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DERIVATIVE ESTIMATES OF SOLUTIONS OF ELLIPTIC SYSTEMS IN NARROW REGIONS

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Abstract. In this paper, we establish C^k estimates for a class of elliptic systems, including linear systems of elasticity, in a narrow region. The problem arises from studies of fiber-reinforced elastic composite materials.

1. Introduction. In this paper we establish local derivative estimates for solutions to a class of elliptic systems arising from studies of fiber-reinforced composite materials. From the structure of the composite, there are a relatively large number of fibers which are touching or nearly touching. The maximal strains can be strongly influenced by the distances between the fibers.

Stimulated by some works on damage analysis of fiber composites ([6]), there have been a number of papers, starting from [9], [15] and [16], on gradient estimates for solutions of elliptic equations and systems with piecewise smooth coefficients which are relevant in such studies. See, e.g. [1–5], [7, 8], [10], [12], [17], [18, 19]. Earlier studies on such and closely related issues can be found in [11], [13, 14].

In a recent paper [2], some gradient estimates were obtained concerning the conductivity problem where the conductivity is allowed to be ∞ (perfect conductor).

Theorem A ([2]). Let B_1 and B_2 be two balls in $\mathbb{R}^3$ with radius R and centered at $(0, 0, \pm R \pm \frac{\epsilon}{2})$, respectively. Let H be a harmonic function in $\mathbb{R}^3$ such that $H(0) = 0$. Define u to be the solutions

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of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_1 \cup B_2}, \\ u = 0 & \text{on } \partial B_1 \cup \partial B_2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to +\infty. \end{cases} \tag{1.1}$$

Then there exists a constant C independent of ϵ such that

$$\|\nabla(u - H)\|_{L^\infty(\mathbb{R}^3 \setminus \overline{B_1 \cup B_2})} \leq C. \tag{1.2}$$

Contrary to scalar equations, less is known about derivative estimates of solutions of systems. In this paper we extend Theorem A to general elliptic systems, including linear systems of elasticity, in all dimensions. Moreover, we allow the two balls in Theorem A to be replaced by any two smooth domains, and we establish a stronger local version.

We use $B_r(0') = \{x' \in \mathbb{R}^{n-1} \mid |x'| < r\}$ to denote a ball in $\mathbb{R}^{n-1}$ centered at the origin $0'$ of radius r. Let h_1 and h_2 be smooth functions in $B_1(0')$ satisfying

$$h_1(0') = h_2(0') = 0, \quad \nabla h_1(0') = \nabla h_2(0') = 0,$$

and

$$-\frac{\epsilon}{2} + h_2(x') < \frac{\epsilon}{2} + h_1(x'), \quad \text{for } |x'| < 1.$$

For $0 < r \leq 1$, we define

$$\Omega_r := \left\{ x \in \mathbb{R}^n \;\Big|\; -\frac{\epsilon}{2} + h_2(x') < x_n < \frac{\epsilon}{2} + h_1(x'),\ x' \in B_r(0') \right\}.$$

Its lower and upper boundaries are, respectively,

$$\Gamma_r^- = \left\{ x \in \mathbb{R}^n \;\Big|\; x_n = -\frac{\epsilon}{2} + h_2(x'),\ |x'| \leq r \right\}, \quad \Gamma_r^+ = \left\{ x \in \mathbb{R}^n \;\Big|\; x_n = \frac{\epsilon}{2} + h_1(x'),\ |x'| \leq r \right\}.$$

Let $u = (u^1, \cdots, u^N)$ be a vector-valued function. We consider the following boundary value problems:

$$\begin{cases} \partial_\alpha \left(A_{ij}^{\alpha\beta}(x) \partial_\beta u^j + B_{ij}^\alpha u^j \right) + C_{ij}^\beta \partial_\beta u^j + D_{ij} u^j = 0 & \text{in } \Omega_1, \\ u = 0 & \text{on } \Gamma_1^+ \cup \Gamma_1^-. \end{cases} \tag{1.3}$$

We use the usual summation convention: α and β are summed from 1 to n, while i and j are summed from 1 to N. For $0 < \lambda < \Lambda < \infty$, we assume that the coefficients $A_{ij}^{\alpha\beta}(x)$ are measurable and bounded,

$$|A_{ij}^{\alpha\beta}| \leq \Lambda, \tag{1.4}$$

and satisfy the rather weak ellipticity condition

$$\int_{\Omega_1} A_{ij}^{\alpha\beta} \partial_\alpha \psi^i \partial_\beta \psi^j \geq \lambda \int_{\Omega_1} |\nabla \psi|^2, \quad \forall\, \psi \in H_0^1(\Omega_1, \mathbb{R}^N). \tag{1.5}$$

