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DERIVATION OF THE HYDRODYNAMICAL EQUATION FOR THE ZERO-RANGE INTERACTION PROCESS

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We first recall the concept of hydrodynamical equation of an infinite particle system. We then prove that the equation associated to the zero-range process is a first order nonlinear partial differential equation.

1. Introduction. In this paper we are interested in the hydrodynamical equations for a particular infinite particle system, the so-called zero-range interaction process introduced by Spitzer [10]. In order to make contact with physics we want to give a rigorous meaning to the following sentences:

"Consider a fluid in $\mathbb{R}^d$ and look at a small volume around a point x. This small volume is so large compared with the distances between molecules that it contains an infinite number of them which are in equilibrium with given characteristics (density, temperature, $\cdots$). Of course these parameters vary with x (say $p(x)$, $T(x)$, $\cdots$). If we look at the system after a time t, the system will be still locally in equilibrium with other local characteristics ($p(x, t)$, $T(x, t)$, $\cdots$) varying slowly in time although each particle individually moves very rapidly. We want to derive the equation of evolution of these macroscopical parameters." However, it is surprising that this derivation is almost never based on the analysis of the movement of the particles but rather on "conservation laws." The limiting procedure that we describe below has been used by various authors [2, 8] in efforts to give rigorous treatment of the program alluded to above.

Consider a system of infinitely many particles in $\mathbb{R}^d$ (or $\mathbb{Z}^d$) moving according to a deterministic or random motion (e.g. the Hamilton equation). Suppose that this evolution has infinitely many extremal invariant measures ν_p (e.g. the Gibbs states) characterized by p in a set of real parameters P (e.g. the inverse temperature, the density, etc.). Let ε be a parameter which will later tend to zero and consider a sequence of measures μ^ε on the space X of configurations of particles which satisfy:

(A) local equilibrium: There exists a function from $\mathbb{R}^d$ into P; $x \to p(x)$ such that for every x in $\mathbb{R}^d$ (all the limits considered in this paper are in the sense of weak convergence of measures):

$$\lim_{\varepsilon\to 0} \tau_{x/\varepsilon}\mu^\varepsilon = \nu_{p(x)}$$

where τ_a denotes the shift by a induced on X by the translations of $\mathbb{R}^d$ (in $\mathbb{Z}^d$ one needs to look at $\tau_{[x/\varepsilon]}\mu^\varepsilon$, where $[a]$ denotes the integer part of a—but we will omit the symbol [ ] whenever it creates no confusion).

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Now let T_t be the flow induced on the set of measures by our evolution. In order to take care of the rapid macroscopic movement of the particles, we will perform a time rescaling and we will say that we have conservation of local equilibrium if

(B) There exists a $p(x, t)$ such that for every x and t:

$$\lim_{\varepsilon\to 0} \tau_{x/\varepsilon} \mu^\varepsilon T_{t/\varepsilon} = \nu_{p(x,t)}.$$

Of course $p(x, t)$ must satisfy $p(x, 0) = p(x)$ and it is generally expected that $p(x, t)$ satisfies a partial differential equation, which will be called the hydrodynamical equation of the process.

It is readily seen from this framework that the problem makes sense even if the evolution is not of physical nature and also may be applied to random evolutions.

In the past few years this problem has been intensively studied by several authors and (B) has been proved to hold with various restrictions on the initial condition $p(x)$ and/or the seqeuence μ^ε for the Euler [2; 9] or the Navier-Stokes [2; 3; 5; 8] equation. In this paper we are interested in the one-dimensional asymmetric zero-range process which can be described intuitively the following way: At each site z of $\mathbb{Z}$, we have a certain number of particles $\eta(z)$. Each of the particles sitting at z moves to the right one unit at speed $1/\eta(z)$. Alternatively, this system may be described by saying that every site z possesses an independent bell that rings after exponentially distributed intervals and that when the bell rings, *one* (if any) of the particles present at z is moved to $z + 1$. It can be proved that there exists a unique Markov process taking values in $\mathbb{N}^{\mathbb{Z}}$ corresponding to this description [7]. Its generator is defined on cylindrical functions by:

