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DERIVATIONS ON SEMIPRIME RINGS

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The main result: Let R be a 2-torson free semiprime ring and let $D\colon R \to R$ be a derivation. Suppose that $[[D(x), x], x] = 0$ holds for all $x \in R$. In this case $[D(x), x] = 0$ holds for all $x \in R$.

Throughout, R will represent an associative ring with centre $Z(R)$. The commutator $xy - yx$ will be denoted by $[x, y]$. We make extensive use of the basic commutator identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation D is inner if there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. Recall that R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is called semiprime in case $aRa = (0)$ implies $a = 0$. A mapping F of R into itself is called centralising on R if $[F(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[F(x), x] = 0$ holds for all $x \in R$, the mapping F is said to be commuting on R. The study of centralising and commuting mappings was initiated by the classical result of Posner [**6**], which states that the existence of a nonzero centralising derivation on a prime ring forces the ring to be commutative (Posner's second theorem). A lot of work has been done during the last twenty years in this field (see [**2, 3, 4**] where further references can be found). It is our aim in this paper to present some results which can be considered as a contribution to the theory of commuting and centralising mappings in semiprime rings.

THEOREM 1. *Let R be a 2-torsion free semiprime ring. Suppose there exists a derivation $D\colon R \to R$, such that the mapping $x \mapsto [D(x), x]$ is commuting on R. In this case D is commuting on R.*

PROOF: Let us introduce a mapping $B(.,.)\colon R \times R \to R$ by the relation $B(x, y) = [D(x), y] + [D(y), x]$, $x, y \in R$. Obviously, $B(.,.)$ is symmetric (that is $B(x, y) = B(y, x)$ for all $x, y \in R$) and additive in both arguments. A routine calculation shows that the relation

$$B(xy, z) = B(x, z)y + xB(y, z) + D(x)[y, z] + [x, z]D(y) \tag{1}$$

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holds for all $x, y, z \in R$. We introduce also a mapping f from R to R by $f(x) = B(x, x)$. We have $f(x) = 2[D(x), x]$, $x \in R$. The mapping f satisfies the relation

$$f(x+y) = f(x) + f(y) + 2B(x, y), \quad x, y \in R. \tag{2}$$

Throughout the paper we shall use the mappings B and f, as well as the relations (1) and (2) without specific references. The assumption of the theorem can now be written in the form

$$[f(x), x] = 0, \ x \in R. \tag{3}$$

The linearisation of (3) gives

$$[f(x), y] + [f(y), x] + 2[B(x, y), x] + 2[B(x, y), y] = 0, \quad x, y \in R.$$

Substituting in the above relation x by $-x$ and comparing the relation so obtained with the above relation we arrive at

$$[f(x), y] + 2[B(x, y), x] = 0, \quad x, y \in R. \tag{4}$$

Replace y in (4) by xy. Then $0 = [f(x), xy] + 2[B(xy, x), x] = [f(x), xy] + 2[f(x)y + xB(x, y) + D(x)[y, x], x] = [f(x), x]y + x[f(x), y] + 2[f(x), x]y + 2f(x)[y, x] + 2x[B(x, y), x] + f(x)[y, x] + 2D(x)[[y, x], x]$, which reduces to

$$3f(x)[y, x] + 2D(x)[[y, x], x] = 0, \quad x, y \in R, \tag{5}$$

according to (3) and (4). In the same fashion one obtains

$$3[y, x]f(x) + 2[[y, x], x]D(x) = 0, \quad x, y \in R, \tag{6}$$

replacing y in (4) by yx. Let us prove the relation

$$3f(x)D(x) = D(x)f(x), \quad x \in R. \tag{7}$$

For this purpose we substitute (5) yz for y in (5). Then $0 = 3f(x)[yz, x] + 2D(x)[[yz, x], x] = 3f(x)[yz, x] + 2D(x)[[y, x]z + y[z, x], x] = 3f(x)[y, x]z + 3f(x)y[z, x] + 2D(x)[[y, x], x]z + 4D(x)[y, x][z, x] + 2D(x)y[[z, x], x]$, whence it follows that

$$3f(x)y[z, x] + 4D(x)[y, x][z, x] + 2D(x)y[[z, x], x] = 0, \quad x, y, z \in R, \tag{8}$$

and in particular for $y = D(x)$, $z = y$

$$3f(x)D(x)[y, x] + 2D(x)f(x)[y, x] + 2D(x)^2[[y, x], x] = 0, \quad x, y \in R. \tag{9}$$

