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DERIVATIVE BLOW-UP AND BEYOND FOR QUASILINEAR PARABOLIC EQUATIONS

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Dedicated to the memory of Peter Hess

Abstract. L^∞-blow-up of solutions of semilinear parabolic equations has received considerable interest. Several major problems like sufficient conditions for blow-up, the form of the blow-up set, the profile of the solution near a blow-up point or the existence after the blow-up time have been studied. The aim of this paper is to deal with similar questions for a related phenomenon, namely blow-up of the spatial derivative while the solution itself stays bounded. We proceed via the maximum and comparison principles.

1. Introduction. Finite time blow up in the L^∞ norm for solutions of semilinear parabolic equations has received considerable interest. Several major problems like sufficient conditions for blow-up, the form of the blow-up set, the profile of the solutions near a blow-up point, and existence of a (suitable extension of the) solution past the blow-up time have been studied. The aim of this paper is to deal with similar questions for a related phenomenon, namely, blow-up of the spatial derivative while the solution stays bounded.

Specifically, we consider the problem

$$u_t = u_{xx} + f(u_x), \qquad 0 < x < L, \quad t > 0, \tag{1.1}$$

$$u(0,t) = u(L,t) = 0, \qquad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad 0 \le x \le L, \tag{1.3}$$

where $f \in C^2(\mathbb{R})$ satisfies the conditions

$$f(v) > 0 \text{ for all } v, \;\; f'(v) \ge 0 \;\text{ for }\; v \;\text{ large enough}, \tag{1.4}$$

$$\limsup_{v\to\infty} f'(v)/f(v) < \infty, \tag{1.5}$$

$$\int^\infty \frac{v\,dv}{f(v)} < \infty, \tag{1.6}$$

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and either

$$\int_{-\infty} \frac{dv}{f(v)} < \infty \quad \text{and} \quad \int_{-\infty} \frac{v\,dv}{f(v)} = -\infty \tag{1.7}$$

or

$$\int_{-\infty} \frac{dv}{f(v)} = \infty. \tag{1.8}$$

For examples, $f(v) = e^v$ satisfies (1.4)–(1.6) and (1.8), while $f(v) = e^v + v^2$ satisfies (1.4)–(1.7).

It is known (see [10]) that the conditions

$$\int^{\infty} \frac{v\,dv}{f(v)} = -\int_{-\infty} \frac{v\,dv}{f(v)} = \infty$$

imply that gradient blow-up cannot occur while the form of (1.1)–(1.3) implies that L^∞ blow-up cannot occur. On the other hand, there are examples of gradient blow-up on the boundary under conditions like (1.6) (see e.g. [3,6–8,10]). However, nonhomogeneous (in most cases time-dependent) Dirichlet boundary conditions were used to construct the examples there.

As far as we know, no results concerning the blow-up profile have been established previously. Concerning the continuation beyond the blow-up time, an existence and uniqueness result (in the viscosity sense) was announced in [7].

We show that for the problem (1.1)–(1.3) there is $L_0 > 0$ (depending on f) such that if $L > L_0$ then for any u_0 there is a $T \in (0, \infty)$ such that

$$u_x(0,t) \to \infty \qquad \text{as } t \to T, \tag{1.9}$$

$$\lim_{t\to T} u_x(x,t) \quad \text{exists and is finite for } 0 < x \le L, \tag{1.10}$$

$$\sup_{Q_T} |u(x,t)| < \infty, \qquad Q_T = [0,L] \times [0,T]. \tag{1.11}$$

It is easy to see that (1.11) holds because $\min u_0$ is a subsolution and $f(0)t + \max u_0$ is a supersolution. The number L_0 is related to the solvability of the corresponding stationary problem

$$\varphi'' + f(\varphi') = 0, \qquad 0 < x < L, \tag{1.12a}$$

$$\varphi(0) = \varphi(L) = 0. \tag{1.12b}$$

Namely,

$$L_0 = \sup\{L > 0 : \text{ there is a solution of (1.12) on } [0,L]\}. \tag{1.13}$$

At time $t = T$ the solution ceases to exist in the classical sense because the boundary condition $u(0, t) = 0$ ceases to be satisfied. We construct a unique continuation U which satisfies the equation (1.1) classically for all $t > 0$ and also $U(L, t) = 0$. Instead of $u(0, t) = 0$, the continuation U satisfies

$$U(0, t) > 0 \quad \text{and} \quad U_x(0, t) = \infty \quad \text{for } t > T.$$

This suggests an alternative interpretation of the solution as an interval-valued function at the point $x = 0$ ($u(0, t) = [0, U(0, t)]$).

