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DERIVATIONS OF A RESTRICTED WEYL TYPE ALGEBRA ON A LAURENT EXTENSION

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Abstract. Several authors find all the derivations of an algebra [1], [3], [7]. A Weyl type non-associative algebra and its subalgebra are defined in the paper [2], [3], [10]. All the derivations of the non-associative algebra $\overline{WN_{0,0,s_1}}$ is found in this paper [4]. We find all the derivations of the non-associative algebra $\overline{WN_{0,s,0_1}}$ in this paper [5].

1. Introduction

Generally, there is an infinite dimensional simple algebra with outer derivation. Thus it is an interesting problem to find all the derivations of an infinite dimensional (non-)associative algebra [2]. The Weyl type non-associative algebras are defined in the papers [3], [12]. All the derivations of the restricted Weyl type non-associative algebras $\overline{WN_{0,0,1_1}}$ and $\overline{WN_{0,0,2_1}}$ are found in the paper [1] and [2]. All the derivations of the non-associative algebra $\overline{WN_{0,0,s_1}}$ is found in this paper [4]. In this paper, we find all the derivations of the restricted Weyl type non-associative algebras $\overline{WN_{0,s,0_1}}$.

2. Preliminaries

Let $\mathbf{F}$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbf{N}$ and $\mathbf{Z}$ will denote the non-negative

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integers and the integers, respectively. Let $\mathbf{F}[x_1, \dots, x_{m+s}]$ be the polynomial ring with the variables $x_1, \dots, x_{m+s}$. Let $g_1, \dots, g_n$ be given polynomials in $\mathbf{F}[x_1, \dots, x_{m+s}]$. For $n, m, s \in \mathbf{N}$, let us define the commutative, associative $\mathbf{F}$-algebra $F_{g_n,m,s} = \mathbf{F}[e^{\pm g_1}, \dots, e^{\pm g_n}, x_1^{\pm 1}, \cdot, x_m^{\pm 1}, x_{m+1}, \dots, x_{m+s}]$ in the formal power series ring $\mathbf{F}[[x_1, \dots, x_{m+s}]]$ which is called a stable algebra in the paper [7] with the standard basis

$$\begin{aligned} \mathbf{B} = \{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \mid\ & a_1, \dots, a_n, i_1, \dots, i_m \in \mathbf{Z}, \\ & i_{m+1}, \dots, i_{m+s} \in \mathbf{N}\} \end{aligned} \tag{1}$$

and with the obvious addition and the multiplication [3], [7], [11], [13], [14] where we take $g_1, \dots, g_n$ so that $\mathbf{B}$ is the standard basis of $F_{g_n,m,s}$. ∂_w, $1 \le w \le m+s$, denotes the usual partial derivative with respect to x_w on $F_{g_n,m,s}$. For partial derivatives $\partial_u, \dots, \partial_v$ of $F_{g_n,m,s}$, the composition $\partial_u^{j_u} \circ \cdots \circ \partial_v^{j_v}$ of them is denoted $\partial_u^{j_u} \cdots \partial_v^{j_v}$ where $j_u, \dots, j_v \in \mathbf{N}$. Let us define the vector space $WN(g_n, m, s)$ over $\mathbf{F}$ which is spanned by the standard basis

$$\begin{aligned} &\{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} | e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \in \mathbf{B}, \\ &j_u, \dots, j_v \in \mathbf{N}, 1 \le u, \dots, v \le m+s\}. \end{aligned} \tag{2}$$

Thus we can define the multiplication $*$ on $WN(g_n, m, s)$ as follows:

$$\begin{aligned} & e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} \\ * \quad & e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w} \\ = \quad & e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} \\ & (e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}}) \partial_h^{j_h} \cdots \partial_w^{j_w} \end{aligned} \tag{3}$$

for any basis elements $e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v}$ and $e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w} \in WN(g_n, m, s)$. Thus we can define the Weyl-type non-associative algebra $\overline{WN_{g_n,m,s}}$ with the multiplication $*$ in (3) and with the set $WN(g_n, m, s)$ [10], [11], [12]. For $r \in \mathbf{N}$, let us define the restricted Weyl type non-associative subalgebra $\overline{WN_{g_n,m,s_r}}$ of the non-associative algebra $\overline{WN_{g_n,m,s}}$ spanned by

$$\begin{aligned} &\{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} | e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \in \mathbf{B}, \\ &j_u, \dots, j_v \in \mathbf{N}, j_u + \cdots + j_v \le r, 1 \le u, \dots, v \le m+s\}. \end{aligned} \tag{4}$$

