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DERIVATIONS OF SIMPLE C^*-ALGEBRAS, III

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1. In the previous paper [8], the author introduced the notion of the derived C^*-algebra of a simple C^*-algebra into the study of derivations on C^*-algebras — i. e. let A be a simple C^*-algebra. Then there exists one and only one primitive C^*-algebra $D(A)$ with unit (called the derived C^*-algebra of A) satisfying the following conditions. (1) A is a two-sided ideal of $D(A)$; (2) for every derivation δ on A, there is an element d (unique modulo scalar multiples of unit) in $D(A)$ such that $\delta(x)=[d, x]$ $(x \in A)$; (3) every derivation of $D(A)$ is inner.

If A has a unit, then $A=D(A)$, so that $D(A)/A=(0)$.

In the present paper, we shall show that for an arbitrary finite-dimensional C^*-algebra B, there exists a simple C^*-algebra A such that $D(A)/A=B$. In particular, there is a simple C^*-algebra A such that $D(A)/A$ is one dimensional and so there is a simple C^*-algebra without unit in which all derivations are inner.

Also, some problems on derived C^*-algebras are stated.

2. Construction of examples. Let A be a simple C^*-algebra, and let L be a closed left ideal of A. Then $L \cap \widetilde{L}$ is a C^*-subalgebra of A, where $\widetilde{L}=\{x^* \mid x \in L\}$.

PROPOSITION 1. *$L \cap \widetilde{L}$ is a simple C^*-algebra.*

PROOF. Let A^* be the dual Banach space of A, and let A^{**} be the second dual of A. Then A^{**} is a W^*-algebra and A is a $\sigma(A^{**}, A^*)$-dense C^*-subalgebra of A^{**}, when A is canonically embedded into A^{**} (cf. [9]). Let $L^{\circ\circ}$ (resp. $(L \cap \widetilde{L})^{\circ\circ}$) be the bipolar of L (resp. $(L \cap \widetilde{L})$) in A^{**}. Then $L^{\circ\circ}$ is a $\sigma(A^{**}, A^*)$-closed left ideal of A^{**}; hence there is a projection e in A^{**} such that $L^{\circ\circ}=A^{**}e$. For $x \in L$, $x^*x \in L \cap \widetilde{L}$ and so $(L \cap \widetilde{L})^{\circ\circ}=eA^{**}e$. In fact, it is clear that $(L \cap \widetilde{L})^{\circ\circ} \subset eA^{**}e$. Suppose that $(L \cap L)^{\circ\circ} \subsetneqq eA^{**}e$; then there exists a self-adjoint element f of A^* such that $f(L \cap \widetilde{L})=0$, but $f(eA^{**}e) \neq (0)$. Since $f(x^*x)=0$ for $x \in L$ and since $y^*x=(1/4)\{(y+x)^*(y+x)-(y-x)^*(y-x)-i(y+ix)^*(y+ix)+i(y-ix)^*(y-ix)\}$

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for x, $y \in L$, $f(y^*x)=0$ for $x,y \in L$.

Take a directed set (x_α) in L such that $\sigma(A^{**}, A^*)$-$\lim x_\alpha = e$; then $f(y^*e)=0$ for $y \in L$ and so $f(\widetilde{L}e)=0$, so that $f(eA^{**}e)=0$, a contradiction.

Now suppose that $L\cap\widetilde{L}$ is not simple; then there exists a non-zero proper closed ideal I of $L\cap\widetilde{L}$. Then the bipolar $I^{\circ\circ}$ of I in A^{**} is a $\sigma(A^{**}, A^*)$-closed ideal of $eA^{**}e$; hence there exists a central projection p of $eA^{**}e$ such that $I^{\circ\circ}=eA^{**}ep$. On the other hand, the center of $eA^{**}e=Ze$, where Z is the center of A^{**}; hence there exists a central projection z of A^{**} such that $I^{\circ\circ}=eA^{**}ez$. Therefore the bipolar $(AIA)^{\circ\circ}$ of AIA is contained in $A^{**}z$, where AIA is the closed linear subspace of A generated by $\{axb \mid a, b \in A, x \in I\}$. Since AIA is a non-zero ideal of A and A is simple, $AIA=A$ and so $z=1$; this implies that $I^{\circ\circ}=eA^{**}e$ and so $I=L\cap\widetilde{L}$, a contradiction. This completes the proof.

