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Derivation of the Cubic NLS and Gross–Pitaevskii Hierarchy from Manybody Dynamics in $d = 3$ Based on Spacetime Norms

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Abstract. We derive the defocusing cubic Gross–Pitaevskii (GP) hierarchy in dimension $d = 3$, from an N-body Schrödinger equation describing a gas of interacting bosons in the GP scaling, in the limit $N \to \infty$. The main result of this paper is the proof of convergence of the corresponding BBGKY hierarchy to a GP hierarchy in the spaces introduced in our previous work on the well-posedness of the Cauchy problem for GP hierarchies (Chen and Pavlović in Discr Contin Dyn Syst 27(2):715–739, 2010; http://arxiv.org/abs/0906.2984; Proc Am Math Soc 141:279–293, 2013), which are inspired by the solution spaces based on space-time norms introduced by Klainerman and Machedon (Comm Math Phys 279(1):169–185, 2008). We note that in $d = 3$, this has been a well-known open problem in the field. While our results do not assume factorization of the solutions, consideration of factorized solutions yields a new derivation of the cubic, defocusing nonlinear Schrödinger equation (NLS) in $d = 3$.

1. Introduction

We derive the defocusing cubic Gross–Pitaevskii (GP) hierarchy from an N-body Schrödinger equation in dimension $d = 3$ describing a gas of interacting bosons in the GP scaling, as $N \to \infty$. The main result of this paper is the proof of convergence in the spaces introduced in our previous work on the well-posedness of the Cauchy problem for GP hierarchies [7–9], which are inspired by the solution spaces based on space-time norms introduced by Klainerman and Machedon [25]. In dimension 3, this problem has so far remained a key open problem, while in dimensions 1 and 2, it was solved in [6,26] for the cubic and quintic case.

The derivation of nonlinear dispersive PDEs, such as the nonlinear Schrödinger (NLS) or nonlinear Hartree (NLH) equations, from manybody quantum dynamics is a central topic in mathematical physics, and has been approached by many authors in a variety of ways; see [16–18,25,26,31] and the references therein, and also [1,3,11,12,15,19–24,30,33]. This problem is closely related to the phenomenon of Bose–Einstein condensation (BEC) in systems of interacting bosons, which was first experimentally verified in 1995 [4,14]. For the mathematical study of BEC, we refer to the fundamental works [2,27–29] and the references therein.

1.1. The Gross–Pitaevskii Limit for Bose Gases

As a preparation for our analysis in the present paper, we will outline some main ingredients of the approach due to L. Erdös, B. Schlein, and H.-T. Yau. In an important series of works [16–18], these authors developed a powerful method to derive the cubic nonlinear Schrödinger equation (NLS) from the dynamics of an interacting Bose gas in the Gross–Pitaevskii limit. We remark that the defocusing quintic NLS can be derived from a system of bosons with repelling three body interactions, see [6].

1.1.1. From *N*-Body Schrödinger to BBGKY Hierarchy. We consider a quantum mechanical system consisting of N bosons in $\mathbb{R}^3$ with wave function $\Phi_N \in L^2(\mathbb{R}^{3N})$. According to Bose–Einstein statistics, Φ_N is invariant under the permutation of particle variables,

$$\Phi_N(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(N)}) = \Phi_N(x_1, x_2, \ldots, x_N) \qquad \forall \pi \in S_N, \quad (1.1)$$

where S_N is the Nth symmetric group. We denote by $L^2_{\mathrm{sym}}(\mathbb{R}^{3N})$ the subspace of $L^2(\mathbb{R}^{3N})$ of elements obeying (1.1). The dynamics of the system is determined by the N-body Schrödinger equation

$$i\partial_t \Phi_N \;=\; H_N \Phi_N. \quad (1.2)$$

The Hamiltonian H_N is given by a self-adjoint operator acting on the Hilbert space $L^2_{\mathrm{sym}}(\mathbb{R}^{3N})$, of the form

$$H_N \;=\; \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N}\sum_{1\le i<j\le N} V_N(x_i - x_j), \quad (1.3)$$

where $V_N(x) = N^{3\beta}V(N^\beta x)$ with $V \ge 0$ spherically symmetric, sufficiently regular, and for $0 < \beta < \frac{1}{4}$.

Since the Schrödinger equation (1.2) is linear and H_N self-adjoint, the global well-posedness of solutions is evident. To perform the infinite particle number limit $N \to \infty$, we outline the strategy developed in [16,17] as follows.

One introduces the density matrix

$$\gamma_{\Phi_N}(t, \underline{x}_N, \underline{x}'_N) = |\Phi_N(t, \underline{x}_N)\,\rangle\langle\,\Phi_N(t, \underline{x}'_N)| := \Phi_N(t, \underline{x}_N)\overline{\Phi_N(t, \underline{x}'_N)}$$

where $\underline{x}_N = (x_1, x_2, \ldots, x_N)$ and $\underline{x}'_N = (x'_1, x'_2, \ldots, x'_N)$. Furthermore, one considers the associated sequence of k-particle marginal density matrices $\gamma^{(k)}_{\Phi_N}(t)$, for $k = 1, \ldots, N$, as the partial traces of γ_{Φ_N} over the degrees of freedom associated to the last $(N-k)$ particle variables,

$$\gamma_{\Phi_N}^{(k)} = \mathrm{Tr}_{k+1,k+2,\ldots,N}|\Phi_N\rangle\langle\Phi_N|.$$

Here, $\mathrm{Tr}_{k+1,k+2,\ldots,N}$ denotes the partial trace with respect to the particles indexed by $k+1, k+2, \ldots, N$. Accordingly, $\gamma_{\Phi_N}^{(k)}$ is explicitly given by

$$\begin{aligned}\gamma_{\Phi_N}^{(k)}(\underline{x}_k, \underline{x}_k') &= \int \mathrm{d}\underline{x}_{N-k}\gamma_{\Phi_N}(\underline{x}_k, \underline{x}_{N-k}; \underline{x}_k', \underline{x}_{N-k}) \\ &= \int \mathrm{d}\underline{x}_{N-k}\Phi_N(\underline{x}_k, \underline{x}_{N-k})\overline{\Phi_N(\underline{x}_k', \underline{x}_{N-k})}. \qquad (1.4)\end{aligned}$$

It follows immediately from the definitions that the property of *admissibility* holds,

$$\gamma_{\Phi_N}^{(k)} = \mathrm{Tr}_{k+1}(\gamma_{\Phi_N}^{(k+1)}) , \qquad k = 1, \ldots, N-1, \qquad (1.5)$$

for $1 \leq k \leq N-1$, and that $\mathrm{Tr}\gamma_{\Phi_N}^{(k)} = \|\Phi_N\|_{L_s^2(\mathbb{R}^{3N})}^2 = 1$ for all N, and all $k = 1, 2, \ldots, N$.

Moreover, $\gamma_{\Phi_N}^{(k)} \geq 0$ is positive semidefinite as an operator $\mathcal{S}(\mathbb{R}^{3k}) \times \mathcal{S}(\mathbb{R}^{3k}) \to \mathbb{C}, (f,g) \mapsto \int \mathrm{d}\underline{x}\mathrm{d}\underline{x}' f(\underline{x})\gamma(\underline{x}; \underline{x}')\overline{g(\underline{x}')}$.

The time evolution of the density matrix γ_{Φ_N} is determined by the Heisenberg equation

$$i\partial_t\gamma_{\Phi_N}(t) = [H_N, \gamma_{\Phi_N}(t)], \qquad (1.6)$$

which has the explicit form

$$\begin{aligned}&i\partial_t\gamma_{\Phi_N}(t, \underline{x}_N, \underline{x}_N') \\ &\quad = -(\Delta_{\underline{x}_N} - \Delta_{\underline{x}_N'})\gamma_{\Phi_N}(t, \underline{x}_N, \underline{x}_N') \\ &\qquad + \frac{1}{N}\sum_{1\leq i<j\leq N}[V_N(x_i - x_j) - V_N(x_i' - x_j')]\gamma_{\Phi_N}(t, \underline{x}_N, \underline{x}_N'). \qquad (1.7)\end{aligned}$$

Accordingly, the k-particle marginals satisfy the BBGKY hierarchy

$$\begin{aligned}&i\partial_t\gamma_{\Phi_N}^{(k)}(t, \underline{x}_k; \underline{x}_k') \\ &\quad = -(\Delta_{\underline{x}_k} - \Delta_{\underline{x}_k'})\gamma_{\Phi_N}^{(k)}(t, \underline{x}_k, \underline{x}_k') \\ &\qquad + \frac{1}{N}\sum_{1\leq i<j\leq k}[V_N(x_i - x_j) - V_N(x_i' - x_j')]\gamma_{\Phi_N}^{(k)}(t, \underline{x}_k; \underline{x}_k') \qquad (1.8) \\ &\qquad + \frac{N-k}{N}\sum_{i=1}^{k}\int \mathrm{d}x_{k+1}[V_N(x_i - x_{k+1}) - V_N(x_i' - x_{k+1})] \\ &\qquad\qquad \gamma_{\Phi_N}^{(k+1)}(t, \underline{x}_k, x_{k+1}; \underline{x}_k', x_{k+1}) \qquad (1.9)\end{aligned}$$

where $\Delta_{\underline{x}_k} := \sum_{j=1}^{k}\Delta_{x_j}$, and similarly for $\Delta_{\underline{x}_k'}$. We note that the number of terms in (1.8) is $\approx \frac{k^2}{N} \to 0$, and the number of terms in (1.9) is $\frac{k(N-k)}{N} \to k$ as $N \to \infty$. Accordingly, for fixed k, (1.8) disappears in the limit $N \to \infty$ described below, while (1.9) survives.

1.1.2. From BBGKY Hierarchy to GP Hierarchy. It is proven in [16–18] that, for asymptotically factorized initial data, and in the weak topology on the space of marginal density matrices, one can extract convergent subsequences $\gamma^{(k)}_{\Phi_N} \to \gamma^{(k)}$ as $N \to \infty$, for $k \in \mathbb{N}$, which satisfy the infinite limiting hierarchy

$$i\partial_t \gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) = -(\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) + \kappa_0 \sum_{j=1}^{k} \left(B_{j,k+1}\gamma^{k+1}\right)(t, \underline{x}_k; \underline{x}'_k), \tag{1.10}$$

which is referred to as the *Gross–Pitaevskii (GP) hierarchy*. Here,

$$\begin{aligned}(B_{j,k+1}\gamma^{k+1})&(t, \underline{x}_k; \underline{x}'_k) \\ := \int & \mathrm{d}x_{k+1}\mathrm{d}x'_{k+1} \left[\delta(x_j - x_{k+1})\delta(x_j - x'_{k+1}) \right. \\ & \left. -\delta(x'_j - x_{k+1})\delta(x'_j - x'_{k+1})\right] \gamma^{(k+1)}(t, \underline{x}_k, x_{k+1}; \underline{x}'_k, x'_{k+1}).\end{aligned}$$

The coefficient κ_0 is the *scattering length* if $\beta = 1$ (see [16,28] for the definition), and $\kappa_0 = \int V(x)\mathrm{d}x$ if $\beta < 1$ (corresponding to the Born approximation of the scattering length). For $\beta < 1$, the interaction term is obtained from the weak limit $V_N(x) \to \kappa_0\delta(x)$ in (1.9) as $N \to \infty$. The proof for the case $\beta = 1$ is much more difficult, and the derivation of the scattering length in this context is a breakthrough result obtained in [16,17]. For notational convenience, we will mostly set $\kappa_0 = 1$ in the sequel.

Some key properties satisfied by the solutions of the GP hierarchy are:

- The solution of the GP hierarchy obtained in [16,17] exists *globally* in t.
- It satisfies the property of admissibility,

$$\gamma^{(k)} = \mathrm{Tr}_{k+1}(\gamma^{(k+1)}), \quad \forall\, k \in \mathbb{N}, \tag{1.11}$$

  which is inherited from the system at finite N.
- There exists a constant b'_1 depending on the initial data only, such that the *a priori energy bound*

$$\mathrm{Tr}(\,|S^{(k,1)}\gamma^{(k)}(t)|\,) < (b'_1)^k \tag{1.12}$$

  is satisfied for all $k \in \mathbb{N}$, and for all $t \in \mathbb{R}$, where

$$S^{(k,\alpha)} := \prod_{j=1}^{k} \langle \nabla_{x_j}\rangle^{\alpha} \langle \nabla_{x'_j}\rangle^{\alpha}. \tag{1.13}$$

  This is obtained from energy conservation in the original N-body Schrödinger system.
- Solutions of the GP hierarchy are studied in spaces of k-particle marginals $\{\gamma^{(k)} \mid \|\gamma^{(k)}\|_{\mathfrak{h}^1} < \infty\}$ with norms

$$\|\gamma^{(k)}\|_{\mathfrak{h}^\alpha} := \mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|). \tag{1.14}$$

  This is in agreement with the a priori bounds (1.12).

1.1.3. Factorized Solutions of GP and NLS. The NLS emerges as the mean field dynamics of the Bose gas for the very special subclass of solutions of the GP hierarchy that are *factorized*. Factorized k-particle marginals at time $t = 0$ have the form

$$\gamma_0^{(k)}(\underline{x}_k;\underline{x}'_k) = \prod_{j=1}^{k} \phi_0(x_j)\overline{\phi_0(x'_j)},$$

where we assume that $\phi_0 \in H^1(\mathbb{R}^3)$. One can easily verify that

$$\gamma^{(k)}(t,\underline{x}_k;\underline{x}'_k) = \prod_{j=1}^{k} \phi(t,x_j)\overline{\phi(t,x'_j)},$$

is a solution (usually referred to as a factorized solution) of the GP hierarchy (1.10) with $\kappa_0 = 1$, if $\phi(t) \in H^1(\mathbb{R}^3)$ solves the defocusing cubic NLS,

$$i\partial_t\phi = -\Delta_x\phi + |\phi|^2\phi, \tag{1.15}$$

for $t \in I \subseteq \mathbb{R}$, and $\phi(0) = \phi_0 \in H^1(\mathbb{R}^3)$.

1.1.4. Uniqueness of Solutions of GP Hierarchies. While the existence of factorized solutions can be easily verified in the manner outlined above, the proof of the *uniqueness of solutions* of the GP hierarchy (which encompass non-factorized solutions) is the most difficult part in this analysis. The proof of uniqueness of solutions to the GP hierarchy was originally achieved by Erdös et al. [16–18] in the space $\{\gamma^{(k)} \mid \|\gamma^{(k)}\|_{\mathfrak{h}^1} < \infty\}$, for which the authors developed highly sophisticated Feynman graph expansion methods.

Klainerman and Machedon [25] introduced an alternative method for proving uniqueness in a space of density matrices defined by the Hilbert–Schmidt type Sobolev norms

$$\|\gamma^{(k)}\|_{H_k^\alpha} := \|S^{(k,\alpha)}\gamma^{(k)}\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} < \infty. \tag{1.16}$$

While this is a different (strictly larger) space of marginal density matrices than the one considered by Erdös et al. [16,17], the authors of [25] impose an additional a priori condition on space-time norms of the form

$$\|B_{j;k+1}\gamma^{(k+1)}\|_{L_t^1 H_k^1} < C^k, \tag{1.17}$$

for some arbitrary but finite C independent of k. The strategy in [25] developed to prove the uniqueness of solutions of the GP hierarchy (1.10) in $d = 3$ involves the use of certain space-time bounds on density matrices (of generalized Strichartz type), and crucially employs the reformulation of a combinatorial result in [16,17] into a "board game" argument. The latter is used to organize the Duhamel expansion of solutions of the GP hierarchy into equivalence classes of terms which leads to a significant reduction of the complexity of the problem.

Subsequently, Kirkpatrick et al. [26] proved that the a priori spacetime bound (1.17) is satisfied for the cubic GP hierarchy in $d = 2$, locally in time. Their argument is based on the conservation of energy in the original N-body Schrödinger system, and a related a priori H^1-bounds for the BBGKY hierarchy in the limit $N \to \infty$ derived in [16,17], combined with a generalized Sobolev inequality for density matrices.

1.2. Cauchy Problem for GP Hierarchies

In [7], we began investigating the well-posedness of the Cauchy problem for GP hierarchies, with both focusing and defocusing interactions. We do so independently of the fact that it is currently not known how to rigorously derive a GP hierarchy from the $N \to \infty$ limit of a BBGKY hierarchy with L^2-supercritical, attractive interactions. In [7], we introduced the notions of *cubic*, *quintic*, *focusing*, or *defocusing GP hierarchies*, according to the type of NLS obtained from factorized solutions.

In [7], we introduced the following topology on the Banach space of sequences of k-particle marginal density matrices

$$\mathfrak{G} = \{ \Gamma = (\gamma^{(k)}(x_1, \dots, x_k; x_1', \dots, x_k'))_{k\in\mathbb{N}} \,|\, \mathrm{Tr}\gamma^{(k)} < \infty \}. \tag{1.18}$$

Given $\xi > 0$, we defined the space

$$\mathcal{H}_\xi^\alpha = \{\Gamma \,|\, \| \Gamma \|_{\mathcal{H}_\xi^\alpha} < \infty \} \tag{1.19}$$

with the norm

$$\| \Gamma \|_{\mathcal{H}_\xi^\alpha} := \sum_{k\in\mathbb{N}} \xi^k \| \gamma^{(k)} \|_{H^\alpha}, \tag{1.20}$$

where

$$\|\gamma^{(k)}\|_{H_k^\alpha} := \|S^{(k,\alpha)}\gamma^{(k)}\|_{L^2} \tag{1.21}$$

is the norm (1.16) considered in [25]. If $\Gamma \in \mathcal{H}_\xi^\alpha$, then ξ^{-1} an upper bound on the typical H^α-energy per particle; this notion is made precise in [7]. We note that small energy results are characterized by large $\xi > 1$, while results valid without any upper bound on the size of the energy can be proven for arbitrarily small values of $\xi > 0$; in the latter case, one can assume $0 < \xi < 1$ without any loss of generality. The GP hierarchy can then be written in the form

$$i\partial_t\Gamma + \widehat{\Delta}_\pm\Gamma = B\Gamma, \tag{1.22}$$

with $\Gamma(0) = \Gamma_0$, where the components of $\widehat{\Delta}\Gamma$ and $B\Gamma$ can be read off from (1.10). Here we have set $\kappa_0 = 1$.

In [7], we prove the local well-posedness of solutions for energy subcritical focusing and defocusing cubic and quintic GP hierarchies in a subspace of $\mathcal{H}_\xi^\alpha$ defined by a condition related to (1.17). The parameter α determines the regularity of the solution,

$$\alpha \in \mathfrak{A}(d,p) := \begin{cases} \left(\frac{1}{2}, \infty\right) & \text{if } d = 1 \\ \left(\frac{d}{2} - \frac{1}{2(p-1)}, \infty\right) & \text{if } d \geq 2 \text{ and } (d,p) \neq (3,2) \\ \left[1, \infty\right) & \text{if } (d,p) = (3,2), \end{cases} \tag{1.23}$$

where $p = 2$ for the cubic, and $p = 4$ for the quintic GP hierarchy. Our result is obtained from a Picard fixed point argument, and holds for various dimensions d, without any requirement on factorization. The parameter $\xi > 0$ is determined by the initial condition, and it sets the energy scale of the given Cauchy problem. In addition, we prove lower bounds on the blowup rate for blowup solutions of focusing GP hierarchies in [7]. The Cauchy problem for GP hierarchies was also analyzed by the authors of [13], and the cubic GP hierarchy was derived in [12] with the presence of an external trapping potential in 2D.