For any basis element $e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v}$ of $\overline{WN_{g_n,m,s}}$, let us define the degree $deg_r(e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v})$ of x_r as i_r and the total degree $deg_{tot}(e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v})$ of $e^{a_1 g_1} \cdots$

$e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v}$ as $i_1 + \cdots + i_s$. Thus for any element l of $\overline{WN_{g_n,m,s}}$, we define $deg_r(l)$ of x_r as the highest degree of x_r in the basis terms of l and $deg_{tot}(l)$ as the highest total degree of basis terms of l. It is well known that the non-associative algebra $\overline{WN_{g_n,m,s}}$ is simple, even though it has the right annihilator [7], [8]. For any element of l of $\overline{WN_{g_n,m,s}}$, an element l_1 (resp. l_2) is right (resp. left) identity of l if $l * l_1 = l$ (resp. $l_2 * l = l$).

3. Derivations of $\overline{WN_{0,s,0_1}}$

LEMMA 1. *For any derivation D of the non-associative algebra $\overline{WN_{0,3,0_1}}$, then we have that*

$$\begin{aligned} D(x_1^i \partial_1) &= (1-i)a_{1,0,0} x_1^i \partial_1 - i b_{1,0,0} x_1^{i-1} x_2 \partial_1 - i c_{1,0,0} x_1^{i-1} x_3 \partial_1 \\ &\quad + i d_{1,0,0} x_1^{i-1} \partial_1 + a_{2,0,0} x_1^i \partial_2 + a_{3,0,0} x_1^i \partial_3 \end{aligned}$$

and $D(x_u^j \partial_u)$ has the similar formula as $D(x_1^i \partial_1)$ where $a_{1,0,0}, \ldots, a_{3,0,0} \in$ $\mathbf{F}$, $2 \leq u \leq 3$, and $i, j \in \mathbf{Z}$.

PROOF. The proof of the lemma is similar to the proof of Lemma by considering $\mathbf{Z}$ instead of $\mathbf{N}$ in the paper [4]. Let us omit it. □

LEMMA 2. *For any derivation D of the non-associative algebra $\overline{WN_{0,3,0_1}}$,*

$$\begin{aligned} D(x_1 \partial_2) &= -a_{1,0,0} x_1 \partial_2 + b_{2,0,0} x_1 \partial_2 - b_{1,0,0} x_2 \partial_2 - c_{1,0,0} x_3 \partial_2 \\ &\quad + d_{1,0,0} \partial_2 + b_{1,0,0} x_1 \partial_1 + b_{3,0,0} x_1 \partial_3 \end{aligned}$$

hold with appropriate scalars. $D(x_u \partial_v)$ has the similar formula as $D(x_1 \partial_2)$ where $1 \leq u \neq v \leq 3$.

PROOF. The proof of the lemma comes from the similar comments of the proof of Lemma 1. Let us omit it. □

LEMMA 3. *For any derivation D of the non-associative algebra $\overline{WN_{0,3,0_1}}$*

$$\begin{aligned} D(x_1 x_2 \partial_1) &= -a_{2,0,0} x_1^2 \partial_1 - b_{1,0,0} x_2^2 \partial_1 - b_{2,0,0} x_1 x_2 \partial_1 - c_{1,0,0} x_2 x_3 \partial_1 \\ &\quad - c_{2,0,0} x_1 x_3 \partial_1 + d_{1,0,0} x_2 \partial_1 + h_{2,0,0} x_1 \partial_1 + a_{2,0,0} x_1 x_2 \partial_2 \\ &\quad + a_{3,0,0} x_1 x_2 \partial_3 \\ D(x_1 x_3 \partial_1) &= -a_{3,0,0} x_1^2 \partial_1 - b_{1,0,0} x_2 x_3 \partial_1 - b_{3,0,0} x_1 x_2 \partial_1 - c_{1,0,0} x_3^2 \partial_1 \end{aligned}$$