We remark that $v = u_x$ satisfies

$$v_t = v_{xx} + f'(v)v_x, \qquad 0 < x < L, \qquad 0 < t < T, \tag{1.14}$$

$$v_x = -f(v), \qquad x = 0, L, \qquad 0 < t < T, \tag{1.15}$$

$$v(\cdot, 0) = u_0', \qquad 0 \le x \le L, \tag{1.16}$$

and v blows up in L^∞ at $t = T$ and $x = 0$. Blow-up of the solution on the boundary for linear and semilinear equations with nonlinear boundary conditions was studied previously in [2,4,5], for example. But we don't know of any results on continuation beyond the blow-up time.

On the other hand, the function $w = G(u_x)$, where G is defined by

$$G(s) = \int_s^\infty \frac{dv}{f(v)}, \tag{1.17}$$

satisfies

$$w_t = w_{xx} + f'(G^{-1}(w))(w_x - w_x^2), \qquad 0 < x < L, \quad 0 < t < T, \tag{1.18}$$

$$w_x(0, t) = w_x(L, t) = 1, \qquad 0 < t < T, \tag{1.19}$$

$$w(x, 0) = G(u_0'(x)), \qquad 0 \le x \le L. \tag{1.20}$$

Since $f'(v)$ is unbounded as $v \to \infty$ and $G^{-1}(s) \to \infty$ as $s \to 0$, the problem (1.18)–(1.19) might be viewed as a quenching problem as $w \to 0$. By a quenching problem we here mean a problem where a term in the equation becomes unbounded while the solution itself stays bounded. (The original problem for u is also a quenching problem in this sense.) Regularity of solutions of equations similar to (1.18) near an interior zero was studied in [1]. In our case, the solution w can reach zero only on the boundary and we have a term linear in w_x instead of a negative constant (cf. equations (1.6) and (1.7) in [1]).

Now we describe our results concerning the behavior of solutions for $t \ge T$. We show that for any $u_0 \in C^1[0, L]$ there is a constant $k \in (0, 1)$ such that

$$U_x(x, t) \le G^{-1}(kx), \qquad 0 < x \le L, \quad t \ge T. \tag{1.21}$$

On the other hand, there are initial functions for which

$$G^{-1}(x) \leq U_x(x, t) \quad \text{if } x \text{ is small enough and } t \geq T. \tag{1.22}$$

At the other endpoint $x = L$ we get

$$|U_x(L, t)| < \infty \qquad \text{for } t \geq T \tag{1.23}$$

and the bound is uniform in t if (1.8) holds but it may depend on t under the assumption (1.7).

We are also interested in long time behavior of U. Under the assumption (1.8) we have for any u_0 that

$$U(x, t) \to \psi(x) = \int_x^L G^{-1}(s) ds \qquad \text{uniformly as } t \to \infty. \tag{1.24}$$

Here ψ is the unique solution of (1.12a) with boundary conditions

$$\psi'(0) = \infty, \qquad \psi(L) = 0. \tag{1.25}$$

Under the assumption (1.7) there is a number

$$L_1 = \int_{-\infty}^{\infty} \frac{dv}{f(v)} > L_0 \tag{1.26}$$

such that for $L \in (L_0, L_1)$ we get convergence to a unique singular steady state as before, while for $L \geq L_1$ the global continuation U blows up in L^∞ as $t \to \infty$ everywhere in $[0, L)$ for any u_0. The last statement follows from the fact that the equation (1.1) with the boundary conditions $u_x(0, t) = \infty$, $u(L, t) = 0$ admits a Lyapunov functional but for $L \geq L_1$ there is no solution to (1.12) satisfying $\varphi(L) = 0$. We give an alternative proof via comparison in Section 4.