$$Lf(\eta) = \textstyle\sum_{x\in Z} 1_{(\eta(x)\neq 0)}[f(\eta^x) - f(\eta)]$$

where

$$\eta^x(y) = \begin{cases} \eta(y) & \text{if } \; y \neq x \text{ or } x+1 \\ \eta(x) - 1 & \text{if } \; y = x \\ \eta(x+1) + 1 & \text{if } \; y = x+1. \end{cases}$$

Moreover, the set of its extremal invariant measures is the set of product measures ν_p such that $\nu_p(\eta(x) = k) = p^k(1-p)$ where p is a constant belonging to $[0, 1[$. (The proof in [1] does not cover the case we consider here but can be easily adapted).

In this paper given $p(x)$: $\mathbb{R} \to [0, 1[$ we always take the sequence μ^ε to be a sequence of product measures with

$$\mu^\varepsilon(\eta(x) = k) = p(x\varepsilon)^k(1 - p(x\varepsilon)).$$

We prove that (B) holds in this case for several initial conditions, where $p(x, t)$ is the solution of the nonlinear P.D.E.:

$$\frac{\partial p}{\partial t} = -(1-p)^2 \frac{\partial p}{\partial x}.$$

We must emphasize that we prove convergence to the ν_p with the correct parameter (which proves (B)), while the exact correct parameter was guessed via the following intuitive argument:

Suppose that you *a priori* know that conservation of equilibrium takes place; then, because of the form of the ν_p we must have:

$$E_{\mu^\varepsilon}(\eta_{t/\varepsilon}(x/\varepsilon)) \simeq \frac{p(x, t)}{1 - p(x, t)}$$

$$\mathbb{P}_{\mu_\varepsilon}(\eta_{t/\varepsilon}(x/\varepsilon) \geq 1) \simeq p(x, t).$$

And applying the generator to the function $f(\eta) = \eta(x/\varepsilon)$, we get:

$$\frac{\partial}{\partial t}\left(\frac{p(x, t)}{1 - p(x, t)}\right) \simeq \frac{1}{\varepsilon}(p(x - \varepsilon, t) - p(x, t)) \simeq -\frac{\partial p}{\partial x}.$$

Solving this nonlinear P.D.E. by means of characteristics, we can determine the value of $p(x, t)$, or characterize $p(x, t)$ in terms of certain inequalities which are used later to prove the convergence. However this equation is readily seen to present shock-wave phenomena for increasing initial data. Since the jump condition of the generalized solution [6] is not conserved by a change of variables and also because the total number of particles in our system is a conserved quantity, we rather choose to put it in terms of a conservation law after performing the change of variables $u = p/(1 - p)$ where the equation takes the form:

$$\frac{\partial u}{\partial t} = -\frac{1}{(1 + u)^2}\frac{\partial u}{\partial x}.$$

We then prove that $p(x, t)$ is the natural solution when $p(x)$ is decreasing and C^1 or an increasing step-function.

2. Hydrodynamical behavior for some increasing initial conditions. In this section we want to define a coupling of two versions of the zero-range process which will be essential in the sequel. The state space of the coupled process is $(\mathbb{N}^{\mathbb{Z}})^2$ and its generator is defined on cylindrical functions by

$$\bar{L}f(\eta, \xi) = \textstyle\sum_x 1_{(\eta(x)\geq 1)}[f(\eta^x, \xi) - f(\eta, \xi)] + \sum 1_{(\eta(x)=0, \xi(x)\geq 1)}[f(\eta, \xi^x) - f(\eta, \xi)].$$