$$
\begin{aligned}
& -c_{3,0,0}x_1x_3\partial_1 + d_{1,0,0}x_3\partial_1 + g_{3,0,0}x_1\partial_1 + a_{2,0,0}x_1x_3\partial_2 \\
& +a_{3,0,0}x_1x_3\partial_3 \\
D(x_2x_3\partial_1) \ &= \ a_{1,0,0}x_2x_3\partial_1 - a_{2,0,0}x_1x_3\partial_1 - a_{3,0,0}x_1x_2\partial_1 - b_{2,0,0}x_2x_3\partial_1 \\
& -b_{3,0,0}x_2^2\partial_1 - c_{2,0,0}x_3^2\partial_1 - c_{3,0,0}x_2x_3\partial_1 + h_{2,0,0}x_3\partial_1 \\
& +g_{3,0,0}x_2\partial_1 + a_{2,0,0}x_2x_3\partial_2 + a_{3,0,0}x_2x_3\partial_3 \qquad (5)
\end{aligned}
$$

hold with appropriate scalars. We have the similar formulas of $D(x_ux_v\partial_w)$ *as* (5) *where* $1 \le u, v \le 3$ *and* $2 \le w \le 3$.

PROOF. The proof of the lemma comes from the similar comments of the proof of Lemma 1. Let us omit it. □

The proof of the following lemma is similar to the proof of Lemma 4 in [4], but let us put it for the readers.

LEMMA 4. *For any derivation* D *of the non-associative algebra* $\overline{WN_{0,3,0_1}}$ *and for any basis element* $x_1^n x_2^m x_3^k \partial_u$, $1 \le u \le 3$, *of* $\overline{WN_{0,3,0_1}}$, *then we have that*

$$
\begin{aligned}
D(x_1^nx_2^mx_3^k\partial_1) \ &= \ (1-n)a_{1,0,0}x_1^nx_2^mx_3^k\partial_1 - mb_{2,0,0}x_1^nx_2^mx_3^k\partial_1 \\
& -kc_{3,0,0}x_1^nx_2^mx_3^k\partial_1 + nd_{1,0,0}x_1^{n-1}x_2^mx_3^k\partial_1 \\
& +mh_{2,0,0}x_1^nx_2^{m-1}x_3^k\partial_1 + kg_{3,0,0}x_1^nx_2^mx_3^{k-1}\partial_1 \\
& -ma_{2,0,0}x_1^{n+1}x_2^{m-1}x_3^k\partial_1 - ka_{3,0,0}x_1^{n+1}x_2^mx_3^{k-1}\partial_1 \\
& -nb_{1,0,0}x_1^{n-1}x_2^{m+1}x_3^k\partial_1 - kb_{3,0,0}x_1^nx_2^{m+1}x_3^{k-1}\partial_1 \\
& -nc_{1,0,0}x_1^{n-1}x_2^mx_3^{k+1}\partial_1 - mc_{2,0,0}x_1^nx_2^{m-1}x_3^{k+1}\partial_1 \\
& +a_{2,0,0}x_1^nx_2^mx_3^k\partial_2 + a_{3,0,0}x_1^nx_2^mx_3^k\partial_3, \qquad (6)
\end{aligned}
$$

where $a_{i,0,0}, b_{i,0,0}, c_{i,0,0} \in \mathbf{F}$, $1 \le i \le 3$, *and* $d_{1,0,0}, h_{2,0,0}, g_{3,0,0} \in \mathbf{F}$. *We have the similar formulas of* $D(x_1^nx_2^mx_3^k\partial_2)$ *and* $D(x_1^nx_2^mx_3^k\partial_3)$ *as* (6).

