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DERIVATIVE FREE COMBINED METHOD FOR THE SIMULTANEOUS INCLUSION OF POLYNOMIAL ZEROS

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ABSTRACT. **A combined method for the simultaneous inclusion of complex zeros of a polynomial, composed of two circular arithmetic methods, is presented. This method does not use polynomial derivatives and has the order of convergence equals four. Computationally verifiable initial conditions that guarantee the convergence are also stated. Two numerical example are included to demonstrate the convergence speed of the presented method.**

1. INTRODUCTION

A great importance of the problem of finding polynomial zeros in the theory and practice (e.g., in the theory of control systems, stability of systems, analysis of transfer functions, various mathematical models, differential and difference equations, eigenvalue problems and other branches) has led to the development of a great number of root-finding methods. As stressed by Wilkinson [11], the aim of any numerical algorithm is to improve the approximate result and also to give error bound for the improved approximation. During the last four decades, various techniques for *a posteriori* error estimates for the determination of polynomial zeros were developed, for instance, procedures based on Gerschgorin's theorem, Rouché's theorem, rational approximations, fixed point relations, and so on.

Gargantini and Henrici proposed in [4] a quite different approach to error estimates for a given set of approximate zeros, developing an algorithm in circular complex interval arithmetic. Resulting disks contain the wanted zeros, which automatically provides the information about upper error bounds

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of approximations to the zeros. After this fundamental paper, a lot of iterative methods for the simultaneous inclusion of complex zeros of polynomials were constructed (for more details, see the books [2], [7], [9]).

In this paper our attention will be devoted to a derivative free combined method for the simultaneous inclusion of complex zeros. A detailed convergence analysis, including computationally verifiable initial conditions for the convergence, and numerical results will be presented.

In Section 1 we give a short review of definitions and operations of circular arithmetic. Cubically convergent combined algorithm, consisting of the second order method of Weierstrass' type and the third order Börsch-Supan-like inclusion method, are presented in Section 2. Convergence analysis of this algorithm is the subject of Section 3. To demonstrate the convergence rate of the proposed method, we present numerical results in Section 4.

First, we give some properties of circular complex arithmetic, which are necessary to carry out the convergence analysis of the presented combined method. A disk Z with center $c := \text{mid } Z$ and radius $r := \text{rad } Z$ will be denoted by the parametric notation $Z = \{c; r\}$. The basic circular arithmetic operations are defined as follows:

$$\{c_1; r_1\} \pm \{c_2; r_2\} = \{c_1 \pm c_2; r_1 + r_2\},$$

$$\{c_1; r_1\} \cdot \{c_2; r_2\} := \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\}, \tag{1.1}$$

$$Z^{-1} = \{c; r\}^{-1} = \frac{\{\bar{c}; r\}}{|c|^2 - r^2} \quad (|c| > r,\ \text{i.e. } 0 \notin Z) \quad (\textit{exact inversion}), \tag{1.2}$$

$$Z^I := \{c; r\}^I = \left\{\frac{1}{c}; \frac{r}{|c|(|c| - r)}\right\} \quad (|c| > r,\ \text{i.e. } 0 \notin Z) \quad (\textit{centered inversion}). \tag{1.3}$$

Using (1.2) and (1.3) the division is defined as

$$Z_1 : Z_2 := Z_1 \cdot Z_2^{-1} \quad \text{or} \quad Z_1 : Z_2 := Z_1 \cdot Z_2^I \qquad (0 \notin Z_2).$$

It is easy to show that $Z^{-1} \subset Z^I$. In the sequel, INV (Z) will denote one of the two inversions Z^{-1}, Z^I. We will use the following estimates

$$|\text{mid } (\text{INV } (Z))| \le \frac{|z|}{|z|^2 - r^2}, \tag{1.4}$$

$$\text{rad } (\text{INV } (Z)) \le \frac{r}{|z|(|z| - r)}. \tag{1.5}$$

Using (1.1) it can be proved that the product of n disks is given by

$$\prod_{k=1}^{n} \{c_k; r_k\} = \Big\{ \prod_{k=1}^{n} c_k; \prod_{k=1}^{n} (|c_k| + r_k) - \prod_{k=1}^{n} |c_k| \Big\}. \tag{1.6}$$