Left multiplication of (5) by $D(x)$ gives

$$(10)\qquad 3D(x)f(x)[y,x]+2D(x)^2[[y,x],x]=0,\quad x,y\in R.$$

From (9) and (10) it follows that $(3f(x)D(x)-D(x)f(x))[y,x]=0,\quad x,y\in R,$ which gives

$$(11)\qquad (3f(x)D(x)-D(x)f(x))y[z,x]=0.\quad x,y,z\in R.$$

By putting $z=2D(x)$ in (11) we obtain

$$(12)\qquad (3f(x)D(x)-D(x)f(x))yf(x)=0,\quad x,y\in R.$$

Right multiplication of (12) by $3D(x)$ gives

$$(13)\qquad (3f(x)D(x)-D(x)f(x))y3f(x)D(x)=0,\quad x,y\in R.$$

Putting $yD(x)$ for y in (12) we arrive at

$$(14)\qquad (3f(x)D(x)-D(x)f(x))yD(x)f(x)=0,\quad x,y\in R.$$

By subtracting (14) from (13) we obtain

$$(15)\qquad (3f(x)D(x)-D(x)f(x))y(3f(x)D(x)-D(x)f(x))=0,\quad x,y\in R,$$

which proves (7) by semiprimeness of R. In the same fashion one obtains

$$(16)\qquad 3D(x)f(x)=f(x)D(x),\quad x\in R,$$

starting from (6). From (15) and (16) it follows immediately

$$(17)\qquad f(x)D(x)=0,\quad x\in R,$$

and

$$(18)\qquad D(x)f(x)=0,\quad x\in R.$$

From (17) one obtains easily

$$(19)\qquad f(x)D(y)+2B(x,y)D(x)=0,\quad x,y\in R.$$

By substituting xy for y in (19) we obtain

$$\begin{aligned}0&=f(x)D(xy)+2B(xy,x)D(x)\\&=f(x)D(x)y+f(x)xD(y)+2f(x)yD(x)+2xB(x,y)D(x)+2D(x)[y,x]D(x).\end{aligned}$$

According to (18) we can write $-f(x)D(y)$ instead of $2B(x, y)D(x)$ in the above calculation which, gives $[f(x), x]D(y) + 2f(x)yD(x) + 2D(x)[y, x]D(x) = 0$ whence it follows that

$$f(x)yD(x) + D(x)[y, x]D(x) = 0, \quad x, y \in R, \tag{20}$$

according to (3). Replacing in (20) y by yx we obtain

$$f(x)yxD(x) + D(x)[y, x]xD(x) = 0, \quad x, y \in R. \tag{21}$$

Right multiplication of (20) by x gives

$$f(x)yD(x)x + D(x)[y, x]D(x)x = 0, \quad x, y \in R. \tag{22}$$

By subtracting (21) from (22) we obtain

$$f(x)yf(x) + D(x)[y, x]f(x) = 0, \quad x, y \in R. \tag{23}$$

By putting $z = 2D(x)$ in (8), we obtain

$$3f(x)yf(x) + 4D(x)[y, x]f(x) = 0, \quad x, y \in R. \tag{24}$$

From (23) and (24) it follows that $f(x)yf(x) = 0$, $x, y \in R$, whence $[D(x), x] = 0$, $x \in R$. The proof of the theorem is complete. ∎

In our earlier paper [7] one can find an extension of Posner's second theorem which states that there is no nonzero derivation on a noncommutative prime ring of characteristic different from two satisfying the relation $[[D(x), x], x] = 0$ for all $x \in R$. This result has been generalised by Brešar [2] and Lanski [5]. Since in noncommutative semiprime rings there exist nonzero commuting derivations, the assumptions of Theorem 1 do not imply that $D = 0$. However, in the special case when D is an inner derivation, one can prove the following result.

Corollary 2. *Let R be a 2-torsion free noncommutative semiprime ring. Suppose there exists an inner derivation $D\colon R \to R$, such that the mapping $x \mapsto [D(x), x]$ is commuting on R. In this case $D = 0$.*

Proof: An immediate consequence of Theorem 1 and Corollary 5 in [1]. ∎

We continue with the following result.

Theorem 3. *Let R be a 2-torsion free and 3-torsion free semiprime ring and $D\colon R \to R$ a derivation. Suppose that the mapping $x \mapsto [D(x), x]$ is centralising on R. In this case D is commuting on R.*

The result above is not a generalisation of Theorem 1, since in Theorem 3 we have an additional assumption that R is 3-torsion free. We feel that this assumption can be avoided, but unfortunately we were unable to do it.

PROOF OF THEOREM 3: According to the assumption of the theorem we have

$$[f(x),\,x] \in Z(R), \quad x \in R. \tag{25}$$

Similarly, as in the proof of identity (4) one can prove that

$$[f(x),\,y] + 2[B(x,\,y),\,x] \in Z(R), \quad x,\,y \in R. \tag{26}$$

The substitution $y = x^2$ in (16) gives $[f(x),\,x^2] + 2[f(x)x + xf(x),\,x] \in Z(R)$, and $[f(x),\,x]x + x[f(x),\,x] + 2[f(x),\,x] + 2[f(x),\,x]x + 2x[f(x),\,x] \in Z(R)$. Thus we have

$$6[f(x),\,x]x \in Z(R), \quad x \in R. \tag{27}$$

From (24) and (26) it follows $6[f(x),\,x][y,\,x] = 0$, $x,\,y \in R$, which gives

$$[f(x),\,x][y,\,x] = 0, \; x,\,y \in R,$$

since we have assumed that R is 2-torsion free and 3-torsion free. By putting $yf(x)$ for y in the above relation we obtain

$$[f(x),\,x]y[f(x),\,x] = 0, \; x,\,y \in R,$$

whence it follows that $[f(x),\,x] = 0, \quad x \in R$ which completes the proof of the theorem since all the requirements of Theorem 1 are fulfilled. ∎

Our next result was inspired by Posner's first theorem [6], which asserts that if R is a prime ring of characteristic different from two, and D, G are nonzero derivations on R, then DG cannot be a derivation.