The plan of this paper is very simple. In Section 2, we develop the properties of the function φ, which solves (1.12a) with initial conditions $\varphi'(0) = \infty$, $\varphi(0) = 0$, and use these properties to show that u_x blows up in finite time. Behavior of the solution at the blow-up time and existence of a global continuation are discussed in Section 3. Finally Section 4 studies the behavior of the continuation as $t \to \infty$.

2. Derivative blow-up. To show that the derivative does blow up if $L > L_0$, we need first to study the singular steady-state problem.

Lemma 2.1. *Assume that $f(v) > 0$ for all v and that* (1.6) *holds together with one of the conditions* (1.7), (1.8) *and let L_1 be given by* (1.26). *Then the function φ defined by*

$$\varphi(x) = \int_0^x G^{-1}(s)\, ds \tag{2.1}$$

(G^{-1} *is the inverse of* G *from* (1.17)) *satisfies*

$$\varphi'' + f(\varphi') = 0 \qquad in\ (0, L_1), \quad 0 < L_1 \le \infty, \tag{2.2}$$
$$\varphi(0) = 0, \tag{2.3}$$
$$\varphi'(0) = \infty. \tag{2.4}$$

Moreover, $\varphi(x) \to -\infty$ *as* $x \to L_1$ *and* $L_1 = \infty$ *if* (1.8) *holds but* $L_1 < \infty$ *if* (1.7) *holds. The number* L_0 *from* (1.13) *is positive and finite, it is the unique positive root of* φ.

Proof. It is easy to see that φ is a solution of (2.2), (2.4) if and only if

$$\int_{\varphi'(x)}^{\infty} \frac{dv}{f(v)} = x \tag{2.5}$$

or

$$\varphi'(x) = G^{-1}(x). \tag{2.6}$$

The range of G is $(0, L_1)$. Therefore φ' is defined by (2.6) for $x \in (0, L_1)$. Integrating (2.6) and using (2.3) we obtain (2.1). If we make the substitution $s = G(v)$ in (2.1) we get

$$\varphi(x) = \int_{G^{-1}(x)}^{\infty} \frac{v\,dv}{f(v)}.$$

The assumption (1.6) implies that φ is defined (and finite) for $x \in (0, L_1)$. Also $\varphi(x) \to -\infty$ as $x \to L_1$. Since $\varphi''(x) < 0$ and $\varphi'(0) = -\varphi'(L_1) = \infty$, we obtain that φ' has a unique zero. This implies that φ has a unique positive root $a \in (0, L_1)$.

If $L \le a$ then the equation $\varphi(L + b) = \varphi(b)$ has a unique nonnegative solution b (because $\varphi'' < 0$ and $\varphi'(G(0)) = 0$) and $z(x) = \varphi(x + b) - \varphi(b)$ is the unique solution of (1.12). On the other hand, if $L > a$ then there is no solution of (1.12). Hence $L_0 = a$. □

We are now ready to show the blow-up of the derivative in finite time.

Theorem 2.2. *Let* f *be a positive function satisfying* (1.6) *together with*

$$\liminf_{v \to \infty} f(v) > 0 \tag{2.7}$$

and either (1.7) *or* (1.8). *If* $L > L_0$ *then for any* $u_0 \in C^1[0, L]$ *there is a* $T \in (0, \infty)$ *such that*

$$\limsup_{t \to T} u_x(0, t) = \infty. \tag{2.8}$$

Proof. We first construct a suitable subsolution to show that there is a $T < \infty$ such that

$$\limsup_{t \to T} \max_{0 \le x \le L} |u_x(x, t)| = \infty. \tag{2.9}$$

Then we show that $u_x(0,t)$ is bounded from below and $u_x(L,t)$ is bounded for $0<t<T$. This will imply (2.8) since $\max|u_x|$ must occur on the boundary because $v=u_x$ satisfies a linear parabolic equation of the form $v_t=v_{xx}+g(x,t)v_x$ with g locally bounded.