2. Cubically convergent combined method

Let us consider a monic complex polynomial P of degree $n \geq 3$ with simple zeros $\zeta_1, \ldots, \zeta_n$,

$$P(z) = \prod_{j=1}^{n} (z - \zeta_j),$$

wherefrom we obtain the fixed point relations

$$\zeta_i = z - \frac{P(z)}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (z - \zeta_j)} = z - P(z) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{z - \zeta_j}. \tag{2.1}$$

Let us assume that we have found disks $Z_1, \ldots, Z_n$ such that $\zeta_i \in Z_i$, $i \in I_n = \{1, \ldots, n\}$, where I_n is the index set, and let $z_i = \operatorname{mid} Z_i$. Setting $z = z_i$ and replacing the exact zero ζ_j $(j \neq i)$ by its inclusion disk Z_j in (2.1), we obtain the quadratically convergent method for the simultaneous inclusion of simple complex zeros of the polynomial P,

$$\hat{Z}_i = z_i - \frac{P(z_i)}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (z_i - Z_j)} \quad (i \in I_n), \tag{2.2}$$

or in an alternative form

$$\hat{Z}_i = z_i - P(z_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{z_i - Z_j} \quad (i \in I_n), \tag{2.3}$$

where $\hat{Z}_i$ is a new inclusion disk for ζ_i. The last two formulas define the so-called Weierstrass' inclusion method in circular complex arithmetic, considered in details in [5] and [10]. If $Z_1, \ldots, Z_n$ are real intervals containing real simple zeros, then (2.2) reduces to the interval iterative formula studied by Alefeld and Herzberger in [1]. The name comes from the similarity with the classical Weierstrass method

$$\hat{z}_i = z_i - W_i, \quad W_i = \frac{P(z_i)}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)} \quad (i \in I_n) \tag{2.4}$$

for the simultaneous approximation of polynomial zeros in ordinary complex arithmetic (see [12]). The quantity W_i is often called the Weierstrass correction.

A faster derivative free algorithm for the simultaneous inclusion of polynomial zeros was developed in [6]. Using the fixed point relation of the form

$$\zeta_i = z_i - \frac{W_i}{1 + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_j}{\zeta_i - z_j}} \quad (i \in I_n) \tag{2.5}$$

(see [3]), and taking the inclusion disk Z_i instead of the zero ζ_i in (2.5), the following third order inclusion method was obtained [6],

$$\hat{Z}_i = z_i - \frac{W_i}{1 + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_j}{Z_i - z_j}} \quad (i \in I_n), \tag{2.6}$$

where $\hat{Z}_i$ denotes a new outer circular approximation for ζ_i, that is, $\zeta_i \in \hat{Z}_i$ $(i \in I_n)$. Let us note that the iterative methods (2.2), (2.3), (2.4) and (2.6) do not use the derivatives of the polynomial P.

In this paper we combine the inclusion methods (2.2) and (2.6) to obtain an efficient two-stage method. In this construction we first apply formula (1.6) for the product of disks

$$\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - Z_j) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \{z_i - z_j; r_j\} = \{c_i; \eta_i\},$$

and the centered inversion (1.3) to the iterative formula (2.2), and get

$$Z_i^* = z_i - \frac{P(z_i)}{\{c_i; \eta_i\}} = z_i - \left\{ \frac{P(z_i)}{c_i}; \frac{|P(z_i)|\eta_i}{|c_i|(|c_i| - \eta_i)} \right\} = z_i - \{W_i; R_i^*\},$$

where

$$c_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j), \quad \eta_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} |z_i - z_j| - \prod_{\substack{j=1 \\ j \neq i}}^{n} (|z_i - z_j| - r_j), \tag{2.7}$$

$$R_i^* = \frac{|P(z_i)|\eta_i}{|c_i|(|c_i| - \eta_i)}. \tag{2.8}$$

In this manner we construct the two-stage method for the simultaneous inclusion of all zeros of P,

$$\begin{cases} Z_i^* = z_i - \{W_i; R_i^*\}, \\ \hat{Z}_i = z_i - W_i \left[1 + \sum_{\substack{j=1 \\ j \neq i}}^{n} W_j \, \mathrm{INV}\ (Z_i^* - z_j) \right]^{-1} \quad (i \in I_n). \end{cases} \tag{2.9}$$

The combined iterative formula (2.9) can be implemented by employing either the exact inversion (1.2) or the centered inversion (1.3). From (2.9) we notice that the already calculated quantities W_j $(j \in I_n)$ are again applied in the second formula of (2.9), which decreases the computational cost.