THEOREM 1. *Let N be a type II_1-factor or a countably decomposable type III-factor, and let M be a maximal left ideal of M. Then $M\cap\widetilde{M}$ is a simple C^*-algebra without unit and the quotient C^*-algebra $D(M\cap\widetilde{M})/M\cap\widetilde{M}$ is one-dimensional, where $\widetilde{M}=\{x^* \mid x \in M\}$.*

PROOF. It is well known that N is a simple C^*-algebra with unit. Therefore by Proposition 1, $M\cap\widetilde{M}$ is a simple C^*-algebra. $M\cap\widetilde{M}$ does not have a unit; in fact, if $M\cap\widetilde{M}$ has a unit e, then e is a projection of M. Since $Ne=M$, $(1-e)N(1-e)$ is one-dimensional and so N is a type I-factor, a contradiction. Let ρ be the identical mapping of $M\cap\widetilde{M}$ in $D(M\cap\widetilde{M})$ onto $M\cap\widetilde{M}$ in N. Since $M\cap\widetilde{M}$ is a two-sided ideal of $D(M\cap\widetilde{M})$, ρ can be extended to a *-homomorphism (denoted again by ρ) of $D(M\cap\widetilde{M})$ into N (cf. [1], [8]). Since $D(M\cap\widetilde{M})$ is primitive and $M\cap\widetilde{M}$ is simple, the extended ρ must be a *-isomorphism. Therefore we may identify $D(M\cap\widetilde{M})$ with $\rho(D(M\cap\widetilde{M}))$; then we have $M\cap\widetilde{M}\subset D(M\cap\widetilde{M})\subset N$. If $D(M\cap\widetilde{M})/M\cap\widetilde{M}$ is not one-dimensional, there is a non-zero commutative C^*-subalgebra C of $D(M\cap\widetilde{M})/M\cap\widetilde{M}$ which does not contain the unit of $D(M\cap\widetilde{M})/M\cap\widetilde{M}$. Let C_1 be the inverse image of C in $D(M\cap\widetilde{M})$. Then C_1 is a C^*-subalgebra of N which does not contain the unit of N. Since $1 \notin C_1$, $\|1-x\| \geqq 1$ for $x \in C_1$; hence there exists a bounded linear functional φ on N such that $\varphi(C_1)=0$ and $\varphi(1)=\|\varphi\|=1$. Then φ is a state (cf. [1]). Let $M_\varphi=\{x \mid \varphi(x^*x)=0, x \in N\}$; then $M\cap\widetilde{M}\subset C_1\subset M_\varphi$. For $x\in M$, $x^*x \in M\cap\widetilde{M}$, so that $x^*x \in M_\varphi$; hence $\varphi(x^*x) \leqq \varphi(1)^{1/2}\,\varphi((x^*x)^2)^{1/2}=0$. Therefore $M\subset M_\varphi$. Since M is maximal, $M=M_\varphi$ and so $C_1=M\cap\widetilde{M}$, a contradiction. Hence $D(M\cap\widetilde{M})/M\cap\widetilde{M}$ is one-dimensional. This completes the proof.

The above C^*-algebra $M\cap\widetilde{M}$ has the following remarkable properties.

COROLLARY 1. *Let A be a C^*-algebra. Suppose that $(M\cap\widetilde{M})\otimes A$ is $*$-isomorphic to $M\cap\widetilde{M}$; then A is the field of all complex numbers, where $\otimes$ is the C^*-tensor product.*

PROOF. Since $(M\cap\widetilde{M})\otimes A$ is $*$-isomorphic to $M\cap\widetilde{M}$, A is simple. Clearly, $D((M\cap\widetilde{M})\otimes A)\supset D(M\cap\widetilde{M})\otimes A\supset 1\otimes A$. Hence we have $1\otimes A=1\otimes(\lambda 1)$ (λ, complex numbers) and so A is the field of complex numbers. This completes the proof.