Furthermore, we assume that $A_{ij}^{\alpha\beta}$, B_{ij}^α, C_{ij}^β, D_{ij}, h_1 and h_2 are in $C^k(\Omega_1)$ for some $k \geq 0$, denote

$$\|A\|_{C^k(\Omega_1)} + \|B\|_{C^k(\Omega_1)} + \|C\|_{C^k(\Omega_1)} + \|D\|_{C^k(\Omega_1)} \leq \beta_k,$$

and

$$\|h_1\|_{C^k(\Omega_1)} + \|h_2\|_{C^k(\Omega_1)} \leq \gamma_k,$$

where β_k and γ_k are some positive constants. Hypotheses (1.4) and (1.5) are satisfied by linear systems of elasticity (see [20]).

We give local estimates of weak solutions u of (1.3); that is, $u \in H^1(\Omega_1, \mathbb{R}^N)$, $u = 0$ on $\Gamma_1^+ \cup \Gamma_1^-$ a.e., and satisfies

$$\int_{\Omega_1} \left(A_{ij}^{\alpha\beta}(x)\partial_\beta u^j + B_{ij}^{\alpha} u^j\right)\partial_\alpha \zeta^i - C_{ij}^{\beta}\partial_\beta u^j \zeta^i - D_{ij}u^j\zeta^i = 0$$

for every vector-valued function $\zeta = (\zeta^1, \cdots, \zeta^N) \in C_c^\infty(\Omega_1, \mathbb{R}^N)$, and hence for every $\zeta \in H_0^1(\Omega_1, \mathbb{R}^N)$.

Theorem 1.1. Assume the above and let $u \in H^1(\Omega_1, \mathbb{R}^N)$ be a weak solution of (1.3). Then for $k \geq 0$, there exist constants $0 < \mu < 1$ and C, depending only on $n, N, \lambda, \Lambda, k, \beta_{k+1+\frac{n}{2}}$ and $\gamma_{k+1+\frac{n}{2}}$, such that

$$|\nabla^k u(x)| \leq C\mu^{\frac{1}{\sqrt{\epsilon+|x'|}}}\|u\|_{L^2(\Omega_1)}, \quad \text{for all } x = (x', x_n) \in \Omega_{\frac{1}{2}}.$$

In particular,

$$\max_{-\frac{\epsilon}{2}+h_2(0')<x_n<\frac{\epsilon}{2}+h_1(0')} \left|\nabla^k u(0', x_n)\right| \to 0, \qquad \text{as } \epsilon \to 0.$$

A consequence of Theorem 1.1 is an extension of Theorem A to all dimensions and to any smooth domains.

Corollary 1.1. Let D_1 and D_2 be two disjoint bounded open sets in $\mathbb{R}^n$, $n \geq 2$, with C^k boundaries for $k = \left[\frac{n}{2}+2\right]$, and $\mathrm{dist}(\partial D_1, \partial D_2) = \epsilon \in (0,1)$. Let H be a harmonic function in $\mathbb{R}^n \setminus (D_1 \cup D_2)$. Assume that u satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{D_1 \cup D_2}, \\ u = 0 & \text{on } \partial D_1 \cup \partial D_2, \\ \liminf\limits_{|x|\to\infty} |u(x) - H(x)| \leq K, & \text{for some } K > 0. \end{cases} \tag{1.6}$$

Then there exists a constant C, depending only on K, $\|H\|_{L^\infty(\partial D_1 \cup \partial D_2)}$ and the C^k norms and diameters of D_1 and D_2 (but independent of ϵ), such that

$$\|\nabla(u - H)\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_1 \cup D_2})} \leq C. \tag{1.7}$$

2. Proof of Theorem 1.1.

In this section, we derive the C^k estimates for solutions of elliptic systems (1.3). In the following, we first show that the energy in Ω_r decays exponentially as r tends to 0. Unless otherwise stated, we use C to denote some positive constants, whose values may vary from line to line, which depend only on $n, N, \lambda, \Lambda, \beta_0$ and γ_2, but is independent of ϵ.