For simplicity, in Section 3 we will write $\sum_{j\neq i}$ and $\prod_{j\neq i}$ instead of $\sum_{\substack{j=1\\ j\neq i}}^{n}$ and $\prod_{\substack{j=1\\ j\neq i}}^{n}$, respectively.

3. Convergence of combined method

In the sequel we will assume that $n \geq 3$. Also, for any $i \in I_n$ we introduce the abbreviations

$$\begin{aligned}
\varepsilon_i &= z_i - \zeta_i, \quad v_{ij} = z_i - z_j - W_i, \quad r_i = \text{rad }\ Z_i, \quad R_i^* = \text{rad }\ Z_i^*, \\
r &= \max_{i\in I_n} r_i, \quad R^* = \max_{i\in I_n} R_i^*, \\
d &= \min_{\substack{i,j\in I_n\\ i\neq j}} |\text{mid }\ Z_i - \text{mid }\ Z_j|, \\
H_i &= 1 + \sum_{j\neq i} W_j \,\text{INV }\ (Z_i^* - z_j), \quad h_i = \text{mid }\ H_i, \quad R_i = \text{rad }\ H_i.
\end{aligned}$$

First, we give some results necessary for the convergence analysis.

Lemma 3.1. *Let* $\alpha = e^{2/7}$ $(\cong 1.331 < 4/3)$. *Then, under the condition*

$$d > \frac{7}{2}(n-1)r, \tag{3.1}$$

the following inequalities are valid for every $i \in I_n$:

(i) $|W_i| < \alpha|\varepsilon_i| \leq \alpha r_i \leq \alpha r < \frac{4}{3}r$;

(ii) $R_i^* < \dfrac{r_i}{2}$;

(iii) $R^* < \dfrac{7\alpha r^2}{d}\left(\left(\dfrac{7}{6}\right)^{n-1} - 1\right)$.

Proof of (i). The sequence $\{a_k\}$, defined by $a_k = \left(1 + \frac{2}{7k}\right)^k$, is bounded and monotonically increasing so that

$$a_k < \lim_{k\to\infty} a_k = e^{2/7} = \alpha$$

for all $k \in \mathbb{N}$. According to this, for any $i \in I_n$ we have

$$\begin{aligned}
|W_i| &= \frac{|P(z_i)|}{\prod\limits_{j \neq i} |z_i - z_j|} = |z_i - \zeta_i| \cdot \prod_{j \neq i} \frac{|z_i - \zeta_j|}{|z_i - z_j|} \\
&\le |\varepsilon_i| \cdot \prod_{j \neq i} \frac{|z_i - z_j| + r_j}{|z_i - z_j|} \le |\varepsilon_i| \Big(1 + \frac{r}{d}\Big)^{n-1} \\
&< |\varepsilon_i| \Big(1 + \frac{2}{7(n-1)}\Big)^{n-1} < \alpha |\varepsilon_i| \le \alpha r_i \le \alpha r < \frac{4}{3} r,
\end{aligned}$$

which proves (i).