THEOREM 4. *Let R be a 2-torsion free semiprime ring and $D\colon R \to R$, $G\colon R \to R$ derivations. Suppose that the mapping $x \mapsto D^2(x) + G(x)$ is centralising on R. In this case D and G are both commuting on R.*

PROOF OF THEOREM 4: By Proposition 3.1 in [4], the mapping $x \mapsto D^2(x) + G(x)$ is commuting on R. Thus we have

$$[F(x),\,x] = 0, \quad x \in R, \tag{28}$$

where $F(x)$ stands for $D^2(x) + G(x)$. The linearisation of (28) leads to

$$[F(x),\,y] + [F(y),\,x] = 0, \quad x,\,y \in R. \tag{29}$$

By substituting for y in (29) yx and noting that

$$F(yx) = F(y)x + yF(x) + 2D(y)D(x), \quad x,\,y \in R,$$

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we then obtain

$$\begin{aligned}0 &= [F(x), yx] + [F(yx), x] = [F(x), yx] + [F(y)x + yF(x) + 2D(y)D(x), x]\\ &= [F(x), y]x + y[F(x), x] + [F(y), x]x + [y, x]F(x) + y[F(x), x]\\ &\quad + 2[D(y), x]D(x) + 2D(y)[D(x), x].\end{aligned}$$

According to (28) and (29) the above relation reduces to

(30) $$[y, x]F(x) + 2[D(y), x]D(x) + 2D(y)[D(x), x] = 0, \quad x, y \in R.$$

By replacing y by xy in (30) we obtain

$$\begin{aligned}0 &= [xy, x]F(x) + 2[D(xy), x]D(x) + 2D(xy)[D(x), x]\\ &= [xy, x]F(x) + 2[D(x)y + xD(y), x]D(x) + 2D(xy)[D(x), x]\\ &= x[y, x]F(x) + 2[D(x), x]yD(x) + 2D(x)[y, x]D(x) + 2x[D(y), x]D(x)\\ &\quad + 2D(x)y[D(x), x] + 2xD(y)[D(x), x].\end{aligned}$$

Using (30) it follows from the above calculation that

(31) $$[D(x), x]yD(x) + D(x)[y, x]D(x) + D(x)y[D(x), x] = 0, \quad x, y \in R.$$

Let us replace y by $yD(x)z$ in (31). Then

$$\begin{aligned}0 &= [D(x), x]yD(x)zD(x) + D(x)[yD(x)z, x]D(x) + D(x)yD(x)z[D(x), x]\\ &= [D(x), x]yD(x)zD(x) + D(x)[y, x]D(x)zD(x) + D(x)y[D(x), x]zD(x)\\ &\quad + D(x)yD(x)[z, x]D(x) + D(x)yD(x)z[D(x), x],\end{aligned}$$

which because of (31) reduces to $D(x)yD(x)[z, x]D(x) + D(x)yD(x)z[D(x), x] = 0$, and using again (31) to

$$D(x)y[D(x), x]zD(x) = 0, \quad x, y, x \in R.$$

Right multiplication of the above relation by $y[D(x), x]$ give

$$D(x)y[D(x), x]zD(x)y[D(x), x] = 0, \quad x, y, x \in R,$$

whence it follows that

(32) $$D(x)y[D(x), x] = 0, \quad x, y \in R,$$

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because of the semiprimeness of R. By replacing y by xy in (32) we obtain

(33) $$D(x)xy[D(x), x] = 0, \quad x, y \in R.$$

Left multiplication of (32) by x gives

(34) $$xD(x)y[D(x), x] = 0, \quad x, y \in R.$$

From (33) and (34) it follows that

$$[D(x), x]y[D(x), x] = 0, \quad x, y \in R,$$

which means that

(35) $$[D(x), x] = 0, \quad x \in R.$$

The proof of the first part of the theorem is complete. The linearisation of (35) gives $[D(x), y] + [D(y), x] = 0$, $x, y \in R$ and in particular for $y = D(x)$

$$[D^2(x), x] = 0, \quad x \in R.$$

Using the above relation in (28) we obtain that $[G(x), x] = 0$, $x \in R$, which completes the proof of the theorem. □

The result below is an immediate consequence of Theorem 4 and Corollary 5 in [**1**].

Corollary 5. *Let R be a 2-torsion free noncommutative semiprime ring and let $D\colon R \to R$, $G\colon R \to R$ be derivations. Suppose that the mapping $x \mapsto D^2(x)+G(x)$ is centralising on R. If D is inner we have that $D = 0$. If G is inner then $G = 0$.*

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