PROOF. Let D be the derivation in the lemma. By Lemma 1 and $D(\partial_1 * x_1x_2x_3\partial_1) = D(x_2x_3\partial_1)$, we have that $\partial_1 * D(x_1x_2x_3\partial_1) = -2a_{2,0,0}x_1x_3\partial_1 - c_{2,0,0}x_3^2\partial_1 - 2a_{3,0,0}x_1x_2\partial_1 - b_{2,0,0}x_2x_3\partial_1 - b_{3,0,0}x_2^2\partial_1 - c_{3,0,0}x_2x_3\partial_1 + h_{2,0,0}x_3\partial_1 + g_{3,0,0}x_2\partial_1 + a_{2,0,0}x_2x_3\partial_2 + a_{3,0,0}x_2x_3\partial_3$. This implies that

$$
\begin{aligned}
D(x_1x_2x_3\partial_1) \ &= \ -a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{2,0,0}x_1x_2x_3\partial_1 \\
& -b_{3,0,0}x_1x_2^2\partial_1 - c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1
\end{aligned}
$$

$$+h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2$$
$$+a_{3,0,0}x_1x_2x_3\partial_3 + \sum_{i,j\geq 0} s_{1,i,j}x_2^ix_3^j\partial_1$$
$$(7)\qquad + \sum_{i,j\geq 0} s_{2,i,j}x_2^ix_3^j\partial_2 + \sum_{i,j\geq 0} s_{3,i,j}x_2^ix_3^j\partial_3,$$

where $s_{1,i,j}, s_{2,i,j}, s_{3,i,j} \in \mathbf{F}$, $i, j \geq 0$. By $D(\partial_2 * x_1x_2x_3\partial_1) = D(x_1x_3\partial_1)$, we have that

$$\begin{aligned} & \sum_{i\geq 1, j\geq 0} is_{1,i,j}x_2^{i-1}x_3^j\partial_1 + \sum_{i\geq 1, j\geq 0} is_{2,i,j}x_2^{i-1}x_3^j\partial_2 \\ & + \sum_{i\geq 1, j\geq 0} is_{3,i,j}x_2^{i-1}x_3^j\partial_3 \\ = & -2b_{1,0,0}x_2x_3\partial_1 - c_{1,0,0}x_3^2\partial_1 + d_{1,0,0}x_3\partial_1. \end{aligned}$$

So we have that

$$\begin{aligned} D(x_1x_2x_3\partial_1) = & -a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{1,0,0}x_2^2x_3\partial_1 \\ & -b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 - c_{1,0,0}x_2x_3^2\partial_1 \\ & -c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + d_{1,0,0}x_2x_3\partial_1 \\ & +h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2 \\ & +a_{3,0,0}x_1x_2x_3\partial_3 + \sum_{j\geq 0} s_{1,0,j}x_3^j\partial_1 + \sum_{j\geq 0} s_{2,0,j}x_3^j\partial_2 \\ (8)\qquad & + \sum_{j\geq 0} s_{3,0,j}x_3^j\partial_3. \end{aligned}$$

By $D(\partial_3 * x_1x_2x_3\partial_1) = D(x_1x_2\partial_1)$ and (8), we have that

$$\begin{aligned} D(x_1x_2x_3\partial_1) = & -a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{1,0,0}x_2^2x_3\partial_1 \\ & -b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 - c_{1,0,0}x_2x_3^2\partial_1 \\ & -c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + d_{1,0,0}x_2x_3\partial_1 \\ & +h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2 \\ (9)\qquad & +a_{3,0,0}x_1x_2x_3\partial_3 + s_{1,0,0}\partial_1 + s_{2,0,0}\partial_2 + s_{3,0,0}\partial_3. \end{aligned}$$

Since $x_1\partial_1$ is a left identity of $x_1x_2x_3\partial_1$, by Lemma 1 and (9), we have that

$$D(x_1x_2x_3\partial_1) = -a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{1,0,0}x_2^2x_3\partial_1$$

$$-b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 - c_{1,0,0}x_2x_3^2\partial_1$$
$$-c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + d_{1,0,0}x_2x_3\partial_1$$
$$+h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2$$
$$+a_{3,0,0}x_1x_2x_3\partial_3.$$