In the joint work [10] with N. Tzirakis, we identify a conserved energy functional $E_1(\Gamma(t)) = E_1(\Gamma_0)$ describing the average energy per particle, and we prove virial identities for solutions of GP hierarchies. In particular, we use these ingredients to prove that for L^2-critical and supercritical focusing GP hierarchies, blowup occurs whenever $E_1(\Gamma_0) < 0$ and the variance is finite. We note that prior to [10], no exact conserved energy functional on the level of the GP hierarchy was identified in any of the previous works, including [26] and [16,17].

In [8], we discovered an infinite family of multiplicative energy functionals and prove that they are conserved under time evolution; their existence is a consequence of the mean field character of GP hierarchies. Those conserved energy functionals allow us to prove global well-posedness for H^1 subcritical defocusing GP hierarchies, and for L^2 subcritical focusing GP hierarchies.

In the paper [9], we prove the *existence* of solutions to the GP hierarchy, without the assumption of the Klainerman–Machedon condition. This is achieved via considering a truncated version of the GP hierarchy (for which existence of solutions can be easily obtained) and showing that the limit of solutions to the truncated GP hierarchy exists as the truncation parameter goes to infinity, and that this limit is a solution to the GP hierarchy. Such a "truncation-based" proof of existence of solutions to the GP hierarchy motivated us to try to implement a similar approach at the level of the BBGKY hierarchy, which is what we do in this paper.

1.3. Main Results of this Paper

As noted above, our results in [7] prove the local well-posedness of solutions for spaces

$$\mathfrak{W}^\alpha_\xi(I) := \{\, \Gamma \in L^\infty_{t\in I}\mathcal{H}^\alpha_\xi \mid B\Gamma \in L^2_{t\in I}\mathcal{H}^\alpha_\xi \,\}, \quad \alpha \in \mathfrak{A}(d,p), \tag{1.24}$$

where the condition on the boundedness of the $L^2_{t\in I}\mathcal{H}^\alpha_\xi$ spacetime norm corresponds to the condition (1.17) used by Klainerman and Machedon [25].

This is a different solution space than that considered by Erdös et al. [16,17]. As a matter of fact, it has so far not been known if the limiting solution to the GP hierarchy constructed by Erdös et al. is an element of (1.24) or not in dimension $d \geq 3$ (for $d \leq 2$, it is known to be the case [6,26]). This is a central open question surrounding the well-posedness theory for GP hierarchies in the context of our approach developed in [7–9,25].

In this paper, we answer this question in the affirmative. We give a derivation of the cubic GP hierarchy from the BBGKY hierarchy in dimensions $d = 3$ based on the spacetime norms used in [7,25]. The main result can be formulated as follows:

Let $d = 3$, and $\delta > 0$ be an arbitrary, small, fixed number. Moreover, let

$$0 < \beta < \frac{1}{4 + 2\delta}. \tag{1.25}$$

Suppose that the pair potential $V_N(x) = N^{3\beta} V(N^\beta x)$, for $V \in L^1(\mathbb{R}^3)$, is spherically symmetric, positive, and $\widehat{V} \in C^\delta(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ decays rapidly outside the unit ball.

Let $(\Phi_N)_N$ denote a sequence of solutions to the N-body Schrödinger equation (1.2) for which we have that for some $0 < \xi' < 1$, and every $N \in \mathbb{N}$,

$$\Gamma^{\Phi_N}(0) \;=\; (\gamma^{(1)}_{\Phi_N}(0), \ldots, \gamma^{(N)}_{\Phi_N}(0), 0, 0, \ldots) \in \mathcal{H}^{1+\delta}_{\xi'}$$

holds at initial time $t = 0$, and moreover, that the strong limit

$$\Gamma_0 = \lim_{N\to\infty} \Gamma^{\Phi_N}(0) \;\in\; \mathcal{H}^{1+\delta}_{\xi'} \tag{1.26}$$

exists. We emphasize that Γ_0 does not need to be of factorized form. The additional δ amount of regularity is introduced to control the convergence of certain terms, see Sect. 5.

We denote by

$$\Gamma^{\Phi_N}(t) := (\gamma^{(1)}_{\Phi_N}(t), \ldots, \gamma^{(N)}_{\Phi_N}(t), 0, 0, \ldots, 0, \ldots) \tag{1.27}$$

the solution to the associated BBGKY hierarchy (1.8) – (1.9), trivially extended by $\gamma^{(n)}_{\Phi_N} \equiv 0$ for $n > N$.

We define the truncation operator $P_{\leq K}$ by

$$P_{\leq K}\Gamma = (\gamma^{(1)}, \ldots, \gamma^{(K)}, 0, 0, \ldots), \tag{1.28}$$

and let

$$K(N) := b_0 \log N \tag{1.29}$$

for a suitable constant $b_0 > 0$.

Then, the following hold for sufficiently small $0 < \xi < 1$:

1. There exists $\Gamma \in L^\infty_{t\in[0,T]} \mathcal{H}^1_\xi$ such that the limit

$$s - \lim_{N\to\infty} P_{\leq K(N)} \Gamma_{\Phi_N} = \Gamma \tag{1.30}$$

   holds strongly in $L^\infty_{t\in[0,T]} \mathcal{H}^1_\xi$.

2. Moreover, the limit

$$s - \lim_{N\to\infty} B_N P_{\leq K(N)} \Gamma_{\Phi_N} = B\Gamma \tag{1.31}$$

   holds strongly in $L^2_{t\in[0,T]} \mathcal{H}^1_\xi$.

3. The limit point $\Gamma \in L^\infty_{t\in[0,T]} \mathcal{H}^1_\xi$ is a mild solution to the cubic GP hierarchy with initial data Γ_0, satisfying

$$\Gamma(t) = U(t)\,\Gamma_0 + i \int_0^t U(t-s)\, B\, \Gamma(s)\, \mathrm{d}s, \tag{1.32}$$

   with $\Gamma(0) = \Gamma_0$, and $U(t) := \mathrm{e}^{it\widehat{\Delta}_\pm}$.

An outline of our proof is given in Sect. 3 below.

Remark 1.1. We emphasize the following:

- The results stated above imply that the N-BBGKY hierarchy (truncated by $P_{\leq K(N)}$ with a suitable choice of $K(N)$) has a limit in the space introduced in [7], which is based on the space considered by Klainerman and Machedon [25]. For factorized solutions, this provides the derivation of the cubic defocusing NLS in those spaces.
- In [16,17,26], the limit $\gamma_{\Phi_N}^{(k)} \rightharpoonup \gamma^{(k)}$ of solutions to the BBGKY hierarchy to solutions to the GP hierarchy holds in the weak, subsequential sense, for an arbitrary but fixed k. In our approach, we prove strong convergence for a sequence of suitably truncated solutions to the BBGKY hierarchy, in an entirely different space of solutions. An important ingredient for our construction is that this convergence is in part controlled by use of the parameter $\xi > 0$, which is not available in [16,17,26]. Moreover, we assume initial data that are slightly more regular than of class $\mathcal{H}^1_{\xi'}$.
- We assume that the initial data has a limit, $\Gamma^{\phi_N}(0) \to \Gamma_0 \in \mathcal{H}^{1+\delta}_{\xi'}$ as $N \to \infty$, which does not need to be factorized. We note that in [16,17], the initial data is assumed to be asymptotically factorized.
- The method based on spacetime norms developed in this paper works for the cubic case in $d = 2, 3$, and is expected to have a straightforward generalization for the quintic case in $d = 2$. Our result is completely new for the cubic case in $d = 3$; the other cases (of cubic and quintic GP in $d \leq 2$) were covered in [6,26]; however the mode of convergence proven here is different and the initial data in this paper do not need to be of factorized form. A main obstacle in treating the quintic GP hierarchy in $d = 3$ is the fact that the currently available Strichartz estimates are not good enough for the quintic GP hierarchy [6].

2. Definition of the Model

In this section, we introduce the mathematical model that will be studied in this paper. Most notations and definitions are adopted from [7], and we refer to [7] for additional motivations and details.

2.1. The *N*-Body Schrödinger System

We consider the N-boson Schrödinger equation

$$i\partial_t \Phi_N = \left(-\sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N} \sum_{1\leq j<\ell\leq N} V_N(x_j - x_\ell) \right) \Phi_N \tag{2.1}$$

on $L^2_{\mathrm{Sym}}(\mathbb{R}^{3N})$, with initial data $\Phi_N(0) = \Phi_{0,N} \in L^2_{\mathrm{Sym}}(\mathbb{R}^{3N})$. Here, $V_N(x) = N^{3\beta} V(N^\beta x)$ for $V \in L^1(\mathbb{R}^3)$ spherically symmetric, and positive. Moreover, we assume that $\widehat{V} \in C^1(\mathbb{R}^3)$ with rapid decay outside the unit ball. The parameter $0 < \beta < 1$ is assumed to satisfy the smallness condition (3.1).

Let

$$\gamma_{\Phi_N}^{(k)} := \mathrm{Tr}_{k+1,\ldots,N}(|\Phi_N\rangle\langle\Phi_N|). \tag{2.2}$$

It is proved in [16,17,26] that for V satisfying the above assumptions,

$$\Big\langle \Phi_N, \, (N+H_N)^K \, \Phi_N \Big\rangle \geq C^K \, N^K \, \mathrm{Tr}(S^{(1,K)} \gamma^{(K)}_{\Phi_N}) \tag{2.3}$$

for some positive constant $C < \infty$ independent of N, K. This a priori bound makes use of energy conservation in the N-body Schrödinger equation satisfied by Φ_N, and will be used in the proof of our main results.

2.2. The Solution Spaces

We recall the space introduced in [7]

$$\mathfrak{G} \, := \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})$$

of sequences of marginal density matrices

$$\Gamma \, := \, (\, \gamma^{(k)} \,)_{k \in \mathbb{N}}$$

where $\gamma^{(k)} \geq 0$, $\mathrm{Tr}\gamma^{(k)} = 1$, and where every $\gamma^{(k)}(\underline{x}_k, \underline{x}'_k)$ is symmetric in all components of $\underline{x}_k$, and in all components of $\underline{x}'_k$, respectively, i.e.

$$\gamma^{(k)}(x_{\pi(1)}, \ldots, x_{\pi(k)}; x'_{\pi'(1)}, \ldots, x'_{\pi'(k)}) = \gamma^{(k)}(x_1, \ldots, x_k; x'_1, \ldots, x'_k) \tag{2.4}$$

holds for all $\pi, \pi' \in S_k$.

For brevity, we will write $\underline{x}_k := (x_1, \cdots, x_k)$, and similarly, $\underline{x}'_k := (x'_1, \cdots, x'_k)$.

The k-particle marginals are assumed to be hermitian,

$$\gamma^{(k)}(\underline{x}_k; \underline{x}'_k) = \overline{\gamma^{(k)}(\underline{x}'_k; \underline{x}_k)}. \tag{2.5}$$

We call $\Gamma = (\gamma^{(k)})_{k \in \mathbb{N}}$ admissible if $\gamma^{(k)} = \mathrm{Tr}_{k+1} \gamma^{(k+1)}$, that is,

$$\gamma^{(k)}(\underline{x}_k; \underline{x}'_k) = \int \mathrm{d}x_{k+1} \, \gamma^{(k+1)}(\underline{x}_k, x_{k+1}; \underline{x}'_k, x_{k+1})$$

for all $k \in \mathbb{N}$.

Let $0 < \xi < 1$. We define

$$\mathcal{H}^{\alpha}_{\xi} := \Big\{ \Gamma \in \mathfrak{G} \, \Big| \, \|\Gamma\|_{\mathcal{H}^{\alpha}_{\xi}} < \infty \Big\} \tag{2.6}$$

where

$$\|\Gamma\|_{\mathcal{H}^{\alpha}_{\xi}} \; = \sum_{k=1}^{\infty} \xi^k \| \, \gamma^{(k)} \, \|_{H^{\alpha}_k(\mathbb{R}^{3k} \times \mathbb{R}^{3k})},$$

with

$$\|\gamma^{(k)}\|_{H^{\alpha}_k} := \|S^{(k,\alpha)} \gamma^{(k)}\|_{L^2} \tag{2.7}$$

where $S^{(k,\alpha)} := \prod_{j=1}^{k} \big\langle \nabla_{x_j} \big\rangle^{\alpha} \big\langle \nabla_{x'_j} \big\rangle^{\alpha}$.

2.3. The GP Hierarchy

The main objective of the paper at hand will be to prove that, in the limit $N \to \infty$, solutions of the BBGKY hierarchy converge to solutions of an infinite hierarchy, referred to as the Gross–Pitaevskii (GP) hierarchy. In this section, we introduce the necessary notations and definitions, adopting them from [7].

The cubic GP (Gross–Pitaevskii) hierarchy is given by

$$i\partial_t \gamma^{(k)} = \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma^{(k)}] + \kappa_0 B_{k+1} \gamma^{(k+1)} \tag{2.8}$$

in d dimensions, for $k \in \mathbb{N}$. Here,

$$B_{k+1}\gamma^{(k+1)} = B^{+}_{k+1}\gamma^{(k+1)} - B^{-}_{k+1}\gamma^{(k+1)}, \tag{2.9}$$

where

$$B^{+}_{k+1}\gamma^{(k+1)} = \sum_{j=1}^{k} B^{+}_{j;k+1}\gamma^{(k+1)}, \tag{2.10}$$

and

$$B^{-}_{k+1}\gamma^{(k+1)} = \sum_{j=1}^{k} B^{-}_{j;k+1}\gamma^{(k+1)}, \tag{2.11}$$

with

$$\begin{aligned}
&\left(B^{+}_{j;k+1}\gamma^{(k+1)}\right)(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k) \\
&\quad = \int \mathrm{d}x_{k+1}\mathrm{d}x'_{k+1}\delta(x_j - x_{k+1})\delta(x_j - x'_{k+1})\gamma^{(k+1)}(t, x_1, \ldots, x_{k+1}; x'_1, \ldots, x'_{k+1}),
\end{aligned}$$

and

$$\begin{aligned}
&\left(B^{-}_{j;k+1}\gamma^{(k+1)}\right)(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k) \\
&\quad = \int \mathrm{d}x_{k+1}\mathrm{d}x'_{k+1}\delta(x'_j - x_{k+1})\delta(x'_j - x'_{k+1})\gamma^{(k+1)}(t, x_1, \ldots, x_{k+1}; x'_1, \ldots, x'_{k+1}).
\end{aligned}$$

We remark that for factorized initial data,

$$\gamma^{(k)}(0; \underline{x}_k; \underline{x}'_k) = \prod_{j=1}^{k} \phi_0(x_j)\, \overline{\phi_0(x'_j)}, \tag{2.12}$$

the corresponding solutions of the GP hierarchy remain factorized,

$$\gamma^{(k)}(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k) = \prod_{j=1}^{k} \phi(t, x_j)\, \bar{\phi}(t, x'_j). \tag{2.13}$$

if the corresponding 1-particle wave function satisfies the defocusing cubic NLS

$$i\partial_t \phi = -\Delta\phi + \kappa_0 |\phi|^2 \phi.$$

The GP hierarchy can be rewritten in the following compact manner:

$$\begin{aligned}
i\partial_t \Gamma + \widehat{\Delta}_{\pm}\Gamma &= \kappa_0\, B\Gamma \\
\Gamma(0) &= \Gamma_0,
\end{aligned} \tag{2.14}$$

where

$$\widehat{\Delta}_{\pm}\Gamma := (\,\Delta_{\pm}^{(k)}\gamma^{(k)}\,)_{k\in\mathbb{N}}, \qquad \text{with } \Delta_{\pm}^{(k)} = \sum_{j=1}^{k}\Big(\Delta_{x_j} - \Delta_{x'_j}\Big),$$

and

$$B\Gamma := (\,B_{k+1}\gamma^{(k+1)}\,)_{k\in\mathbb{N}}. \tag{2.15}$$

We will also use the notation

$$B^{+}\Gamma := (\,B_{k+1}^{+}\gamma^{(k+1)}\,)_{k\in\mathbb{N}},$$
$$B^{-}\Gamma := (\,B_{k+1}^{-}\gamma^{(k+1)}\,)_{k\in\mathbb{N}}.$$

2.4. The BBGKY Hierarchy

In analogy to the compact notation for the GP hierarchy described above, we introduce a similar notation for the cubic defocusing BBGKY hierarchy.

We consider the cubic defocusing BBGKY hierarchy for the marginal density matrices, given by

$$\begin{aligned} i\partial_t\gamma_N^{(k)}(t) &= \sum_{j=1}^{k}\Big[-\Delta_{x_j}, \gamma_N^{(k)}(t)\Big] + \frac{1}{N}\sum_{1\le j<k}\Big[V_N(x_j-x_k), \gamma_N^{(k)}(t)\Big] \\ &\quad + \frac{(N-k)}{N}\sum_{1\le j\le k}\mathrm{Tr}_{k+1}\Big[V_N(x_j-x_{k+1}), \gamma_N^{(k+1)}(t)\Big], \end{aligned} \tag{2.16}$$

for $k = 1, \ldots, n$. We extend this finite hierarchy trivially to an infinite hierarchy by adding the terms $\gamma_N^{(k)} = 0$ for $k > N$. This will allow us to treat solutions of the BBGKY hierarchy on the same footing as solutions to the GP hierarchy.