Consider first the case when (1.8) holds and define

$$\varphi_\sigma(x)=\int_0^x G^{-1}(\sigma s)ds=\frac{1}{\sigma}\varphi(\sigma x).$$

Then for any $L>L_0$ there is a $\sigma\in(0,1)$ such that $\varphi_\sigma(L)<0$ and an $M_\sigma>0$ such that $\varphi_\sigma'(x)\geq -M_\sigma$ for $x\in(0,L]$. Choose $\delta\in(0,(1-\sigma)\eta)$ where $\eta>0$ is such that $f(v)\geq\eta$ if $v\geq -M_\sigma$ and set

$$K=\sup_{0\leq x\leq L}(\varphi_\sigma(x)-u_0(x)),\qquad \phi(x,t)=\varphi_\sigma(x)-K+\delta t.$$

Then $\phi(x,0)\leq u_0(x)$ and ϕ satisfies:

$$\begin{aligned}
&\phi_t\leq\phi_{xx}+f(\phi_x) \qquad \text{on } (0,L)\times(0,\infty)\\
&\phi_x(0,t)=\infty\\
&\phi(0,t)<0,\quad \phi(L,t)<0 \quad \text{for } 0\leq t<\frac{K}{\delta},\\
&\phi(0,\frac{K}{\delta})=0,\quad \phi(L,\frac{K}{\delta})<0.
\end{aligned}$$

By comparison, $u\geq\phi$ as long as u exists, therefore u ceases to exist at a finite time $T\leq\frac{K}{\delta}$. Since u itself stays bounded, $\sup|u_x(\cdot,t)|$ must become unbounded as $t\to T$.

Now, $v=u_x$ satisfies

$$\begin{aligned}
&v_t=v_{xx}+f'(v)v_x &&\text{in } (0,L)\times(0,T), &&(2.10a)\\
&v_x=-f(v) &&\text{for } x=0,L \text{ and } t\in(0,T), &&(2.10b)\\
&v(\cdot,0)=u_0' &&\text{in } [0,L] &&(2.10c)
\end{aligned}$$

and $\max|v|$ must occur on the boundary.

If we take ℓ_1 large enough, then $G^{-1}(x+\ell_1)\leq u_0'(x)$ for $x\in[0,L]$ and $G^{-1}(x+\ell_1)$ is a stationary solution of (2.10a,b). Hence, u_x is bounded from below. On the other hand, for any u_0 there is an $\ell_2\in(0,L)$ such that $G^{-1}(x-\ell_2)\geq u_0'(x)$ for $x\in(\ell_2,L)$. Therefore $u_x(L,t)\leq G^{-1}(L-\ell_2)$ for $0\leq t<T$ and (2.8) follows.

Consider now the case when (1.7) holds. Then the only difference in the proof is that G^{-1} is now defined only for $x\in(0,L_1)$. Now we take $\ell_1<L_1$ and comparison with $G^{-1}(x+\ell_1)$ yields a lower bound for $u_x(0,t)$. An upper bound for $u_x(L,t)$ is obtained as before and we shall show that $u_x(L,t)\geq -k(T)$ for $t<T$.

The supersolution $f(0)t + \max u_0$ yields that

$$u(x,t) \le F = f(0)T + \max u_0 \qquad \text{for } t < T.$$

If we choose ε, ℓ_3 and M positive such that $\varphi(L - \varepsilon - \ell_3) + M \ge F$ and $\varphi(L - \ell_3) + M = 0$, then

$$\begin{aligned} &u(x,t) \le \varphi(x - \ell_3) + M && \text{in } [L-\varepsilon, L] \times [0, T) \\ &u_x(L,t) \ge \varphi'(L - \ell_3) && \text{for } 0 < t < T. \quad \square \end{aligned} \tag{2.11}$$

Note that the lower bound on $u_x(L,t)$ can be taken independent of t if (1.7) holds and $L < L_1$ because we can then use as supersolution $\varphi(x,t) - \min\{0, \varphi(L)\} + \max u_0$.

