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Derived Azumaya algebras and generators for twisted derived categories

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Abstract We introduce a notion of derived Azumaya algebras over ring and schemes generalizing the notion of Azumaya algebras of Grothendieck (Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses. Dix Exposés sur la Cohomologie des Schémas, pp. 46–66, North-Holland, Amsterdam, 1968). We prove that any such algebra B on a scheme X provides a class $\phi(B)$ in $H^1_{et}(X, \mathbb{Z}) \times H^2_{et}(X, \mathbb{G}_m)$. We prove that for X a quasi-compact and quasi-separated scheme ϕ defines a bijective correspondence, and in particular that any class in $H^2_{et}(X, \mathbb{G}_m)$, torsion or not, can be represented by a derived Azumaya algebra on X. Our result is a consequence of a more general theorem about the existence of compact generators in *twisted derived categories, with coefficients in any local system of reasonable dg-categories*, generalizing the well known existence of compact generators in derived categories of quasi-coherent sheaves of Bondal and Van Den Bergh (Mosc. Math. J. 3(1):1–36, 2003). A huge part of this paper concerns the treatment of twisted derived categories, as well as the proof that the existence of compact generator locally for the fppf topology implies the existence of a global compact generator. We present explicit examples of derived Azumaya algebras that are not represented by classical Azumaya algebras, as well as applications of our main result to the localization for twisted algebraic K-theory and to the stability of saturated *dg*-categories by direct push-forwards along smooth and proper maps.

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1 Introduction

In his seminal paper [7], Grothendieck studied Azumaya algebras over schemes, which are locally free sheaves of $\mathcal{O}$-algebras A, such that the natural morphism

$$A \otimes_{\mathcal{O}} A^{op} \longrightarrow \underline{End}_{\mathcal{O}}(A, A)$$

is an isomorphism. He constructed, for any such algebra A on a scheme X, a cohomology class $\gamma(A) \in H^2_{et}(X, \mathbb{G}_m)$. The class $\gamma(A)$ is always a torsion element, and Grothendieck asked the question wether all torsion classes in $H^2_{et}(X, \mathbb{G}_m)$ arise by this construction. The importance of this question lies in the fact that Azumaya algebras have description purely in terms of vector bundles on X, never mentioning the étale topology. A positive answer to his question would thus provide a description of the torsion part of the group $H^2_{et}(X, \mathbb{G}_m)$ uniquely in terms envolving vector bundles on the scheme X.

Since then, this question has been studied by several authors and many results, positive or negative, have been proven. The most important of these results is certainly the theorem of Gabber [4, 6], stating that any torsion class in $H^2_{et}(X, \mathbb{G}_m)$ can be realized by an Azumaya algebra when X is a (quasi-compact and separated) scheme admitting a ample line bundle (e.g. a quasi-projective scheme over some ring k). On the side of negative results, there are known examples of non-separated schemes for which not all torsion classes in $H^2_{et}(X, \mathbb{G}_m)$ come from Azumaya algebras (see [5, Corollary 3.11]). There are also schemes X with non-torsion $H^2_{et}(X, \mathbb{G}_m)$, such as the famous example of Mumford with X a normal complex algebraic surface, for which non-torsion elements persist locally for the Zariski topology on X (see [7]). Non-reduced scheme, such as $X[\epsilon]$ for X a smooth variety, also provide plenty of examples of schemes with non-torsion $H^2_{et}(X, \mathbb{G}_m)$ (e.g. first-order deformations of the trivial Azumaya algebra, which are classified by $H^2(X, \mathcal{O}_X)$).