By $D(x_1^n\partial_1 * x_1x_2x_3\partial_1) = D(x_1^nx_2x_3\partial_1)$, Lemma 1, and (10), we have that

$$\begin{aligned} D(x_1^nx_2x_3\partial_1) &= (1-n)a_{1,0,0}x_1^nx_2x_3\partial_1 - a_{2,0,0}x_1^{n+1}x_3\partial_1 \\ &\quad -a_{3,0,0}x_1^{n+1}x_2\partial_1 - nb_{1,0,0}x_1^{n-1}x_2^2x_3\partial_1 \\ &\quad -b_{2,0,0}x_1^nx_2x_3\partial_1 - b_{3,0,0}x_1^nx_2^2\partial_1 \\ &\quad -nc_{1,0,0}x_1^{n-1}x_2x_3^2\partial_1 - c_{2,0,0}x_1^nx_3^2\partial_1 \\ &\quad -c_{3,0,0}x_1^nx_2x_3\partial_1 + nd_{1,0,0}x_1^{n-1}x_2x_3\partial_1 \\ &\quad +h_{2,0,0}x_1^nx_3\partial_1 + g_{3,0,0}x_1^nx_2\partial_1 \\ &\quad +a_{2,0,0}x_1^nx_2x_3\partial_2 + a_{3,0,0}x_1^nx_2x_3\partial_3. \end{aligned} \tag{10}$$

By $D(x_2^m\partial_2 * x_1^nx_2x_3\partial_1) = D(x_1^nx_2^mx_3\partial_1)$, Lemma 1, and (10), we have that

$$\begin{aligned} D(x_1^nx_2^mx_3\partial_1) &= (1-n)a_{1,0,0}x_1^nx_2^mx_3\partial_1 - ma_{2,0,0}x_1^{n+1}x_2^{m-1}x_3\partial_1 \\ &\quad -a_{3,0,0}x_1^{n+1}x_2^m\partial_1 - nb_{1,0,0}x_1^{n-1}x_2^{m+1}x_3\partial_1 \\ &\quad -mb_{2,0,0}x_1^nx_2^mx_3\partial_1 - b_{3,0,0}x_1^nx_2^{m+1}\partial_1 \\ &\quad -nc_{1,0,0}x_1^{n-1}x_2^mx_3^2\partial_1 - mc_{2,0,0}x_1^nx_2^{m-1}x_3^2\partial_1 \\ &\quad -c_{3,0,0}x_1^nx_2^mx_3\partial_1 + nd_{1,0,0}x_1^{n-1}x_2^mx_3\partial_1 \\ &\quad +mh_{2,0,0}x_1^nx_2^{m-1}x_3\partial_1 + g_{3,0,0}x_1^nx_2^m\partial_1 \\ &\quad +a_{2,0,0}x_1^nx_2^mx_3\partial_2 + a_{3,0,0}x_1^nx_2^mx_3\partial_3. \end{aligned} \tag{11}$$

By $D(x_3^k\partial_3 * x_1^nx_2^mx_3\partial_1) = D(x_1^nx_2^mx_3^k\partial_1)$, Lemma 1, and (11), we have that

$$\begin{aligned} D(x_1^nx_2^mx_3^k\partial_1) &= (1-n)a_{1,0,0}x_1^nx_2^mx_3^k\partial_1 - mb_{2,0,0}x_1^nx_2^mx_3^k\partial_1 \\ &\quad -kc_{3,0,0}x_1^nx_2^mx_3^k\partial_1 + nd_{1,0,0}x_1^{n-1}x_2^mx_3^k\partial_1 \\ &\quad +mh_{2,0,0}x_1^nx_2^{m-1}x_3^k\partial_1 + kg_{3,0,0}x_1^nx_2^mx_3^{k-1}\partial_1 \\ &\quad -ma_{2,0,0}x_1^{n+1}x_2^{m-1}x_3^k\partial_1 - ka_{3,0,0}x_1^{n+1}x_2^mx_3^{k-1}\partial_1 \\ &\quad -nb_{1,0,0}x_1^{n-1}x_2^{m+1}x_3^k\partial_1 - kb_{3,0,0}x_1^nx_2^{m+1}x_3^{k-1}\partial_1 \\ &\quad -nc_{1,0,0}x_1^{n-1}x_2^mx_3^{k+1}\partial_1 - mc_{2,0,0}x_1^nx_2^{m-1}x_3^{k+1}\partial_1 \\ &\quad +a_{2,0,0}x_1^nx_2^mx_3^k\partial_2 + a_{3,0,0}x_1^nx_2^mx_3^k\partial_3. \end{aligned} \tag{12}$$

