(\nabla^2+\frac{\partial^2}{\partial z^2}\right)\end{bmatrix}, \quad (8)$$

and $\delta\mathbf{D}$ is the matrix with one nonzero element δR in the place of R_c in Eq. (8). Also N is the nonlinear term

$$\mathbf{N}(\mathbf{V},\mathbf{V}')=-(\mathbf{V}\cdot\boldsymbol{\partial})\mathbf{V}'. \quad (9)$$

First consider the linearized version of Eq. (6),

$$\dot{\mathbf{V}}=\mathbf{D}\mathbf{V}-\boldsymbol{\partial}P. \quad (10)$$

together with Eq. (7). The solution is the sum of eigenvectors with time dependences $\exp(\lambda_{\mathbf{q}}^{(i)}t)$ with $\lambda_{\mathbf{q}}^{(i)}$ the corresponding eigenvalue, where $\mathbf{q}$ is the transverse wave vector and i runs over the discrete set of z eigenfunctions (satisfying the relevant boundary conditions) for each $\mathbf{q}$. For small Rayleigh numbers all eigenvalues are negative of order -1 (for these purposes σ is taken to be of order 1). However, as R passes through R_c a set of eigenvalues ($i=1$ say, $|q|=q_0$) passes through zero. For R just above R_c a set of slow modes exists

$$\lambda_{\mathbf{q}}^{(1)}=O(\varepsilon)-O(q-q_0)^2. \quad (11)$$

The amplitude equation is derived as the projection of Eqs. (4) and (5) onto these "critically slowed" modes, just as the hydrodynamic equations are themselves the projection of the microscopic equations of motion onto the slow hydrodynamic modes. The velocity field may be synthesized from these components. The components projected onto the other modes are small, but can be calculated, if required, as adiabatically following the slow order parameter motion.

The actual procedure is vastly simplified by noticing[7] that for vectors satisfying Eq. (7) the operator $\mathbf{D}^0$ is self-adjoint:

$$\langle\mathbf{V},\mathbf{D}^0\mathbf{V}'\rangle=\langle\mathbf{V}',\mathbf{D}^0\mathbf{V}\rangle^*, \quad (12)$$

where the inner product $\langle\mathbf{V},\mathbf{V}'\rangle$ is defined as

$$[\sigma\theta^*\theta'+R_c(\mathbf{u}^*\cdot\mathbf{u}'+w^*w')]_m,$$

with $[\;]_m$ signifying averaging over the layer. The condition Eq. (12) will implicitly be assumed in all the following statements. It is true for both stress-free boundaries ($\theta=\partial\mathbf{u}/\partial z=w=0$ at $z=\pm\frac{1}{2}$) and rigid boundaries ($\theta=\mathbf{u}=w=0$ at $z=\pm\frac{1}{2}$). The normalized plane wave eigenvectors $\mathbf{e}_{\mathbf{q}}^{(i)}$ and eigenvalues $\lambda_{\mathbf{q}}^{(i)}$ of $\mathbf{D}^0$ then have the following properties:

(i) $\lambda_{\mathbf{q}}^{(i)}$ are real;

(ii) It is convenient to choose $\mathbf{e}_{\mathbf{q}}^{(i)}\propto\exp[i(\mathbf{q}\cdot\mathbf{r})]$, and then, $\mathbf{e}_{-\mathbf{q}}^{(i)}=\mathbf{e}_{\mathbf{q}}^{*(i)}$;

(iii) The eigenvectors $\mathbf{e}_{\mathbf{q}}^{(i)}$ form a complete orthonormal set

$$\mathbf{V}=\sum_{\mathbf{q},i}V_{\mathbf{q}}^{(i)}\mathbf{e}_{\mathbf{q}}^{(i)}$$

with

$$V_{\mathbf{q}}^{(i)}=\langle\mathbf{e}_{\mathbf{q}}^{(i)},\mathbf{V}\rangle. \quad (13)$$