COROLLARY 2. *Let A_1, A_2 be two C^*-algebras. Suppose that $M\cap\widetilde{M}=A_1\otimes A_2$. Then A_1 or A_2 is the field of complex numbers.*

PROOF. Clearly, A_1 and A_2 are simple; moreover either of them is a C^*-algebra without unit. Suppose that A_1 does not have a unit. Since $D(M\cap\widetilde{M})=D(A_1\otimes A_2)\supset D(A_1)\otimes D(A_2)\supsetneqq A_1\otimes D(A_2)\supset A_1\otimes A_2=M\cap\widetilde{M}$. Hence $A_1\otimes D(A_2)=A_1\otimes A_2$; therefore $D(A_2)=A_2$. If A_2 is not one-dimensional, $\dim(D(A_1\otimes A_2)/A_1\otimes A_2)\geqq\dim(1\otimes A_2)$, a contradiction. This completes the proof.

The following problem is interesting.

PROBLEM 1. Let A be an infinite-dimensional simple C^*-algebra with unit, and let M be a maximal left ideal of A. Then can we conclude that $D(M\cap\widetilde{M})/\widetilde{M}\cap M$ is one-dimensional?

If A is an infinite-dimensional simple C^*-algebra with unit, then it is not a type I C^*-algebra and so it has a type III-factor $*$-representation ([3], [6]). If the following problem is affirmative, the problem 1 is affirmative.

PROBLEM 2. Let B be an arbitrary C^*-algebra which contains the C^*-algebra A in the problem 1 as a proper C^*-subalgebra. Then, can we conclude that there exists a $*$-representation $\{\pi,\mathfrak{H}\}$ of B on a Hilbert space $\mathfrak{H}$ such that $\overline{\pi(A)}$ is a type II (or III) W^*-algebra and $\overline{\pi(A)}\subsetneqq\overline{\pi(B)}$, where $\overline{\pi(A)}$ (resp. $\overline{\pi(B)}$) is the weak closure of $\pi(A)$ (resp. $\pi(B)$)?

Next we shall construct a simple C^*-algebra A such that $D(A)/A$ is a type I_n-factor ($n=1,2,\cdots$).

PROPOSITION 2. *Let B_n be a type I_n-factor ($n=1,2,\cdots$), and let A be a simple C^*-algebra. Then $D(A\otimes B_n)=D(A)\otimes D(B_n)$.*

PROOF. It is clear that $D(A\otimes B_n)\supset D(A)\otimes D(B_n)=D(A)\otimes B_n$. Let $\{\pi,\mathfrak{H}\}$ be an irreducible $*$-representation of A on a Hilbert space $\mathfrak{H}$. Then $\overline{\pi(A)}\otimes B_n$ is a W^*-algebra, where $\overline{\pi(A)}$ is the weak closure of $\pi(A)$; hence $\overline{\pi(A)}\otimes B_n$

$\supset D(\pi(A)\otimes B_n)$ ([5]). Since $\overline{\pi(A)}\otimes B_n$ can be considered as the matrix algebra of all $n\times n$ matrices over the algebra $\overline{\pi(A)}$, for $d\in D(\pi(A)\otimes B_n)$ there is an element $(a_{ij})(a_{ij}\in\overline{\pi(A)})$ in $\overline{\pi(A)}\otimes B_n$ such that $[d,(x_{ij})]=[(a_{ij}),(x_{ij})]$, where $x_{ij}\in\pi(A)$. Put $x_{ij}=\delta_{ij}a$ $(a\in\pi(A))$, where δ_{ij} is the Kronecker symbol; then $[d,(\delta_{ij}a)]=[(a_{ij}),(\delta_{ij}a)]=([a_{ij},a])$. Hence $[a_{ij},a]\in\pi(A)$ $(i,j=1,2,\cdots,n)$ and so $a_{ij}\in D(\pi(A))$. This completes the proof.