Similarly, we can prove that

$$\begin{aligned}
D(x_1^n x_2^m x_3^k \partial_2) &= -na_{1,0,0}x_1^n x_2^m x_3^k \partial_2 + (1-m)b_{2,0,0}x_1^n x_2^m x_3^k \partial_2 \\
&\quad -kc_{3,0,0}x_1^n x_2^m x_3^k \partial_2 + nd_{1,0,0}x_1^{n-1} x_2^m x_3^k \partial_2 \\
&\quad +mh_{2,0,0}x_1^n x_2^{m-1} x_3^k \partial_2 + kg_{3,0,0}x_1^n x_2^m x_3^{k-1} \partial_2 \\
&\quad -ma_{2,0,0}x_1^{n+1} x_2^{m-1} x_3^k \partial_2 - ka_{3,0,0}x_1^{n+1} x_2^m x_3^{k-1} \partial_2 \\
&\quad -kb_{3,0,0}x_1^n x_2^{m+1} x_3^{k-1} \partial_2 - nc_{1,0,0}x_1^{n-1} x_2^m x_3^{k+1} \partial_2 \\
&\quad -mc_{2,0,0}x_1^n x_2^{m-1} x_3^{k+1} \partial_2 - nb_{1,0,0}x_1^{n-1} x_2^{m+1} x_3^k \partial_2 \\
&\quad +b_{1,0,0}x_1^n x_2^m x_3^k \partial_1 + b_{3,0,0}x_1^n x_2^m x_3^k \partial_3
\end{aligned}$$

$$\begin{aligned}
D(x_1^n x_2^m x_3^k \partial_3) &= -na_{1,0,0}x_1^n x_2^m x_3^k \partial_3 - mb_{2,0,0}x_1^n x_2^m x_3^k \partial_3 \\
&\quad +(1-k)c_{3,0,0}x_1^n x_2^m x_3^k \partial_3 + nd_{1,0,0}x_1^{n-1} x_2^m x_3^k \partial_3 \\
&\quad +mh_{2,0,0}x_1^n x_2^{m-1} x_3^k \partial_3 + kg_{3,0,0}x_1^n x_2^m x_3^{k-1} \partial_3 \\
&\quad -ma_{2,0,0}x_1^{n+1} x_2^{m-1} x_3^k \partial_3 - ka_{3,0,0}x_1^{n+1} x_2^m x_3^{k-1} \partial_3 \\
&\quad -nb_{1,0,0}x_1^{n-1} x_2^{m+1} x_3^k \partial_3 - kb_{3,0,0}x_1^n x_2^{m+1} x_3^{k-1} \partial_3 \\
&\quad -nc_{1,0,0}x_1^{n-1} x_2^m x_3^{k+1} \partial_3 - mc_{2,0,0}x_1^n x_2^{m-1} x_3^{k+1} \partial_3 \\
&\quad +c_{1,0,0}x_1^n x_2^m x_3^k \partial_1 + c_{2,0,0}x_1^n x_2^m x_3^k \partial_2
\end{aligned}$$

as (12). This completes the proof of the lemma. □

Note 1. For any basis element $x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u$, $1 \le u \le 3$, of $\overline{WN_{0,3,0_1}}$, we define **F**-maps D_v, $1 \le v \le 3$, $D_{v,w}$, $1 \le v \ne w \le 3$, and D_{9+v}, $1 \le v \le 3$, of $\overline{WN_{0,3,0_1}}$ as follows:

$$\begin{aligned}
&D_v(x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u) = (\delta_{u,v} - n_v) x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u \quad \text{for } 1 \le v \le 3, \\
&D_{v,w}(x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u) = -n_w x_v^{n_v+1} x_w^{n_w-1} x_k^{n_k} \partial_u + \delta_{u,v} x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_w \\
&\qquad\qquad \text{for } 1 \le v \ne w \le 3, k \notin \{v, w\}, \\
&D_{9+v}(x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u) = n_v x_v^{n_v-1} x_k^{n_k} x_w^{n_w} \partial_u \\
&\qquad\qquad \text{for } 1 \le v \le 3 \text{ and } 1 \le k \ne w \le 3, v \notin \{k, w\},
\end{aligned}$$