Techniques similar to those of [7] prove the existence and uniqueness of this process. It is easy to verify that the first coordinate η is itself a zero-range process and that $\zeta = \eta + \xi$ (meaning componentwise addition) is also a zero-range process. It has also a simple natural interpretation: When the bell rings at site x, particles of η have priority and jump forward. When no particle of η is present, then one of the ξ-particles is allowed to jump forward if any is present at x. We will denote this priority of η over ξ by the notation $\eta \vdash \xi$. This coupling is useful to verify some properties such as Lemma 2.1 below. First recall the definition of stochastic order for measures on a partially ordered set X: if μ and ν are two probability measures on X, we say that μ is larger that ν iff there exists a probability M on $X \times X$ such that its first (resp. second) marginal is μ (resp. ν) and M is concentrated on the (x, y) such that $x \geq y$. In our case we endow $\mathbb{N}^{\mathbb{Z}}$ with the partial order defined by $\eta \geq \eta'$ iff for all $x \in \mathbb{Z}$, $\eta(x) \geq \eta'(x)$. From the definition and the coupling of the processes defined above it is clear that if $\mu \geq \nu$ then for all t's, μP_t is larger than νP_t. This and the observation that the semi-group commutes with the shift on $\mathbb{Z}$ proves:

LEMMA 2.1. *If μ is a product measure such that the marginal law of $\eta(x)$ is monotone in x then for every $t \geq 0$, the marginal laws of $\eta_t(x)$ are also monotonic in the same order.*

We will also use the following lemma which can be deduced easily from Burke's theorem using finite approximations to our process. A proof of Burke's theorem can be found in Kelly [4].

LEMMA 2.2. *Under $\mathbb{P}_{\nu_p}$, the successive instants when a particle jumps across a given fixed bond $(x, x+1)$ is a Poisson process with intensity p.*

In this section as well as in the next one, we give intuitive proofs for all the theorems and formalize these arguments for the first theorem. The other proofs can be made rigorous by using the same techniques.

We will now use Lemma 2.2 to prove the following two theorems which clearly show, on the macroscopic scale, the existence of shock waves as predicted by the integral form of the equation $\partial u/\partial x = -(1-u)^2(\partial u/\partial t)$.

THEOREM 2.3. *If $0 \leq \alpha < 1$ and $\mu_{0,\alpha}$ is a product measure with*

$$\mu_{0,\alpha}(\eta(z) = k) = \begin{cases} \nu_0(\eta(z) = k) & \text{if } z < 0 \\ \nu_\alpha(\eta(z) = k) & \text{if } z \geq 0 \end{cases}$$

then:

$$\lim_{\varepsilon \to 0} \tau_{x/\varepsilon} \mu_{0,\alpha} T_{t/\varepsilon} = \begin{cases} \nu_0 & \text{if } x < (1-\alpha)t \\ \nu_\alpha & \text{if } x > (1-\alpha)t. \end{cases}$$

REMARK. It should be noted that at the shock, i.e. at $x = (1-\alpha)t$, we do not know whether we are in an extremal equilibrium state (and probably this last property is *not* true at $x = (1-\alpha)t$!).

PROOF. In order to prove the theorem we will couple our initial $\mu_{0,\alpha}$ with ν_α by completing at the left of zero with ξ-particles so that η has distribution $\mu_{0,\alpha}$ and $\eta + \xi$ distribution ν_α as follows. First choose independently for each x in $\mathbb{Z}$ an integer $\rho(x)$ with common probability $P(\rho(x) = k) = \alpha^k(1-\alpha)$. Then define η and ξ by:

$$\xi(x) = \rho(x) \quad \text{and} \quad \eta(x) = 0 \qquad \text{if } x < 0$$
$$\xi(x) = 0 \quad \text{and} \quad \eta(x) = \rho(x) \quad \text{if } x \geq 0.$$

Clearly the distribution of η is $\mu_{0,\alpha}$ and that of $\eta + \xi$ is ν_α. Let now (η, ξ) evolve according to the coupled process with $\eta \vdash \xi$.