Lemma 2.1. Let $u \in H^1(\Omega_1, \mathbb{R}^N)$ be a weak solution of (1.3); then there exist $0 < \mu_0 < 1$ and C, depending only on $n, N, \lambda, \Lambda, \beta_0$ and γ_2, such that, for any $\sqrt{\epsilon} \leq r < \frac{1}{2}$,

$$\int_{\Omega_r} |\nabla u|^2 dx \leq C(\mu_0)^{\frac{1}{r}} \int_{\Omega_1} |\nabla u|^2. \tag{2.1}$$

Proof. Without loss of generality, we can assume that $\int_{\Omega_1} |\nabla u|^2 = 1$. For any $0 < t < s \leq 1$, we introduce a cutoff function $\eta \in C^\infty(\Omega_1)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ in Ω_t, $\eta = 0$ in $\Omega_1 \setminus \Omega_s$, and $|\nabla \eta| \leq \frac{2}{s-t}$. Multiplying $(u\eta^2)$ on both sides of the equation in (1.3) and integrating by parts, we have

$$\int_{\Omega_1} \left(A_{ij}^{\alpha\beta}(x)\partial_\beta u^j + B_{ij}^{\alpha}u^j\right)\partial_\alpha(u^i\eta^2) - C_{ij}^{\beta}\partial_\beta u^j(u^i\eta^2) - D_{ij}u^j(u^i\eta^2) = 0.$$

Since

$$
\begin{aligned}
&\int_{\Omega_s} \left(A_{ij}^{\alpha\beta}(x)\partial_\beta u^j + B_{ij}^{\alpha} u^j\right)\partial_\alpha(u^i\eta^2)\\
&= \int_{\Omega_s} A_{ij}^{\alpha\beta}(x)\partial_\beta(\eta\, u^j)\partial_\alpha(\eta\, u^i) - \int_{\Omega_s} A_{ij}^{\alpha\beta}(x)(u^j\partial_\beta\eta)\partial_\alpha(\eta\, u^i) + \int_{\Omega_s} B_{ij}^{\alpha}\eta\, u^j\partial_\alpha(u^i\eta)\\
&\quad + \int_{\Omega_s} A_{ij}^{\alpha\beta}(x)\partial_\beta(\eta\, u^j)u^i\partial_\alpha\eta - \int_{\Omega_s} A_{ij}^{\alpha\beta}(x)(u^j\partial_\beta\eta)(u^i\partial_\alpha\eta) + \int_{\Omega_s} B_{ij}^{\alpha}(\eta\, u^j)(\partial_\alpha\eta\, u^i),
\end{aligned}
$$

it follows, in view of (1.5), that

$$
\begin{aligned}
&\lambda\int_{\Omega_s} |\nabla(u\eta)|^2 dx\\
&\le \int_{\Omega_s} A_{ij}^{\alpha\beta}\partial_\beta(u^j\eta)\partial_\alpha(u^i\eta)dx\\
&= \int_{\Omega_s} A_{ij}^{\alpha\beta}(x)(u^j\partial_\beta\eta)\partial_\alpha(\eta\, u^i) - \int_{\Omega_s} B_{ij}^{\alpha}\eta\, u^j\partial_\alpha(u^i\eta) - \int_{\Omega_s} A_{ij}^{\alpha\beta}(x)\partial_\beta(\eta\, u^j)u^i\partial_\alpha\eta\\
&\quad + \int_{\Omega_s} A_{ij}^{\alpha\beta}(x)(u^j\partial_\beta\eta)(u^i\partial_\alpha\eta) - \int_{\Omega_s} B_{ij}^{\alpha}(\eta\, u^j)(\partial_\alpha\eta\, u^i)\\
&\quad + \int_{\Omega_s} C_{ij}^{\beta}\partial_\beta(u^j\eta)(u^i\eta) - \int_{\Omega_s} C_{ij}^{\beta}(u^j\partial_\beta\eta)(u^i\eta) + \int_{\Omega_s} D_{ij}u^j(u^i\eta^2)\\
&\le \frac{\lambda}{4}\int_{\Omega_s} |\nabla(u\eta)|^2 dx + C\int_{\Omega_s} |u\nabla\eta|^2 dx + C\int_{\Omega_s} |u\eta|^2 dx.
\end{aligned}
$$

Since $u\eta = 0$ on Γ_1^-, by the Hölder inequality, it follows that

$$
\begin{aligned}
\int_{\Omega_s} |u\eta|^2 dx &= \int_{\Omega_s} \left(\int_{-\frac{\epsilon}{2}+h_2(x')}^{x_n} \partial_n(u\eta)(x', x^n)dx_n\right)^2 dx\\
&\le \int_{\Omega_s} \left((x_n + \frac{\epsilon}{2} - h_2(x'))\int_{-\frac{\epsilon}{2}+h_2(x')}^{x_n} |\partial_n(u\eta)|^2 dx_n\right)dx\\
&\le C(\epsilon + s^2)^2\int_{\Omega_s} |\nabla(u\eta)|^2 dx.
\end{aligned}
$$