Proof of (ii) and (iii). Using (3.1) and (i) we obtain

$$\begin{aligned}
R_i^* &= \frac{|P(z_i)|\eta_i}{|c_i|(|c_i| - \eta_i)} = |W_i| \frac{\eta_i}{|c_i| - \eta_i} < \alpha r_i \Bigg(\prod_{j \neq i} \frac{|z_i - z_j|}{|z_i - z_j| - r_j} - 1 \Bigg) \\
&= \alpha r_i \Bigg(\prod_{j \neq i} \Big(1 + \frac{r_j}{|z_i - z_j| - r_j} \Big) - 1 \Bigg) \le \alpha r_i \Big(\Big(1 + \frac{r}{d - r}\Big)^{n-1} - 1 \Big) \\
&= \frac{\alpha r_i r}{d} \cdot \frac{1}{1 - \frac{r}{d}} \cdot \Bigg(\Big(1 + \frac{1}{d/r - 1}\Big)^{n-2} + \Big(1 + \frac{1}{d/r - 1}\Big)^{n-3} + \cdots + 1 \Bigg) \\
&< \frac{\alpha r_i r}{d} \cdot \frac{7}{6} \Bigg(\Big(\frac{7}{6}\Big)^{n-2} + \Big(\frac{7}{6}\Big)^{n-3} + \cdots + 1 \Bigg) = \frac{7 \alpha r_i r}{d} \Big(\Big(\frac{7}{6}\Big)^{n-1} - 1 \Big).
\end{aligned}$$

Hence, it follows

$$R_i^* < \frac{r_i}{2}$$

and

$$R_i^* \le R^* < \frac{7 \alpha r^2}{d} \Big(\Big(\frac{7}{6}\Big)^{n-1} - 1 \Big). \quad \square$$

Lemma 3.2. *If* (3.1) *holds and* $\zeta_i \in Z_i$ *for all* $i \in I_n$, *then the inversions in* (2.9) *exist, i.e.,*

(i) $0 \notin \{c_i; \eta_i\}$;
(ii) $0 \notin \{v_{ij}; R_i^*\}$;
(iii) $0 \notin H_i = \{h_i; R_i\}$.

Proof of (i). It follows from (2.7) and the fact that the disks $Z_1, \dots, Z_n$ are disjoint, that is, $|z_i - z_j| > r_j$ holds for any pair $i, j \in I_n$.

Proof of (ii). According to (i) of Lemma 3.1, we have

(3.2)
$$|v_{ij}| = |z_i - z_j - W_i| \ge |z_i - z_j| - |W_i| > d - \alpha r > d - \frac{4}{3}r.$$

Hence, by (3.1) and (ii) of Lemma 3.1,

$$|v_{ij}| > d - \frac{4}{3}r > r \ge r_i > R_i^*,$$

which proves that $0 \notin \{v_{ij}; R_i^*\}$.

Proof of (iii). Using (1.5), (3.1), (3.2) and the assertions of Lemma 3.1, we find

$$\begin{aligned} R_i &= \text{rad } H_i = \sum_{j\ne i} |W_j| \,\text{rad } \Big(\text{INV } \big(z_i - z_j - \{W_i; R_i^*\}\big)\Big) \\ &\le \sum_{j\ne i} |W_j| \frac{R_i^*}{|z_i - z_j - W_i|(|z_i - z_j - W_i| - R_i^*)} \\ &< \alpha r \sum_{j\ne i} \frac{R^*}{(d-\alpha r)(d - \alpha r - R^*)} \\ &< \frac{\frac{4}{3}r \cdot (n-1)\frac{r}{2}}{\Big(d - \frac{4r}{3}\Big)\Big(d - \frac{4r}{3} - \frac{r}{2}\Big)} = \frac{\frac{2}{3}(n-1)}{\Big(\frac{d}{r} - \frac{4}{3}\Big)\Big(\frac{d}{r} - \frac{11}{6}\Big)} < \frac{1}{20}\,. \end{aligned}$$

Let us bound the center h_i of the disk H_i. According to (1.4) and (3.1) we get

$$\begin{aligned} |h_i| &\ge 1 - \sum_{j\ne i} |W_j| |\text{mid } \big(\text{INV } (z_i - z_j - \{W_i; R_i^*\})\big)| \\ &\ge 1 - \sum_{j\ne i} \frac{|W_j| \cdot |v_{ij}|}{|v_{ij}|^2 - R_i^{*2}} > 1 - \frac{\frac{4}{3}r(n-1)\Big(d - \frac{4r}{3}\Big)}{\Big(d - \frac{4r}{3}\Big)^2 - \frac{r^2}{4}} \\ &= 1 - \frac{\frac{4}{3}(n-1)\Big(\frac{d}{r} - \frac{4}{3}\Big)}{\Big(\frac{d}{r} - \frac{4}{3}\Big)^2 - \frac{1}{4}} > \frac{13}{25}. \end{aligned}$$

Hence

(3.3)
$$|h_i|^2 - {R_i}^2 > \left(\frac{13}{25}\right)^2 - \left(\frac{1}{20}\right)^2 > \frac{1}{4}$$

and $|h_i| > R_i$, which means that $0 \notin \{h_i; R_i\}$. □

The following assertion is concerned with the convergence of the combined method (2.9).