We next introduce the following compact notation for the BBGKY hierarchy.

$$i\partial_t\gamma_N^{(k)} = \sum_{j=1}^{k}\Big[-\Delta_{x_j}, \gamma_N^{(k)}\Big] + \mu(B_N\Gamma_N)^{(k)} \tag{2.17}$$

for $k \in \mathbb{N}$. Here, we have $\gamma_N^{(k)} = 0$ for $k > N$, and we define

$$(B_N\Gamma_N)^{(k)} := \begin{cases} B_{N;k+1}^{\mathrm{main}}\gamma_N^{(k+1)} + B_{N;k}^{\mathrm{error}}\gamma_N^{(k)} & \text{if } k \le N \\ 0 & \text{if } k > N \end{cases} \tag{2.18}$$

The interaction terms on the right hand side are defined by

$$B_{N;k+1}^{\mathrm{main}}\gamma_N^{(k+1)} = B_{N;k+1}^{+,\mathrm{main}}\gamma_N^{(k+1)} - B_{N;k+1}^{-,\mathrm{main}}\gamma_N^{(k+1)}, \tag{2.19}$$

and

$$B_{N;k}^{\mathrm{error}}\gamma_N^{(k)} = B_{N;k}^{+,\mathrm{error}}\gamma_N^{(k)} - B_{N;k}^{-,\mathrm{error}}\gamma_N^{(k)}, \tag{2.20}$$

where

$$B_{N;k+1}^{\pm,\mathrm{main}}\gamma_N^{(k+1)} := \frac{N-k}{N}\sum_{j=1}^{k}B_{N;j;k+1}^{\pm,\mathrm{main}}\gamma_N^{(k+1)}, \tag{2.21}$$

and

$$B_{N;k}^{\pm,\mathrm{error}}\gamma_N^{(k)} := \frac{1}{N}\sum_{i<j}^{k} B_{N;i,j;k}^{\pm,\mathrm{error}}\gamma_N^{(k)}, \tag{2.22}$$

with

$$\Big(B_{N;j;k+1}^{+,\mathrm{main}}\gamma_N^{(k+1)}\Big)(t,x_1,\ldots,x_k;x_1',\ldots,x_k')$$
$$=\int \mathrm{d}x_{k+1}V_N(x_j-x_{k+1})\gamma_N^{(k+1)}(t,x_1,\ldots,x_k,x_{k+1};x_1',\ldots,x_k',x_{k+1}) \tag{2.23}$$

and

$$\Big(B_{N;i,j;k}^{+,\mathrm{error}}\gamma_N^{(k)}\Big)(t,x_1,\ldots,x_k;x_1',\ldots,x_k')$$
$$= V_N(x_i-x_j)\gamma^{(k)}(t,x_1,\ldots,x_k;x_1',\ldots,x_k'). \tag{2.24}$$

Moreover,

$$\Big(B_{N;j;k+1}^{-,\mathrm{main}}\gamma_N^{(k+1)}\Big)(t,x_1,\ldots,x_k;x_1',\ldots,x_k')$$
$$= \int \mathrm{d}x_{k+1}V_N(x_j'-x_{k+1})\gamma_N^{(k+1)}(t,x_1,\ldots,x_k,x_{k+1};x_1',\ldots,x_k',x_{k+1}).$$

and

$$\Big(B_{N;i,j;k}^{-,\mathrm{error}}\gamma_N^{(k)}\Big)(t,x_1,\ldots,x_k;x_1',\ldots,x_k')$$
$$= V_N(x_i'-x_j')\gamma^{(k)}(t,x_1,\ldots,x_k;x_1',\ldots,x_k').$$

The advantage of this notation will be that we can treat the BBGKY hierarchy and the GP hierarchy on the same footing. We remark that in all of the above definitions, we have that $B_{N;k}^{\pm,\mathrm{main}}$, $B_{N;k}^{\pm,\mathrm{error}}$, etc. are defined to be given by multiplication with zero for $k>N$.

As a consequence, we can write the BBGKY hierarchy compactly in the form

$$\begin{aligned} i\partial_t\Gamma_N + \widehat{\Delta}_\pm\Gamma_N &= B_N\Gamma_N \\ \Gamma_N(0) &\in \mathcal{H}_\xi^\alpha, \end{aligned} \tag{2.25}$$

where

$$\widehat{\Delta}_\pm\Gamma_N := (\,\Delta_\pm^{(k)}\gamma_N^{(k)}\,)_{k\in\mathbb{N}}, \quad \text{with } \Delta_\pm^{(k)} = \sum_{j=1}^{k}\Big(\Delta_{x_j}-\Delta_{x_j'}\Big),$$

and

$$B_N\Gamma_N := (\,B_{N;k+1}\gamma_N^{(k+1)}\,)_{k\in\mathbb{N}}. \tag{2.26}$$

In addition, we introduce the notation

$$B_N^+\Gamma_N := (\,B_{N;k+1}^+\gamma_N^{(k+1)}\,)_{k\in\mathbb{N}}$$
$$B_N^-\Gamma_N := (\,B_{N;k+1}^-\gamma_N^{(k+1)}\,)_{k\in\mathbb{N}}$$

which will be convenient.

3. Statement of Main Results and Outline of Proof Strategy

The main result proven in this paper is the following theorem.

Theorem 3.1. *Let $\delta > 0$ be an arbitrary small, fixed number, and assume that*

$$0 < \beta < \frac{1}{4+2\delta}. \tag{3.1}$$

Assume that Φ_N solves the N-body Schrödinger equation (2.1) with initial condition $\Phi_N(t=0) = \Phi_{0,N} \in L^2(\mathbb{R}^{3N})$, where the pair potential $V_N(x) = N^{3\beta}V(N^\beta x)$, for $V \in L^1(\mathbb{R}^3)$, is spherically symmetric, positive, and $\widehat{V} \in C^\delta(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ with rapid decay outside the unit ball.

Let

$$\Gamma^{\Phi_N} = \left(\gamma^{(1)}_{\Phi_N}, \dots, \gamma^{(N)}_{\Phi_N}, 0, 0, \dots\right) \tag{3.2}$$

denote the associated sequence of marginal density matrices (trivially extended by zeros), which solves the N-BBGKY hierarchy,

$$\Gamma^{\Phi_N}(t) = U(t)\,\Gamma^{\Phi_N}(0) + i\int_0^t U(t-s)\,B_N\Gamma^{\Phi_N}(s)\,\mathrm{d}s, \tag{3.3}$$

where $U(t) := \mathrm{e}^{it\widehat{\Delta}_\pm}$. Furthermore, we assume that $\Gamma^{\Phi_{0,N}} \in \mathcal{H}^{1+\delta}_{\xi'}$ for all N, and that

$$\Gamma_0 = \lim_{N\to\infty} \Gamma^{\Phi_{0,N}} \in \mathcal{H}^{1+\delta}_{\xi'} \tag{3.4}$$

exists for some $0 < \xi' < 1$.

Define the truncation operator $P_{\leq K}$ by

$$P_{\leq K}\Gamma = \left(\gamma^{(1)}, \dots, \gamma^{(K)}, 0, 0, \dots\right), \tag{3.5}$$

and observe that

$$P_K\Gamma^{\Phi_N}(t) = U(t)\,P_K\Gamma^{\Phi_N}(0) + i\int_0^t U(t-s)\,P_K B_N\Gamma^{\Phi_N}(s)\,\mathrm{d}s. \tag{3.6}$$

Writing $\beta =: \frac{1-\delta'}{4}$, let

$$K(N) = \frac{\delta'}{2\log C_0}\log N, \tag{3.7}$$

and assume that

$$\xi < \min\left\{\eta^3\xi',\ \frac{1}{b_1}\mathrm{e}^{-\frac{2}{\delta'}(1-4\delta')\log C_0}\right\} \tag{3.8}$$

where the constant b_1 is as in Lemma 7.2, C_0 is as in Lemma B.2, and η is as in Lemma B.3, below.

Then, there exists $\Gamma \in L^\infty_{t\in I}\mathcal{H}^1_\xi$ with $B\Gamma \in L^2_{t\in I}\mathcal{H}^1_\xi$ such that the limits

$$\lim_{N\to\infty} \|\, P_{\leq K(N)}\Gamma^{\Phi_N} - \Gamma\,\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} = 0 \tag{3.9}$$

and

$$\lim_{N\to\infty} \| B_N P_{\leq K(N)} \Gamma^{\Phi_N} - B\Gamma \|_{L^2_{t\in I}\mathcal{H}^1_\xi} = 0 \tag{3.10}$$

hold, for $I = [0, T]$ *with* $0 < T < T_0(\xi)$.

In particular, Γ *solves the cubic GP hierarchy,*

$$\Gamma(t) = U(t)\,\Gamma_0 + i \int_0^t U(t-s)\,B\,\Gamma(s)\,\mathrm{d}s, \tag{3.11}$$

with initial data Γ_0.

We note that the limits $K \to \infty$ and $N \to \infty$ are taken simultaneously, and that the smallness of the parameter $\xi > 0$ is used (since small $\xi > 0$ corresponds to large energy per particle, this does not lead to any loss of generality).

In our proof, we will significantly make use of our work [9] which proves the unconditional existence of solutions $\Gamma \in L^\infty_{t\in I}\mathcal{H}^\alpha_\xi$ of GP hierarchies, without assuming $B\Gamma \in L^2_{t\in I}\mathcal{H}^\alpha_\xi < \infty$.

3.1. Outline of the Proof Strategy

The proof contains the following main steps:

- *Step* 1: In a first step, we construct a solution to the N-BBGKY hierarchy with truncated initial data.

First, we recall that the N-BBGKY hierarchy is given by

$$i\partial_t \gamma_N^{(k)} = \sum_{j=1}^k [-\Delta_{x_j}, \gamma_N^{(k)}] + B_{N,k+1}\gamma_N^{(k+1)} \tag{3.12}$$

for all $k \leq N$.

Given K, we let $P_{\leq K}$ denote the projection operator

$$P_{\leq K} \,:\, \mathfrak{G} \to \mathfrak{G}$$
$$\Gamma_N = (\gamma_N^{(1)}, \gamma_N^{(2)}, \ldots, \gamma_N^{(N)}, 0, 0, \ldots) \mapsto (\gamma_N^{(1)}, \ldots, \gamma_N^{(K)}, 0, 0, \ldots), \tag{3.13}$$

and $P_{>K} = 1 - P_{\leq K}$, as well as $P_K := P_{\leq K} - P_{\leq K-1}$.

Instead of considering the solution obtained from Φ_N, we consider (3.12) with truncated initial data $\Gamma^K_{0,N} := P_{\leq K}\Gamma_{0,N}$, for some fixed K. We will refer to solutions of this system as the K-truncated N-BBGKY hierarchy, or (K, N)-BBGKY hierarchy in short. We note that in contrast, Γ^{Φ_N} solves (3.12) with un-truncated initial data $\Gamma_{0,N}$.

Next, we prove via a fixed point argument that there exists a unique solution of the (K, N)-BBGKY hierarchy for every initial condition $\Gamma^K_{0,N} \in \mathcal{H}^{1+\delta}_{\xi'}$ in the space

$$\{\, \Gamma^K_N \in L^\infty_{t\in I_K}\mathcal{H}^{1+\delta}_{\xi'} \mid B_N\Gamma^K_N \in L^2_{t\in I_K}\mathcal{H}^{1+\delta}_{\xi'} \,\}. \tag{3.14}$$

To this end, we re-interpret $\Gamma^K_{0,N}$ as an infinite sequence, extended by zeros for elements $(\Gamma^K_{0,N})^{(k)} = 0$ with[1] $k > K$.

To obtain this result, we need to require that given K, N is large enough that the condition

$$K < \frac{\delta'}{\log C_0} \log N \tag{3.16}$$

is satisfied. Clearly, the choice (3.7) complies with this condition. The condition (3.16) is needed by the Lemma B.3 proven in the Appendix, which is crucial for the fixed point argument used in this part of the proof. It uses the Klainerman–Machedon boardgame argument to account for the part B_N^{main} in the interaction operator B_N; to also accommodate the part B_N^{error}, the condition (3.16) is used.

Hence, we have obtained solutions $\Gamma^K_N(t)$ of the BBGKY hierarchy,

$$i\partial_t \Gamma^K_N = \widehat{\Delta}_\pm \Gamma^K_N + B_N \Gamma^K_N, \tag{3.17}$$

for the truncated initial data

$$\Gamma^K_N(0) = P_{\leq K}\Gamma_N(0) = (\gamma^{(1)}_N(0), \dots, \gamma^{(K)}_N(0), 0, 0, \dots) \tag{3.18}$$

for an arbitrary, large, fixed $K \leq N$, and where component $(\Gamma^K_N)^{(m)}(t) = 0$ for the mth component, for all $m > K$. By the Duhamel formula, the solution of (3.17) is given by

$$\Gamma^K_N(t) = U(t)\Gamma^K_N(0) + i\int_0^t U(t-s)\, B_N \Gamma^K_N(s)\, \mathrm{d}s \tag{3.19}$$

for initial data $\Gamma^K_N(0) = P_{\leq K}\Gamma_N(0)$.

• *Step* 2: In this step, we take the limit $N \to \infty$ of the solution $\Gamma^{K(N)}_N$ to (3.17) which was obtained in Step 1, for some sequence $K(N) \to \infty$ as $N \to \infty$ that could be quite arbitrary. However, to comply with other parts of the proof, the choice (3.7) is made for $K(N)$.

To this end, we invoke the solution Γ^K of the GP hierarchy with truncated initial data, $\Gamma^K(t = 0) = P_{\leq K}\Gamma_0 \in \mathcal{H}^1_\xi$. In [9], we proved the existence of a solution Γ^K that satisfies the K-truncated GP hierarchy in integral form,

$$\Gamma^K(t) = U(t)\Gamma^K(0) + i\int_0^t U(t-s)\, B\Gamma^K(s)\, \mathrm{d}s \tag{3.20}$$

where $(\Gamma^K)^{(k)}(t) = 0$ for all $k > K$. Moreover, it is shown in [9] that this solution satisfies $B\Gamma^K \in L^2_{t\in I}\mathcal{H}^1_\xi$.

[1] We observe that then, (3.12) determines a closed, infinite sub-hierarchy, for initial data $\gamma^{(k)}_N(0) = 0$, for $k > K$, which has the trivial solution

$$(\gamma^K_N)^{(k)}(t) = 0, \quad t \in I = [0, T], \quad k > K. \tag{3.15}$$

We then prove the following convergence:

(a) In the limit $N \to \infty$, $\Gamma_N^{K(N)}$ satisfies

$$\lim_{N\to\infty} \|\Gamma_N^{K(N)} - \Gamma^{K(N)}\|_{L_t^\infty \mathcal{H}_\xi^1} = 0. \tag{3.21}$$

(b) In the limit $N \to \infty$, $B_N\Gamma_N^{K(N)}$ satisfies

$$\lim_{N\to\infty} \|B_N\Gamma_N^{K(N)} - B\Gamma^{K(N)}\|_{L_t^2 \mathcal{H}_\xi^1} = 0. \tag{3.22}$$

The proof of these limits makes use of the δ amount of extra regularity of the initial data $\Gamma_0, \Gamma_{0,N} \in \mathcal{H}_{\xi'}^{1+\delta}$ beyond $\mathcal{H}_{\xi'}^1$.

• *Step* 3: We compare the solution Γ_N^K of the K-truncated N-BBGKY hierarchy to the truncated solution $P_{\leq K(N)}\Gamma^{\Phi_N}$ of the N-BBGKY hierarchy. Notably, both have the same value at $t = 0$, given by $P_{\leq K(N)}\Gamma_{0,N}$. We prove that

$$\lim_{N\to\infty} \|\Gamma_N^{K(N)} - P_{\leq K(N)}\Gamma^{\Phi_N}\|_{L_{t\in I}^\infty \mathcal{H}_\xi^1} = 0. \tag{3.23}$$

and

$$\lim_{N\to\infty} \|B_N\Gamma_N^{K(N)} - B_N P_{\leq K(N)}\Gamma^{\Phi_N}\|_{L_{t\in I}^2 \mathcal{H}_\xi^1} = 0. \tag{3.24}$$

The proof of this limit involves the a priori energy bounds for the N-body Schrödinger system (2.3) established in [16,17,26].

The norm differences considered here can be bounded by $O(K^{a_1}\xi^K N^{a_2})$ at finite N, for some positive constants a_1, a_2. We may choose $K(N) = b\log N$, with the constant b small enough to satisfy (3.16); this is accomplished by (3.7). We then make use of the freedom to choose the parameter $\xi > 0$ to be sufficiently small; the condition (3.8) suffices to obtain $O((K(N))^{a_1}\xi^{K(N)}N^{a_2}) \to 0$ as $N \to \infty$.

• *Step* 4: Finally, we determine the limit $N \to \infty$ of $\Gamma^{K(N)}$ from Step 2, obtaining that:

(i) The strong limit $\lim_{N\to\infty} \Gamma^{K(N)}$ exists in $L_t^\infty \mathcal{H}_\xi^1$, and satisfies

$$\lim_{N\to\infty} \Gamma^{K(N)} = \Gamma \in L_t^\infty \mathcal{H}_\xi^1, \tag{3.25}$$

where Γ is a solution to the full GP hierarchy (2.8) with initial data Γ_0.

(ii) In addition, the strong limit $\lim_{N\to\infty} B\Gamma^{K(N)}$ exists in $L_t^2 \mathcal{H}_\xi^1$, and satisfies

$$\lim_{N\to\infty} B\Gamma^{K(N)} = B\Gamma \quad \in L_t^2 \mathcal{H}_\xi^1. \tag{3.26}$$

The results of Step 4 were proven in our earlier work [9].

4. Local Well-Posedness for the (K, N)-BBGKY Hierarchy

In this section, we prove the local well-posedness of the Cauchy problem for the K-truncated N-BBGKY hierarchy, which we refer to as the (K, N)-BBGKY hierarchy for brevity. In the sequel, we will have $d = 2, 3$.

Lemma 4.1. *Assume that N is sufficiently large, and in particular, given $K \in \mathbb{N}$ and $\beta =: \frac{1-\delta'}{4}$, that*

$$K < \frac{\delta'}{\log C_0} \log N \tag{4.1}$$

holds. Assume that $\Gamma_{0,N}^K = P_{\leq K}\Gamma_{0,N} \in \mathcal{H}_{\xi'}^{1+\delta}$ for some $0 < \xi' < 1$ and $\delta \geq 0$.

Then, there exists a unique solution $\Gamma_N^K \in L_{t\in I}^\infty \mathcal{H}_\xi^{1+\delta}$ of (3.19) for $I = [0,T]$ with $T > 0$ sufficiently small, and independent of K, N. In particular, $B_N\Gamma_N^K \in L_{t\in I}^2 \mathcal{H}_\xi^{1+\delta}$. Moreover,

$$\|\Gamma_N^K\|_{L_{t\in I}^\infty \mathcal{H}_\xi^{1+\delta}} \leq C_2(T,\xi,\xi')\, \|\Gamma_{0,N}^K\|_{\mathcal{H}_{\xi'}^{1+\delta}} \tag{4.2}$$

and

$$\|B_N\Gamma_N^K\|_{L_{t\in I}^2 \mathcal{H}_\xi^{1+\delta}} \leq C_2(T,\xi,\xi')\, \|\Gamma_{0,N}^K\|_{\mathcal{H}_{\xi'}^{1+\delta}} \tag{4.3}$$

hold for $0 < \xi < \xi'$ sufficiently small (it is sufficient that $0 < \xi < \eta\xi'$ with η specified in Lemma B.3 below). The constant $C_2 = C_2(T,\xi,\xi')$ is independent of K, N.

Furthermore, $(\Gamma_N^K(t))^{(k)} = 0$ for all $K < k \leq N$, and all $t \in I$.