The main purpose of this work is to continue further the study of the cohomology group $H^2_{et}(X, \mathbb{G}_m)$, and more particularly of the part which is not represented by Azumaya algebras, including the non-torsion part. For this, we introduce a notion of *derived Azumaya algebras*, which are differential graded analogs of Azumaya algebras, which, as a first rough approximation, can be thought as *Azumaya algebra objects in the perfect derived category*. By definition, a derived Azumaya algebra over a scheme X, is a sheaf of associative $\mathcal{O}_X$-*dg*-algebras B, satisfying the following two conditions:

1. The underlying complex of $\mathcal{O}_X$-modules is a perfect complex on X, and we have $B \otimes^{\mathbb{L}}_{\mathcal{O}_X} k(x) \neq 0$ for all point $x \in X$ (which is equivalent to say that the restriction of B over any affine open sub-scheme *Spec* $A \subset X$, provides a compact generator of the derived category $D(A)$, see Remark 2.2).
2. The natural morphism of complexes of $\mathcal{O}_X$-modules

$$B \otimes^{\mathbb{L}}_{\mathcal{O}_X} B^{op} \longrightarrow \mathbb{R}\underline{\mathcal{H}om}_{\mathcal{O}_X}(B, B)$$

   is a quasi-isomorphism.

Our first main result state that derived Azumaya algebras are locally trivial for the étale topology (see Proposition 2.14), and that the Morita equivalence classes of derived Azumaya algebras on a scheme X embedds into $H^1_{et}(X, \mathbb{Z}) \times H^2_{et}(X, \mathbb{G}_m)$ (see Corollary 3.12). Any derived Azumaya algebra B on a scheme (or more generally stack) X then possesses a characteristic class $\phi(B) \in H^2_{et}(X, \mathbb{G}_m)$. Our second main result is the following theorem.

Theorem 1.1 (See Corollary 4.8) *If X is a quasi-compact and quasi-separated scheme, then any class in $H^2_{et}(X, \mathbb{G}_m)$ is of the form $\phi(B)$ for some derived Azumaya algebra B over X.*

More precisely, there exists a bijection

$$\phi : dBr(X) \simeq H^1_{et}(X, \mathbb{Z}) \times H^2_{et}(X, \mathbb{G}_m),$$

where $dBr(X)$ is the group of Morita equivalence classes of derived Azumaya algebras over X.

As a first consequence we get interesting derived Azumaya algebras in some concrete examples. To start with we can take X the normal surface considered by Mumford, and A its local ring at the singular point. It is a normal local $\mathbb{C}$-algebra of dimension 2, with a very big $H^2_{et}(Spec\,A, \mathbb{G}_m)$. For any class $\alpha \in H^2_{et}(Spec\,A, \mathbb{G}_m)$, torsion or not, we do get by the previous theorem a derived Azumaya algebra B_α over A representing this class. The *dg*-algebra B is Morita equivalent to an actual (non derived) Azumaya algebra if and only if α is torsion. Another example is to start with X a smooth and projective complex manifold, with $H^2(X, \mathcal{O}_X) \neq 0$. Any non-zero element $\alpha \in H^2(X, \mathcal{O}_X)$, can be interpreted as a class in $H^2_{et}(X[\epsilon], \mathbb{G}_m)$ restricting to zero on X, and this provides by our theorem a derived Azumaya algebra B_α over $X[\epsilon]$, which is a first order deformation of $\mathcal{O}_X$, up to Morita equivalence. In both cases, we obtain examples of derived Azumaya algebras which are not equivalent to Azumaya algebras, but represent interesting, or at least natural, cohomology classes.

The proof of Theorem 1.1 is done in two independant steps. The first step is the construction of the map ϕ, that associates a cohomology class to a given derived Azumaya algebra. For this, we start with a local study of derived Azumaya algebras over a base commutative ring, and we prove that any derived Azumaya algebra B is locally Morita equivalent, for the étale topology, to the trivial derived Azumaya algebra $\mathcal{O}$ (see Proposition 2.14). This local triviality property is deduced from a deformation theory argument: we first show that any derived Azumaya algebras is trivial over an algebraically closed field, and then extends this result over an arbitrary strictly Henselian ring by proving that the stack of trivialization of a given derived Azumaya algebra is algebraic and smooth. This last step uses the existence of a reasonable moduli stack of *dg*-modules over *dg*-algebras and is based on results from [31]. It is the only non-purely algebraic part of the local theory. Once the local triviality is known, the construction of the class ϕ is rather formal. We compute the self-equivalence group of the trivial derived Azumaya algebra, and find that it is the group stack $\mathbb{Z} \times K(\mathbb{G}_m, 1)$. As a consequence the stack associated to the prestack of derived Azumaya algebras is $K(\mathbb{Z}, 1) \times K(\mathbb{G}_m, 2)$, and it follows formally that any such *dg*-algebra B over some scheme X provides a class $\phi(B)$ in $H^1_{et}(X, \mathbb{Z}) \times H^2_{et}(X, \mathbb{G}_m)$. The injectivity of the construction $\phi(B)$ is formal in view of the definition of Morita equivalences between derived Azumaya algebras.