Theorem 3.3. *Let* $(Z_1, \ldots, Z_n) = (Z_1^{(0)}, \ldots, Z_n^{(0)})$ *be initial disks such that* $\zeta_i \in Z_i$ $(i \in I_n)$, *let* $\{Z_i^{(m)}\}$ $(i \in I_n)$ *denote the sequences of disks produced by* (2.9), *and let*

$$r^{(m)} = \max_{i \in I_n} \text{rad } (Z_i^{(m)}), \quad d^{(m)} = \min_{\substack{i,j \in I_n \\ i \neq j}} |\text{mid } (Z_i^{(m)}) - \text{mid } (Z_j^{(m)})|,$$

where $m = 0, 1, 2, \ldots$ *is the iteration index. If the condition*

$$d^{(0)} > \frac{7}{2}(n-1)r^{(0)} \tag{3.4}$$

is satisfied, then for any $i \in I_n$ *and* $m = 0, 1, 2, \ldots$ *there holds*

$$\zeta_i \in Z_i^{(m)}$$

and the sequences of radii $\{\text{rad } (Z_i^{(m)})\}$ $(i \in I_n)$ *tend monotonically towards 0.*

Proof. We will prove Theorem 3.3 by induction and we start with $m = 0$. For simplicity, all indices are omitted and all quantities in the next iteration are denoted by the additional mark $\hat{}$. We define

$$\hat{r} = \max_{i \in I_n} \hat{r}_i \quad \text{and} \quad \hat{d} = \min_{\substack{i,j \in I_n \\ i \neq j}} |\hat{z}_i - \hat{z}_j|.$$

The second formula of (2.9) may be written in the form

$$\hat{Z}_i = z_i - W_i H_i^{-1} = z_i - W_i \{h_i; R_i\}^{-1},$$

whence

$$\hat{r}_i = \text{rad } \hat{Z}_i = |W_i| \frac{R_i}{|h_i|^2 - {R_i}^2}$$

and

$$\hat{z}_i = \text{mid } \hat{Z}_i = z_i - \frac{W_i \bar{h}_i}{|h_i|^2 - R_i^2}.$$

By virtue of Lemma 3.1(i) and the estimates $R_i < 1/20$ and $|h_i| > 13/25$ (see the proof of Lemma 3.2), we get

$$\hat{r}_i < |W_i| \frac{R_i}{|h_i|^2 - {R_i}^2} < \frac{4}{15} r_i, \tag{3.5}$$

for all $i \in I_n$.

On the other hand, according to the inclusion property we obtain $\zeta_i \in \hat{Z}_i$, wherefrom

$$|\hat{z}_i - \zeta_i| \leq \hat{r}_i < \frac{4}{15} r.$$

Since $\zeta_i \in Z_i$, that is $|z_i - \zeta_i| \le r$, we have

$$|\hat{z}_i - z_i| \le |\hat{z}_i - \zeta_i| + |z_i - \zeta_i| < \frac{4}{15}r + r = \frac{19}{15}r.$$

Now, for any pair $i, j \in I_n,\ i \ne j$, we find

$$\hat{d} = |\hat{z}_i - \hat{z}_j| \ge |z_i - z_j| - |z_i - \hat{z}_i| - |z_j - \hat{z}_j| > d - \frac{38}{15}r. \tag{3.6}$$

Using (3.4), (3.5) and (3.6), there follows

$$\frac{\hat{r}}{\hat{d}} < \frac{\frac{4}{15}r}{d\Big(1 - \frac{38}{15}\cdot\frac{r}{d}\Big)} < \frac{1}{2}\cdot\frac{r}{d} < \frac{2}{7(n-1)}.$$

Hence, we conclude by induction that the initial condition (3.4) implies the inequality

$$d^{(m)} > \frac{7}{2}(n-1)r^{(m)}$$

for every $m = 0, 1, \dots$. For this reason, the assertions of Lemmas 3.1 and 3.2 are valid for all $m = 0, 1, \dots$. In particular, following (3.5), we have

$$r^{(m+1)} < \frac{4}{15}r^{(m)} \tag{3.7}$$

for every $m = 1, 2, \dots$.