REMARK. In Proposition 2, we can not replace the algebra B_n by an arbitrary simple C^*-algebra – for example, let $C(\mathfrak{H})$ be the C^*-algebra of all compact operators on an infinite-dimensional Hilbert space $\mathfrak{H}$; then $D(C(\mathfrak{H}))=B(\mathfrak{H})$, where $B(\mathfrak{H})$ is the C^*-algebra of all bounded operators on $\mathfrak{H}$, and $C(\mathfrak{H})\otimes C(\mathfrak{H})=C(\mathfrak{H}\otimes\mathfrak{H})$. On the other hand, $D(C(\mathfrak{H})\otimes C(\mathfrak{H}))=B(\mathfrak{H}\otimes\mathfrak{H})$ and $D(C(\mathfrak{H}))\otimes D(C(\mathfrak{H}))=B(\mathfrak{H})\otimes B(\mathfrak{H})$.

The following problem is interesting.

PROBLEM 3. Let A be a simple C^*-algebra with unit. Then, can we conclude that $D(A\otimes B)=D(A)\otimes D(B)$, where B is a simple C^*-algebra?

COROLLARY 3. *Let $M\cap\widetilde{M}$ be the simple C^*-algebra in Theorem 1, and let B_n be a type I_n-factor $(n=1,2,\cdots)$. Then $D((M\cap\widetilde{M})\otimes B_n)/(M\cap\widetilde{M})\otimes B_n$ is a type I_n-factor $(n=1,2,\cdots)$.*

PROOF. By Proposition 2, $D((M\cap\widetilde{M})\otimes B_n)=D(M\cap\widetilde{M})\otimes B_n$. Hence $D((M\cap\widetilde{M})\otimes B_n)/(M\cap\widetilde{M})\otimes B_n=1\otimes B_n$. This completes the proof.

Now we shall show a generalization of Theorem 1.

THEOREM 2. *Let N be a type II_1-factor or a countably decomposable type III-facor, and let $\{\pi_i,\mathfrak{H}_i\}$ $(i=1,2,\cdots,n)$ be a finite family of mutually inequivalent irreducible *-representations of N. Let $\mathfrak{K}_1,\mathfrak{K}_2,\cdots,\mathfrak{K}_n$ be finite dimensional linear subspaces of $\mathfrak{H}_1,\mathfrak{H}_2,\cdots,\mathfrak{H}_n$ respectively, and let $L=\{x\,|\,\pi_i(x)\mathfrak{K}_i=0,\ i=1,2,\cdots,n;\ x\in N\}$. Then $L\cap\widetilde{L}$ is a simple C^*-algebra such that $D(L\cap\widetilde{L})/L\cap\widetilde{L}=\sum_{i=1}^{n}\oplus B(\mathfrak{K}_i)$, where $B(\mathfrak{K}_i)$ is the C^*-algebra of all bounded operators on $\mathfrak{K}_i$.*

PROOF. Let $\mathfrak{H}=\sum_{i=1}^{n}\oplus\mathfrak{H}_i$, $\mathfrak{K}=\sum_{i=1}^{n}\oplus\mathfrak{K}_i$ and $\pi=\sum_{i=1}^{n}\pi_i$, and let E be the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{K}$. Let $A=\{x\,|\,\pi(x)E=E\pi(x),x\in N\}$; then A is a C^*-subalgebra of N with unit. If $x\in A$ with $\pi(x)E=0$ and $x^*=x$, then $x\in L\cap\widetilde{L}$; conversely if $x\in L\cap\widetilde{L}$ with $x^*=x$, then $\pi(x)E=0$ and so $E\pi(x)=(\pi(x)E)^*=0$, so that $x\in A$. Therefore $L\cap\widetilde{L}=\{x\,|\,\pi(x)E=0,x\in A\}$. Moreover if $x \in A$, then $\pi(y)\pi(x)E = \pi(y)E\pi(x) = 0$ for $y \in L \cap \widetilde{L}$; hence $yx \in L \cap \widetilde{L}$, and analogously $xy \in L \cap \widetilde{L}$. Therefore $L \cap \widetilde{L}$ is a two-sided ideal of A. On the other hand, $D(L \cap \widetilde{L})$ can be realized as a C^*-subalgebra of N, since $L \cap \widetilde{L}$ is a two-sided ideal of $D(L \cap \widetilde{L})$.