Our next step, which sets the stage for investigation of the extension, is to prove a pointwise upper bound for u_x.

Theorem 2.3. *Assume that* (1.4) *holds and that* $\int^\infty \frac{dv}{f(v)} < \infty$. *Then for any* $u_0 \in C^1[0, L]$ *there is a constant* $k \in (0,1)$ *which depends on* u_0, f *and* L *such that*

$$u_x(x,t) \le G^{-1}(kx) \quad \textit{for } x \in (0, L], \quad 0 \le t < T. \tag{2.12}$$

Proof. The function $w(x,t) = G(u_x(x,t))$ satisfies (1.18)–(1.20). We show by comparison that $w(x,t) \ge kx$ for k small enough.

Since G^{-1} is decreasing and $G^{-1}(s) \to \infty$ as $s \to 0$, it is possible to choose $k \in (0,1)$ such that $G^{-1}(kL) \ge \max\{v_0, \sup u_0'\}$, where v_0 is such that $f'(v) \ge 0$ for $v \ge v_0$. If $z(x,t) = kx$, then $w(x,0) \ge z(x,0)$ and $w(0,t) > z(0,t)$, $w_x(L,t) = 1 > k = z_x(L,t)$. Our choice of k guarantees that $f'(G^{-1}(z)) \ge 0$ in $(0, L)$. Therefore z is a subsolution of (1.18)–(1.20) and $w(x,t) \ge z(x,t)$ in $[0, L] \times [0, T)$. $\square$

Under special circumstances, the reverse inequality to (2.12) can be proved with $k = 1$. Clearly, this will be the case if $u_{0,x} \ge G^{-1}$, but the inequality holds more generally.

Theorem 2.4. *Let* f *and* L *be as in Theorem* 2.2. *If there is an* $\varepsilon_0 > 0$ *such that* $G^{-1}(x + \varepsilon) - u_0'(x)$ *changes sign exactly once for* $\varepsilon \in (0, \varepsilon_0]$, *then*

$$G^{-1}(x) \le u_x(x,T) \quad \textit{for } x \in (0, L^*), \quad L^* = \min\left\{L, \int_{-\infty}^{\infty} \frac{dv}{f(v)}\right\}. \tag{2.13}$$

Proof. We follow an idea from [5]. Since u_x and $G^{-1}(x+\varepsilon)$ both satisfy (1.14), (1.15) and $G^{-1}(x+\varepsilon) - u_0'(x)$ changes sign once, the number of sign changes of $G^{-1}(x+\varepsilon) - u_x(x,t)$ is less than or equal to one for $t \in (0, T)$ and it does not increase. But $u_0'(0) < G^{-1}(\varepsilon)$ if ε is small enough and $\limsup_{t\to T} u_x(0, T) = \infty$, hence there is a $t_0 < T$ such that

$$u_x(x, t_0) > G^{-1}(x+\varepsilon) \quad \text{for } x \in (0, L^* - \varepsilon). \qquad \square$$

Remark. There are functions u_0 for which there is an $\varepsilon_0 > 0$ such that $G^{-1}(x+\varepsilon) - u_0'(x)$ has one sign change for $\varepsilon \in (0, \varepsilon_0]$; for example, $u_0 \equiv 0$.

3. Behavior of the solution near and after blow-up. The key ingredient in our analysis is the function w from (1.18)–(1.20). We first notice that

$$f'(G^{-1}(w))(1-w_x) = \frac{f'(u_x)}{f(u_x)}(u_{xx} + f(u_x)) \tag{3.1}$$

by direct calculation. Since $v = u_t$ satisfies

$$\begin{aligned} &v_t = v_{xx} + f'(u_x)v_x && 0 < x < L, \quad 0 < t < T, \\ &v(0, t) = v(L, t) = 0 && 0 < t < T, \end{aligned}$$