The second step in the proof of Theorem 1.1 consists in proving that the map ϕ is surjective, which is, by very far, the hardest part of the result. The general strategy is to deduce the surjectivity from a more general statement stating the existence of compact generators in *twisted derived categories*. Our approach is then very similar to the approach presented in [4], re-interpreting the existence of Azumaya algebras as the existence of a certain object in a category of twisted sheaves. Our approach has to be done at the level of derived categories as well as with a much more general notion of *twisted sheaves*. We introduce in our second section the notion of *locally presentable dg-categories over schemes*, and study the stack of those. These are algebraic families of *dg*-categories, parametrized by some schemes, satisfying some set theory conditions (being locally presentable, or equivalently *well generated* in the sense of triangulated category, see [15]), as well as some descent condition with respect to the flat topology. One key statement is that these locally presentable *dg*-categories form a stack in ∞-categories $\mathbb{D}g^{lp,desc}$ (i.e. locally presentable *dg*-categories can be glued together with respect to the flat topology, see Theorem 3.4). To each scheme X, and to any locally presentable *dg*-category α over X, we can construct its global section $L_\alpha(X)$, which is a *dg*-category obtained by integration of α over all affine open sub-schemes in X. By definition, $L_\alpha(X)$ is called the *twisted derived category of X with coefficients in α*. On the other hand, we construct the stack (again of ∞-categories) of (quasi-coherent) *dg*-algebras $\mathbb{D}g\mathbb{A}lg$, parametrizing quasi-coherent *dg*-algebras over schemes. There exists a morphism of stacks

$$\phi : \mathbb{D}g\mathbb{A}lg \longrightarrow \mathbb{D}g^{lp,desc}$$

simply obtained by sending a given *dg*-algebra to the corresponding *dg*-category of *dg*-modules. The morphism ϕ provides, for any *dg*-algebra B over some scheme X, a locally presentable *dg*-category $\phi(B)$ over X, and thus a corresponding twisted derived category $L_{\phi(B)}(X)$, which is nothing else that the *dg*-category of sheaves of quasi-coherent B-*dg*-modules over X. The main observation is that we can recognize the locally presentable *dg*-categories α over X of the form $\phi(B)$ for some *dg*-algebra B, by a simple criterion concerning the existence of an object $E \in L_\alpha(X)$ whose restriction to each open affine $U = Spec\, A \subset X$ is a compact generator in $L_\alpha(U)$. The major result of this work is then the following existence theorem, which is a far reaching generalization of the main result of [3].

Theorem 1.2 (See Theorem 4.7) *Let X be a quasi-compact and quasi-separated scheme, and α a locally presentable dg-category over X. Assume that there exists a fppf covering $X' \longrightarrow X$ such that $L_\alpha(X')$ admits a compact generator. Then, there exists a compact generator $E \in L_\alpha(X)$ which also a compact local generator.*

Theorem 1.1 is now a direct consequence of the previous result. Indeed, the group of self equivalences of the trivial locally presentable *dg*-category **1** on X (which is the family of *dg*-categories of quasi-coherent complexes over open sub-schemes of X), is equivalent to $\mathbb{Z} \times K(\mathbb{G}_m, 1)$ (shifts and tensoring with a line bundle are the only A-linear self equivalences of the dg-category of A-dg-modules, for A a commutative ring). Therefore, a class $\alpha \in H^1_{et}(X, \mathbb{Z}) \times H^2_{et}(X, \mathbb{G}_m)$ provides a twisted form of $\mathbf{1}$, that is a locally presentable dg-category α over X locally equivalent, for the fppf or étale topology, to $\mathbf{1}$. By Theorem 1.2, $L_\alpha(X)$ possesses a compact local generator, and thus α is of the form $\phi(B)$ for a certain dg-algebra B over X, which is necessarly a derived Azumaya algebras (because it is so locally). We would like to mention, however, that Theorem 1.2 seems to us somehow more important than its consequence Theorem 1.1. Indeed, there exists many interesting consequences of Theorem 1.2, outside of the general question of existence of Azumaya algebras. Some will be given in our last section, and includes for instance a localization sequence for twisted K-theory, as well as construction of smooth and proper dg-categories by direct images of smooth and proper locally presentable dg-categories over smooth and proper schemes.