where $\delta_{u,v}$ is Kronecker delta. The **F**-maps D_v, $1 \le v \le 3$, $D_{v,w}$, $1 \le v \ne w \le 3$, and D_{9+v}, $1 \le v \le 3$, of $\overline{WN_{0,3,0_1}}$ can be linearly extended to non-associative algebra derivations of $\overline{WN_{0,3,0_1}}$.

THEOREM 1. *The additive group $Der(\overline{WN_{0,3,0_1}})$ of the non-associative algebra $\overline{WN_{0,3,0_1}}$ is spanned by D_v, $1 \le v \le 3$, $D_{v,w}$, $1 \le v \ne w \le 3$, and D_{9+v}, $1 \le v \le 3$, which are defined in Note 1, and which are inner* [5].

PROOF. By Lemma 4 and Note 1, any derivation D of $\overline{WN_{0,3,0_1}}$ can be written as linear sum of D_v, $1 \le v \le 3$, $D_{v,w}$, $1 \le v \ne w \le 3$, and D_{9+v}, $1 \le v \le 3$, which are defined in Note 1. This completes the proof of the theorem. □

Note 2. For any $x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_u$, $1 \le u \le s$, of $\overline{WN_{0,s,0_1}}$, we define **F**-maps $D_1, \dots, D_{s^2+s}$ of $\overline{WN_{0,s,0_1}}$ as follows:

$$
\begin{aligned}
&D_v(x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_u) = (\delta_{u,v} - n_v)x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_u \\
&\qquad\qquad\qquad\qquad \text{for } 1 \le v \le s \\
&D_{v,w}(x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_u) = -n_w x_1^{n_1}\cdots x_v^{n_v+1}x_{v+1}^{n_{v+1}}\cdots x_w^{n_w-1} \\
&\qquad\qquad \times x_{w+1}^{n_{w+1}}\cdots x_s^{n_s}\partial_u + \delta_{u,v}x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_w \\
&\qquad\qquad\qquad\qquad \text{for } 1 \le v \ne w \le s \\
&D_{s^2+v}(x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_u) = n_v x_1^{n_1}\cdots x_v^{n_v-1}x_{v+1}^{n_{v+1}}\cdots x_s^{n_s}\partial_u \\
&\qquad\qquad\qquad\qquad \text{for } 1 \le v \le s.
\end{aligned} \tag{13}
$$

Then the **F**-maps $D_1, \dots, D_s$, $D_{v,w}$, $1 \le v \ne w \le s$, $D_{s^2+v}, 1 \le v \le s$, can be linearly extended to non-associative algebra derivations of $\overline{WN_{0,s,0_1}}$.

THEOREM 2. *The additive group $Der(\overline{WN_{0,s,0_1}})$ of the non-associative algebra $\overline{WN_{0,s,0_1}}$ is spanned by D_u, $1 \le u \le s$, $D_{v,w}$, $1 \le v \ne w \le s$, and D_{s^2+v}, $1 \le v \le s$, which are defined in Note 2.*

PROOF. Let D be any derivation $Der(\overline{WN_{0,s,0_1}})$. For any element $x_q^{n_q}\partial_v$ of $\overline{WN_{0,s,0_1}}$, it is easy to prove that

$$D(\partial_v) = \sum_{1\le t\le s} c_t\partial_t \tag{14}$$

$$
\begin{aligned}
D(x_q^{n_q}\partial_v) \;=\; & a_{q,v}(\delta_{q,v} - n_q)x_q^{n_q}\partial_v + b_{q,v}n_q x_q^{n_q-1}\partial_v \\
& + \sum_{1\le w\le s}(1-\delta_{w,q})c_{q,w,v}n_q x_q^{n_q-1}x_w\partial_v \\
& + \sum_{1\le t\le s}(1-\delta_{t,v})d_{q,t,v}x_q^{n_q}\partial_t.
\end{aligned} \tag{15}
$$