it follows that $|v|$ attains its maximum over $[0, L] \times [\varepsilon, T]$ for any $\varepsilon > 0$ when $t = \varepsilon$. (In fact if $u_0 \in C^2$ and $u_0 \in C^2$ and $u_{0,xx} = -f(u_{0,x})$ at $x = 0, L$, then $|v| \le \max|u_{0,xx} + f(u_{0,x})|$. If u_0 is not this regular, we use the local existence of a smooth solution to infer that $u(\cdot, \varepsilon) \in C^2$ with $u_{xx}(0, \varepsilon) + f \circ u_x(0, \varepsilon) = u_{xx}(L, \varepsilon) + f \circ u_x(L, \varepsilon) = 0$ for all sufficiently small positive ε.) Therefore $|v| \le c_1$ on $(0, L) \times [\varepsilon, T]$ for some computable constants ε and c_1. It follows from (1.5), (1.1), (2.11), (1.18), and (3.1) that w solves a linear equation with bounded coefficients in $(0, L) \times (\varepsilon, T)$. It then follows from [11, Theorem 1.2] (or a suitable modification of the linear theory in [9, Chapter IV]) that $w \in C^{1+\alpha,(1+\alpha)/2}$. In particular, w is continuous at $(0, T)$, so $\lim_{x\to 0, t\to T^-} w(x, t) = 0$, and $\lim_{x\to y, t\to T^-} w(x, t)$ exists and is positive if $y > 0$. Converting back to the original variables gives the following result.

Theorem 3.1. *Let u be a solution of* (1.1)–(1.3), *and suppose $f \in C^2(\mathbb{R})$ satisfies conditions* (1.4)–(1.6) *and* (1.7) *or* (1.8). *If $L > L_0$, the constant from* (1.13), *and T is the time at which u ceases to exist, then* (1.9) *and* (1.10) *hold.*

Our next step is to construct a continuation for w after time T. For this construction, we suppose first that (1.8) holds and we consider the problem

$$\begin{aligned} &w_t = w_{xx} + F(w)(w_x - w_x^2) && 0 < x < L, \quad t > 0 \\ &w_x(0, t) = 1, \quad w_x(L, t) = 1 && t > 0 \\ &w(x, 0) = w_0(x) && 0 < x < L, \end{aligned}$$

where $F \in C^1(0, \infty)$ and $\lim_{w\to 0^+} F(w) = \infty$. Standard local existence results show that this problem has a solution until $w = 0$. Specifically, either w is global and positive or there is a $T > 0$ such that $w(\cdot, t) > 0$ for $t < T$ and $\liminf_{t\to T^-} w(\cdot, t) = 0$. In our case, $F = f' \circ G^{-1}$ so the solution is not global and the argument preceding Theorem 3.1 implies that $\lim_{t\to T^-,\, x\to 0} w(x, t) = 0$.

For each small $\varepsilon > 0$, choose t_ε so that $w(0, t_\varepsilon) = \varepsilon$ and $t_\varepsilon \uparrow T$ as $\varepsilon \to 0$. Define w^ε by the same differential equation and initial condition as w but with boundary conditions

$$\begin{aligned} &w^\varepsilon_x(L, t) = 1 && t > 0, \\ &w^\varepsilon_x(0, t) = 1 && 0 < t < t_\varepsilon, \\ &w^\varepsilon(0, t) = \varepsilon && t \ge t_\varepsilon. \end{aligned}$$

Since the constant ε is a subsolution, w^ε exists for all time. As in Theorem 2.3, we have $w^\varepsilon \ge kx$. Since $x + \max w_0$ is a supersolution, we also have a uniform upper bound for w^ε. To continue, we define

$$u^\varepsilon(x, t) = \int_x^L G^{-1}(w^\varepsilon(\xi, t))d\xi$$

and note that

$$\begin{aligned} &u^\varepsilon_t = u^\varepsilon_{xx} + f(u^\varepsilon_x) && 0 < x < L, \quad t > 0 \\ &u^\varepsilon(0, t) = 0 && 0 < t \le t_\varepsilon, \\ &u^\varepsilon_x(0, t) = G^{-1}(\varepsilon) && t > t_\varepsilon, \\ &u^\varepsilon(L, t) = 0 && t > 0, \\ &u^\varepsilon(x, 0) = u_0(x) && 0 < x < L. \end{aligned}$$