We then have for all x and t:

$$(\sum_{z\geq x} \xi_t(z) > 0) \subset (\sum_{z\leq x-1} \eta_t(z) = 0) \tag{1}$$

since this relation is true for $t = 0$ by construction, and since no ξ-particle can ever overtake an η-particle.

The formula (1) means that there is a sharp interface between ξ- and η-particles and that both types of particles can coexist at one site at most.

Denote now by $\gamma_t^+ = \sup\{x: \sum_{z\geq x} \xi_t(z) > 0\}$ and $\gamma_t^- = \inf\{x: \sum_{z\leq x} \eta_t(x) > 0\}$. By (1) we have $\gamma_t^+ \leq \gamma_t^-$, and we now want to prove that if $z > 1 - \alpha$:

$$\lim_{\varepsilon\to 0}\mathbb{P}_{\mu_{0,\alpha}}(\gamma_{t/\varepsilon}^+ > z.t/\varepsilon) = 0.$$

By Lemma 2.2 we know that $\varepsilon/t \sum_{z>0}\xi_{t/\varepsilon}(x) \to \alpha$ almost surely, and since $\eta_t + \xi_t$ has distribution ν_α it follows that:

$$\varepsilon/t \sum_{0<y<zt/\varepsilon} \eta_{t/\varepsilon}(y) + \xi_{t/\varepsilon}(y) \to z\alpha/1 - \alpha > \alpha$$

in probability, hence $\varepsilon/t \sum_{0<y<zt/\varepsilon} \eta_{t/\varepsilon}(y) > 0$ with probability close to one so that:

$$P_{\mu_{0,\alpha}}(\gamma_{t/\varepsilon}^- \leq zt/\varepsilon) < P_{\mu_{0,\alpha}}(\gamma_{t/\varepsilon}^+ \leq zt/\varepsilon) \to 1.$$

To see that this implies the theorem, notice that by coupling we have for any cylindrical f:

$$a(\varepsilon) = \int f(\rho)\tau_{zt/\varepsilon}\mu_{0,\alpha}T_{t/\varepsilon}(d\rho) = E(f(\tau_{zt/\varepsilon}[\eta_{t/\varepsilon}]))$$

and

$$b(\varepsilon) = \int f(\rho)\nu_\alpha(d\rho) = E(f(\tau_{zt/\varepsilon}[\eta_{t/\varepsilon} + \xi_{t/\varepsilon}])).$$

Hence, if we denote by $S(f)$ the finite subset of $\mathbb{Z}$ of those coordinates on which f depends, we have:

$$| a(\varepsilon) - b(\varepsilon) | \leq \sup |f| \times \mathbb{P}(\exists u \in S(f) + z.t/\varepsilon : \xi_{t/\varepsilon}(u) > 0).$$

Taking now any $\bar{z}$ such that $1 - \alpha < \bar{z} < z$ we have:

$$| a(\varepsilon) - b(\varepsilon) | \leq \sup |f| \cdot \mathbb{P}(\gamma_{t/\varepsilon}^+ > \bar{z}\cdot t/\varepsilon) \to 0.$$

A similar argument shows that if $z < 1 - \alpha$ the sequence $\tau_{zt/\varepsilon}\mu_{0,\alpha}T_{t/\varepsilon}$ converges to ν_0. □

THEOREM 2.4. *If $\mu_{\alpha,\beta}$ is a product measure such that:*

$$\mu_{\alpha,\beta}\{\eta(x) = k\} = \begin{cases} \nu_\alpha(\eta(x) = k) & \text{if } x < 0 \\ \nu_\beta(\eta(x) = k) & \text{if } x \geq 0 \end{cases}$$

with $\beta > \alpha$, then:

$$\lim_{\varepsilon\to 0}\tau_{[x/\varepsilon]}\mu_{\alpha,\beta}T_{[t/\varepsilon]} = \begin{cases} \nu_\alpha & \text{if } x < t(1 - \alpha)(1 - \beta) \\ \nu_\beta & \text{if } x > t(1 - \alpha)(1 - \beta). \end{cases}$$