Proof. To obtain local well-posedness of the Cauchy problem for the (K,N)-BBGKY hierarchy, we consider the map

$$\mathcal{M}_N^K(\widetilde{\Theta}^{K-1}) := B_N U(t)\Gamma_{N,0}^K + i\int_0^t B_N U(t-s)\widetilde{\Theta}^{K-1}(s), \tag{4.4}$$

where $P_{\leq K-1}\widetilde{\Theta}^{K-1} = \widetilde{\Theta}^{K-1}$ on the subspace $\mathrm{Ran}(P_{\leq K}) \cap L_{t\in I}^2\mathcal{H}_\xi^{1+\delta} \subset L_{t\in I}^2\mathcal{H}_\xi^{1+\delta}$. Using the K-truncated Strichartz estimate in Proposition A.2, we find that

$$\begin{aligned}
&\|\mathcal{M}_N^K(\widetilde{\Theta}_1^{K-1}) - \mathcal{M}_N^K(\widetilde{\Theta}_2^{K-1})\|_{L_{t\in I}^2\mathcal{H}_\xi^{1+\delta}} \\
&\quad\leq \left\| \int_0^t \mathrm{d}s \left\| B_N U(t-s)(\widetilde{\Theta}_1^{K-1} - \widetilde{\Theta}_2^{K-1})(s) \right\|_{\mathcal{H}_\xi^{1+\delta}} \right\|_{L_{t\in I}^2} \\
&\quad\leq \int_0^T \mathrm{d}s \left\| B_N U(t-s)(\widetilde{\Theta}_1^{K-1} - \widetilde{\Theta}_2^{K-1})(s) \right\|_{L_{t\in I}^2\mathcal{H}_\xi^{1+\delta}} \\
&\quad\leq C_2(K)\,\xi^{-1} \int_0^T \mathrm{d}s \left\| (\widetilde{\Theta}_1^{K-1} - \widetilde{\Theta}_2^{K-1})(s) \right\|_{\mathcal{H}_\xi^{1+\delta}} \\
&\quad\leq C_2(K)\,\xi^{-1}\, T^{\frac{1}{2}} \left\| \widetilde{\Theta}_1^{K-1} - \widetilde{\Theta}_2^{K-1} \right\|_{L_{t\in I}^2\mathcal{H}_\xi^{1+\delta}}.
\end{aligned} \tag{4.5}$$

Thus, for $(T(K))^{\frac{1}{2}} < \frac{\xi}{2C_2(K)}$, we find that $\mathcal{M}_N^K$ is a contraction on $L_{t\in I}^2\mathcal{H}_\xi^{1+\delta}$. By the fixed point principle, we obtain a unique solution $\Theta_N^{K-1} \in L_{t\in I}^2\mathcal{H}_\xi^{1+\delta}$

with $\Theta_N^{K-1} = P_{\leq K-1}\Theta_N^{K-1}$ satisfying

$$\Theta_N^{K-1}(t) = B_N U(t)\Gamma_{N,0}^K + i\int_0^t B_N U(t-s)\Theta_N^{K-1}(s)\,\mathrm{d}s. \tag{4.6}$$

In particular,

$$\begin{aligned}\|\Theta_N^{K-1}\|_{L^2_{t\in I}\mathcal{H}_\xi^{1+\delta}} &\leq \|B_N U(t)\Gamma_{N,0}^K\|_{L^2_{t\in I}\mathcal{H}_\xi^{1+\delta}} \\ &\quad + C_2(K)\,\xi^{-1}\,T^{\frac{1}{2}}\,\|\Theta_N^{K-1}\|_{L^2_{t\in I}\mathcal{H}_\xi^{1+\delta}}\end{aligned} \tag{4.7}$$

and use of Proposition A.2 implies that

$$\|\Theta_N^{K-1}\|_{L^2_{t\in I}\mathcal{H}_\xi^{1+\delta}} \leq \frac{C_2(K)\,\xi^{-1}}{1 - C_2(K)\,\xi^{-1}\,T^{\frac{1}{2}}}\,\|\Gamma_{0,N}^K\|_{\mathcal{H}_\xi^{1+\delta}} \tag{4.8}$$

holds.

Next, we let

$$\Gamma_N^K(t) := U(t)\Gamma_{N,0}^K + i\int_0^t U(t-s)\Theta_N^{K-1}(s)\,\mathrm{d}s. \tag{4.9}$$

Clearly,

$$\begin{aligned}\|\Gamma_N^K\|_{L^\infty_{t\in I}\mathcal{H}_\xi^{1+\delta}} &\leq \|\Gamma_{0,N}^K\|_{L^\infty_{t\in I}\mathcal{H}_\xi^{1+\delta}} + T^{\frac{1}{2}}\,\|\Theta_N^{K-1}\|_{L^2_{t\in I}\mathcal{H}_\xi^{1+\delta}} \\ &\leq \frac{1}{1 - C_2(K)\,\xi^{-1}\,T^{\frac{1}{2}}}\,\|\Gamma_{0,N}^K\|_{\mathcal{H}_\xi^{1+\delta}}\end{aligned} \tag{4.10}$$

from (4.8). Comparing the right hand sides of $B_N\Gamma_N^K$ and Θ_N^{K-1}, we conclude that

$$B_N\Gamma_N^K = \Theta_N^{K-1} \tag{4.11}$$

holds, and that

$$\Gamma_N^K(t) = U(t)\Gamma_{N,0}^K + i\int_0^t U(t-s)B_N\Gamma_N^K(s)\,\mathrm{d}s \tag{4.12}$$

is satisfied, with $B_N\Gamma_N^K \in L^2_{t\in I}\mathcal{H}_\xi^{1+\delta}$. So far, we have established well-posedness of solutions of the (K,N)-BBGKY hierarchy for $t \in [0,T]$ with $T < T_0(K,\xi)$. We can piece those solutions together, in order to extend them to longer time intervals.

In particular, we can prove that (4.10) can be enhanced to an estimate with both C_2 and T_0 independent of K, provided N is large enough for (4.1) to hold. In this case, we observe that applying B_N to (4.12), we find

$$B_N\Gamma_N^K(t) = B_N U(t)\Gamma_{N,0}^K + i\int_0^t B_N U(t-s)B_N\Gamma_N^K(s)\,\mathrm{d}s. \tag{4.13}$$

It is easy to verify that the assumptions of Lemma B.3 in the Appendix are satisfied for

$$\widetilde{\Theta}_N^K := B_N \Gamma_N^K, \quad \Xi_N^K := B_N U(t) \Gamma_{0,N}^K. \tag{4.14}$$

We assume that

$$\xi < \eta \xi'' < \eta^2 \xi' \tag{4.15}$$

where $0 < \eta < 1$ is as in Lemma B.3. Then, Lemma B.3 implies that

$$\begin{aligned} \| B_N \Gamma_N^K \|_{L^2_{t\in I} \mathcal{H}_\xi^{1+\delta}} &\le C(T,\xi,\eta) \, \| B_N U(t) \Gamma_{0,N}^K \|_{L^2_{t\in I} \mathcal{H}_{\xi''}^{1+\delta}} \\ &\le C_2(T,\xi,\eta) \, \| \Gamma_{0,N}^K \|_{\mathcal{H}_{\xi'}^{1+\delta}} \end{aligned} \tag{4.16}$$

holds for a constant $C_2 = C_2(T,\xi,\eta)$ independent of K, N, and for $T < T_0(\xi,\eta)$, if N is sufficiently large.

It remains to prove that $(\Gamma_N^K(t))^{(k)} = 0$ for all $K < k \le N$, and all $t \in I$. To this end, we first note that

$$(B_N P_{\le K} - P_{\le K-1} B_N) \, \Gamma_K^N = 0, \tag{4.17}$$

as one easily verifies based on the componentwise definition of B_N in (2.23) and (2.24). Hence, in particular,

$$(P_{>K} B_N - B_N P_{>K+1}) \, \Gamma_K^N = 0,$$

thanks to which we observe that $P_{>K}\Gamma_N^K$ by itself satisfies a closed sub-hierarchy of the N-BBGKY hierarchy,

$$i\partial_t (P_{>K}\Gamma_N^K) = \widehat{\Delta}_\pm (P_{>K}\Gamma_N^K) + B_N (P_{>K+1}\Gamma_N^K), \tag{4.18}$$

where clearly,

$$P_{>K+1}\Gamma_N^K = P_{>K+1}(P_{>K}\Gamma_N^K), \tag{4.19}$$

with initial data

$$(P_{>K}\Gamma_N^K)(0) = P_{>K}(\Gamma_N^K(0)) = 0. \tag{4.20}$$

Here we recall that the initial data is truncated for $k > K$.

Accordingly, by the same argument as above, there exists a unique solution $(P_{>K}\Gamma_N^K) \in L^\infty_{t\in I}\mathcal{H}_\xi^{1+\delta}$ with $B_N(P_{>K+1}\Gamma_N^K) \in L^2_{t\in I}\mathcal{H}_\xi^{1+\delta}$ such that

$$\| B_N (P_{>K+1}\Gamma_N^K) \|_{L^2_{t\in I}\mathcal{H}_\xi^{1+\delta}} \le C_2(T,\xi,\eta) \, \| (P_{>K}\Gamma_N^K)(0) \|_{\mathcal{H}_{\xi'}^{1+\delta}} = 0, \tag{4.21}$$

for $\xi < \eta^2 \xi'$. Moreover,

$$\| P_{>K}\Gamma_N^K \|_{L^\infty_{t\in I}\mathcal{H}_\xi^{1+\delta}} \le C_1(T,\xi,\eta) \, \| (P_{>K}\Gamma_N^K)(0) \|_{\mathcal{H}_{\xi'}^{1+\delta}} = 0. \tag{4.22}$$

This implies that $(P_{>K}\Gamma_N^K)(t) = 0$ for $t \in I$, as claimed. □

5. From (K, N)-BBGKY to K-Truncated GP Hierarchy

In this section, we control the limit $N \to \infty$ of the truncated BBGKY hierarchy, at fixed K.

Proposition 5.1. *Assume that $V_N(x) = N^{3\beta} V(N^\beta x)$ with $\widehat{V} \in C^\delta \cap L^\infty$ for some arbitrary but fixed, small $\delta > 0$. Moreover, assume that $\Gamma^K \in \mathfrak{W}_\xi^{1+\delta}(I)$ (see (1.24)) is the solution of the GP hierarchy with truncated initial data $\Gamma_0^K = P_{\leq K}\Gamma_0 \in \mathcal{H}_\xi^{1+\delta}$ constructed in [9].*

Let Γ_N^K solve the (K, N)-BBGKY hierarchy with initial data $\Gamma_{0,N}^K := P_{\leq K}\Gamma_{0,N} \in \mathcal{H}_{\xi'}^{1+\delta}$. Let, for $\beta = \frac{1-\delta'}{4}$,

$$K(N) := \frac{\delta'}{2\log C_0}\,\log N \tag{5.1}$$

so that (4.1) is satisfied. Then, as $N \to \infty$, the strong limits

$$\lim_{N\to\infty} \| \, \Gamma_N^{K(N)} - \Gamma^{K(N)} \, \|_{L^\infty_{t\in[0,T]}\mathcal{H}^1_\xi} = 0 \tag{5.2}$$

and

$$\lim_{N\to\infty} \| \, B_N\Gamma_N^{K(N)} - B\Gamma^{K(N)} \, \|_{L^2_{t\in[0,T]}\mathcal{H}^1_\xi} = 0 \tag{5.3}$$

hold, for $0 < T < T_0(\xi)$.

Proof. In [9], we constructed a solution Γ^K of the full GP hierarchy with truncated initial data, $\Gamma(0) = \Gamma_0^K \in \mathcal{H}_\xi^{1+\delta}$, satisfying the following: For an arbitrary fixed K, Γ^K satisfies the GP hierarchy in integral representation,

$$\Gamma^K(t) = U(t)\Gamma_0^K + i\int_0^t U(t-s)\, B\Gamma^K(s)\,\mathrm{d}s, \tag{5.4}$$

and in particular, $(\Gamma^K)^{(k)}(t) = 0$ for all $k > K$.

Accordingly, we have

$$\begin{aligned} B_N\Gamma_N^K - B\Gamma^K &= B_N U(t)\Gamma_{0,N}^K - BU(t)\Gamma_0^K \\ &\quad + i\int_0^t \big(B_N U(t-s) B_N\Gamma_N^K - BU(t-s)B\Gamma^K \big)(s)\mathrm{d}s \\ &= (B_N - B)U(t)\Gamma_{0,N}^K + BU(t)(\Gamma_{0,N}^K - \Gamma_0^K) \\ &\quad + i\int_0^t \big(B_N - B\big)U(t-s)B\Gamma^K(s)\,\mathrm{d}s \\ &\quad + i\int_0^t B_N U(t-s)\big(B_N\Gamma_N^K - B\Gamma^K\big)(s)\,\mathrm{d}s. \end{aligned} \tag{5.5}$$

Here, we observe that for N sufficiently large, we can apply Lemma B.3 with

$$\widetilde{\Theta}_N^K := B_N\Gamma_N^K - B\Gamma^K \tag{5.6}$$

and

$$\Xi_N^K := (B_N - B)U(t)\Gamma_{0,N}^K + BU(t)(\Gamma_{0,N}^K - \Gamma_0^K) + i\int_0^t \big(B_N - B\big)U(t-s)B\Gamma^K(s)\,\mathrm{d}s. \tag{5.7}$$

Given ξ', we introduce parameters ξ, ξ'', ξ''' satisfying

$$\xi < \eta\,\xi'' < \eta^2\,\xi''' < \eta^3\xi' \tag{5.8}$$

where $0 < \eta < 1$ is as in Lemma B.3. Accordingly, Lemma B.3 implies that

$$\begin{aligned}\|B_N\Gamma_N^K - B\Gamma\|_{L^2_{t\in I}\mathcal{H}^1_\xi} &\le C_2(T,\xi,\xi'')\Big(\,\|BU(t)(\Gamma_{0,N}^K - \Gamma_0^K)\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}} + R^K(N)\Big)\\ &\le C_1(T,\xi,\xi',\xi'')\Big(\,\|\Gamma_{0,N}^K - \Gamma_0^K\|_{L^2_{t\in I}\mathcal{H}^1_{\xi'}} + R^K(N)\Big),\end{aligned} \tag{5.9}$$

where we used Lemma A.1 to pass to the last line. Here,

$$R^K(N) = R_1^K(N) + R_2^K(N), \tag{5.10}$$

with

$$R_1^K(N) := \|(B_N - B)U(t)\Gamma_{0,N}^K\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}} \tag{5.11}$$

and

$$R_2^K(N) := \left\| \int_0^t \big(B_N - B\big)U(t-s)B\Gamma^K(s)\,\mathrm{d}s \right\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}}. \tag{5.12}$$

Next, we consider the limit $N \to \infty$ with $K(N)$ as given in (5.1). We choose $K(N)$ in this manner for it to be compatible with (4.1), which is necessary for results in other sections.

To begin with, we note that

$$\lim_{N\to\infty} \|\Gamma_{0,N} - \Gamma_0\|_{\mathcal{H}^{1+\delta}_{\xi'}} = 0. \tag{5.13}$$

Including the truncation at $K(N)$, it is easy to see that

$$\begin{aligned}\lim_{N\to\infty} \|\Gamma_{0,N}^{K(N)} - \Gamma_0^{K(N)}\|_{\mathcal{H}^{1+\delta}_{\xi'}} &= \lim_{N\to\infty} \|\,P_{\le K(N)}\,(\,\Gamma_{0,N} - \Gamma_0\,)\,\|_{\mathcal{H}^{1+\delta}_{\xi'}}\\ &\le \lim_{N\to\infty} \|\,\Gamma_{0,N} - \Gamma_0\,\|_{\mathcal{H}^{1+\delta}_{\xi'}}\\ &= 0\end{aligned} \tag{5.14}$$

follows.

To control $R^{K(N)}(N)$, we invoke Lemma 5.2 below, which implies that for an arbitrary but fixed $\delta > 0$,

$$\lim_{N\to\infty} R_1^{K(N)}(N) \le \lim_{N\to\infty} C_{V,\delta}\,\xi^{-1}\,N^{-\delta\beta}\,\|\Gamma_{0,N}^{K(N)}\|_{L^2_{t\in I}\mathcal{H}^{1+\delta}_{\xi'}} = 0, \tag{5.15}$$

for a constant $C_{V,\delta}$ that depends only on V and δ, since

$$\lim_{N\to\infty} \|\Gamma_{0,N}^{K(N)} - \Gamma_0\|_{\mathcal{H}^{1+\delta}_{\xi'}} = 0, \tag{5.16}$$

and $\|\Gamma_0\|_{\mathcal{H}^{1+\delta}_{\xi'}} < \infty$.

Moreover, invoking Lemma 5.3 below, we find

$$\lim_{N\to\infty} R_2^{K(N)}(N) \leq \lim_{N\to\infty} C_{V,\delta}\, \xi^{-1}\, N^{-\delta\beta}\, \|B\Gamma^{K(N)}\|_{L^2_{t\in I}\mathcal{H}^{1+\delta}_{\xi'''}} = 0, \quad (5.17)$$

because

$$\|B\Gamma^{K(N)}\|_{L^2_{t\in I}\mathcal{H}^{1+\delta}_{\xi'''}} < C(T, \xi''', \xi')\, \|\Gamma_0\|_{\mathcal{H}^{1+\delta}_{\xi'}} \quad (5.18)$$

is uniformly bounded in N, as shown in [9]. □

Lemma 5.2. *Let $\delta > 0$ be an arbitrary, but fixed, small number. Assume that $V_N(x) = N^{3\beta}V(N^\beta x)$ with $\widehat{V} \in C^\delta \cap L^\infty$. Then, with $\xi < \eta\,\xi''$ as in (5.8),*

$$\|(B_N - B)U(t)\Gamma^K_{0,N}\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^1_\xi} < C_{V,\delta}\, \xi^{-1}\, N^{-\delta\beta}\, \|\Gamma^K_{0,N}\|_{\mathcal{H}^{1+\delta}_{\xi''}} \quad (5.19)$$

for a constant $C_{V,\delta}$ depending only on V and δ, but not on K or N.

Proof. In a first step, we prove that

$$\|(B^+_{N;k+1} - B^+_{k+1})\, U^{(k+1)}(t)\gamma_0^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H^1} \leq C\, k^2\, N^{-\delta\beta}\, \|\gamma_0^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H^{1+\delta}} \quad (5.20)$$

holds, for $\widehat{V} \in C^\delta \cap L^\infty$ with $\delta > 0$.

To this end, we note that

$$\widehat{V_N}(\xi) = \widehat{V}(N^{-\beta}\xi), \quad \widehat{V_N}(0) = \int V_N(x)\mathrm{d}x = \int V(x)\mathrm{d}x = \widehat{V}(0) = 1,$$

and we define

$$\chi_N(\xi) := \frac{N-k}{N}\widehat{V_N}(\xi) - \widehat{V}(0), \quad (5.21)$$

We have

$$\chi_N(q-q') = \chi^1_N(q-q') + \chi^2_N(q-q') \quad (5.22)$$

where

$$\chi^1_N(q-q') := \widehat{V_N}(q-q') - \widehat{V_N}(0), \quad \chi^2_N(q-q') := \frac{k}{N}\widehat{V_N}(q-q'). \quad (5.23)$$

Clearly, we have that for $\delta > 0$ small, δ-Holder continuity of $\widehat{V}$ implies

$$\begin{aligned} |\chi^1_N(q-q')| &\leq \|\widehat{V}\|_{C^\delta}\, N^{-\delta\beta}|q-q'|^\delta \\ &\leq \|\widehat{V}\|_{C^\delta}\, N^{-\delta\beta}\,(\,|q|^\delta + |q'|^\delta\,), \end{aligned} \quad (5.24)$$

and

$$\chi^2_N(q-q')| \leq \|\widehat{V}\|_{L^\infty} kN^{-1} \quad (5.25)$$

is clear.