In view of the fixed point relations (2.1) and (2.5), the chain of inclusions

$$\zeta_i \in Z_i^{(m)} \implies \zeta_i \in Z_i^{*(m)} \implies \zeta_i \in Z_i^{(m+1)},$$

and the assumption $\zeta_i \in Z_i^{(0)}$, by induction we find that $\zeta_i \in Z_i^{(m)}$ for any $i \in I_n$ and $m = 0, 1, \dots$ if (3.4) holds.

¿From Lemma 3.2 we conclude that the inversions in (2.9) exist in each iterative step so that the combined method (2.9) is feasible. Besides, since $d^{(m)} > 2r^{(m)}$ it follows that the disks $Z_1^{(m)}, \dots, Z_n^{(m)}$ are pairwise disjoint. The inequality (3.7) shows that the sequences of radii $\{r_i^{(m)}\}$ $(i \in I_n)$ converge monotonically to 0. □

We are now able to determine the order of convergence of the combined method (2.9).

Theorem 3.4. *Let the interval sequences* $\{Z_i^{(m)}\}$ $(i \in I_n)$ *be defined by the iterative formula* (2.9). *Then, under the condition* (3.4), *the following is* *valid:*

$$r^{(m+1)} < \frac{112(n-1)\Big(\Big(\frac{7}{6}\Big)^{n-1}-1\Big)(r^{(m)})^4}{\Big(d^{(0)}-\frac{38}{11}r^{(0)}\Big)^3} \quad (m=0,1,2,\dots).$$

Proof. We give the proof for the case when the centered inversion is applied in (2.9). Starting from the disks

$$Z_i^* - z_j = \{v_{ij}; R_i^*\} \subseteq \{v_{ij}; R^*\},$$

we find

$$(Z_i^* - z_j)^I \subseteq \Big\{\frac{1}{v_{ij}}; \frac{R^*}{|v_{ij}|(|v_{ij}|-R^*)}\Big\},$$

so that, by (1.3) and Lemma 3.1(i),

$$\sum_{j\neq i} W_j (Z_i^* - z_j)^I \subseteq \Big\{\sum_{j\neq i}\frac{W_j}{v_{ij}}; \frac{(n-1)\alpha r R^*}{|v_{ij}|(|v_{ij}|-R^*)}\Big\}.$$

Having in mind this inclusion, we start from (2.9) and find that the radius of the disk

$$\hat{Z}_i = z_i - W_i\Big[1+\sum_{j\neq i} W_j (Z_i^* - z_j)^I\Big]^{-1}$$

is bounded by

$$\hat{r}_i = \operatorname{rad}\ \hat{Z}_i \le \frac{\dfrac{(n-1)\alpha^2 r^2 R^*}{|v_{ij}|(|v_{ij}|-R^*)}}{\Big|1+\sum_{j\neq i}\dfrac{W_j}{v_{ij}}\Big|^2 - \Big(\dfrac{(n-1)\alpha r R^*}{|v_{ij}|(|v_{ij}|-R^*)}\Big)^2}. \tag{3.8}$$

Now, using Lemma 3.1(iii), we estimate the numerator in (3.8):

$$\frac{(n-1)\alpha^2 r^2 R^*}{|v_{ij}|(|v_{ij}|-R^*)} < \frac{\alpha^2 r^2 (n-1)\dfrac{7\alpha r^2}{d}\Big(\Big(\frac{7}{6}\Big)^{n-1}-1\Big)}{\Big(d-\frac{4}{3}r\Big)\Big(d-\frac{11}{6}r\Big)}$$

$$= \frac{7\alpha^3 r^4 (n-1)\Big(\Big(\frac{7}{6}\Big)^{n-1}-1\Big)}{d^3\Big(1-\frac{4}{3}\cdot\frac{r}{d}\Big)\Big(1-\frac{11}{6}\cdot\frac{r}{d}\Big)} < \frac{28(n-1)\Big(\Big(\frac{7}{6}\Big)^{n-1}-1\Big)r^4}{d^3},$$

because of

$$\frac{7\alpha^3}{\Big(1-\frac{4}{3}\cdot\frac{r}{d}\Big)\Big(1-\frac{11}{6}\cdot\frac{r}{d}\Big)} < 28.$$