PROOF OF 2.4. We use the following coupling: η-particles are distributed according to ν_α, ξ-particles are added to the η-particles on all non-negative sites in such a way that the distribution of $\eta + \xi$ is $\mu_{\alpha,\beta}$. Finally ζ-particles are added in such a way that the distribution of $\eta + \xi + \zeta$ is ν_β.

We now couple η, ξ and ζ by $\eta \vdash \xi \vdash \zeta$ and so the generator of the process we consider is:

$$\begin{aligned} \bar{L} f(\eta, \xi, \zeta) = & \textstyle\sum_x 1_{(\eta(x)\geq 1)}[f(\eta^x, \xi, \zeta) - f(\eta, \xi, \zeta)] \\ & + \textstyle\sum_x 1_{(\eta(x)=0, \xi(x)>0)}[f(\eta, \xi^x, \zeta) - f(\eta, \xi, \zeta)] \\ & + \textstyle\sum_x 1_{(\eta(x)=0, \xi(x)=0, \zeta(x)>0)}[f(\eta, \xi, \zeta^x) - f(\eta, \xi, \zeta)]. \end{aligned}$$

In this way the η-particles are in equilibrium. As before, there is a sharp interface between ζ and ξ-particles. To determine the position γ of the interface note that the rate at which ζ-particles enter the interval $[1, \infty]$ is $\beta - \alpha$. This shows that

$$\beta - \alpha = \left(\frac{\beta}{1-\beta} - \frac{\alpha}{1-\alpha}\right) \cdot \gamma;$$

therefore

$$\gamma = (1-\beta)(1-\alpha). \quad \square$$

3. Hydrodynamical behavior for decreasing initial conditions. In order to study the general hydrodynamical behavior of decreasing initial conditions, we will first need an upper estimate for the number of particles in an interval $(0, x/\varepsilon)$ for an initial distribution of the form $\mu_{\alpha,0}$ corresponding to the function $\alpha 1_{(-\infty,0)}$.

NOTATION. (FOR A SEQUENCE X_t of random variables and a constant c we write $\limsup X_t \leq c$ for $\lim P(X_t \geq c + \delta) = 0$ for all $\delta > 0$.)

LEMMA 3.1. (Rost [9]). *The number of particles in the interval* $[0, x/\varepsilon]$ *for* $\mu_{\alpha,0} T_{t/\varepsilon}$ *called* $R_\varepsilon(t, x, \alpha)$ *satisfies*:

$$\limsup_{\varepsilon \to 0} \varepsilon\, R_\varepsilon(t, x, \alpha) \leq \begin{cases} \dfrac{\alpha}{1-\alpha}\, x & \text{if } 0 < x < t(1-\alpha)^2 \\ \alpha t - (\sqrt{t} - \sqrt{x})^2 & \text{if } t(1-\alpha)^2 \leq x < t \\ \alpha t & \text{if } t < x. \end{cases}$$

PROOF. Call $N_\varepsilon(t, x, \alpha)$ the number of particles that have passed x/ε at time t/ε. Notice the identity:

$$R_\varepsilon(t, x, \alpha) = N_\varepsilon(t, 0, \alpha) - N_\varepsilon(t, x, \alpha).$$

On one hand, a coupling argument and the fact that each ν_α is an equilibrium state shows that:

$$\limsup \varepsilon\, R_\varepsilon(t, x, \alpha) \leq \frac{\alpha}{1-\alpha}\, x.$$