Taking $0 < \delta_0 < 1$ such that $C^2(\delta_0 + \delta_0^2)^2 = \frac{\lambda}{4}$, then we have

$$
\int_{\Omega_s} |\nabla u|^2\eta^2 dx \le C\int_{\Omega_s} u^2|\nabla\eta|^2 dx, \quad \text{for } \epsilon, s < \delta_0. \tag{2.2}
$$

Again using $u = 0$ on Γ_1^-, and by the Hölder inequality, we have

$$
\int_{\Omega_s} u^2 dx \le C(\epsilon + s^2)^2\int_{\Omega_s} |\nabla u|^2 dx. \tag{2.3}
$$

Combining (2.2) and (2.3), we have

$$
\int_{\Omega_t} |\nabla u|^2 dx \le C\left(\frac{\epsilon + s^2}{s - t}\right)^2\int_{\Omega_s} |\nabla u|^2 dx, \quad \text{for } s < \delta_0. \tag{2.4}
$$

For simplicity of notation, we denote

$$
F(t) = \int_{\Omega_t} |\nabla u|^2 dx.
$$

Then (2.4) can be written as

$$F(t) \le C\left(\frac{\epsilon + s^2}{s-t}\right)^2 F(s). \tag{2.5}$$

For $\sqrt{\epsilon} \le t < s \le \delta_0$, we have the following iterative formula:

$$F(t) \le \left(\frac{C_0 s^2}{s-t}\right)^2 F(s),$$

where C_0 is a fixed constant, depending only on $n, N, \lambda, \Lambda, \beta_0$ and γ_2. Let $\delta = \min\{\frac{1}{8C_0}, \delta_0\}$ and $t_0 = r < \delta$, $t_{i+1} = 2\delta(1 - \sqrt{1 - t_i/\delta})$ if $t_i \le \delta$. Then

$$\frac{C_0 t_{i+1}^2}{t_{i+1} - t_i} = \frac{1}{2}, \tag{2.6}$$

and $\{t_i\}$ is an increasing sequence. It is easy to see that for some i, $t_i > \delta$. Let k be the integer satisfying $t_k \le \delta$ and $t_{k+1} > \delta$. Clearly $t_{k+1} \le 2\delta$. Then for any $0 \le i \le k$, we have

$$F(t_i) \le \left(\frac{C_0 t_{i+1}^2}{t_{i+1} - t_i}\right)^2 F(t_{i+1}) = \frac{1}{4} F(t_{i+1}), \tag{2.7}$$

Iterating (2.7) k times, we have

$$F(t_0) \le (\frac{1}{4})^{k+1} F(t_{k+1}) \le (\frac{1}{4})^{k+1} F(2\delta) \le (\frac{1}{4})^{k+1}. \tag{2.8}$$

Now we estimate k. From (2.6) it follows that

$$\frac{1}{2C_0 t_i} = \frac{1}{2C_0 t_{i+1}} + \frac{1}{1 - 2C_0 t_{i+1}}, \quad \text{for } 0 \le i \le k.$$

Then summing it from $i = 0$ to $i = k$, we have

$$\frac{1}{2C_0 t_0} = \frac{1}{2C_0 t_{k+1}} + \sum_{i=1}^{k+1} \frac{1}{1 - 2C_0 t_i}.$$

Since $0 < t_i \le \delta \le \frac{1}{8C_0}$ for $1 \le i \le k$, it follows that

$$1 < \frac{1}{1 - 2C_0 t_i} \le \frac{4}{3}.$$

Then

$$k + 1 < \frac{1}{2C_0}\left(\frac{1}{t_0} - \frac{1}{t_{k+1}}\right) < \frac{4}{3}(k+1).$$

Recalling $t_0 = r$, and $\delta < t_{k+1} \le 2\delta$, we have

$$\frac{3}{8C_0 r} - 3 \le k + 1 < \frac{1}{2C_0 r} - 2.$$

Therefore, from (2.8),

$$F(r) = F(t_0) \le \left(\frac{1}{4}\right)^{k+1} \le \left(\frac{1}{4}\right)^{\frac{3\delta}{r} - 3}.$$

The proof is completed. ∎

Proof of Theorem 1.1. Given a point $z = (z', z_n) \in \Omega_1$, define

$$\widehat{\Omega}_s(z) := \left\{ x = (x', x_n) \in \Omega_1 \Big| -\frac{\epsilon}{2} + h_2(x') < x_n < \frac{\epsilon}{2} + h_1(x'),\ |x' - z'| < s \right\}. \tag{2.9}$$