Since $L \cap \widetilde{L}$ is weakly dense in the W^*-algebra $N, A \subset D(L \cap \widetilde{L})$. Since the weak closure of $\pi(L \cap \widetilde{L})$ on $\mathfrak{H}$ is $(1_{\mathfrak{H}} - E)\overline{\pi(N)}(1_{\mathfrak{H}} - E)$, where $1_{\mathfrak{H}}$ is the identity operator on $\mathfrak{H}$ and $\overline{\pi(N)}$ is the weak closure of $\pi(N)$ on $\mathfrak{H}$, and since $L \cap \widetilde{L}$ is a two-sided ideal of $D(L \cap \widetilde{L})$, for $y \in D(L \cap \widetilde{L})$, $\pi(y)(1_{\mathfrak{H}} - E)$, $(1_{\mathfrak{H}} - E)\pi(y) \in (1_{\mathfrak{H}} - E) \cdot \overline{\pi(N)}(1_{\mathfrak{H}} - E)$, and so $(1_{\mathfrak{H}} - E)\pi(y)(1_{\mathfrak{H}} - E) = \pi(y)(1_{\mathfrak{H}} - E) = (1_{\mathfrak{H}} - E)\pi(y)$; hence $y \in A$ and so $D(L \cap \widetilde{L}) = A$.

Now by Kadison's theorem [1], for an arbitrary self-adjoint element H of $\sum_{i=1}^{n} \oplus B(\mathfrak{K}_i)$, there exists a self-adjoint element h in N such that $\pi(h)E = HE$. Since $EHE = HE$, $(\pi(h)E)^* = E\pi(h) = \pi(h)E$; hence $h \in A$. Therefore the $*$-homomorphism $y \to \pi(y)E$ of A into $\sum_{i=1}^{n} \oplus B(\mathfrak{K}_i)$ is onto, and its kernel is $L \cap \widetilde{L}$. Hence $D(L \cap \widetilde{L}) / L \cap \widetilde{L} = \sum_{i=1}^{n} \oplus B(\mathfrak{K}_i)$. This completes the proof.

COROLLARY 4. *For an arbitray finite-dimensional C^*-algebra B, there exists a simple C^*-algebra A such that $D(A)/A = B$.*

Since the algebra N in Theorem 2 has uncountably many inequivalent irreducible $*$-representations, this is clear.

Now the following problems are interesting.

PROBLEM 4. In Theorem 2, can we replace the algebra N by an arbitrary infinite-dimensional simple C^*-algebra with unit ?

PROBLEM 5. For an arbitrary commutative C^*-algebra C with unit, does there exist a simple C^*-algebra A such that $D(A)/A = C$?

PROBLEM 6. For an arbitrary simple C^*-algebra B with unit, does there exist a simple C^*-algebra A such that $D(A)/A = B$?

This problem is closely related to Problem 3.

PROBLEM 7. For an arbitrary C^*-algebra B with unit, does there exist a simple C^*-algebra A such that $D(A)/A = B$?

PROBLEM 8. Investigate the derived C^*-algebras of matroid C^*-algebras (cf. [2]).

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After writing this paper, the author found that the problems 1, 2 and 4 are negative for arbitrary uniformly hyperfinite C^*-algebra. Next, G. Elliot proved more generally that the problems 1, 2 and 4 are negative for arbitrary infinite-dimensional separable simple C^*-algebra with unit.

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