In regard to (3.3), the inequality (3.8) becomes

$$\hat{r}_i \le \hat{r} < \frac{112(n-1)\Big(\big(\frac{7}{6}\big)^{n-1} - 1\Big) r^4}{d^3}\,.$$

By induction we prove that the inequality

$$r^{(m+1)} < \frac{112(n-1)\Big(\big(\frac{7}{6}\big)^{n-1} - 1\Big) {r^{(m)}}^4}{{d^{(m)}}^3} \tag{3.9}$$

is valid for every $m = 0, 1, 2, \ldots$.

According to the geometric construction and the fact that the disks $Z_i^{(m)}$ and $Z_i^{(m+1)}$ have at least one point in common (the zero ζ_i), the following relation can be derived

$$d^{(m+1)} \ge d^{(m)} - 2r^{(m)} - 2r^{(m+1)}\,. \tag{3.10}$$

Let us put $\beta = 4/15$. By successive application of (3.7) and (3.10), one obtains

$$\begin{aligned}
\frac{1}{2}d^{(m)} &> \frac{1}{2}d^{(m-1)} - r^{(m-1)} - \beta r^{(m-1)} = \frac{1}{2}d^{(m-1)} - r^{(m-1)}(1+\beta) \\
&> \frac{1}{2}d^{(m-2)} - r^{(m-2)} - r^{(m-1)} - (1+\beta) r^{(m-1)} \\
&= \frac{1}{2}d^{(m-2)} - r^{(m-2)}(1 + 2\beta + 2\beta^2 - \beta^2) \\
&\;\;\vdots \\
&> \frac{1}{2}d^{(0)} - r^{(0)}(1 + 2\beta + 2\beta^2 + \cdots + 2\beta^m - \beta^m) \\
&> \frac{1}{2}d^{(0)} - \frac{19}{11} r^{(0)},
\end{aligned}$$

so that

$$d^{(m)} > d^{(0)} - \frac{38}{11} r^{(0)}.$$

According to the last inequality and (3.9) we finally obtain

$$r^{(m+1)} < \frac{112(n-1)\Big(\big(\frac{7}{6}\big)^{n-1} - 1\Big)(r^{(m)})^4}{\Big(d^{(0)} - \frac{38}{11} r^{(0)}\Big)^3} \quad (m = 0, 1, \ldots). \tag{3.11}$$

The proof in the case of the exact inversion is derived in a quite similar way so that will be omitted. Note that the constant 112 in the previous relation can be replaced with 104 when the exact inversion is used. The last relation (3.11) shows that the order of convergence of the method (2.9) is *four.* □

4. Numerical results

To test the convergence properties of the presented combined algorithm (2.9), we applied this method to polynomials of various degrees. We realized the corresponding algorithms on PC PENTIUM IV using the programming package *Mathematica* 4.1 with multiple precision arithmetic to save all significant digits. For the comparison purpose, we also tested the Börsch-Supan-like method with Weierstrass' correction

(4.1)
$$\hat{Z}_i = z_i - W_i \cdot \left[1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} W_j \cdot \text{INV } \left(z_j - Z_i + W_i \right) \right]^{-1} \quad (i \in I_n),$$

proposed in [8]. It was proved in [8] that the R-order of convergence of the method (4.1) is $(3+\sqrt{17})/2 \cong 3.562$ if INV $= ()^{-1}$ and 4 if INV $= ()^{I}$.

In all tested examples the choice of initial disks was carried out under weaker condition than (3.4). The type of inversion is stressed by the subscript indices "E" (*exact*) and "C" (*centered*); for instance, $(2.9)_E$ and $(2.9)_C$ denote two versions of the inclusion method (2.9) where the exact inversion $()^{-1}$ and the centered inversion $()^{I}$ are applied. For demonstration, we present two examples.