The projection of the differential equation onto the slow modes is then given simply by the scalar product with $\mathbf{e}_{\mathbf{q}}^{(1)}$

$$\dot{V}_{\mathbf{q}}^{(1)}=\langle\mathbf{e}_{\mathbf{q}}^{(1)},\dot{\mathbf{V}}\rangle=\langle\mathbf{e}_{\mathbf{q}}^{(1)},\mathbf{D}^0\mathbf{V}+\delta\mathbf{D}\mathbf{V}-\boldsymbol{\partial}P+\mathbf{N}(\mathbf{V},\mathbf{V})\rangle. \quad (14)$$

The order parameter $\psi_{\mathbf{q}}$ is taken proportional to $V_{\mathbf{q}}^{(1)}$ with the proportionality constant chosen so that the normalization condition in Eq. (3) is satisfied.

The amplitude equation is now derived by treating each term on the right-hand side of Eq. (14) in turn repeatedly using properties (i)–(iii).

The first term becomes

$$\langle\mathbf{e}_{\mathbf{q}}^{(1)},\mathbf{D}^0\mathbf{V}\rangle=\lambda_{\mathbf{q}}^{(1)}V_{\mathbf{q}}^{(1)}. \quad (15)$$

Furthermore, since by definition of $R_c, q=q_0$ minimizes the eigenvalue at zero, $\lambda_{\mathbf{q}}^{(1)}\propto(q-q_0)^2$ for q close to q_0. This gives the coefficient ξ_0 of Eq. (2).

For the second term we find

$$\langle\mathbf{e}_{\mathbf{q}}^{(1)},\delta\mathbf{D}\mathbf{V}\rangle=\sum_i\langle\mathbf{e}_{\mathbf{q}}^{(1)},\delta\mathbf{D}\mathbf{e}_{\mathbf{q}}^{(i)}\rangle\langle\mathbf{e}_{\mathbf{q}}^{(i)},\mathbf{V}\rangle$$
$$\approx\langle\mathbf{e}_{\mathbf{q}}^{(1)}\delta\mathbf{D}\mathbf{e}_{\mathbf{q}}^{(1)}\rangle V_{\mathbf{q}}^{(1)}. \quad (16)$$

The first equality follows from completeness and the fact that $\delta\mathbf{D}$ does not mix wave vectors. Since $\delta\mathbf{D}$ already contains the small parameter ε, to the accuracy required it is sufficient to restrict the sum over i to $i=1$ leading to the last expression. Then, in evaluating $\langle\mathbf{e}_{\mathbf{q}}^{(1)},\delta\mathbf{D}\mathbf{e}_{\mathbf{q}}^{(1)}\rangle$ q may be set equal to q_0. The eigenvector $\mathbf{e}_{\mathbf{q}}^{(1)}$ for $q=q_0$ is known explicitly for both rigid and free boundaries, and this gives the linear growth rate of the convection pattern for all cases very directly.

The third term disappears,

$$\langle\mathbf{e}_{\mathbf{q}}^{(1)},\boldsymbol{\partial}P\rangle=0, \quad (17)$$

upon integrating by parts.

As might be expected, calculating the nonlinear coupling coefficient $g(\mathbf{q},\mathbf{q}',\mathbf{q}'',\mathbf{q}''')$ presents the most difficulty. To the accuracy required, it is sufficient to calculate g with all wave vector arguments of magnitude q_0. The first step is again to use the completeness condition

$$\langle\mathbf{e}_{\mathbf{q}}^{(1)},\mathbf{N}(\mathbf{V},\mathbf{V})\rangle=\sum_{\substack{ij\\kk'}}\langle\mathbf{e}_{\mathbf{q}}^{(1)},N(\mathbf{e}_{\mathbf{k}}^{(i)},\mathbf{e}_{\mathbf{k}'}^{(j)})\rangle V_{\mathbf{k}}^{(i)}V_{\mathbf{k}'}^{(j)}. \quad (18)$$