Now $u = u^\varepsilon$ for $t \le t_\varepsilon$ and hence $v^\varepsilon = u^\varepsilon_t$ is uniformly bounded because it is continuous at $(0, t_\varepsilon)$ and $v^\varepsilon_x(0, t) = 0$ for $t > t_\varepsilon$. The argument at the beginning of the section now provides a uniform $C^{1+\alpha,(1+\alpha)/2}$ bound on the family (w^ε) over any finite time interval (and the bound is independent of the time interval). It follows that (w^ε) converges uniformly to a global, classical solution of

$$\begin{aligned} &w_t = w_{xx} + F(w)(w_x - w_x^2) && 0 < x < L, \quad t > 0, \\ &w_x(0, t) = 1 && 0 < t \le T, \\ &w(0, t) = 0 && t > T, \\ &w_x(L, t) = 1 && t > 0, \\ &w(x, 0) = w_0(x) && 0 < x < L. \end{aligned}$$

Our continuation U of u is defined by

$$U(x, t) = \int_x^L G^{-1}(w(\xi, t))d\xi.$$

Because $U_x^\varepsilon(0,t) \geq 0$ (so U^ε cannot attain its maximum at $x = 0$), U is bounded from above by virtue of the proof of (1.11).

Moreover, (1.6) implies that $\limsup_{v\to\infty} f'(v) = \infty$ and $\lim_{x\to 0} G^{-1}(w(x,t)) = \infty$ for $t \geq T_\varepsilon$. Hence, for any $t > T_\varepsilon$, there is a sequence (x_n) with $x_n \to 0$ such that $f'(G^{-1}(w(x_n,t))) \to \infty$. But

$$f'(G^{-1}(w))(1 - w_x) = \frac{f'(u_x)}{f(u_x)} u_t$$

is bounded, so $1 - w_x(x_n, t) \to 0$. It follows from the continuity of w_x that, in fact, $w_x(0,t) = 1$ for all $t > 0$. This additional boundary condition can be viewed as justification of our continuation.

When (1.7) is assumed rather than (1.8), the arguments of this section need only slight modification. As long as $w < L_1$, we can follow our previous arguments exactly. But if u^ε exists for $t < t_0$, we can use the lower bound for $u_x^\varepsilon(L,t)$ from Section 2. This bound is also valid up to time t_0 and hence $w^\varepsilon(L,t)$ is bounded away from L_1, so the maximum principle implies that w is bounded away from L_1 in $[0,L] \times [0, L-0]$. Therefore w^ε exists for all times and the uniform lower bound on $u_x^\varepsilon(L,t)$ allows all of our arguments to go through. In particular, w remains bounded away from L_1 for all time.

4. Asymptotic behavior of U. When $L_0 < L < L_1$, the asymptotic behavior of our continuation is easily studied. To simplify our notation, we set $\varphi_0 = -\varphi(L)$. In this case, from the proof of Theorem 2.2, we see that, for any $\sigma < 1$ with $1 - \sigma$ sufficiently small, there are positive numbers K and δ such that

$$\phi(x,t) = \frac{1}{\sigma}\varphi(\sigma x) - K + \delta t + \varphi_0 \tag{4.1}$$

is a supersolution (even for our continuation by [10, Lemma 3.1]) as long as $\phi(L,t) \leq 0$. Now, for any $\varepsilon > 0$, there is $\sigma \in (0,1)$ (with $\sigma \to 1$ as $\varepsilon \to 0$) such that

$$\left|\frac{1}{\sigma}\varphi(\sigma x) - \varphi(x)\right| < \varepsilon$$

for $0 < x < L$, so, with this choice of σ and $t_* = (K-\varepsilon)/\delta$, we have $\phi(L,t_*) \leq 0$. It follows from the comparison principle, applied to u and ϕ, that $U(x,t_*) \geq \phi(x,t_*) \geq \varphi(x) - \varepsilon + \varphi_0$. Applying the comparison principle now to U and $\varphi - \varepsilon + \varphi_0$ shows that $U \geq \varphi - \varepsilon + \varphi_0$. It follows that