We consider the following scaling in $\widehat{\Omega}_{\frac{1}{2}(\epsilon + h_1(z') - h_2(z'))}(z)$,

$$\begin{cases} Ry' + z' = x', \\ Ry_n - \dfrac{\epsilon}{2} + h_2(z') = x_n, \end{cases}$$

where $R = (\epsilon + h_1(z') - h_2(z'))$. Denote

$$\widehat{h}_1(y') := \frac{1}{R}\left(\epsilon - h_2(z') + h_1\left(z' + Ry'\right)\right),$$

$$\widehat{h}_2(y') := \frac{1}{R}\left(-h_2(z') + h_2\left(z' + Ry'\right)\right).$$

Then

$$\widehat{h}_1(0') = 1, \quad \widehat{h}_2(0') = 0,$$

and

$$\widehat{h}_2(y') < \widehat{h}_1(y'), \quad |\nabla^l \widehat{h}_1(y')|, |\nabla^l \widehat{h}_2(y')| \leq C_l, \text{ for } |y'| \leq 1,\ l \geq 1.$$

Let

$$\widehat{u}(y', y_n) = u\left(Ry' + z', Ry_n - \frac{\epsilon}{2} + h_2(z')\right).$$

Then $\widehat{u}(y)$ satisfies

$$\partial_\alpha\left(\widehat{A}_{ij}^{\alpha\beta}(y)\partial_\beta \widehat{u}^j(y) + \widehat{B}_{ij}^{\alpha}(y)\widehat{u}^j(y)\right) + \widehat{C}_{ij}^{\beta}(y)\partial_\beta \widehat{u}^j(y) + \widehat{D}_{ij}(y)\widehat{u}^j(y) = 0 \quad \text{in } Q_1, \tag{2.10}$$

where

$$\widehat{A}(y) = A\left(Ry' + z', Ry_n - \frac{\epsilon}{2} + h_2(z')\right), \quad \widehat{B}(y) = RB\left(Ry' + z', Ry_n - \frac{\epsilon}{2} + h_2(z')\right),$$

$$\widehat{C}(y) = RC\left(Ry' + z', Ry_n - \frac{\epsilon}{2} + h_2(z')\right), \quad \widehat{D}(y) = R^2 D\left(Ry' + z', Ry_n - \frac{\epsilon}{2} + h_2(z')\right),$$

and for $r < 1$,

$$Q_r := \left\{ (y', y_n) \in \mathbb{R}^n \,\Big|\, \widehat{h}_2(y') < y_n < \widehat{h}_1(y'),\ |y'| < r \right\}.$$

Using L^2 estimates for elliptic systems (2.10) and by the Sobolev imbedding theorems, we have

$$\max_{0 \leq y_n \leq 1} |\nabla^k \widehat{u}(0', y_n)| \leq C\|\nabla \widehat{u}\|_{L^2(Q_1)},$$

where C depends only on $n, N, \lambda, \Lambda, k, \beta_{k+\frac{n}{2}+1}$ and $\gamma_{k+\frac{n}{2}+1}$. It follows, in view of Lemma 2.1, that

$$|\nabla^k u(z)| \leq C\left(\epsilon + |z'|^2\right)^{1-k-\frac{n}{2}} \|\nabla u\|_{L^2\left(\Omega_{|z'|+\frac{R}{2}}\right)} \leq C\left(\epsilon + |z'|^2\right)^{1-k-\frac{n}{2}} (\mu_0)^{\frac{1}{\max\{\sqrt{\epsilon},\ |z'|+\frac{R}{2}\}}}, \tag{2.11}$$

where $\mu_0 < 1$ was defined in Lemma 2.1, and C depends only on $n, N, \lambda, \Lambda, k, \beta_{k+\frac{n}{2}+1}$ and $\gamma_{k+\frac{n}{2}+1}$. The proof is completed. □

Proof of Corollary 1.1. Without loss of generality, we assume that D_1 and D_2 are separated by the plane $x_n = 0$, with $(0', \frac{\epsilon}{2}) \in \partial D_1$ and $(0', -\frac{\epsilon}{2}) \in \partial D_2$. Since

$$\Delta(u - H) = 0 \text{ in } \mathbb{R}^n \setminus \overline{D_1 \cup D_2},$$

we have, after applying the maximum principle to $u - H$, that

$$|u - H| \leq \|H\|_{L^\infty(\partial D_1 \cup \partial D_2)} + K, \quad \text{in } \mathbb{R}^n \setminus \overline{D_1 \cup D_2}.$$

Corollary 1.1 then follows from Theorem 1.1. □

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