Next, we let $(\tau, \underline{u}_k, \underline{u}'_k)$, q and q' denote the Fourier conjugate variables corresponding to $(t, \underline{x}_k, \underline{x}'_k)$, x_{k+1}, and x'_{k+1}, respectively. Without any loss of generality, we may assume that $j = 1$ in $B_{N;j;k+1}$ and $B_{j;k+1}$. Then, abbreviating

$$\delta(\cdots) := \delta(\,\tau + (u_1 + q - q')^2 + \sum_{j=2}^k u_j^2 + q^2 - |\underline{u}'_k|^2 - (q')^2\,) \quad (5.26)$$

we find

$$\begin{aligned}&\Big\| \, S^{(k,1)}(B_{N;1;k+1}-B_{1;k+1})U^{(k+1)}(t)\gamma_{0,N}^{(k+1)} \, \Big\|^2_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\ &\quad = \int_{\mathbb{R}} \mathrm{d}\tau \int \mathrm{d}\underline{u}_k \mathrm{d}\underline{u}_k' \prod_{j=1}^k \big\langle u_j \big\rangle^2 \big\langle u_j' \big\rangle^2 \\ &\qquad \Big(\int \mathrm{d}q \mathrm{d}q' \, \delta(\cdots) \, \chi_N(q-q') \, \widehat{\gamma}^{(k+1)}(\tau, u_1+q-q', u_2, \dots, u_k, q; \underline{u}_k', q') \Big)^2 ,\end{aligned} \tag{5.27}$$

similarly as in [25,26]. Using the Schwarz inequality, this is bounded by

$$\begin{aligned}&\leq \int_{\mathbb{R}} \mathrm{d}\tau \int \mathrm{d}\underline{u}_k \mathrm{d}\underline{u}_k' \, J(\tau, \underline{u}_k, \underline{u}_k') \int \mathrm{d}q \mathrm{d}q' \, \delta(\cdots) \\ &\quad \big\langle u_1+q-q' \big\rangle^2 \big\langle q \big\rangle^2 \big\langle q' \big\rangle^2 \prod_{j=2}^k \big\langle u_j \big\rangle^2 \prod_{j'=1}^k \big\langle u_{j'}' \big\rangle^2 \, | \, \chi_N(q-q') \, |^2 \\ &\quad \Big| \, \widehat{\gamma}^{(k+1)}(\tau, u_1+q-q', u_2, \dots, u_k, q; \underline{u}_k', q') \, \Big|^2\end{aligned} \tag{5.28}$$

where

$$J(\tau, \underline{u}_k, \underline{u}_k') := \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{d}q \, \mathrm{d}q' \, \frac{\delta(\cdots) \, \big\langle u_1 \big\rangle^2}{\big\langle u_1+q-q' \big\rangle^2 \big\langle q \big\rangle^2 \big\langle q' \big\rangle^2} . \tag{5.29}$$

The boundedness of

$$C_J := \Big(\sup_{\tau, \underline{u}_k, \underline{u}_k'} J(\tau, \underline{u}_k, \underline{u}_k') \Big)^{\frac{1}{2}} < \infty \tag{5.30}$$

is proven in [25] for dimension 3, and in [7,26] for dimension 2.

Using (5.24) and (5.25), we obtain, from the Schwarz inequality, that

$$\begin{aligned}(5.28) &\leq C_{V,J} \int_{\mathbb{R}} \mathrm{d}\tau \int \mathrm{d}\underline{u}_k \mathrm{d}\underline{u}_k' \int \mathrm{d}q \mathrm{d}q' \, \big(N^{-2\delta\beta}(\, |q|^{2\delta} + |q'|^{2\delta} \,) + k^2 N^{-2} \, \big) \\ &\quad \big\langle u_1+q-q' \big\rangle^2 \big\langle q \big\rangle^2 \big\langle q' \big\rangle^2 \prod_{j=2}^k \big\langle u_j \big\rangle^2 \prod_{j'=1}^k \big\langle u_{j'}' \big\rangle^2 \\ &\quad \Big| \, \widehat{\gamma}^{(k+1)}(\tau, u_1+q-q', u_2, \dots, u_k, q; \underline{u}_k', q') \, \Big|^2\end{aligned} \tag{5.31}$$

where $C_{V,J} := C_V C_J$, and C_V is a finite constant depending on V. Hence,

$$\begin{aligned}&\Big\| \, S^{(k,1)}(B_{N;1;k+1}-B_{1;k+1})U^{(k+1)}(t)\gamma_{0,N}^{(k+1)} \, \Big\|^2_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\ &\quad \leq C_{V,J} \, N^{-2\delta\beta} \, \|\gamma_{0,N}^{(k+1)}\|^2_{H^{1+\delta}} + C_{V,J} \, k^2 \, N^{-2} \, \|\gamma_{0,N}^{(k+1)}\|^2_{H^1} \\ &\quad \leq C_{V,J} \, k^2 \, N^{-2\delta\beta} \, \|\gamma_{0,N}^{(k+1)}\|^2_{H^{1+\delta}}\end{aligned} \tag{5.32}$$

follows, given that $\delta\beta < 1$ for $\delta > 0$ sufficiently small.

Therefore, we conclude that

$$\begin{aligned}
&\| (B_N - B)\, U(t)\, \Gamma_0^K \|_{L^2_{t\in\mathbb{R}}\mathcal{H}^1_\xi} \\
&= \sum_{k=1}^{K} \xi^k \, \|(B^+_{N;k+1} - B^+_{k+1})\, U^{(k+1)}(t)\gamma_0^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H^1} \\
&\leq C\, N^{-\delta\beta}\, \xi^{-1} \sum_{k=1}^{K} k^2\, \xi^{k+1}\, \|\, \gamma_0^{(k+1)}\, \|_{H^{1+\delta}_{k+1}} \\
&\leq C\, N^{-\delta\beta} \Big(\sup_k k^2 \Big(\frac{\xi}{\xi''}\Big)^k \Big)\, \xi^{-1}\, \|\Gamma_0^K\|_{\mathcal{H}^{1+\delta}_{\xi''}} \\
&\leq C\, N^{-\delta\beta}\, \xi^{-1}\, \|\Gamma_0^K\|_{\mathcal{H}^{1+\delta}_{\xi''}},
\end{aligned} \tag{5.33}$$

for $\xi < \xi''$. This proves the Lemma. □

Lemma 5.3. *Assume that* $V_N(x) = N^{3\beta}V(N^\beta x)$ *with* $\widehat{V} \in C^\delta \cap L^\infty$*. Then,*

$$\begin{aligned}
&\left\| \int_0^t \big(B_N - B\big)U(t-s)B\Gamma^K(s)\,\mathrm{d}s \right\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}} \\
&\quad < C_{V,\delta}\, \xi^{-1}\, T^{\frac{1}{2}}\, N^{-\delta\beta}\, \|B\Gamma^K\|_{L^2_{t\in I}\mathcal{H}^{1+\delta}_{\xi'''}}
\end{aligned} \tag{5.34}$$

where the constant $C_{V,\delta} > 0$ *depends only on* V *and* δ*, and* $\xi'' < \eta\,\xi'''$ *as in* (5.8).

Proof. Using Lemma 5.2,

$$\begin{aligned}
&\left\| \int_0^t \big(B_N - B\big)U(t-s)B\Gamma^K(s)\,\mathrm{d}s \right\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}} \\
&\leq \int_0^T \left\| \big(B_N - B\big)U(t-s)B\Gamma^K(s)\,\mathrm{d}s \right\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^{1+\delta}_{\xi''}} \\
&\leq C_{V,\delta}\, \xi^{-1}\, N^{-\delta\beta} \int_0^T \|\, B\Gamma^K(s)\,\mathrm{d}s\, \|_{\mathcal{H}^{1+\delta}_{\xi'''}} \\
&< C_{V,\delta}\, \xi^{-1}\, T^{\frac{1}{2}}\, N^{-\delta\beta}\, \|B\Gamma^K\|_{L^2_{t\in I}\mathcal{H}^{1+\delta}_{\xi'''}},
\end{aligned} \tag{5.35}$$

for $C_{V,\delta}$ as in the previous lemma. This proves the claim. □

6. Comparing the (K,N)-BBGKY with the Full N-BBGKY Hierarchy

In this section, we compare solutions Γ_N^K of the (K,N)-BBGKY hierarchy to solutions Γ^{Φ_N} to the full N-BBGKY hierarchy obtained from Φ_N which solves the N-body Schrödinger equation (2.1).

Lemma 6.1. *Assume that N is sufficiently large, and that in particular, (4.1) holds. Then, there is a finite constant $C(T,\xi)$ independent of K,N such that the estimate*

$$\|B_N\Gamma_N^K - B_N P_{\leq K}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \leq C(T,\xi)\,(\xi')^K\,K\,\|(B_N\Gamma^{\Phi_N})^{(K)}\|_{L^2_{t\in I}H^1} \tag{6.1}$$

holds, where $(B_N\Gamma^{\Phi_N})^{(K)}$ is the Kth component of $B_N\Gamma^{\Phi_N}$ (and the only non-vanishing component of $P_K B_N\Gamma^{\Phi_N}$), and $\xi' = \xi/\eta$ with $\eta < 1$ as specified in Lemma B.3 in the Appendix.

Proof. We have already shown that $B_N\Gamma_N^K \in L^2_{t\in I}\mathcal{H}^1_\xi$. Moreover, it is easy to see that

$$\|B_N\Gamma_N^K\|_{L^2_{t\in I}\mathcal{H}^1_\xi} < C(N,K,T). \tag{6.2}$$

The easiest way to see this is to use the trivial bound $\|V_N\|_{L^\infty} < c(N)$, and the fact that $I = [0,T]$ is finite.

Thus,

$$B_N\Gamma_N^K - B_N P_{\leq K}\Gamma^{\Phi_N} \in L^2_{t\in I}\mathcal{H}^1_\xi \tag{6.3}$$

follows.

Next, we observe that

$$\begin{aligned}
&(B_N\Gamma_N^K - B_N P_{\leq K}\Gamma^{\Phi_N})(t)\\
&\quad= (B_N\Gamma_N^K - P_{\leq K-1}B_N\Gamma^{\Phi_N})(t) \qquad\qquad (6.4)\\
&\quad= B_N U(t)\Gamma_N^K(0) - P_{\leq K-1}B_N U(t)\Gamma_N(0)\\
&\qquad + i\int_0^t B_N U(t-s)B_N\Gamma_N^K(s)\,\mathrm{d}s - i\int_0^t P_{\leq K-1}B_N U(t-s)B_N\Gamma^{\Phi_N}(s)\,\mathrm{d}s\\
&\quad= (B_N P_{\leq K} - P_{\leq K-1}B_N)\,U(t)\Gamma_N(0)\\
&\qquad + i\,(B_N P_{\leq K} - P_{\leq K-1}B_N)\int_0^t U(t-s)B_N\Gamma^{\Phi_N}(s)\,\mathrm{d}s\\
&\qquad + i\int_0^t B_N U(t-s)B_N\Gamma_N^K(s)\,\mathrm{d}s\\
&\qquad - i\,B_N P_{\leq K}\int_0^t U(t-s)B_N\Gamma^{\Phi_N}(s)\,\mathrm{d}s\\
&\quad= (B_N P_{\leq K} - P_{\leq K-1}B_N)\,\Gamma^{\Phi_N}(t)\\
&\qquad + i\int_0^t B_N U(t-s)\big(B_N\Gamma_N^K - P_{\leq K-1}B_N\Gamma^{\Phi_N}\big)(s)\,\mathrm{d}s\\
&\qquad + i(B_N P_{\leq K-1} - B_N P_{\leq K})\int_0^t U(t-s)\,B_N\,\Gamma^{\Phi_N}(s)\,\mathrm{d}s, \qquad\qquad (6.5)
\end{aligned}$$

where to obtain (6.4) we used the fact that

$$(B_N P_{\leq K} - P_{\leq K-1} B_N)\, \Gamma^{\Phi_N} = 0, \tag{6.6}$$

which follows, as (4.17), based on the componentwise definition of B_N in (2.23) and (2.24). Now we notice that $B_N P_{\leq K} - B_N P_{\leq K-1} = B_N P_K$. Hence (6.5) implies that

$$\begin{aligned}
&(B_N \Gamma_N^K - P_{\leq K-1} B_N \Gamma^{\Phi_N})(t) \\
&\quad = (B_N P_{\leq K} - P_{\leq K-1} B_N)\, \Gamma^{\Phi_N}(t) \\
&\qquad -i \int_0^t B_N\, U(t-s)\, P_K B_N\, \Gamma^{\Phi_N}(s)\, \mathrm{d}s \\
&\qquad +i \int_0^t B_N U(t-s) \big(B_N \Gamma_N^K - P_{\leq K-1} B_N \Gamma^{\Phi_N}\big)(s)\, \mathrm{d}s,
\end{aligned} \tag{6.7}$$

which thanks to (6.6) simplifies to

$$\begin{aligned}
&(B_N \Gamma_N^K - P_{\leq K-1} B_N \Gamma^{\Phi_N})(t) \\
&\quad = -i \int_0^t B_N\, U(t-s)\, P_K B_N\, \Gamma^{\Phi_N}(s)\, \mathrm{d}s \\
&\qquad +i \int_0^t B_N U(t-s) \big(B_N \Gamma_N^K \,-\, P_{\leq K-1} B_N \Gamma^{\Phi_N}\big)(s)\, \mathrm{d}s.
\end{aligned} \tag{6.8}$$

We observe that the term in parenthesis on the last line corresponds to (6.4), which is the same as (6.3). Given K, the condition that N is large enough that (4.1) holds, allows us to apply Lemma B.3 with

$$\widetilde{\Theta}_N^K(t) := B_N \Gamma_N^K(t) - P_{\leq K-1} B_N \Gamma^{\Phi_N}(t) \tag{6.9}$$

and

$$\Xi_N^K(t) := -i \int_0^t B_N U(t-s) P_K B_N \Gamma^{\Phi_N}(s)\, \mathrm{d}s. \tag{6.10}$$

We note that for the integral on the rhs of (6.10),

$$\begin{aligned}
&\left\| \int_0^t B_N U(t-s) P_K B_N \Gamma^{\Phi_N}(s)\, \mathrm{d}s \right\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \\
&\qquad \leq C\, T^{\frac{1}{2}}\, K\, \| P_K\, B_N \Gamma^{\Phi_N} \|_{L^2_{t\in I}\mathcal{H}^1_\xi},
\end{aligned} \tag{6.11}$$

for a constant C uniformly in N and K, based on similar arguments as in the proof of Lemma 5.3, and using the Strichartz estimates (A.19) and (A.31).

Accordingly, Lemma B.3 implies that for $N \gg K$ sufficiently large, and in particular satisfying (4.1),

$$\begin{aligned}\|B_N\Gamma_N^K - B_N P_{\leq K}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} &= \|B_N\Gamma_N^K \,-\, P_{\leq K-1}B_N\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \\ &\leq C'(T,\xi)\,\|\Xi_N^K\|_{L^2_{t\in I}\mathcal{H}^1_{\xi'}} \\ &\leq C(T,\xi)\,K\,\|P_K\,B_N\,\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_{\xi'}}, \end{aligned} \tag{6.12}$$

where $P_K = P_{\leq K} - P_{\leq K-1}$, and $\xi' = \xi/\eta$ with η specified in Lemma B.3. This immediately implies the asserted estimate, for T sufficiently small (depending on K). Clearly,

$$\|P_K\,B_N\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_{\xi'}} = (\xi')^K\,\|(B_N\Gamma^{\Phi_N})^{(K)}\|_{L^2_{t\in I}H^1}. \tag{6.13}$$

Therefore,

$$\|B_N\Gamma_N^K - B_N P_{\leq K}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \leq C(T,\xi)\,(\xi')^K\,K\,\|(B_N\Gamma^{\Phi_N})^{(K)}\|_{L^2_{t\in I}H^1}, \tag{6.14}$$

as claimed. Here, we have used the result of Lemma B.3, and used the fact that $P_K\,B_N\Gamma^{\Phi_N}$ has a single nonzero component. □

7. Control of Γ^{Φ_N} and Γ_N^K as $N \to \infty$

In this section, we control the comparison between Γ^{Φ_N} and Γ_N^K in a limit where simultaneously, $N \to \infty$ and $K = K(N) \to \infty$ at a suitable rate.

Proposition 7.1. *Writing $\beta = \frac{1-\delta'}{4}$, let*

$$K(N) = \frac{\delta'}{2\log C_0}\log N, \tag{7.1}$$

and $\xi > 0$ small enough that

$$\xi < \frac{1}{b_1}\mathrm{e}^{-\frac{2}{\delta'}(1-4\delta')\log C_0}, \tag{7.2}$$

with constants b_1 as in Lemma 7.2, and C_0 as in Lemma B.2. Then,

$$\lim_{N\to\infty}\|B_N\Gamma_N^{K(N)} - P_{\leq K(N)-1}B_N\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} = 0 \tag{7.3}$$

holds.

Proof. From Lemma 7.2 below, we have the estimate

$$\|B_N\Gamma_N^K - P_{\leq K-1}B_N\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \leq C(T,\xi)\,N^{4\beta}\,K^2\,(b_1\xi)^K \tag{7.4}$$

where b_1, C_0 are independent of K, N.

One can easily check that the stated assumptions on $K(N)$ and ξ imply that

$$N^{4\beta}\,K^2\,(b_1\xi)^{K(N)} < N^{-\epsilon}, \tag{7.5}$$

for some $\epsilon > 0$.

We note that the given choice of $K(N)$ complies with (5.1) and the hypotheses of Lemma B.3, which are needed for results in previous sections. This immediately implies the claim. □

Lemma 7.2. *The estimate*

$$\|B_N \Gamma_N^K - P_{\leq K-1} B_N \Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \leq C(T,\xi)\ N^{4\beta}\, K^2\, (b_1\xi)^K \tag{7.6}$$

holds for finite constants b_1, $C(T,\xi)$ independent of K and N. The constant b_1 only depends on the initial state $\Phi_N(0)$ of the N-body Schrödinger problem, and on the constant η as defined in Lemma B.3.

Proof. From Lemma 6.1, we have that

$$\|B_N \Gamma_N^K - P_{\leq K-1} B_N \Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \leq\ C(T,\xi)\, (\eta^{-1}\xi)^K\, K\, \|(B_N\Gamma^{\Phi_N})^{(K)}\|_{L^2_{t\in I}H^1} \tag{7.7}$$

holds for a finite constant $C(T,\xi)$ independent of K, N.

The fact that in dimension $d = 3$,

$$\|V_N\|_{C^1} \leq C\, N^{4\beta} \tag{7.8}$$

follows immediately from the definition of V_N. Thus, we have

$$\begin{aligned}
&\|(B_N^+\Gamma^{\Phi_N})^{(K)}\|^2_{L^2_{t\in I}H^1} \\
&\leq C \int_I \mathrm{d}t \int \mathrm{d}\underline{x}_K \mathrm{d}\underline{x}'_K \Big| \sum_{\ell=1}^K \int \Big[\prod_{j=1}^K \langle\nabla_{x_j}\rangle\langle\nabla_{x'_j}\rangle\Big] V_N(x_\ell - x_{K+1})\Phi_N(t,\underline{x}_N) \\
&\quad \times\overline{\Phi_N(t,\underline{x}'_K, x_{K+1},\dots,x_N)}\,\mathrm{d}x_{K+1}\cdots\mathrm{d}x_N\Big|^2 \\
&\leq C\,\|V_N\|^2_{C^1} \int_I \mathrm{d}t \int \mathrm{d}\underline{x}_N \Big|\sum_{\ell=1}^K \int \Big[\prod_{j=1}^K \langle\nabla_{x_j}\rangle\Big]\, \Phi_N(t,\underline{x}_N)\Big|^2 \\
&\quad \times \sup_{t\in I} \int \mathrm{d}\underline{x}'_N \Big|\Big[\prod_{j=1}^K \langle\nabla_{x'_j}\rangle\Big]\, \overline{\Phi_N(t,\underline{x}'_N)}\Big|^2 &(7.9)\\
&\leq C\, T\, N^{8\beta}\, K^2 \sup_{t\in I}\Big(\,\mathrm{Tr}\big(\,S^{(K,1)}\,\gamma_N^{(K)}\,\big)\Big)^2 &(7.10)
\end{aligned}$$

using Cauchy–Schwarz to pass to (7.9), and admissibility to obtain (7.10). We also recall that $\gamma_N^{(K)}$ is positive and self-adjoint.