Related works To finish this introduction, other approaches already exist in order to describe the group $H^2_{et}(X, \mathbb{G}_m)$ and particularly its non-torsion part. In [8], the notion of central separable algebras, as well as the *big Brauer group* is studied. The authors show that the whole group $H^2_{et}(X, \mathbb{G}_m)$ is in bijection with certain equivalence classes of central separable algebras, for any quasi-compact Artin stack with an affine diagonal. The present work is in some sense orthogonal to the approach [8], as our notion of derived Azumaya algebras stay as close as possible to the usual notion of Azumaya algebras, whereas the notion of central separable algebras involves non-unital as well as infinite dimensional algebras. Also, the local triviality for the étale topology is taken as part of the definition in the big Brauer group (see [8, Definition 2.1]), and therefore the results of [8] do not truly make a correspondence between $H^2_{et}(X, \mathbb{G}_m)$ and a purely quasi-coherent structure on X which would not mention the étale topology, contrary to our correspondence Theorem 1.1 for which $dBr(X)$ is defined purely in terms of the theory of perfect complexes on X. Another difference, it is unclear to me how modules over central separable algebras behave, and how far they form a category equivalent to the category of twisted sheaves. One important feature of derived Azumaya algebras is that the corresponding derived category of dg-modules gives back the twisted derived category. There are, however, some similarities between our approach and the work [8]. The key statement we had to prove is the existence of compact object with certain nice local properties in the twisted derived category. In the same way, the authors [8] deduces their result from the existence of a twisted coherent sheaf with nice local property. The existence of this coherent sheaf is in turn a formal consequence of the extension property of coherent sheaves on nice Artin stacks, whereas the existence of a compact local generator requires extensions properties of compact objects along open immersions (Proposition 4.9), as well as certain tricky descent statements of compact generators along étale and fppf coverings (Propositions 4.12, 4.13).

We should also mention the preprint [1], in which the authors define and study a topological analog of our notion of derived Azumaya algebras. The definitions of [1] has been compared with the one of the present paper in [9]. A slight adaptation of our proof of the local triviality for the étale topology (Proposition 2.14) also proves the local triviality for the étale topology over connective ring spectra (this adaptation requires to use *brave new algebraic geometry*, as this is done in [33, Sect. 2.4] for instance). Using these techniques, it is for instance possible to prove that the derived Brauer group of the sphere spectrum is trivial.

Finally, a recent result concerning the existence of compact generators over algebraic spaces has been proved in [13, Theorem 1.5.10]. It can be used in order to extend Theorem 1.2 from schemes to algebraic spaces, at least if the étale topology is used instead of the fppf topology.

2 The local theory

For a simplicial commutative ring A we note $N(A)$ its normalized complex. The shuffle maps of [20] endow the complex $N(A)$ with the structure of a commutative *dg*-algebra (over $\mathbb{Z}$). The category of unbounded *dg*-modules over $N(A)$ will be denoted by A-*dg-mod*, and its objects will be called A-*dg-modules*. The category A-*dg-mod* is a symmetric monoïdal model category, for the monoïdal structure induced by the tensor product of *dg*-modules over $N(A)$, and for which equivalences are quasi-isomorphisms and fibrations are epimorphisms. The monoïdal model category $N(A)$-*dg-mod* satisfies moreover the condition [19], and therefore there induced model structure on monoïds in $N(A)$-*dg-mod* exists (equivalences and fibrations are defined on the underlying objects in A-*dg-mod*). Monoïds in $N(A)$-*dg-mod* will be called A-*dg-algebras*, and their category is noted A-*dg-alg*.