However, $\mathbf{e}_{\mathbf{k}}^{(i)}$ and $\mathbf{e}_{\mathbf{k}'}^{(j)}$ cannot both be slow modes (leading to additional, quadratic terms in the amplitude equation), since the average over the layer involved in the inner product vanishes (the integrand would be an odd function of z). Thus, the amplitude of the fast component, e.g., $V_{\mathbf{k}}^{(i)}$ driven by the slow components must in turn be calculated from its own equation of motion like Eq. (14). To the order required, the time derivative approximately $\lambda^{(1)}$ can be neglected with respect to the fast eigenvalue $\lambda^{(i)}$, $i \neq 1$ (the adiabatic following), δD may be neglected and in N only slow modes need be retained:

$$0=\lambda_{\mathbf{k}}^{(i)} V_{\mathbf{k}}^{(i)}+\sum_{\mathbf{q}'' \mathbf{q}'''}\langle \mathbf{e}_{\mathbf{k}}^{(i)}, \mathrm{N}(\mathbf{e}_{\mathbf{q}''}^{(1)}, \mathbf{e}_{\mathbf{q}'''}^{(1)})\rangle V_{\mathbf{q}''}^{(1)} V_{\mathbf{q}'''}^{(1)} . \tag{19}$$

In principle, this result must be substituted for $V_{\mathbf{k}}^{(i)}$ and $V_{\mathbf{k}'}^{(j)}$ in turn giving two contributions to g. A considerable simplification results from the result proved by Ref. 7:

$$\langle \mathbf{e}_{\mathbf{q}}^{(1)}, \mathrm{N}(\mathbf{V}, \mathbf{e}_{\mathbf{q}'}^{(1)})\rangle=0 \quad \text{for } q=q'=q_0 \tag{20}$$

for any velocity field $\mathbf{V}$ that has zero vertical circulation and satisfies Eq. (7). It is readily proved that $\mathbf{e}_{\mathbf{k}}^{(i)}$ satisfies this condition. This follows since, of the eigenvectors of D^0, only the stirring modes with $\lambda = -\sigma(q^2+q_Z^2)$ have a nonzero vertical circulation, and these do not couple to the convection according to Eq. (19).

Equation (18) and (19) can then be written in the compact form

$$\begin{aligned}\langle \mathbf{e}_{\mathbf{q}}^{(1)}, \mathrm{N}(\mathbf{V}, \mathbf{V})\rangle &= \sum_{\mathbf{q}' \mathbf{q}'' \mathbf{q}'''}\left[\sum_i |\langle \mathbf{e}_{\mathbf{q}+\mathbf{q}'}^{(i)}, \mathrm{N}(\mathbf{e}_{\mathbf{q}}^{(1)}, \mathbf{e}_{\mathbf{q}'}^{(1)})\rangle|^2 \frac{1}{\lambda_{\mathbf{q}+\mathbf{q}'}^{(i)}}\right] \\ &\times V_{\mathbf{q}'}^{(1)*} V_{\mathbf{q}''}^{(1)} V_{\mathbf{q}'''}^{(1)} \delta_{\mathbf{q}+\mathbf{q}', \mathbf{q}''+\mathbf{q}'''} ,\end{aligned} \tag{21}$$

where the relationship found by integrating by parts

$$\langle \mathbf{V}_1, \mathrm{N}(\mathbf{V}_2, \mathbf{V}_3)\rangle=-\langle \mathrm{N}(\mathbf{V}_2^*, \mathbf{V}_1), \mathbf{V}_3\rangle , \tag{22}$$

and

$$V_{\mathbf{q}}^{(1)*}=\langle \mathbf{e}_{\mathbf{q}}^{(1)}, \mathbf{V}\rangle^*=\langle \mathbf{e}_{-\mathbf{q}}^{(1)}, \mathbf{V}\rangle , \tag{23}$$

for the actual velocity fields $\mathbf{V}$, have been used for cosmetic purposes.