It remains to bound the term

$$\mathrm{Tr}\big(\,S^{(K,1)}\,\gamma_N^{(K)}\,\big) \tag{7.11}$$

in (7.10). To this end, we recall energy conservation in the N-body Schrödinger equation satisfied by Φ_N. Indeed, it is proved in [16,17,26] that

$$\Big\langle\, \Phi_N,\, (N + H_N)^K\, \Phi_N\, \Big\rangle \geq C^K\, N^K\, \mathrm{Tr}(S^{(1,K)}\gamma^{(K)}_{\Phi_N}) \tag{7.12}$$

for some positive constant $C > 0$ where

$$\gamma^{(k)}_{\Phi_N} = \mathrm{Tr}_{k+1,\dots,N}(|\Phi_N\rangle\langle\Phi_N|). \tag{7.13}$$

This implies that

$$\mathrm{Tr}\big(\,S^{(K,1)}\,\gamma_N^{(K)}\,\big) < (b'_1)^K \tag{7.14}$$

for some finite constant $b'_1 > 0$. We then define $b_1 := b'_1\eta^{-1}$. □

8. Proof of the Main Theorem 3.1

We may now collect all estimates proven so far, and prove the main result of this paper, Theorem 3.1.

To this end, we recall again the solution Γ^K of the GP hierarchy with truncated initial data, $\Gamma^K(t=0) = P_{\leq K}\Gamma_0 \in \mathcal{H}^1_\xi$. In [9], we proved the existence of a solution Γ^K that satisfies the K-truncated GP hierarchy in integral form,

$$\Gamma^K(t) = U(t)\Gamma^K(0) + i\int_0^t U(t-s)\, B\Gamma^K(s)\,\mathrm{d}s \tag{8.1}$$

where $(\Gamma^K)^{(k)}(t) = 0$ for all $k > K$. Moreover, it is shown in [9] that this solution satisfies $B\Gamma^K \in L^2_{t\in I}\mathcal{H}^1_\xi$.

Moreover, we proved in [9] the following convergence:

(a) The strong limit

$$\Gamma := s-\lim_{K\to\infty}\Gamma^K \quad \in L^\infty_t\mathcal{H}^1_\xi \tag{8.2}$$

exists.

(b) The strong limit

$$\Theta := s-\lim_{K\to\infty}B\Gamma^K \quad \in L^2_t\mathcal{H}^1_\xi. \tag{8.3}$$

exists, and in particular,

$$\Theta = B\Gamma. \tag{8.4}$$

Clearly, we have that

$$\|B\Gamma - B_N P_{\leq K(N)}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi}$$

$$\leq \|B\Gamma - B\Gamma^{K(N)}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \tag{8.5}$$

$$+\|B\Gamma^{K(N)} - B_N\Gamma_N^{K(N)}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \tag{8.6}$$

$$+\|B_N\Gamma_N^{K(N)} - B_N P_{\leq K(N)}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi}. \tag{8.7}$$

In the limit $N\to\infty$, we have that (8.5) $\to 0$ from (8.3) and (8.4).

Moreover, (8.6) $\to 0$ follows from Proposition 5.1.

Finally, (8.7) $\to 0$ follows from Proposition 7.1.

Therefore,

$$\lim_{N\to\infty}\|B\Gamma - B_N P_{\leq K(N)}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} = 0 \tag{8.8}$$

follows.

Moreover, we have that

$$\|P_{\leq K(N)}\Gamma^{\Phi_N} - \Gamma\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} \leq \|P_{\leq K(N)}\Gamma^{\Phi_N} - \Gamma_N^{K(N)}\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} \tag{8.9}$$

$$+\|\Gamma^{K(N)} - \Gamma\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} \tag{8.10}$$

$$+\|\Gamma_N^{K(N)} - \Gamma^{K(N)}\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi}. \tag{8.11}$$

In the limit $N \to \infty$, we have (8.9) $\to 0$, as a consequence of Proposition 7.1. Indeed,

$$\|P_{\leq K(N)}\Gamma^{\Phi_N} - \Gamma_N^{K(N)}\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} \leq T^{\frac{1}{2}}\|B_N\Gamma_N^{K(N)} - B_N P_{\leq K(N)}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \tag{8.12}$$

where the rhs tends to zero as $N \to \infty$, as discussed for (8.7).

Moreover, (8.10) $\to 0$, as a consequence of (8.2).

Finally, (8.11) $\to 0$ follows from Proposition 5.1.

This completes the proof of Theorem 3.1.

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Appendix A. Strichartz Estimates for GP and BBGKY Hierarchies

In Appendices A and B, we will prove certain technical results, which we formulate for dimensions $d \geq 1$, respectively $d \geq 2$.

In Appendix A, motivated by the Strichartz estimate for the GP hierarchy, we establish a Strichartz estimate for the BBGKY hierarchy.

A.1. Strichartz Estimates for the GP Hierarchy

Following [9], we first recall a version of the GP Strichartz estimate for the free evolution $U(t) = \mathrm{e}^{it\widehat{\Delta}_\pm} = (U^{(n)}(t))_{n\in\mathbb{N}}$. The estimate is obtained via reformulating the Strichartz estimate proven by Klainerman and Machedon [25].

Lemma A.1. *Let*

$$\alpha \in \mathfrak{A}(d) = \begin{cases} (\frac{1}{2}, \infty) & \text{if } d = 1 \\ (\frac{d-1}{2}, \infty) & \text{if } d \geq 2 \text{ and } d \neq 3 \\ [1, \infty) & \text{if } d = 3. \end{cases} \tag{A.1}$$

Then, the following hold:

1. Bound for K-truncated case: *Assume that* $\Gamma_0 \in \mathcal{H}_\xi^\alpha$ *for some* $0 < \xi < 1$. *Then, for any* $K \in \mathbb{N}$, *there exists a constant* $C(K)$ *such that the Strichartz estimate for the free evolution*
$$\|BU(t)\Gamma_0^K\|_{L^2_{t\in\mathbb{R}}\mathcal{H}_\xi^\alpha} \leq \xi^{-1}\, C(K)\, \|\Gamma_0^K\|_{\mathcal{H}_\xi^\alpha} \tag{A.2}$$
holds. Notably, the value of ξ *is the same on both the lhs and rhs.*
2. Bound for $K \to \infty$: *Assume that* $\Gamma_0 \in \mathcal{H}_{\xi'}^\alpha$ *for some* $0 < \xi' < 1$. *Then, for any* $0 < \xi < \xi'$, *there exists a constant* $C(\xi, \xi')$ *such that the Strichartz estimate for the free evolution*
$$\|BU(t)\Gamma_0\|_{L^2_{t\in\mathbb{R}}\mathcal{H}_\xi^\alpha} \leq C(\xi, \xi')\, \|\Gamma_0\|_{\mathcal{H}_{\xi'}^\alpha} \tag{A.3}$$
holds.

Proof. From Theorem 1.3 in [25] we have, for $\alpha \in \mathfrak{A}(d,p)$, that

$$\begin{aligned}\|B_{k+1}U^{(k+1)}(t)\gamma_0^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H_k^\alpha} &\leq 2\sum_{j=1}^{k} \|\, B^+_{j;k+1}\, U^{(k+1)}(t)\gamma_0^{(k+1)}\,\|_{L^2_{t\in\mathbb{R}}H_k^\alpha} \\ &\leq C\, k\, \|\,\gamma_0^{(k+1)}\,\|_{H^\alpha_{k+1}}. \end{aligned} \tag{A.4}$$

Then for any $0 < \xi < \xi'$, we have:

$$\begin{aligned}\|BU(t)\Gamma_0\|_{L^2_{t\in\mathbb{R}}\mathcal{H}_\xi^\alpha} &\leq \sum_{k\geq 1}\xi^k\|B_{k+1}U^{(k+1)}(t)\gamma_0^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H_k^\alpha} \\ &\leq C\sum_{k\geq 1} k\,\xi^k\,\|\gamma_0^{(k+1)}\|_{H^\alpha_{k+1}} \\ &= C\,(\xi')^{-1}\sum_{k\geq 1} k\left(\frac{\xi}{\xi'}\right)^k (\xi')^{(k+1)}\,\|\gamma_0^{(k+1)}\|_{H^\alpha_{k+1}} \\ &\leq C\,(\xi')^{-1}\sup_{k\geq 1} k\left(\frac{\xi}{\xi'}\right)^k \sum_{k\geq 1}(\xi')^{(k+1)}\,\|\gamma_0^{(k+1)}\|_{H^\alpha_{k+1}} \\ &\leq C(\xi,\xi')\,\|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi'}}, \end{aligned} \tag{A.5}$$

where we used (A.4) to obtain (A.5).

On the other hand, we have

$$\begin{aligned}\|BU(t)\Gamma_0^K\|_{L^2_{t\in\mathbb{R}}\mathcal{H}_\xi^\alpha} &\leq \sum_{k=1}^{K-1}\xi^k\|B_{k+1}U^{(k+1)}(t)\gamma_0^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H_k^\alpha} \\ &\leq C'\sum_{k=1}^{K-1} k\,\xi^k\,\|\gamma_0^{(k+1)}\|_{H^\alpha_{k+1}} \end{aligned}$$

$$= C'\,K\,(\xi')^{-1}\sum_{k=1}^{K-1}\xi^{(k+1)}\,\|\gamma_0^{(k+1)}\|_{H^\alpha_{k+1}}$$
$$\leq C(K)\,\xi^{-1}\,\|\Gamma_0^K\|_{\mathcal{H}^\alpha_\xi}. \tag{A.6}$$

This proves the Lemma. □

A.2. Strichartz Estimates for the BBGKY Hierarchy

In this subsection, we prove a new Strichartz estimate for the free evolution $U(t) = \mathrm{e}^{it\Delta^\pm}$ in $L^2_{t\in I}\mathcal{H}^\alpha_\xi$, for the BBGKY hierarchy, at the level of finite N. This result parallels the one for the GP hierarchy, which was stated in Lemma A.1.

Proposition A.2. *Let $\alpha \in \mathfrak{A}(d)$ for $d \geq 2$, and*

$$\beta < \frac{1}{d+2\alpha-1}. \tag{A.7}$$

Assume that $V \in L^1(\mathbb{R}^d)$, and that $\widehat{V}$ decays rapidly outside the unit ball. Letting $c_0 := 1-\beta(d+2\alpha-1)$, the following hold:

1. Bound for K-truncated case: *Assume that $\Gamma_0 \in \mathcal{H}^\alpha_\xi$ for some $0<\xi<1$. Then, for any $K \in \mathbb{N}$, there exists a constant $C(K)$ such that the Strichartz estimate for the free evolution*
$$\|B_N^{\mathrm{main}}U(t)P_{\leq K}\Gamma_0\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^\alpha_\xi} \leq \xi^{-1}\,C\,K\,\|\Gamma_0\|_{\mathcal{H}^\alpha_\xi} \tag{A.8}$$
and
$$\|B_N^{\mathrm{error}}U(t)P_{\leq K}\Gamma_0\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^\alpha_\xi} \leq \xi^{-1}\,C\,K^2\,N^{-c_0}\|\Gamma_0\|_{\mathcal{H}^\alpha_\xi}. \tag{A.9}$$
Notably, the value of ξ is the same on both the lhs and rhs.
2. Bound for $K\to\infty$: *Assume that $\Gamma_N(0) \in \mathcal{H}^\alpha_{\xi'}$ for some $0<\xi'<1$. Then, for any $0<\xi<\xi'$, there exists a constant $C(\xi,\xi')$ such that we have the Strichartz estimates for the free evolution*
$$\|B_N^{\mathrm{main}}\widehat{U}(t)\Gamma_N(0)\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^\alpha_\xi} \leq C(\xi,\xi')\,\|\Gamma_N(0)\|_{\mathcal{H}^\alpha_{\xi'}} \tag{A.10}$$
and
$$\|B_N^{\mathrm{error}}\widehat{U}(t)\Gamma_N(0)\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^\alpha_\xi} \leq C(\xi,\xi')\,N^{-c_0}\,\|\Gamma_N(0)\|_{\mathcal{H}^\alpha_{\xi'}}. \tag{A.11}$$

Proof. We recall that B_N contains a main, and an error term. We will see that the error term is small only if the condition (A.7) on the values of β holds. This is an artifact of the L^2-type norms used in this paper; squaring the potential V_N in the error term makes it more singular to a degree that it can only be controlled for sufficiently small β.

(1) The main term. We first consider the main term in $B^\pm_{N;k;k+1}\gamma_N^{(k+1)}$. We have

$$\|B^{+,main}_{N;j;k+1}U^{(k+1)}(t)\gamma_N^{(k+1)}(0)\|^2_{L^2_{t\in\mathbb{R}}H^\alpha} = \|B^{+,main}_{N;k;k+1}U^{(k+1)}(t)\gamma_N^{(k+1)}(0)\|^2_{L^2_{t\in\mathbb{R}}H^\alpha}$$
$$= \int_{\mathbb{R}}\mathrm{d}t\int\mathrm{d}\underline{x}_k\mathrm{d}\underline{x}_k'\,\Big|\,S^{(k,\alpha)}\int\mathrm{d}\underline{u}_{k+1}\mathrm{d}\underline{u}_{k+1}'\int\mathrm{d}x_{k+1}\int\mathrm{d}q\,\widehat{V_N}(q)\,\mathrm{e}^{iq(x_k-x_{k+1})}$$

$$
\begin{aligned}
&\mathrm{e}^{i\sum_{j=1}^{k}(x_j u_j - x_j' u_j')}\ \mathrm{e}^{ix_{k+1}(u_{k+1}-u_{k+1}')} \\
&\mathrm{e}^{it\sum_{j=1}^{k+1}(u_j^2-(u_j')^2)}\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k+1};\underline{u}_{k+1}')\Big|^2 \qquad \text{(A.12)}\\
&= \int_{\mathbb{R}} \mathrm{d}t \int \mathrm{d}\underline{x}_k \mathrm{d}\underline{x}_k' \int \mathrm{d}\underline{u}_{k+1}\mathrm{d}\underline{u}_{k+1}'\mathrm{d}\underline{\widetilde{u}}_{k+1}\mathrm{d}\underline{\widetilde{u}}_{k+1}' \int \mathrm{d}x_{k+1}\,\mathrm{d}\widetilde{x}_{k+1} \int \mathrm{d}q\,\mathrm{d}\widetilde{q} \\
&\Big[\prod_{j=1}^{k-1}\langle u_j\rangle^{\alpha}\langle u_j'\rangle^{\alpha}\langle \widetilde{u}_j\rangle^{\alpha}\langle \widetilde{u}_j'\rangle^{\alpha}\Big]\,\langle u_k+q\rangle^{\alpha}\,\langle \widetilde{u}_k+\widetilde{q}\rangle^{\alpha}\,\langle u_k'\rangle^{\alpha}\,\langle \widetilde{u}_k'\rangle^{\alpha} \\
&\widehat{V_N}(q)\,\overline{\widehat{V_N}(\widetilde{q})}\ \mathrm{e}^{iq(x_k-x_{k+1})\,-\,i\widetilde{q}(x_k-\widetilde{x}_{k+1})} \\
&\mathrm{e}^{i\sum_{j=1}^{k}(x_j u_j - x_j' u_j' - x_j\widetilde{u}_j + x_j'\widetilde{u}_j')}\ \mathrm{e}^{ix_{k+1}(u_{k+1}-u_{k+1}')-i\widetilde{x}_{k+1}(\widetilde{u}_{k+1}+\widetilde{u}_{k+1}')} \\
&\mathrm{e}^{it\sum_{j=1}^{k+1}(u_j^2-(u_j')^2-\widetilde{u}_j^2+(\widetilde{u}_j')^2)} \\
&\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k+1};\underline{u}_{k+1}')\,\overline{\widehat{\gamma_N}^{(k+1)}(0;\underline{\widetilde{u}}_{k+1};\underline{\widetilde{u}}_{k+1}')} \qquad \text{(A.13)}\\
&= \int \mathrm{d}\underline{u}_{k+1}\mathrm{d}\underline{u}_{k+1}'\,\mathrm{d}\widetilde{u}_k\,\mathrm{d}\widetilde{u}_{k+1}\,\mathrm{d}\widetilde{u}_{k+1}'\Big[\prod_{j=1}^{k-1}\langle u_j\rangle^{2\alpha}\langle u_j'\rangle^{2\alpha}\Big]\,\langle u_k'\rangle^{2\alpha} \\
&\int \mathrm{d}q\mathrm{d}\widetilde{q}\ \widehat{V_N}(q)\,\overline{\widehat{V_N}(\widetilde{q})}\,\langle u_k+q\rangle^{\alpha}\langle \widetilde{u}_k+\widetilde{q}\rangle^{\alpha} \\
&\delta(q-\widetilde{q}+u_k-\widetilde{u}_k)\,\delta(-q+u_{k+1}-u_{k+1}')\delta(-\widetilde{q}+\widetilde{u}_{k+1}-\widetilde{u}_{k+1}') \\
&\delta(u_k^2-\widetilde{u}_k^2+u_{k+1}^2-\widetilde{u}_{k+1}^2-(u_{k+1}')^2+(\widetilde{u}_{k+1}')^2) \\
&\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k+1};\underline{u}_{k+1}')\,\overline{\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k-1},\widetilde{u}_k,\widetilde{u}_{k+1};\underline{u}_k',\widetilde{u}_{k+1}')} \qquad \text{(A.14)}
\end{aligned}
$$