The homotopy category of A-*dg*-modules is denoted by $D(A)$, endowed with its natural triangulated structure. Recall that compact objects in $D(A)$ are precisely the perfect (or dualizable) A-*dg*-modules (see [31]), or equivalently the retracts of finite cell A-*dg*-modules. The category $D(A)$ is moreover a closed symmetric monoïdal category for the tensor product of A-*dg*-modules. We will note $\otimes_A^{\mathbb{L}}$ its monoïdal structure and $\mathbb{R}\underline{Hom}_A$ the corresponding internal Hom objects. The complexes underlying $\mathbb{R}\underline{Hom}_A$, by forgetting the A-*dg*-module structure, will be denoted $\mathbb{R}Hom_A$. More generally, if B is a A-*dg*-algebra (or even a A-*dg*-category), the derived category $D(B)$ of B-*dg*-modules is naturally enriched over $D(A)$. We denote by $\mathbb{R}\underline{Hom}_B$ the corresponding Hom's with values in $D(A)$ (note that the base commutative simplicial ring A is then ambiguous in this notation, but in each situation the base should be clear from the context). When B is itself commutative, $\mathbb{R}\underline{Hom}_B$ is then itself a B-dg-module, and the two notations agree.

2.1 Derived Azumaya algebras over simplicial rings

We start by the definition of derived Azumaya algebras over simplicial rings. This notion will be later generalized to a sheaf-like setting in order to consider derived Azumaya over derived schemes and more generally derived stacks. In this section we will concentrate on the basic definition as well as its formal properties.

Definition 2.1 Let A be a commutative simplicial ring. A *derived Azumaya algebra over A* (*a deraz A-algebra* for short) is a A-dg-algebra B satisfying the following two conditions.

(Az-1) The underlying A-dg-module is a compact generator of the triangulated category $D(A)$.

(Az-2) The natural morphism in $D(A)$

$$B \otimes^{\mathbb{L}}_A B^{op} \longrightarrow \mathbb{R}\underline{Hom}_A(B, B),$$

induced by

$$(b, b') \mapsto (c \mapsto bcb')$$

is an isomorphism.

Remark 2.2

1. When A is a nonsimplicial ring, considered as a constant simplicial ring, then Azumaya A-algebras are special cases of deraz A-algebras: they correspond, up to quasi-isomorphism, to deraz A-algebras B whose underlying A-dg-module is flat and concentrated in degree 0. Indeed, any such deraz A-algebra is quasi-isomorphic to a non-dg A-algebra B, which is projective and of finite type as an A-module (condition (Az-1)), and moreover satisfies $B \otimes_A B^{op} \simeq \underline{Hom}_A(B, B)$ (condition (Az-2)). Our notion of derived Azumaya algebras over A is therefore a generalization of the usual notion of Azumaya algebras. We will see later that there are examples of deraz A-algebras non-quasi-isomorphic (and even not Morita equivalent) to underived ones.
2. The condition (Az-1) can be checked using the following criterion: a compact object $E \in D(A)$ is a compact generator if and only if for any prime ideal p of $\pi_0(A)$, with residue field $A \to k(p)$, we have $E \otimes^{\mathbb{L}}_A k(p) \neq 0$ in $D(k(p))$. Indeed, the condition is equivalent to state that the support of E is the whole derived affine scheme $\mathbb{R}\underline{Spec}\, A$. By [25, Lemma 3.14] this implies that A belongs to the thick triangulated sub-category generated by E, and thus that E is a compact generator of $D(A)$.

For any morphism of simplicial commutative rings $A \to A'$, there are base change functors

$$A' \otimes^{\mathbb{L}}_A - : D(A) \longrightarrow D(A'),$$

$$A' \otimes^{\mathbb{L}}_A - : Ho(A\text{-}dg\text{-}alg) \longrightarrow Ho(A'\text{-}dg\text{-}alg).$$

These base change functors will also be denoted by

$$B \mapsto B_{A'} := A' \otimes^{\mathbb{L}}_A B.$$

For the next proposition, remind that $A \to A'$ is faithfully flat if $\pi_0(A) \to \pi_0(A')$ is a faithfully flat morphism of rings, and if

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(A') \simeq \pi_*(A')$$

(see [33, Sect. 2.2.2] for other characterizations).