On the other hand Lemma 2.1 implies that

$$\lim_{\varepsilon\to 0}\varepsilon\, N_\varepsilon(t, 0, \alpha) = \alpha t$$

so that

$$\liminf_{\varepsilon\to 0}\varepsilon\, N_\varepsilon(t, x, \alpha) \geq \left(\alpha t - \frac{\alpha}{1-\alpha}\, x\right)^+ .$$

But in the sense of stochastic order, $N_\varepsilon(t, x, \alpha)$ is increasing in α so that

$$\liminf_{\varepsilon\to 0}\varepsilon\, N_\varepsilon(t, x, \alpha) \geq \sup_{0\leq\beta\leq\alpha}\left(\beta t - \frac{\beta}{1-\beta}\, x\right)^+$$

which is the desired estimate.

REMARK. It is important to notice that if $x > t(1-\alpha)^2$, the number of particles in $[0, x\varepsilon^{-1}]$ is strictly less than $\alpha/(1-\alpha)\, x\varepsilon^{-1}$ with a probability close to one. This fact will be crucial in the rest of the proofs.

THEOREM 3.2. *For every x in $\mathbb{R}$, $\tau_{x\varepsilon^{-1}}\mu_{\alpha,0}\, T_{t\varepsilon^{-1}}$ converges as $\varepsilon \to 0$ to $\nu_{p(x,t)}$ where*

$$p(x, t) = \begin{cases} \alpha & \text{if } \; x < (1-\alpha)^2 t \\ 1 - \sqrt{x/t} & \text{if } \; (1-\alpha)^2 t \leq x < t \\ 0 & \text{if } \; t \leq x. \end{cases}$$

PROOF. Since $\tau_{x/\varepsilon}\mu_{\alpha,0}T_{t/\varepsilon} = \tau_{x/t\cdot t/\varepsilon}\mu_{\alpha,0}T_{t/\varepsilon}$ and $p(x, t) = p(x/t, 1)$ it suffices to show that Theorem for $t = 1$.

We will first use a coupling to prove that we have $\tau_{x\varepsilon^{-1}}\mu_{\alpha,0}T_{\varepsilon^{-1}} \leq \nu_{p(x,1)}$. (This is obvious for $x < (1-\alpha)^2$, so take $x > (1-\alpha)^2$). Now choose $\alpha > p > p(x, 1)$ and couple ν_p and $\mu_{\alpha,0}$ as follows: Let $\eta + \zeta$ have distribution $\mu_{\alpha,0}$, η have distribution $\mu_{p,0}$, and ξ have distribution $\mu_{0,p}$. Let (η, ζ, ξ) evolve according to the process with generator

$$\begin{aligned} Lf(\eta, \zeta, \xi) = {} & \textstyle\sum_x 1_{(\eta(x)\geq 1)}[f(\eta^x, \zeta, \xi) - f(\eta, \zeta, \xi)] \\ & + \textstyle\sum_x 1_{(\eta(x)=0;\zeta(x)\geq 1;\xi(x)\geq 1)}[f(\eta, \zeta^x, \xi^x) - f(\eta, \zeta, \xi)] \\ & + \textstyle\sum_x 1_{(\eta(x)=0;\zeta(x)=0,\xi(x)\geq 1)}[f(\eta, \zeta, \xi^x) - f(\eta, \zeta, \xi)] \\ & + \textstyle\sum_x 1_{(\eta(x)=0;\zeta(x)\geq 1;\xi(x)=0)}[f(\eta, \zeta^x, \xi) - f(\eta, \zeta, \xi)]. \end{aligned}$$

This means that η has priority on both ζ and ξ and that ξ and ζ-particles jump together whenever it is possible. From this we see that $\zeta + \eta$ represents $\mu_{\alpha,0}$ and its evolution and $\xi + \eta$ that of ν_p. It is easy to prove along the same line as in Theorem 2.3 that for all x and t:

$$(\zeta_t(x) + \eta_t(x) > \xi_t(x) + \eta_t(x)) \subset (\textstyle\sum_{z\leq x-1}\xi_t(z) = 0).$$