To the accuracy required, all $\mathbf{q}$ in the brackets in Eq. (21) may be assumed to be of magnitude q_0. Taken together the conditions $q=q'=q''=q'''=q_0$ and $\mathbf{q}+\mathbf{q}'=\mathbf{q}''+\mathbf{q}'''$ require the wave vectors to consist of pairs of antiparallel vectors. Then, the derivation also shows that $g(\mathbf{q}, \mathbf{q}'; \mathbf{q}'', \mathbf{q}''')$ depends only on the angle between $\mathbf{q}$ and $\mathbf{q}'$ (the same as the angle between $\mathbf{q}''$ and $\mathbf{q}'''$), i.e., $g = g(\hat{\mathbf{q}} \cdot \hat{\mathbf{q}}')$.

Thus, we arrive at the amplitude Eq. (2) with expressions for all the parameters in terms of eigenvectors and eigenvalues of $\mathbf{D}^0$.

III. EVALUATION OF THE PARAMETERS

In this section we evaluate the parameter for the two cases of free boundaries and rigid boundaries. First, it is convenient here to calculate the proportionality constant relating $\Psi_{\mathbf{q}}$ to $V_{\mathbf{q}}^{(1)}$. In terms of $V_{\mathbf{q}}^{(1)}$ the Nusselt number is given by

$$\begin{aligned}(N-1) \frac{R}{R_c} &= \frac{(w\theta)_m}{R_c} \simeq \sum_{\mathbf{q}} |V_{\mathbf{q}}^{(1)}|^2 \\ &\times\left(\frac{1}{R_c} \frac{(w_0^* \theta_0)}{[\sigma|\theta_0|^2+R_c(|u_0|^2+|w_0|^2)]_m}\right) ,\end{aligned} \tag{24}$$

where the negligible approximation of replacing q by q_0 in the multiplying factor has been made, and $\mathbf{e}_{\mathbf{q}_0}^{(1)}$ has been written in terms of $(\theta_0, \mathbf{u}_0, w_0)$, the un-normalized eigenvector. Thus, we define $\Psi_{\mathbf{q}}=c V_{\mathbf{q}}^{(1)}$ with

$$c=\left(\frac{1}{R_c} \frac{(w_0^* \theta_0)}{[\sigma|\theta_0|^2+R_c(|u_0|^2+|w_0|^2)]_m}\right)^{1/2} . \tag{25}$$

The linear terms are, of course, given directly by Eqs. (14)–(16) without change from this renormalization

$$\dot{\Psi}_{\mathbf{q}}=(\lambda_{\mathbf{q}}^{(1)}+\langle \mathbf{e}_{\mathbf{q}_0}^{(1)}, \delta D \mathbf{e}_{\mathbf{q}_0}^{(1)}\rangle) \Psi_{\mathbf{q}} . \tag{26}$$

This immediately leads to the results for the coefficients in Eq. (2), as will be listed.

The inner growth rate parameter is given by

$$\tau_0^{-1} \varepsilon=\langle \mathbf{e}_{\mathbf{q}_0}^{(1)}, \delta D \mathbf{e}_{\mathbf{q}_0}^{(1)}\rangle , \tag{27}$$

so that, explicitly,

$$\tau_0^{-1}=\frac{\sigma R_c(\theta_0^* w_0)_m}{[\sigma|\theta_0|^2+R_c(|u_0|^2+|w_0|^2)]_m} , \tag{28}$$

which is easily evaluated in terms of the critical solution.