Example 1. We applied the methods (2.9) and (4.1) for the simultaneous inclusion of the zeros of the polynomial

$$P(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300.$$

The exact zeros of this polynomial are -3, ± 1, $\pm 2i$ and $\pm 2 \pm i$. The centers of the initial disks $Z_i^{(0)} = \{z_i^{(0)}; 0.5\}$ were selected to be $d^{(0)}/r^{(0)} = 4.7$, which is much less than $7(n-1)/2 = 28$.

The entries of the maximal radii of the disk produced in the first three iterations are given in Table 1, where the denotation $X(-p)$ means $X \times 10^{-p}$.

methods	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
$(2.9)_E$	3.21(−2)	5.10(−8)	2.49(−35)
$(2.9)_C$	5.58(−2)	4.06(−7)	8.54(−32)
$(4.1)_E$	1.51(−2)	3.22(−7)	8.85(−25)
$(4.1)_C$	2.15(−2)	2.56(−7)	1.80(−28)

Table 1.

Example 2. The interval methods (2.9) and (4.1) were applied for the determination of the eigenvalues of Hessenberg's matrix

$$H_8 = \begin{bmatrix} 2+3i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4+6i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6+9i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8+12i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10+15i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12+18i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 14+21i & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 16+24i \end{bmatrix},$$

whose characteristic polynomial is

$$\begin{aligned} f_8(\lambda) \;=\; & \lambda^8 - (72+108i)\lambda^7 - (2\,730 - 6\,552i)\lambda^6 + (208\,656 - 40\,824i)\lambda^5 \\ & -(2\,671\,431 + 2\,693\,880i)\lambda^4 - (8\,208\,648 - 40\,168\,548i)\lambda^3 \\ & +(240\,382\,340 - 97\,806\,672i)\lambda^2 - (718\,213\,536 + 487\,539\,216i)\lambda \\ & -9\,636\,481 + 1\,151\,539\,200i. \end{aligned}$$

Gerschgorin's disks which correspond to the matrix $H_8 = [h_{ij}]$ have the form $\{h_{ii}; R_i\}$ $(i = 1, \ldots, n)$, where h_{ii} are the diagonal elements of a matrix H_8 and $R_i = \sum_{j \neq i} |h_{ij}|$. If these disks are mutually nonintersecting, then each of them contains one and only one eigenvalue. In such case these disks are very convenient for the application of inclusion methods. In our example Gerschgorin's disks $\{h_{ii}; R_i\}$ are given by

$$\begin{aligned} &Z_1 = \{2+3i; 1\}, && Z_2 = \{4+6i; 1\}, && Z_3 = \{6+9i; 1\}, \\ &Z_4 = \{8+12i; 1\}, && Z_5 = \{10+15i; 1\}, && Z_6 = \{12+18i; 1\}, \\ &Z_7 = \{14+21i; 1\}, && Z_8 = \{16+24i; 1\}, \end{aligned}$$

and they are mutually disjoint. For this reason we have taken them as initial disks containing the eigenvalues of f_8 in the implementation of the tested methods.

The maximal radii $r^{(m)}$ $(m = 1, 2)$ of the produced disks, which enclose the eigenvalues of H_8, are displayed in Table 2.

methods	$r^{(1)}$	$r^{(2)}$
$(2.9)_E$	$2.24(-19)$	$2.68(-97)$
$(2.9)_C$	$1.34(-20)$	$9.96(-100)$
$(4.1)_E$	$1.16(-13)$	$9.31(-43)$
$(4.1)_C$	$1.46(-13)$	$1.03(-53)$

Table 2.

From Table 2 we notice that the applied methods converge extremely fast. Actually, the eigenvalues of Hessenberg's matrix are very close to the diagonal elements (the dimension of matrix is greater, the approximation

h_{ii} ($\cong \zeta_i$) is better). Since the elements h_{ii} are the centers of initial inclusion disks, this closeness causes the very fast convergence of the sequences of radii of inclusion disks.

The presented examples, and other numerous examples, point to a very good convergence properties of the presented method (2.9) and high computational efficiency.

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