Proposition 2.3 *Let $A \to A'$ be a morphism of simplicial commutative rings.*

1. *If B is deraz A-algebra, then $B_{A'}$ is a deraz A'-algebra.*
2. *If A' is faithfully flat over A, and if $B \in Ho(A\text{-}dg\text{-}alg)$ is such that $B_{A'}$ is a deraz A'-algebra, then B is a deraz A-algebra.*

Proof (1) Let B be a deraz A-algebra. Let $E \in D(A')$, and assume that

$$\mathbb{R}Hom_{A'}(B_{A'}, E) = 0.$$

By adjunction, we have

$$\mathbb{R}Hom_A(B, E) = 0,$$

where E is considered as a A-dg-module throught the morphism $A \to A'$. As B is a compact generator of $D(A)$ we have $E = 0$ in $D(A)$, which implies that $E = 0$ also in $D(A')$. Therefore $B_{A'}$ is compact generator of $D(A')$ and satisfies condition (Az-1) of Definition 2.1.

There exists a natural isomorphism in $D(A')$

$$B_{A'} \otimes^{\mathbb{L}}_{A'} (B_{A'})^{op} \simeq A' \otimes^{\mathbb{L}}_A (B \otimes^{\mathbb{L}}_A B^{op}).$$

Also, as B is a compact A-dg-module, there exists a natural isomorphism in $D(A')$

$$A' \otimes^{\mathbb{L}}_A \mathbb{R}\underline{Hom}_A(B, B) \simeq \mathbb{R}\underline{Hom}_{A'\text{-}dg\text{-}mod}(B_{A'}, B_{A'}).$$

Under these isomorphisms, the morphism of A'-dg-modules

$$B_{A'} \otimes^{\mathbb{L}}_{A'} (B_{A'})^{op} \longrightarrow \mathbb{R}\underline{Hom}_{A'}(B', B'),$$

is the image by the functor $A' \otimes^{\mathbb{L}}_A$—of the morphism

$$B \otimes^{\mathbb{L}}_A B^{op} \longrightarrow \mathbb{R}\underline{Hom}_A(B, B).$$

We thus see that $B_{A'}$ satisfies condition (Az-2) of Definition 2.1.

(2) Let B be a A-dg-algebra such that $B_{A'}$ is a deraz A'-algebra. By [33, Proposition 1.3.7.4] we know that an object $E \in D(A)$ is perfect if and only if $A' \otimes_A^{\mathbb{L}} E$ is so. As perfect A-dg-modules are exactly the compact objects in $D(A)$, we see that B is a compact object in $D(A)$. Assume now that $E \in D(A)$ is such that $\mathbb{R}\underline{Hom}_A(B, E) = 0$. As B is compact, we have

$$A' \otimes_A^{\mathbb{L}} \mathbb{R}\underline{Hom}_A(B, E) \simeq \mathbb{R}\underline{Hom}_{A'}(B_{A'}, E_{A'}).$$

We thus have $\mathbb{R}\underline{Hom}_{A'}(B_{A'}, E_{A'}) = 0$, and as $B_{A'}$ is a compact generator of $D(A')$ we must have $E_{A'} = 0$. As A' is faithfully flat over A, we also have $E = 0$. This shows that B is a compact generator of $D(A)$, and thus that B satisfies condition (Az-1) of Definition 2.1.

Now, as we have seen during the proof of the first point, the morphism

$$B_{A'} \otimes_{A'}^{\mathbb{L}} (B_{A'})^{op} \longrightarrow \mathbb{R}\underline{Hom}_{A'\text{-}dg\text{-}mod}(B', B')$$

is obtained from

$$B \otimes_A^{\mathbb{L}} B^{op} \longrightarrow \mathbb{R}\underline{Hom}_A(B, B)$$

by base change from A to A'. As A' is a fully faithfully flat A-dg-algebra, and as by hypothesis the first of these morphisms is an isomorphism, we see that B must satisfy condition (Az-2) of Definition 2.1, and therefore is a deraz A-algebra. □

We are now going to see that the property of being a derived Azumaya algebra is stable by Morita equivalences. For this we will show that the conditions (Az-1) and (Az-2) can be expressed by stating that B is an invertible object in a certain closed symmetric monoïdal category. A formal consequence of this fact will be the Morita invariance of the notion of deraz algebras, as well as its stability by (derived) tensor products.