For free boundaries θ_0, $\mathbf{u}_0$, and w_0 are known analytically, and $\tau_0^{-1}=\frac{3}{2}\pi^2\sigma/(\sigma+1)$, as found by Newell and Whitehead.[1] For rigid boundaries the critical solution is known in the form

$$\theta_0=\sum_{n=1}^{3} \theta_{0n} \cosh q_n z , \tag{29}$$

with θ_{0n} and q_n known only from numerical calculations.[8] Using the values quoted by Schlüter *et al.*[7] we find, for rigid boundaries,

$$\tau_0^{-1}=\frac{19.65\sigma}{\sigma+.5117} . \tag{30}$$

The integrals involved are given in the Appendix. This result has also been derived by Behringer and Ahlers[9] (quoted in terms of a dimensionless time unit smaller by a factor of $\pi^2/4$) and by Wesfried *et al.*[5]

From the form proved for the amplitude equation it is easiest to derive ξ_0^2 directly as

$$\xi_0^2=\frac{1}{2} R_c^{-1} \frac{d^2}{dq^2} R_c(q)\bigg|_{q=q_0} , \tag{31}$$

where $R_c(q)$ is the critical Rayleigh number for the onset of convection at wave vector q. Again for free boundaries $R_c(q)$ is analytically known, giving $\xi_0^2=8/3\pi^2$. For rigid boundaries $R_c(q)$ has been evaluated for a discrete set of q,[8] and simple interpolation gives $\xi_0^2=0.145 \pm 0.003$. More elegantly, Wesfried *et al.*[5] evaluated Eq. (31) analytically to give the value 0.148.

From Eqs. (2), (14), (21), and (25) the nonlinear coupling constant is found to be

$$g(\hat{\mathbf{q}}\cdot\hat{\mathbf{q}}')=\tau_0 c^{-2}\sum_i |\langle \mathbf{e}^{(i)}_{\mathbf{q}+\mathbf{q}'}, \mathbf{N}(\mathbf{e}^{(1)}_{\mathbf{q}},\mathbf{e}^{(1)}_{\mathbf{q}'})\rangle|^2 \frac{1}{\lambda^{(i)}_{\mathbf{q}+\mathbf{q}'}}, \tag{32}$$

where $q=q'=q_0$. Since all $\lambda^{(i)}_{\mathbf{q}}$, $q\neq q_0$ are positive, this shows immediately that g is positive. For other purposes, although, in principle, the eigenvalues and eigenvectors of $\mathbf{D}^0$ may be calculated, the infinite sum involved makes the expression Eq. (32) unwieldy for direct evaluation. Instead, it may be noticed that the same infinite sum is also involved in the solution $\mathbf{V}_2$ of

$$\mathbf{D}^0\mathbf{V}_2+\mathbf{N}(\mathbf{e}^{(1)}_{\mathbf{q}},\mathbf{e}^{(1)}_{\mathbf{q}'})=0, \tag{33}$$

namely,

$$\mathbf{V}_2=\sum_i \mathbf{e}^{(i)}_{\mathbf{q}+\mathbf{q}'}\langle \mathbf{e}^{(i)}_{\mathbf{q}+\mathbf{q}'}, \mathbf{N}(\mathbf{e}^{(1)}_{\mathbf{q}},\mathbf{e}^{(1)}_{\mathbf{q}'})\rangle(\lambda^{(i)}_{\mathbf{q}+\mathbf{q}'})^{-1}. \tag{34}$$

This is also obvious from the derivation of Eq. (21). For $q=q'=q_0$ as required, Schlüter *et al.* have directly solved Eq. (33) for $\mathbf{V}_2$. In fact, from their expression for J [their Eq. (7.31)], the full expression for g may be written as

$$g(\cos\theta)=\tau_0 c^{-2}\frac{L(\cos\theta,-\cos\theta)}{[\sigma|\theta_0|^2+R_c(|u_0|^2+|w_0|^2)]^2_m}, \tag{35}$$

where L is the function they tabulate and the denominator simply arises from their different choice of normalization. Using the expressions for τ_0 and c we finally arrive at the result

$$g(\cos\theta)=\frac{L(\cos\theta,-\cos\theta)}{\sigma(w_0^*\theta_0)^2_m}, \tag{36}$$