Without loss of generality, we may assume that at least one of c_t, $1 \le t \le s$, is non-zero with appropriate scalars. For any basis element

$x_1^{n_1}\cdots x_s^{n_s}\partial_v$ of $\overline{WN_{0,s,0_1}}$, let us put the following formula:

$$D(x_1^{n_1}\cdots x_s^{n_s}\partial_v) = \sum_{1\le t\le s} a_{t,v}(\delta_{t,v}-n_t)x_1^{n_1}\cdots x_s^{n_s}\partial_v$$
$$+\sum_{1\le t\le s} b_{t,v}n_t x_1^{n_1}\cdots x_t^{n_t-1}x_{t+1}^{n_{t+1}}\cdots x_s^{n_s}\partial_v$$
$$+\sum_{1\le r\ne t\le s} c_{r,t,v}(n_t x_1^{n_1}\cdots x_r^{n_r+1}x_{r+1}^{n_{r+1}}\cdots x_t^{n_t-1}x_{t+1}^{n_{t+1}}\cdots x_s^{n_s}\partial_v$$
$$+\delta_{v,r}x_1^{n_1}\cdots x_s^{n_s}\partial_t) \tag{16}$$

with appropriate scalars. By (14) and (15), we know that D holds (14), (15) and (16) for ∂_u, $1\le u\le s$, and $x_q^{n_q}\partial_v$, $1\le q,v\le s$. Thanks to the similar proofs of Theorem 3.3 in [1], Theorem 1 in [2], and Theorem 1, for any basis element of $\overline{WN_{0,1,0_1}}$, $\overline{WN_{0,2,0_1}}$, and $\overline{WN_{0,3,0_1}}$, D holds (13) appropriately. Thus by induction on s of $\overline{WN_{0,s,0_1}}$, we assume that the theorem holds for $\overline{WN_{0,s-1,0_1}}$. This implies that by (14), (15) and (16), we can assume that the theorem holds for any $x_1^{n_1}\cdots x_s^{n_s}\partial_v$ of $\overline{WN_{0,s,0_1}}$ such that $deg_{tot}(x_1^{n_1}\cdots x_s^{n_s}\partial_v)\in \mathbf{N}$ by induction on the total degree of $x_1^{n_1}\cdots x_s^{n_s}\partial_v$ and a fixed positive integer n_s. We take a basis element $x_1^{n_1}\cdots x_s^{n_s+1}\partial_v$ such that $deg_s(x_1^{n_1}\cdots x_s^{n_s+1}\partial_v) = n_s+1$. By $D(\partial_s * x_1^{n_1}\cdots s_s^{n_s+1}\partial_v) = (n_s+1)D(x_1^{n_1}\cdots x_s^{n_s}\partial_v)$, $D(x_1^{n_1}\cdots x_s^{n_s+1}\partial_w * x_w\partial_v) = D(x_1^{n_1}\cdots\cdots x_s^{n_s+1}\partial_v)$ and by appropriate inductions, we can prove that $D(x_1^{n_1}\cdots x_s^{n_s+1}\partial_v)$ holds (13) routinely where $1\le w\le s$ and wD is the sum of derivations in Note 2 and let us omit the remaining steps of its proof because of routine calculations. □

COROLLARY 1. *If $s_1\ne s_2$, then the non-associative algebras $\overline{WN_{0,s_1,0_1}}$ and $\overline{WN_{0,s_2,0_1}}$ are not isomorphic to each other as non-associative algebras.*

PROOF. By Theorem 2, the dimension of $Der(\overline{WN_{0,s_1,0_1}})$ is $s_1^2+s_1$ and the dimension of $Der(\overline{WN_{0,s_2,0_1}})$ is $s_2^2+s_2$. Thus there is no isomorphism between them. This completes the proof of the corollary. □

COROLLARY 2. *$Der(\overline{WN_{0,s,0_1}})$ is isomorphic to $Der(\overline{WN_{0,0,s_1}})$ as additive groups.*

PROOF. The proof of the corollary is straightforward by Theorem 2 and Theorem 2 in the paper [4]. □

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