Let A be a simplicial commutative ring. We let Hom_A be the homotopy category of A-dg-categories up to derived Morita equivalences. The precise definition of Hom_A is as follows. Its objects are categories enriched into the symmetric monoïdal category of A-dg-modules (which by definition are dg-modules over the commutative dg-algebra $N(A)$), which we simply call *A-dg-categories*. For a A-dg-category T we can form its derived category of T-dg-modules, $D(T)$, which is, by definition, the homotopy category of left T-dg-modules. A morphism of A-dg-categories is simply an enriched functor, and such a functor $f : T \longrightarrow T'$ is a *Morita equivalence* if the induced functor on the level of derived categories of dg-modules

$$f^* : D(T') \longrightarrow D(T)$$

is an equivalence of categories. The category Hom_A is then obtained from the category of A-dg-categories and morphisms of A-dg-categories by inverting the Morita equivalences. The sets of morphisms in Hom_A are usually denoted by $[-,-]$, or $[-,-]_{Hom_A}$. According to [26, Corollary 7.6] these sets of morphisms can be described as follows: for T and T' we form $T \otimes_A^{\mathbb{L}} (T')^{op}$, and consider $D_{pspe}(T \otimes_A^{\mathbb{L}} (T')^{op})$, the full sub-category of $D(T \otimes_A^{\mathbb{L}} (T')^{op})$ consisting of all E such that for any object $x \in T$, $E(x,-)$ is a compact object in $D((T')^{op})$. Then, we have a natural identification

$$[T, T']_{Hom_A} \simeq D_{pspe}(T \otimes_A^{\mathbb{L}} (T')^{op})/isom,$$

between the set of morphisms from T to T' in the category Hom_A, and the set of isomorphism classes of objects in $D_{pspe}(T \otimes_A^{\mathbb{L}} (T')^{op})$.

The category Hom_A is made into a symmetric monoïdal category by the derived tensor product of A-dg-categories

$$\otimes_A^{\mathbb{L}} : Hom_A \times Hom_A \longrightarrow Hom_A.$$

By the results of [26, Corollary 7.6] this monoïdal structure is known to be closed. As in any symmetric monoïdal category, we can therefore make sense of dualizable, as well as invertible, objects in Hom_A, whose definitions are now breifly recalled. An object $T \in Hom_A$ is *dualizable* (or *with duals*) if there is $T' \in Hom_A$ and a morphism

$$ev : T \otimes_A^{\mathbb{L}} T' \longrightarrow \mathbf{1} = A,$$

such that for any two other objects $U, U' \in Hom_A$ the induced map

$$[U, T' \otimes_A^{\mathbb{L}} U'] \xrightarrow{T \otimes_A^{\mathbb{L}}} [T \otimes_A^{\mathbb{L}} U, T \otimes_A^{\mathbb{L}} T' \otimes_A^{\mathbb{L}} U'] \xrightarrow{ev} [T \otimes_A^{\mathbb{L}} U, U']$$

is bijective. This condition is equivalent to ask for the existence of a morphism

$$i : A \longrightarrow T \otimes_A^{\mathbb{L}} T',$$

such that the two composite

$$T' \xrightarrow{id_{T'} \otimes_A^{\mathbb{L}} i} T' \otimes_A^{\mathbb{L}} T \otimes_A^{\mathbb{L}} T' \xrightarrow{ev \otimes_A^{\mathbb{L}} id_{T'}} T', \qquad T \xrightarrow{i \otimes_A^{\mathbb{L}} id_T} T \otimes_A^{\mathbb{L}} T' \otimes_A^{\mathbb{L}} T \xrightarrow{id_T \otimes_A^{\mathbb{L}} ev} T$$

are identities. Such a triple (T, i, ev) will be called a *duality datum for* T. An important fact about dualizable objects is the uniqueness a duality datum. If T is dualizable, with (T', i, ev) and (T'', i', ev') two duality data, then there is a unique isomorphism $\phi : T' \simeq T''$, intertwining (i, ev) and (i', ev'). To finish, an object $T \in Hom_A$ is invertible if there is $T' \in Hom_A$ such that $T \otimes_A^{\mathbb{L}} T'$ is isomorphic to the unit $\mathbf{1} = A$. Invertible objects are dualizable, as the choice of an isomorphism $ev : T \otimes_A^{\mathbb{L}} T' \simeq A$ satisfies the condition for (T', ev^{-1}, ev) to be a duality datum. By uniqueness of these duality datum we see that a dualizable object T, with duality datum (T', ev, i), is invertible if and only if ev or i are isomorphisms.