For free boundaries, Schlüter *et al.*[7] gave an explicit form for L from which we find

$$g=\frac{1}{2}\frac{(1-\cos\theta)^2}{(5+\cos\theta)^3-\frac{27}{4}(1+\cos\theta)}[(5+\cos\theta)^2 + 9(1+\cos\theta)\sigma^{-1}+3(1+\cos\theta)(5+\cos\theta)\sigma^{-2}]. \tag{37}$$

The infinite σ limit can easily be seen to give the same results as originally derived in the coefficient β_{ij} by Newell and Whitehead.[1] For rigid boundaries Schlüter *et al.* calculate L numerically for certain values of $\cos\theta$. From their table we find

$$g=g_0+g_{-1}\sigma^{-1}+g_{-2}\sigma^{-2}. \tag{38}$$

with g_0, g_{-1}, and g_{-2} functions of $\cos\theta$ as tabulated in Table I.

IV. SUMMARY AND DISCUSSION

We have derived, very compactly, the amplitude equation for the evolution of the order parameter $\Psi_{\mathbf{q}}$, $q\approx q_0$ describing the convection pattern near onset for both "rigid" and "free" boundary conditions

$$\tau_0\dot{\Psi}_{\mathbf{q}}=[\varepsilon-\xi_0^2(q-q_0)^2]\Psi_{\mathbf{q}} - \sum_{\mathbf{q}'\mathbf{q}''\mathbf{q}'''} g(\hat{\mathbf{q}}\cdot\hat{\mathbf{q}}')\Psi^*_{\mathbf{q}'}\Psi_{\mathbf{q}''}\Psi_{\mathbf{q}'''}\delta_{\mathbf{q}+\mathbf{q}',\mathbf{q}''+\mathbf{q}'''} \tag{39}$$

with the convective heat transport defining the Nusselt number given by

TABLE I. The parameters giving $g(\cos\theta)$ for the case of rigid boundary conditions as a function of Prandtl number σ [see Eq. (38)].

$8x$	$g_{-2}(x)$	$g_{-1}(x)$	$g_0(x)$
−8	0	0	0.6921
−7	0.0234	0.0350	0.5721
−6	0.0406	0.0557	0.4662
−5	0.0526	0.0659	0.3737
−4	0.0600	0.0684	0.2938
−3	0.0637	0.0657	0.2258
−2	0.0645	0.0596	0.1686
−1	0.0628	0.0515	0.1214
0	0.0595	0.0423	0.0832
1	0.0549	0.0330	0.0532
2	0.0495	0.0240	0.0306
3	0.0436	0.0157	0.0147
4	0.0376	0.0083	0.0049
5	0.0317	0.0021	0.0004
6	0.0262	−0.0030	0.0009
7	0.0211	−0.0068	0.0059
8	0.0166	−0.0094	0.0148

$$(N-1)\frac{R}{R_c}=\sum_{\mathbf{q}}|\Psi_{\mathbf{q}}|^2. \tag{40}$$

The parameters are tabulated for rigid and free upper and lower boundaries in Table II. This is not the place to describe applications of this amplitude equation. However, it may be useful to briefly show the relationship of the parameter $g(\hat{\mathbf{q}}\cdot\hat{\mathbf{q}}')$ to already known quantities for two simple cases: firstly, that of single rolls at the critical wave vector, and then of the hexagonal pattern, both in an infinite region. The latter is of interest since small perturbations, such as non-Boussinesq effects or a nonlinear conducting profile, which may be readily included in the formalism, lead to an additional quadratic term in the amplitude equation that favors the hexagonal pattern.