$$\liminf_{t\to\infty} U(x,t) \geq \varphi(x) + \varphi_0.$$

Similarly, if $\sigma > 1$ with $\sigma - 1$ sufficiently small, then

$$\Phi(x,t) = \frac{1}{\sigma}\varphi(\sigma x) + K - \delta t + \varphi_0$$

is a supersolution as long as $\Phi(L, t) \geq 0$. In this way, we see that also

$$\limsup_{t\to\infty} U(x, t) \leq \varphi(x) + \varphi_0.$$

It follows that U converges to $\varphi + \varphi_0$. Moreover, the convergence is uniform by our proof. Standard regularity theory (along with the uniform upper bound on U_x away from $x = 0$) shows that U converges in C^2 norm uniformly on compact subsets of $(0, L]$. Applying the same regularity theory to w, we see that w in fact converges in $C^1[0, L]$ to the steady-state $\psi(x) = x$.

On the other hand, if $L > L_1$, we fix $x_0 \in [0, L - L_1)$ and define $\Phi(x, t) = \phi(x - x_0, t)$, where ϕ is defined by (4.1). If $\sigma \in (L_1/(L - x_0), 1)$ and $1 - \sigma$ is sufficiently small, then Φ is a subsolution on $[x_0, x_0 + L_1/\sigma] \times (0, \infty)$ because $\Phi_x(x_0, t) = \infty$ and $\Phi(x_0 + L_1, t) = -\infty$. Hence U must tend uniformly to infinity on any compact subinterval of $[x_0, x_0 + L_1)$. It follows that U tends uniformly to infinity on any compact subinterval of $[0, L)$.

REFERENCES

[1] H. Bellout, *A regularity result for a quasilinear equation and its consequences for blowing-up solutions of semilinear heat equations,* J. Diff. Eqns., 100 (1992), 162–172.

[2] M. Chipot, M. Fila, and P. Quittner, *Stationary solutions, blow up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions,* Acta Math. Univ. Comenianae, 60 (1991), 35–103.

[3] T. Dlotko, *Examples of parabolic problems with blowing-up derivatives,* J. Math. Anal. Appl., 154 (1991), 226–237.

[4] M. Fila and P. Quittner *The blow-up rate for the heat equation with a nonlinear boundary condition,* Math. Meth. Appl. Sci., 14 (1991), 197–205.

[5] V.A. Galaktionov, S.P. Kurdyumov, and A.A. Samarskij, *On the method of stationary states for quasilinear parabolic equations,* Mat. Sb., 180 (1989), 995–1016, English transl. in Math. USSR-Sb., 67 (1990) 449-471.

[6] N. Kutev, *Global solvability and boundary gradient blow for one dimensional parabolic equations,* Progress in Partial Differential Equations: Elliptic and Parabolic Problems, (C. Bandle, et al., eds.), Longman, 1992, 176–181.

[7] N. Kutev, *Gradient blow ups and global solvability after the blow up time for nonlinear parabolic equations,* Proceedings of the Third International Conference on Evolution Equations, Control Theory and Biomathematics, (Ph. Clement and G. Lumer, eds.), Marcel Dekker, 1994, 301–306.

[8] N. Kutev, *On the solvability of Dirichlet's problem for a class of nonlinear elliptic and parabolic equations,* Proceedings of Equadiff '91, Vol. 2, (ed., C. Perello, et al.), World Scientific, 1991, 666–670.

[9] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural′ceva, "Linear and Quasilinear Equations of Parabolic Type," Amer. Math. Soc., Providence, R.I., 1967.

[10] G.M. Lieberman, *The first initial-boundary value problem for quasilinear second order parabolic equations,* Ann. Scuola Norm. Sup. Pisa, 13 (1986), 347–387.

[11] G.M. Lieberman, *Hölder continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions,* Ann. Mat. Pura Appl., 148 (1987), 77–99, 397–398.