But $\varepsilon/x \sum_{0<z<x/\varepsilon}\xi_{1/\varepsilon}(z) + \eta_{1/\varepsilon}(z)$ converges in probability to $p/(1-p)$ and by the

previous lemma lim sup $\varepsilon/x \sum_{0<z<x/\varepsilon} \eta_{1/\varepsilon}(z)$ is less than or equal to $p/(1-p)$. Therefore for all $p > p(x, 1)$ we conclude that every weak limit of $\tau_{x/\varepsilon}\mu_{\alpha,0}T_{1/\varepsilon}$ is inferior or equal to ν_p. By monotonicity of ν_p as p decreases, we have the desired inequality.

To prove the equality, we argue by contradiction: suppose that for a given x a weak accumulation point of $\tau_{x/\varepsilon}\mu_{0,\alpha}T_{1/\varepsilon}$ is strictly smaller than $\nu_{p(x,1)}$ (clearly $0 < x < 1$) and let ε_n be the corresponding sequence. Then

$$\lim \int \eta(0)\ d(\tau_{x/\varepsilon_n}\mu_{\alpha,0}T_{1/\varepsilon_n}) = a < \frac{p(x, 1)}{1-p(x, 1)}.$$

By Lemma 2.1, the integral above is nonincreasing in x for fixed n. Also,

$$\limsup \int \eta(0)\ d(\tau_{y/\varepsilon_n}\mu_{\alpha,0}T_{1/\varepsilon_n}) \le \frac{p(y, 1)}{1-p(y, 1)}$$

for every y. Hence, for k large enough there exists a subsequence of ε_n, which will be again denoted by ε_n, such that

$$\lim \int \eta(0)\ d(\tau_{y/\varepsilon_n}\mu_{\alpha,0}T_{1/\varepsilon_n}) \quad \text{exists for all} \quad y = \frac{j}{k}, \quad j = 1, \cdots, k-1$$

and

$$\text{(2)} \qquad \textstyle\sum_{i=0}^{\infty} \frac{1}{k} \lim \int \eta(0)\ d(\tau_{[i/k\varepsilon_n]}\mu_{\alpha,0}T_{1/\varepsilon_n}) < \int_0^{\infty} \frac{p(u, 1)}{1-p(u, 1)}\, du = \alpha.$$

Note that if $i \ge k$, the limit of the term in the left hand side is zero by Theorem 2.3 and that this series is in fact a finite sum. But if $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}_{\mu_{\alpha,0}}$:

$$\begin{aligned}\limsup E[\varepsilon_n \textstyle\sum_{x>0} \eta_{1/\varepsilon_n}(x)] &= \limsup \varepsilon_n \textstyle\sum_{j=0}^{\infty} \sum_{j/k \le x\varepsilon_n \le (j+1)/k} E(\eta_{1/\varepsilon_n}(x)) \\ &\le \textstyle\sum_{j=0}^{\infty} \limsup \varepsilon_n \sum_{j/k \le x\varepsilon_n \le (j+1)/k} E(\eta_{1/\varepsilon_n}(x)) \\ &\le \textstyle\sum_{j=0}^{\infty} \limsup \frac{1}{k} E(\eta_{1/\varepsilon_n}(j/k\varepsilon_n)) < \alpha \quad \text{by (2)}.\end{aligned}$$

On the other hand, by Lemma 2.1

$$E(\varepsilon_n \textstyle\sum_{x>0} \eta_{1/\varepsilon_n}(x)) = \varepsilon_n E(N_{\varepsilon_n}(1, 0, \alpha)) = \alpha$$

which is a contradiction.

Now take $p(x)$ to be strictly decreasing C^1 function of $\mathbb{R}$ into $[0, 1[$.