For single rolls at wave vector $\mathbf{q}_1=q_0\hat{\mathbf{q}}_1$ we may write

$$\Psi_{\mathbf{q}}=\frac{A}{\sqrt{2}}(\delta_{\mathbf{q},\mathbf{q}_1}+\delta_{\mathbf{q},-\mathbf{q}_1}), \tag{41}$$

with A the amplitude of the roll pattern. This leads to

$$\tau_0\dot{A}=(\varepsilon-\bar{g}A^2)A;\quad (N-1)(R/R_c)=A^2, \tag{42}$$

where

$$\bar{g}=\tfrac{1}{2}[g(+1)+2g(-1)]. \tag{43}$$

The static solution gives the linear dependence of the convected heat transport on ε as found previously.[7,8]

For the hexagonal pattern we may write

$$\Psi_{\mathbf{q}}=\frac{A}{\sqrt{6}}(\delta_{\mathbf{q},\mathbf{q}_1}+\delta_{\mathbf{q},-\mathbf{q}_1}+\delta_{\mathbf{q},\mathbf{q}_2} + \delta_{\mathbf{q},-\mathbf{q}_2}+\delta_{\mathbf{q},\mathbf{q}_3}+\delta_{\mathbf{q},-\mathbf{q}_3}), \tag{44}$$

where $\hat{\mathbf{q}}_1$, $\mathbf{q}_2$, and $\hat{\mathbf{q}}_3$ are mutually at 120°. The equation for the amplitude A of the hexagonal pattern is

$$\tau_0\dot{A}=(\varepsilon-\bar{g}A^2)A;\quad (N-1)(R/R_c)=A^2, \tag{45}$$

where

TABLE II. The parameters of the amplitude equation for free and rigid boundary conditions.

	Free	Rigid
τ_0^{-1}	$\frac{3\pi^2}{2}\frac{\sigma}{\sigma+1}$	$\frac{19.65\sigma}{\sigma+0.5117}$
ξ_0^2	$\frac{8}{3\pi^2}$	0.148
$g(x)$	$\frac{1}{2}\frac{(1-x)^2}{(5+x)^3-\frac{27}{4}(1+x)}$ $\times[(5+x)^2+9(1+x)\sigma^{-1}+3(1+x)(5+x)\sigma^{-2}]$	$g_0+g_{-1}\sigma^{-1}+g_{-2}\sigma^{-2}$ (see Table I)

$$\bar{g}=\tfrac{1}{6}[g(1)+6g(-1)+4g(\tfrac{1}{2})+4g(-\tfrac{1}{2})]. \tag{46}$$

Again, the static result is readily seen to be the same as previously calculated for both free[8] and rigid[7] boundaries.

APPENDIX

For rigid boundaries the solutions at the onset wave vector $\mathbf{q}_0$ (the critical solutions) are

$$\exp[i\mathbf{q}_0\cdot\mathbf{r}][\theta_0(z),u_0(z),w_0(z)]$$

with[7]

$$\theta_0(z)=650.68\cosh(i3.974z)+\{(39.277+i0.433)\cosh[(5.195-i2.126)z]+\text{c.c.}\},$$

$$u_0(z)=3.117i\{3.974i\sinh(i3.974z)+[(5.195-i2.126)(-3.076\times10^{-2}+i5.194\times10^{-2})\times\sinh[(5.195-i2.126)z]+\text{c.c.}]\},$$

$$w_0(z)=(3.117)^2\{\cosh(i3.974z)+[(-3.076\times10^{-2}+i5.194\times10^{-2})\times\cosh[(5.195-i2.126)z]+\text{c.c.}]\},$$

with the boundaries at $z=\pm\frac{1}{2}$. The critical Rayleigh number is $R_c=1707.762$. These give for the averages:

$$(|\theta_0|^2)_m=2.524\times10^5,$$

$$R_c(|u_0|^2)_m=7.123\times10^4,$$

$$R_c(|w_0|^2)_m=5.791\times10^4,$$

$$R_c^{1/2}(w_0^*\theta_0)_m=1.200\times10^5.$$

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