To pass from (A.13) to (A.14), we have first integrated out the variables $\underline{x}_{k-1}, \underline{\widetilde{x}}_k$, thus obtaining delta distributions $\prod_{j=1}^{k-1}\delta(u_j-\widetilde{u}_j)\prod_{\ell=1}^{k}\delta(u_\ell'-\widetilde{u}_\ell')$ enforcing momentum constraints, which we subsequently eliminate by integrating over the variables $\widetilde{u}_j$, $\widetilde{u}_\ell'$, for $j=1,\ldots,k-1$, $\ell=1,\ldots,k$. The first delta distribution in (A.14) stems from integration in x_k, the second and third from integrating in x_{k+1} and $\widetilde{x}_{k+1}$, and the fourth from integrating in t (where terms of the form $u_j^2-\widetilde{u}_j^2$ and $(u_\ell')^2-(\widetilde{u}_\ell')^2$ for $j=1,\ldots,k-1$, $\ell=1,\ldots,k$ have canceled, due to the momentum constraints). We note that the expression (A.14) differs from the corresponding ones in [6,7,25] where the Fourier transform in t was first taken before squaring (in particular, the delta implementing energy conservation in (A.14) is simpler). Then we have:

$$
\begin{aligned}
&= \int \mathrm{d}\underline{u}_{k+1}\mathrm{d}\underline{u}_k'\,\mathrm{d}\widetilde{u}_k\,\mathrm{d}\widetilde{u}_{k+1}\ \left[\prod_{j=1}^{k-1}\langle u_j\rangle^{2\alpha}\langle u_j'\rangle^{2\alpha}\right]\ \langle u_k'\rangle^{2\alpha} \qquad \text{(A.15)}\\
&\quad\int \mathrm{d}q\mathrm{d}\widetilde{q}\ \widehat{V_N}(q)\,\overline{\widehat{V_N}(\widetilde{q})}\,\langle u_k+q\rangle^{2\alpha}\,\delta(q-\widetilde{q}+u_k-\widetilde{u}_k) \\
&\quad\delta(u_k^2-\widetilde{u}_k^2+u_{k+1}^2-(u_{k+1}-q)^2-\widetilde{u}_{k+1}^2+(\widetilde{u}_{k+1}-\widetilde{q})^2) \\
&\quad\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k+1};\underline{u}_k',u_{k+1}-q)\,\overline{\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k-1},\widetilde{u}_k,\widetilde{u}_{k+1};\underline{u}_k',\widetilde{u}_{k+1}-\widetilde{q})}
\end{aligned}
$$

$$
\begin{aligned}
&= \int \mathrm{d}\underline{u}_{k+1}\mathrm{d}\underline{u}'_k\, \mathrm{d}\widetilde{u}_k\, \mathrm{d}\widetilde{u}_{k+1} \left[\prod_{j=1}^{k-1}\langle u_j\rangle^{2\alpha}\langle u'_j\rangle^{2\alpha}\right]\, \langle u'_k\rangle^{2\alpha} \qquad (A.16)\\
&\quad \int \mathrm{d}q\ \widehat{V_N}(q)\,\overline{\widehat{V_N}(q+u_k-\widetilde{u}_k)}\,\langle u_k+q\rangle^{2\alpha}\\
&\quad \delta(u_k^2-\widetilde{u}_k^2+u_{k+1}^2-(u_{k+1}-q)^2-\widetilde{u}_{k+1}^2+(\widetilde{u}_{k+1}-(q+u_k-\widetilde{u}_k))^2)\\
&\quad \frac{\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k+1};\underline{u}'_k,u_{k+1}-q)}{\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k-1},\widetilde{u}_k,\widetilde{u}_{k+1};\underline{u}'_k,\widetilde{u}_{k+1}-(q+u_k-\widetilde{u}_k))}\\
&= \int \mathrm{d}\underline{u}_{k+1}\mathrm{d}\underline{u}'_k\, \mathrm{d}\widetilde{u}_k\, \mathrm{d}\widetilde{u}_{k+1} \left[\prod_{j=1}^{k-1}\langle u_j\rangle^{2\alpha}\langle u'_j\rangle^{2\alpha}\right]\, \langle u'_k\rangle^{2\alpha}\\
&\quad \int \mathrm{d}q\ \widehat{V_N}(q+\widetilde{u}_k)\,\overline{\widehat{V_N}(q+u_k)}\,\langle u_k+\widetilde{u}_k+q\rangle^{2\alpha}\\
&\quad \delta(u_k^2-\widetilde{u}_k^2+u_{k+1}^2-(u_{k+1}-q-\widetilde{u}_k)^2-\widetilde{u}_{k+1}^2+(\widetilde{u}_{k+1}-q-u_k))^2)\\
&\quad \frac{\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k+1};\underline{u}'_k,u_{k+1}-q-\widetilde{u}_k)}{\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k-1},\widetilde{u}_k,\widetilde{u}_{k+1};\underline{u}'_k,\widetilde{u}_{k+1}-q-u_k)} \qquad (A.17)
\end{aligned}
$$

where to obtain (A.15) we integrated out the variables $u'_{k+1},\widetilde{u}'_{k+1}$, to obtain (A.16) we integrated out the variable $\widetilde{q}$ and to obtain (A.17) we performed the shift $q\to q+\widetilde{u}_k$. The last expression is manifestly real and non-negative. One immediately finds the upper bound

$$
\begin{aligned}
&\le \|\widehat{V_N}\|_{L^\infty}^2 \int \mathrm{d}\underline{u}_{k+1}\mathrm{d}\underline{u}'_k\, \mathrm{d}\widetilde{u}_k\, \mathrm{d}\widetilde{u}_{k+1}\, \mathrm{d}q\, \langle u_k+\widetilde{u}_k+q\rangle^{2\alpha} \left[\prod_{j=1}^{k-1}\langle u_j\rangle^{2\alpha}\langle u'_j\rangle^{2\alpha}\right]\\
&\quad \langle u'_k\rangle^{2\alpha}\delta(u_k^2-\widetilde{u}_k^2+u_{k+1}^2-(u_{k+1}-q-\widetilde{u}_k)^2-\widetilde{u}_{k+1}^2+(\widetilde{u}_{k+1}-q-u_k))^2)\\
&\quad \frac{\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k+1};\underline{u}'_k,u_{k+1}-q-\widetilde{u}_k)}{\widehat{\gamma_N}^{(k+1)}(0;\underline{u}_{k-1},\widetilde{u}_k,\widetilde{u}_{k+1};\underline{u}'_k,\widetilde{u}_{k+1}-q-u_k)}\\
&= \|\widehat{V_N}\|_{L^\infty}^2\, \|\, B^+_{k;k+1}U^{(k+1)}(t)\gamma_0^{(k+1)}\,\|^2_{L^2_{t\in\mathbb{R}}H^\alpha_k}\\
&\le C\,\|\,\gamma_0^{(k+1)}\,\|^2_{H^\alpha_{k+1}}. \qquad (A.18)
\end{aligned}
$$

Here, we have used $\|\widehat{V_N}\|_{L^\infty}\le\|V_N\|_{L^1_x}=\|V_1\|_{L^1_x}$ uniformly in N, and the Strichartz estimate for the free evolution in the (infinite) GP hierarchy.

Therefore, we conclude that

$$
\begin{aligned}
&\|B^{\pm,\mathrm{main}}_{N;k+1}U^{(k+1)}(t)\gamma_N^{(k+1)}(0)\|_{L^2_{t\in\mathbb{R}}H^\alpha_k}\\
&\quad\le \frac{k(N-k)}{N}\sup_j\|B^{\pm,\mathrm{main}}_{N;j;k+1}U^{(k+1)}(t)\gamma_N^{(k+1)}(0)\|_{L^2_{t\in\mathbb{R}}H^\alpha}\\
&\quad\le C\,(k-\frac{k^2}{N})\,\|\gamma_N^{(k+1)}(0)\|_{H^\alpha}. \qquad (A.19)
\end{aligned}
$$

Hence we have that

$$\sum_{k\geq 1} \xi^k \, \| B^{\pm,\mathrm{main}}_{N;k+1} U^{(k+1)}(t) \gamma_N^{(k+1)}(0) \|_{L^2_{t\in\mathbb{R}} H^\alpha_k}$$

$$\leq C \sum_{k\geq 1} (k - \frac{k^2}{N}) \, \xi^k \, \| \gamma_N^{(k+1)}(0) \|_{H^\alpha_{k+1}} \tag{A.20}$$

$$= C \, (\xi')^{-1} \sum_{k\geq 1} (k - \frac{k^2}{N}) \left(\frac{\xi}{\xi'} \right)^k (\xi')^{(k+1)} \, \| \gamma_N^{(k+1)}(0) \|_{H^\alpha_{k+1}}$$

$$\leq C \, (\xi')^{-1} \sup_{k\geq 1} \left((k - \frac{k^2}{N}) \left(\frac{\xi}{\xi'} \right)^k \right) \sum_{k\geq 1} (\xi')^{(k+1)} \, \| \gamma_N^{(k+1)}(0) \|_{H^\alpha_{k+1}}$$

$$\leq C(\xi, \xi') \, (1 + \frac{1}{N}) \, \| \Gamma_N(0) \|_{\mathcal{H}^\alpha_{\xi'}}, \tag{A.21}$$

where to obtain (A.20) we used (A.19).

(2) The error term. Next, we consider the error terms $B^{\pm,\mathrm{error}}_{N;k} \gamma_N^{(k)}$. By symmetry, we have

$$\| B^{+,\mathrm{error}}_{N;i,j;k} U^{(k)}(t) \gamma_N^{(k)}(0) \|^2_{L^2_{t\in\mathbb{R}} H^\alpha}$$

$$= \| B^{+,\mathrm{error}}_{N;1,2;k} U^{(k)}(t) \gamma_N^{(k)}(0) \|^2_{L^2_{t\in\mathbb{R}} H^\alpha}$$

$$= \int_{\mathbb{R}} \mathrm{d}t \int \mathrm{d}\underline{x}_k \mathrm{d}\underline{x}'_k \, \Big| \, S^{(k,\alpha)} \int \mathrm{d}\underline{u}_k \mathrm{d}\underline{u}'_k \int \mathrm{d}q \, \widehat{V_N}(q) \, \mathrm{e}^{iq(x_1 - x_2)}$$

$$\mathrm{e}^{i \sum_{j=1}^k (x_j u_j - x'_j u'_j)} \mathrm{e}^{it \sum_{j=1}^k (u_j^2 - (u'_j)^2)} \widehat{\gamma_N}^{(k)}(0; \underline{u}_k; \underline{u}'_k) \Big|^2$$

$$= \int \mathrm{d}t \int \mathrm{d}\underline{x}_k \mathrm{d}\underline{x}'_k \, \Big| \int \mathrm{d}\underline{u}_k \mathrm{d}\underline{u}'_k \int \mathrm{d}q \, \widehat{V_N}(q) \, \mathrm{e}^{iq(x_1 - x_2)}$$

$$\langle u_1 + q \rangle^\alpha \langle u_2 - q \rangle^\alpha \langle u'_1 \rangle^\alpha \langle u'_2 \rangle^\alpha \prod_{j=3}^k \langle u_j \rangle^\alpha \langle u'_j \rangle^\alpha$$

$$\mathrm{e}^{i \sum_{j=1}^k (x_j u_j - x'_j u'_j)} \mathrm{e}^{it \sum_{j=1}^k (u_j^2 - (u'_j)^2)} \widehat{\gamma_N}^{(k)}(0; \underline{u}_k; \underline{u}'_k) \Big|^2$$

$$= \int \mathrm{d}\underline{u}_k \mathrm{d}\underline{u}'_k \mathrm{d}\widetilde{u}_1 \mathrm{d}\widetilde{u}_2 \langle u'_1 \rangle^{2\alpha} \langle u'_2 \rangle^{2\alpha} \prod_{j=3}^k \langle u_j \rangle^{2\alpha} \langle u'_j \rangle^{2\alpha}$$

$$\int \mathrm{d}q \, \mathrm{d}\widetilde{q} \, \widehat{V_N}(q) \, \overline{\widehat{V_N}(\widetilde{q})} \, \langle u_1 + q \rangle^\alpha \, \langle u_2 - q \rangle^\alpha \, \langle \widetilde{u}_1 + \widetilde{q} \rangle^\alpha \, \langle \widetilde{u}_2 - \widetilde{q} \rangle^\alpha$$

$$\delta(q - \widetilde{q} + u_1 - \widetilde{u}_1) \, \delta(q - \widetilde{q} - u_2 + \widetilde{u}_2) \, \delta(\, u_1^2 + u_2^2 - \widetilde{u}_1^2 - \widetilde{u}_2^2 \,)$$

$$\widehat{\gamma_N}^{(k)}(0; \underline{u}_k; \underline{u}'_k) \, \overline{\widehat{\gamma_N}^{(k)}(0; \widetilde{u}_1, \widetilde{u}_2, u_3, \ldots, u_k; \underline{u}'_k)}$$

$$= \int \mathrm{d}\underline{u}_k \mathrm{d}\underline{u}'_k \, \langle u'_1 \rangle^{2\alpha} \langle u'_2 \rangle^{2\alpha} \prod_{j=3}^k \langle u_j \rangle^{2\alpha} \langle u'_j \rangle^{2\alpha} \tag{A.22}$$

$$\int \mathrm{d}q\,\mathrm{d}\widetilde{q}\,\widehat{V_N}(q)\,\overline{\widehat{V_N}(\widetilde{q})}\,\langle u_1+q\rangle^{2\alpha}\,\langle u_2-q\rangle^{2\alpha}$$
$$\delta(\,u_1^2+u_2^2-(u_1+q-\widetilde{q})^2-(u_2-q+\widetilde{q})^2\,)$$
$$\widehat{\gamma_N}^{(k)}(0;\underline{u}_k;\underline{u}_k')\,\overline{\widehat{\gamma_N}^{(k)}(0;u_1+q-\widetilde{q},u_2-q+\widetilde{q},u_3,\dots,u_k;\underline{u}_k')}$$
$$=\int \mathrm{d}\underline{u}_k\mathrm{d}\underline{u}_k'\left[\prod_{j=1}^k\langle u_j\rangle^{2\alpha}\langle u_j'\rangle^{2\alpha}\right]\int \mathrm{d}q\,\mathrm{d}\widetilde{q}\,\widehat{V_N}(q)\,\overline{\widehat{V_N}(\widetilde{q})} \tag{A.23}$$
$$\delta(\,(u_1-q)^2+(u_2+q)^2-(u_1-\widetilde{q})^2-(u_2+\widetilde{q})^2\,)$$
$$\widehat{\gamma_N}^{(k)}(0;u_1-q,u_2+q,u_3,\dots,u_k;\underline{u}_k')$$
$$\overline{\widehat{\gamma_N}^{(k)}(0;u_1-\widetilde{q},u_2+\widetilde{q},u_3,\dots,u_k;\underline{u}_k')}$$
$$=\int \mathrm{d}\underline{u}_k\mathrm{d}\underline{u}_k'\int \mathrm{d}q\,\mathrm{d}\widetilde{q}\,\widehat{V_N}(q)\,\overline{\widehat{V_N}(\widetilde{q})}\,\delta(\,2(-u_1+u_2+q+\widetilde{q})\cdot(q-\widetilde{q})\,)$$
$$\left[\prod_{j=1}^k\langle u_j\rangle^{2\alpha}\langle u_j'\rangle^{2\alpha}\right]\,\widehat{\gamma_N}^{(k)}(0;u_1-q,u_2+q,u_3,\dots,u_k;\underline{u}_k')$$
$$\overline{\widehat{\gamma_N}^{(k)}(0;u_1-\widetilde{q},u_2+\widetilde{q},u_3,\dots,u_k;\underline{u}_k')},$$

where to obtain (A.22) we integrated out the variables $\widetilde{u}_1$, $\widetilde{u}_2$ and to obtain (A.23) we performed the shifts $u_1\to u_1-q$ and $u_2\to u_2+q$. Clearly, the last expression is bounded by

$$\|B^{+,\mathrm{error}}_{N;i,j;k}U^{(k)}(t)\gamma_N^{(k)}(0)\|^2_{L^2_tH^\alpha_k}\le C_V(N)\,\|\gamma_N^{(k)}(0)\|^2_{H^\alpha_k} \tag{A.24}$$

where

$$C_V(N):=\sup_{u_1,u_2}\int \mathrm{d}q\,\mathrm{d}\widetilde{q}\,\delta(\,2(-u_1+u_2+q+\widetilde{q})\cdot(q-\widetilde{q})\,)$$
$$\frac{\widehat{V_N}(q)\,\overline{\widehat{V_N}(\widetilde{q})}\,\langle u_1\rangle^{2\alpha}\langle u_2\rangle^{2\alpha}}{\langle u_1-q\rangle^\alpha\langle u_2+q\rangle^\alpha\langle u_1-\widetilde{q}\rangle^\alpha\langle u_2+\widetilde{q}\rangle^\alpha}, \tag{A.25}$$

and $\|\widehat{V_N}\|_{L^\infty}\le\|V_N\|_{L^1}\le C$, uniformly in N. We may assume that $\mathrm{supp}\{\widehat{V_N}\}\subset B_{CN^\beta}(0)$, for some constant C. The modifications for $\widehat{V_N}$ non-vanishing, but decaying rapidly outside $B_{CN^\beta}(0)$ are straightforward.

Then,

$$C_V(N)\le\sup_{u_1,u_2\in\mathbb{R}^d;q,\widetilde{q}\in B_{CN^\beta}(0)}\left[\frac{\langle u_1\rangle^{2\alpha}\langle u_2\rangle^{2\alpha}}{\langle u_1-q\rangle^\alpha\langle u_2+q\rangle^\alpha\langle u_1-\widetilde{q}\rangle^\alpha\langle u_2+\widetilde{q}\rangle^\alpha}\right]$$
$$\sup_{u_1,u_2}\int_{B_{CN^\beta}(0)\times B_{CN^\beta}(0)}\mathrm{d}q\,\mathrm{d}\widetilde{q}\,\delta(\,2(-u_1+u_2+q+\widetilde{q})\cdot(q-\widetilde{q})\,)$$
$$\le C(N^\beta)^{4\alpha}\sup_u\int_{B_{CN^\beta}(0)\times B_{CN^\beta}(0)}dv_+\,dv_-\,\delta(\,2(u+v_+)\cdot v_-\,) \tag{A.26}$$

$$\leq C(N^\beta)^{4\alpha} \sup_u \int_{B_{CN^\beta}(0)\times\mathcal{D}_{N^\beta}} dv_+ \, dv_-^\perp \, \frac{1}{|u+v_+|} \tag{A.27}$$

$$= C(N^\beta)^{4\alpha+d-1} \int_{B_{CN^\beta}(0)} \frac{dv_+}{|v_+|} \tag{A.28}$$

$$\leq C\,(N^\beta)^{4\alpha+2d-2}. \tag{A.29}$$

To pass to (A.27), we used

$$\sup_{u_1,u_2\in\mathbb{R}^d; q,\widetilde{q}\in B_{CN^\beta}(0)} \left[\frac{\langle u_1\rangle^{2\alpha}\langle u_2\rangle^{2\alpha}}{\langle u_1-q\rangle^\alpha\langle u_2+q\rangle^\alpha\langle u_1-\widetilde{q}\rangle^\alpha\langle u_2+\widetilde{q}\rangle^\alpha}\right] < C\,(N^\beta)^{4\alpha} \tag{A.30}$$

where we note that the maximum is attained for configurations similar to $u_1 = q = \widetilde{q} = -u_2, |q| = O(N^\beta)$.

Moreover, we introduced $v_\pm := q \pm \widetilde{q}$ as new variables. Passing to (A.27), we integrated out the delta distribution with the component of v_- parallel to $u+v_+$, for fixed v_+ and $u := u_1 - u_2$. Accordingly, we denoted by $v_-^\perp$ the $(d-1)$-dimensional variable in the hyperplane

$$\mathcal{P} := \{ v \in \mathbb{R}^d \,|\, v \perp (u+v_+) \}$$

perpendicular to $u+v_+$, for u, v_+ fixed.

The integral in v_- is supported on the set $\mathcal{D}_{N^\beta}$, given by the intersection of a ball of radius $O(N^\beta)$ with the hyperplane $\mathcal{P}$. The measure of $\mathcal{D}_{N^\beta}$ is at most $O((N^\beta)^{d-1})$. This is accounted for in passing to (A.28).