THEOREM 3.3. *Let μ^ε be a product measure with*

$$\mu^\varepsilon\{\eta(x) = k\} = p(x\varepsilon)^k[1-p(x\varepsilon)].$$

Then

$$\lim \tau_{x/\varepsilon}\mu^\varepsilon T_{t/\varepsilon} = \nu_{p(x,t)}$$

where $p(x, t)$ is the unique classical solution of the equation

$$\frac{\partial p}{\partial t} = -(1 - p(x, t))^2 \frac{\partial p}{\partial x}$$

with initial condition $p(x, 0) = p(x)$.

PROOF. If we solve this equation using the method of characteristics [6], we see that for each (x, t) there exists a unique $y(x, t)$ such that $p(x, t) = p(y(x, t))$ and that it must satisfy $x - y(x, t) = t(1 - p(y(x, t)))^2$. Fix now x and t and take q such that $p(x, t) < q < 1$. There exists a unique z such that $p(z) = q$ and it must also satisfy $x - z > t(1 - q)^2$. Now couple ν_q and μ^ε as follows: Define

$$q_1(v) = \begin{cases} p(v) & \text{for } v \le z \\ 0 & \text{otherwise} \end{cases} \qquad q_3(v) = \begin{cases} q & \text{for } v \le z \\ 0 & \text{otherwise} \end{cases}$$

$$q_2(v) = \begin{cases} 0 & \text{for } v \le z \\ p(v) & \text{otherwise} \end{cases} \qquad q_4(v) = \begin{cases} 0 & \text{for } v \le z \\ q & \text{otherwise} \end{cases}$$

and define $\mu^\varepsilon_{q_i}$ as a product measure with $\mu^\varepsilon_{q_i}(\eta(x) = k) = q_i(x\varepsilon)^k(1 - q_i(x\varepsilon))$. Take now ρ, ξ, η, ξ such that:

$$\rho + \xi =_d \mu^\varepsilon_{q_1}; \quad \xi =_d \mu^\varepsilon_{q_3}$$

$$\zeta + \eta =_d \mu^\varepsilon_{q_4}; \quad \zeta =_d \mu^\varepsilon_{q_2}$$

and take the following rule of precedence: $\xi \vdash \zeta \vdash \rho$ and η, where ρ and η have to jump together whenever possible. Here $\rho + \xi + \zeta$ represent our system, whereas $\xi + \zeta + \eta$ has distribution ν_q. We now prove that with probability close to one we have, at site x/ε and time t/ε, more $\eta + \xi + \zeta$-particles than $\rho + \xi + \zeta$-particles. Indeed, in the interval $[z/\varepsilon, x/\varepsilon]$ there are $(x - z) \cdot q/(1 - q) \cdot \varepsilon^{-1}$ of $\eta + \xi + \zeta$-particles, but strictly less than this quantity of ξ-particles because we choose $x - z > t(1 - q)^2$ (see remark below Lemma 3.1). Therefore there are ζ- or η-particles in this interval with probability almost one. Now by construction, no ρ-particle can ever overtake a ζ- or a η-particle so no ρ-particle has reached x/ε by time t/ε. By monotonicity this proves that lim sup $\tau_{x/\varepsilon}\mu^\varepsilon T_{t/\varepsilon} \ge \nu_{p(x,t)}$.

On the other hand since $\mu^\varepsilon > \tilde{\mu}^\varepsilon_y$ for all y, where $\tilde{\mu}^\varepsilon_y$ is defined by:

$$\tilde{\mu}^\varepsilon_y(\eta(u) = k) = \begin{cases} \nu_{p(y)}(\eta(u) = k) & \text{for } u < y\varepsilon^{-1} \\ \nu_0(\eta(u) = k) & \text{for } u > y\varepsilon^{-1} \end{cases}$$

we also have by Theorem 3.2: lim $\tau_{x\varepsilon^{-1}}\mu^\varepsilon T_{t\varepsilon^{-1}} \ge \nu_{p(y)}$ for all y's such that $x - y < (1 - p(y))^2 t$. □

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