The integral in v_+ in (A.27) over a ball or radius $O(N^\beta)$ in $\mathbb{R}^d$ yields another factor $O((N^\beta)^{d-1})$ in dimensions $d \geq 2$. To make this evident, we have shifted $v_+ \to v_+ + u$ in (A.28).

Similarly, we can bound the term $\|B_{N;k}^{-,error} U^{(k)}(t)\gamma_N^{(k)}(0)\|_{L^2_{t\in\mathbb{R}}H^\alpha_k}$.

Thus, we conclude that

$$\begin{aligned} &\|B_{N;k}^{\pm,\text{error}} U^{(k)}(t)\gamma_N^{(k)}(0)\|_{L^2_{t\in\mathbb{R}}H^\alpha_k} \\ &\quad \leq \frac{k(k-1)}{N} \sup_j \|B_{N;i,j;k}^{\pm,\text{error}} U^{(k)}(t)\gamma_N^{(k)}(0)\|_{L^2_{t\in\mathbb{R}}H^\alpha} \\ &\quad \leq C\,k(k-1)\,N^{\beta(d+2\alpha-1)-1}\,\|\gamma_N^{(k)}(0)\|_{H^\alpha}. \end{aligned} \tag{A.31}$$

We may now complete the proof.

- *Bound for $K\to\infty$:* We have

$$\begin{aligned} &\sum_{k\geq1} \xi^k \,\|B_{N;k}^{\pm,\text{error}} U^{(k)}(t)\gamma_N^{(k)}(0)\|_{L^2_{t\in\mathbb{R}}H^\alpha_k} \\ &\quad \leq C\,N^{\beta(d+2\alpha-1)-1} \sum_{k\geq1} k(k-1)\,\xi^k\,\|\gamma_N^{(k)}(0)\|_{H^\alpha_{k+1}} \qquad \text{(A.32)} \\ &\quad = C\,N^{\beta(d+2\alpha-1)-1} \sum_{k\geq1} k(k-1)\left(\frac{\xi}{\xi'}\right)^k (\xi')^{(k+1)}\,\|\gamma_N^{(k)}(0)\|_{H^\alpha_{k+1}} \end{aligned}$$

$$\leq C\, N^{\beta(d+2\alpha-1)-1} \sup_{k\geq 1} \left(k(k-1) \left(\frac{\xi}{\xi'}\right)^k \right) \sum_{k\geq 1} (\xi')^{(k+1)} \, \|\gamma_N^{(k)}(0)\|_{H^\alpha_{k+1}}$$
$$\leq C(\xi,\xi')\, N^{\beta(d+2\alpha-1)-1}\, \|\Gamma_N(0)\|_{\mathcal{H}^\alpha_{\xi'}}, \tag{A.33}$$

where we used (A.31) to obtain (A.32).

Summarizing, we combine (A.21) and (A.33) to obtain:

$$\begin{aligned} &\|B_N \widehat{U}(t)\Gamma_N(0)\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^\alpha_\xi} \\ &\quad = \sum_{k\geq 1} \xi^k \, \|(B_N \widehat{U}(t)\Gamma_N(0))^{(k)}\|_{L^2_{t\in\mathbb{R}} H^\alpha_\xi} \\ &\quad \leq \sum_{k\geq 1} \xi^k \, \|B^{\pm,\mathrm{main}}_{N;k+1} U^{(k+1)}(t)\gamma_N^{(k+1)}(0)\|_{L^2_{t\in\mathbb{R}} H^\alpha_\xi} \\ &\qquad + \sum_{k\geq 1} \xi^k \, \|B^{\pm,\mathrm{error}}_{N;k} U^{(k)}(t)\gamma_N^{(k)}(0)\|_{L^2_{t\in\mathbb{R}} H^\alpha_\xi} \\ &\quad \leq \; C(\xi,\xi')\,(1+N^{-1}+N^{\beta(d+2\alpha-1)-1})\, \|\Gamma_N(0)\|_{\mathcal{H}^\alpha_\xi}. \end{aligned} \tag{A.34}$$

• *Bound for finite* K*:* Replacing the infinite sum over indices k in (A.33) by a finite sum with $1 \leq k \leq K$, it is easy to see that one gets

$$\begin{aligned} &\|B_N \widehat{U}(t) P_{\leq K}\Gamma_N(0)\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^\alpha_\xi} \\ &\quad = \sum_{k=1}^K \xi^k \, \|(B_N \widehat{U}(t)P_{\leq K}\Gamma_N(0))^{(k)}\|_{L^2_{t\in\mathbb{R}} H^\alpha_\xi} \\ &\quad \leq \sum_{k=1}^K \xi^k \, \|B^{\pm,\mathrm{main}}_{N;k+1} U^{(k+1)}(t)\gamma_N^{(k+1)}(0)\|_{L^2_{t\in\mathbb{R}} H^\alpha_\xi} \\ &\qquad + \sum_{k=1}^K \xi^k \, \|B^{\pm,\mathrm{error}}_{N;k} U^{(k)}(t)\gamma_N^{(k)}(0)\|_{L^2_{t\in\mathbb{R}} H^\alpha_\xi} \\ &\quad \leq C\, K\, \xi^{-1}\,(1+N^{-1}+KN^{\beta(d+2\alpha-1)-1})\, \|\Gamma_N(0)\|_{\mathcal{H}^\alpha_\xi} \end{aligned} \tag{A.35}$$

(by setting $\xi = \xi'$, and taking $\sup_{1\leq k\leq K}$ in the second last line of (A.33))).

This concludes the proof. □

Remark A.3. We note that the restriction on β is due to the error term in B_N. It stems from the fact that since we are using the L^2-type H^α-norms, the quantity V_N, which is essentially a Dirac function, is squared. We can only expect to get $\beta = 1$ if we use L^1-type trace norms similarly as Erdös et al. [16,17].

The main term in B_N, on the other hand, does allow for the entire range $0 < \beta \leq 1$. This is because in this term, averaging (integration over the variable x_{k+1}, which is part of the argument of $V_N(x_j - x_{k+1})$) is performed before squaring, in order to obtain the H^α-norm.

Appendix B. Iterated Duhamel Formula and Boardgame Argument

The goal of Appendix B is to prove the main Lemma B.3 below, following our earlier work [7,9], where we used analogous estimates to prove well-posedness results for the infinite GP hierarchy. The proof is based on the boardgame strategy introduced in [25] (which is a reformulation of a method introduced in [16,17]).

Definition B.1. *Let* $\widetilde{\Xi} = (\widetilde{\Xi^{(k)}})_{n\in\mathbb{N}}$ *denote a sequence of arbitrary Schwartz class functions* $\widetilde{\Xi^{(k)}} \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^{kd}\times\mathbb{R}^{kd})$*. Then, we define the associated sequence* $\mathrm{Duh}_j(\Xi)$ *of* jth level iterated Duhamel terms *based on* B_N^{main}*, with components given by*

$$\mathrm{Duh}_j(\widetilde{\Xi})^{(k)}(t) := (-i\mu)^j \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{j-1}} \mathrm{d}t_j B^{\mathrm{main}}_{N;k+1} \mathrm{e}^{i(t-t_1)\Delta_\pm^{(k+1)}} B^{\mathrm{main}}_{N;k+2} \mathrm{e}^{i(t_1-t_2)\Delta_\pm^{(k+2)}} B^{\mathrm{main}}_{N;k+2} \cdots\cdots \mathrm{e}^{i(t_{j-1}-t_j)\Delta_\pm^{(k+j)}} (\,\widetilde{\Xi}\,)^{(k+j)}(t_j). \quad \text{(B.1)}$$

for $\mu=\pm1$*, with the conventions* $t_0 := t$*, and*

$$\mathrm{Duh}_0(\widetilde{\Xi})^{(k)}(t) := (\,\widetilde{\Xi}\,)^{(k)}(t) \quad \text{(B.2)}$$

for $j=0$*.*

Here, the definition is given for Schwartz class functions, and can be extended to other spaces by density arguments. The fact that $\mathrm{Duh}_j(\widetilde{\Xi})^{(k)} \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^{kd}\times\mathbb{R}^{kd})$ holds in this situation, for all k, can be easily verified. Using the boardgame strategy of [25] (which is a reformulation of a combinatorial argument developed in [16,17]), one obtains:

Lemma B.2. *Let* $\alpha\in\mathfrak{A}(d)$*. Then, for* $\widetilde{\Xi} = (\widetilde{\Xi^{(k)}})_{k\in\mathbb{N}}$ *as above,*

$$\|\,\mathrm{Duh}_j(\widetilde{\Xi})^{(k)}(t)\,\|_{L^2_{t\in I}H^\alpha(\mathbb{R}^{kd}\times\mathbb{R}^{kd})} \leq k\,C_0^k\,(c_0T)^{\frac{j}{2}} \|\widetilde{\Xi^{(k+j)}}\|_{L^2_{t\in I}H^\alpha(\mathbb{R}^{(k+j)d}\times\mathbb{R}^{(k+j)d})}, \quad \text{(B.3)}$$

where the constants c_0, C_0 *depend only on* d, p*.*

In [7,25], Lemma B.2 is proven for the operator B instead of B_N^{main}, based on the use of Lemma A.1. For the case of B_N^{main}, we invoke Proposition A.2 instead; the argument then proceeds exactly in the same way. We remark, however, that we do not know if the boardgame argument can be adapted to the case of B_N^{error}.[2]

[2] We thank Xuwen Chen for pointing out to us an issue in this regard in an earlier version of this paper.

We then consider solutions Θ_N^K of the integral equation

$$\Theta_N^K(t) = \Xi_N^K(t) + i\int_0^t B_N\, U(t-s)\,\Theta_N^K(s)\mathrm{d}s$$
$$= \widetilde{\Xi_N^K(t)} + i\int_0^t B_N^{\text{main}}\, U(t-s)\,\Theta_N^K(s)\mathrm{d}s \tag{B.4}$$

where $(\Xi_N^K)^{(k)}(t) = 0$ and $(\Theta_N^K)^{(k)}(t) = 0$ for all $k > K$, and all $t \in I = [0, T]$. Moreover,

$$\widetilde{\Xi_N^K(t)} := \Xi^K(t) + i\int_0^t B_N^{\text{error}}\, U(t-s)\,\Theta_N^K(s)\mathrm{d}s. \tag{B.5}$$

By iteration of the Duhamel formula,

$$(\Theta_N^K)^{(k)}(t) = \sum_{j=0}^{\ell-1} \mathrm{Duh}_j(\widetilde{\Xi_N^K})^{(k)}(t) + \mathrm{Duh}_k(\Theta_N^K)^{(k)}(t), \tag{B.6}$$

obtained from iterating the Duhamel formula ℓ times for the kth component of Θ_N^K. Since $(\Theta_N^K)^{(m)}(t) = 0$ for all $m > K$, the remainder term on the rhs is zero whenever $n + \ell > K$. Thus,

$$(\Theta_N^K)^{(k)}(t) = \sum_{j=0}^{N-k} \mathrm{Duh}_j(\widetilde{\Xi_N^K})^{(k)}(t), \tag{B.7}$$

where each term on the right explicitly depends only on $\widetilde{\Xi_N^K}(t)$ (there is no implicit dependence on the solution $\Theta_N^K(t)$).

Lemma B.3. *Assume that N is sufficiently large, and that in particular, defining $\delta' > 0$ by*

$$\beta = \frac{1-\delta'}{d + 2\alpha - 1}, \tag{B.8}$$

the condition

$$K < \frac{\delta'}{\log C_0}\, \log N, \tag{B.9}$$

holds, where the constant C_0 is as in Lemma B.2.

Let Θ_N^K and Ξ_N^K satisfy (B.4).

Assume that $\Xi_N^K \in L^2_{t\in I}\mathcal{H}^\alpha_{\xi'}$ for some $0 < \xi' < 1$, and that ξ is small enough that $0 < \xi < \eta\xi'$, with η specified in (B.16). Then, the estimate

$$\|\Theta_N^K\|_{L^2_{t\in I}\mathcal{H}^\alpha_\xi} \le C_1(T, \xi, \xi')\, \|\Xi_N^K\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi'}} \tag{B.10}$$

holds for a finite constant $C_1(T, \xi, \xi') > 0$ independent of K, N.

Proof. We have

$$(\Theta_N^K)^{(k)}(t) = \sum_{j=0}^{N-k} \mathrm{Duh}_j(\widetilde{\Xi_N^K})^{(k)}(t), \tag{B.11}$$

using the fact that $(\Theta_N^K)^{(k+j)} = 0$ for $j > N-k$, see (B.1).

Using Lemma B.2, we therefore find that

$$\begin{aligned}\|(\Theta_N^K)^{(k)}\|_{L^2_{t\in I}H^\alpha} &\leq \sum_{j=0}^{N-k} \|(\mathrm{Duh}_j(\widetilde{\Xi_N^K})^{(k+1)}(t)\|_{L^2_{t\in I}H^\alpha} \\ &\leq \sum_{j=0}^{N-k} k\, C_0^k\, (c_0 T)^{\frac{j}{2}}\, \|(\widetilde{\Xi_N^K})^{(k+j)}\|_{L^2_{t\in I}H^\alpha} \\ &\leq (I)_k \,+\, (II)_k \end{aligned} \tag{B.12}$$

where

$$\begin{aligned}(I)_k &:= \xi^{-k} k\, C_0^k\, (\xi/\xi')^k \sum_{j=0}^{N-k} (c_0 T(\xi')^{-2})^{\frac{j}{2}} (\xi')^{k+j} \|(\Xi_N^K)^{(k+j)}\|_{L^2_{t\in I}H^\alpha} \\ (II)_k &:= \xi^{-k} k C_0^k \sum_{j=0}^{N-k} (c_0 T \xi^{-2})^{\frac{j}{2}} \xi^{k+j} \\ &\quad \times \Big\| \int_0^t (B_N^{\mathrm{error}} U(t-s) \Theta_N^K)^{(k+j)} \mathrm{d}s \Big\|_{L^2_{t\in I}H^\alpha}, \end{aligned} \tag{B.13}$$

recalling (B.5).

We have

$$(I)_k \leq (\xi)^{-k} k\, C_0^k\, (\xi/\xi')^k\, C(T,\xi')\, \|\Xi_N^K\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi'}}, \tag{B.14}$$

for $T > 0$ sufficiently small so that $c_0 T(\xi')^{-2} \leq 1$. Hence,

$$\begin{aligned}\sum_{k\in\mathbb{N}} \xi^k (I)_k &\leq C(T,\xi') \Big(\sum_{k\in\mathbb{N}} k\, C_0^k\, (\xi/\xi')^k \Big) \|\Xi_N^K\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi'}} \\ &\leq C'(T,\xi,\xi')\, \|\Xi_N^K\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi'}}, \end{aligned} \tag{B.15}$$

for $\xi < \eta\xi'$ where

$$\eta < (\max\{1, C_0\})^{-1} \tag{B.16}$$

noting that $C_0 = C_0(d,p)$.

To bound $(II)_k$, we note that

$$\begin{aligned}&\Big\| \int_0^t (B_N^{\mathrm{error}}\, U(t-s)\, \Theta_N^K(s))^{(k+j)} \mathrm{d}s \Big\|_{L^2_{t\in I}H^\alpha} \\ &\quad \leq \Big\| \int_0^t \|(B_N^{\mathrm{error}}\, U(t-s)\, \Theta_N^K)^{(k+j)}(s)\|_{H^\alpha} \mathrm{d}s \Big\|_{L^2_{t\in I}}\end{aligned}$$

$$\leq \int_0^T \|(B_N^{\text{error}}\, U(t-s)\,\Theta_N^K)^{(k+j)}(s)\|_{L^2_{t\in I}H^\alpha}\mathrm{d}s$$

$$\leq C(k+j)^2 N^{\beta(d+2\alpha-1)-1}\int_0^T \|(\Theta_N^K)^{(k+j)}\|_{H^\alpha}\mathrm{d}s$$

$$\leq T^{\frac{1}{2}}\, C(k+j)^2\, N^{\beta(d+2\alpha-1)-1}\,\|(\Theta_N^K)^{(k+j)}\|_{L^2_{t\in I}H^\alpha} \tag{B.17}$$

using (A.31) to pass from the third to the fourth line. This implies that for $T>0$ small enough that $c_0T\xi^{-2}\leq 1$,

$$\begin{aligned}\sum_{k=1}^K \xi^k (II)_k &\leq T^{\frac{1}{2}}\, KCC_0^K\, N^{\beta(d+2\alpha-1)-1}\\ &\quad\times\sum_{j=0}^{N-k}(c_0T\xi^{-2})^{\frac{j}{2}}\xi^{k+j}(k+j)^2\|(\Theta_N^K)^{(k+j)}\|_{L^2_{t\in I}H^\alpha}\\ &\leq T^{\frac{1}{2}}\, K^3CC_0^K\, N^{\beta(d+2\alpha-1)-1}\sum_{j=1}^{K-k}\xi^{k+j}\|(\Theta_N^K)^{(k+j)}\|_{L^2_{t\in I}H^\alpha}\\ &\leq T^{\frac{1}{2}}\, K^3CC_0^K\, N^{\beta(d+2\alpha-1)-1}\|\Theta_N^K\|_{L^2_{t\in I}\mathcal{H}^\alpha_\xi}.\end{aligned} \tag{B.18}$$

Here, we used that $(\Theta_N^K)^{(k+j)}=0$ for $k+j>K$ to pass to the third line.

Summarizing,

$$\begin{aligned}\|\Theta_N^K\|_{L^2_{t\in I}\mathcal{H}^\alpha_\xi} &= \sum_{k\in\mathbb{N}}\xi^k\|(\Theta_N^K)^{(k)}\|_{L^2_{t\in I}H^\alpha}\\ &\leq \sum_{k\in\mathbb{N}}\xi^k\Big(\,(I)_k\;+\;(II)_k\Big)\\ &\leq C_{N,K}(T,\xi,\xi')\,\|\Xi_N^K\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi'}},\end{aligned} \tag{B.19}$$

using (B.15) and (B.18), where

$$C_{N,K}(T,\xi,\xi') := \frac{C'(T,\xi,\xi')}{1-T^{\frac{1}{2}}\,K^3CC_0^K\,N^{\beta(d+2\alpha-1)-1}}. \tag{B.20}$$

Letting

$$\begin{aligned}\beta &= \frac{1-\delta'}{d+2\alpha-1}\qquad \delta'\,>\,0,\\ K &< \frac{\delta'}{\log C_0}\,\log N,\end{aligned} \tag{B.21}$$

we find that, writing $K=\frac{\delta'-\epsilon}{\log C_0}\log N$ for $\epsilon>0$,

$$T^{\frac{1}{2}}\,K^3CC_0^K\,N^{\beta(d+2\alpha-1)-1} < C''(T,\xi)\,(\log N)^3\,N^{-\epsilon}\,<\,\frac{1}{2} \tag{B.22}$$

for sufficiently large N, where the constant $C''(T,\xi)$ is independent of N.

We conclude that given (B.21), and all N sufficiently large, we have that

$$C_{N,K}(T,\xi,\xi') < 2C'(T,\xi,\xi'). \tag{B.23}$$

This proves the claim. □

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