An A-dg-algebra B defines canonically an object $B \in Hom_A$, by considering B as a A-dg-category with a unique object $*$, and B as being the endomorphisms of $*$. This constructions defines a functor

$$Ho(A\text{-}dg\text{-}alg) \longrightarrow Hom_A,$$

which is compatible with base change and forgetful functors, as well as with the derived tensor product $\otimes^{\mathbb{L}}_A$.

Definition 2.4 Two deraz A-algebras are *Morita equivalent* if their images in Hom_A are isomorphic.

We will use often the following easy criterion (whose proof is left to the reader): two A-dg-algebras B and B' are isomorphic in Hom_A if and only if there exists a compact generator $E \in D(B')$ whose derived endomorphism A-dg-algebra $\mathbb{R}\underline{End}_{B'}(E)$ is quasi-isomorphic to B.

We now have the following proposition, characterizing deraz A-algebras as invertible objects in Hom_A. As a side result we also show that dualizable objects in Hom_A are precisely the smooth and proper A-dg-algebras (as stated without proofs in [29, Proposition 5.4.2]).

Proposition 2.5 *Let A be a simplicial commutative ring.*

1. *An object T in Hom_A is dualizable if and only if it is isomorphic to* (*the image of*) *an A-dg-algebra B satisfying the following two conditions.*
   (a) *B is a compact object in $D(A)$* (*i.e. B is proper over A*).
   (b) *B is a compact object in $D(B \otimes^{\mathbb{L}}_A B^{op})$* (*i.e. B is smooth over A*).
2. *An object T in Hom_A is invertible if and only if is isomorphic to* (*the image of*) *a deraz A-algebra.*

Proof (1) We introduce a bigger symmetric monoïdal category $\widetilde{Hom}_A$ containing Hom_A as a monoïdal sub-category. The objects of $\widetilde{Hom}_A$ are A-dg-categories. The set of morphisms from T to T' in $\widetilde{Hom}_A$ is by definition the set of isomorphism classes of objects in $D(T \otimes^{\mathbb{L}}_A (T')^{op})$. The composition of morphisms is then induced by the (derived) convolution tensor product construction

$$- \otimes^{\mathbb{L}}_{T'} - : D(T \otimes^{\mathbb{L}}_A (T')^{op}) \times D(T' \otimes^{\mathbb{L}}_A (T'')^{op}) \longrightarrow D(T \otimes^{\mathbb{L}}_A (T'')^{op}).$$

The reader might worry about the fact that the Hom sets of the category $\widetilde{Hom}_A$ are not small sets. One obvious solution to this problem is to work with universes: if our dg-categories belongs to a universe $\mathbb{U}$, then $\widetilde{Hom}_A$ will be a $\mathbb{V}$-small category, for $\mathbb{V}$ a universe with $\mathbb{U} \in \mathbb{V}$. Another solution, avoiding to be forced to believe that universes exist, is to check that we are going to use a very small part of the category $\widetilde{Hom}_A$, which consists of all the tensor powers of a given dg-category and its opposite, as well as certain diagonal dg-modules. All of these obviously live in a small sub-category of $\widetilde{Hom}_A$ in which the argument below can be done.

The category $\widetilde{Hom}_A$ is endowed with a symmetric monoïdal structure induced by the derived tensor product of dg-categories. The nice property of the symmetric monoïdal category $\widetilde{Hom}_A$ is that all of its objects are dualizable.

Indeed, for any A-dg-category T, the $T \otimes_A^{\mathbb{L}} T^{op}$ dg-module $(a,b) \mapsto T(b,a)$, provides an object $T \in D(T \otimes_A^{\mathbb{L}} T^{op})$, and therefore two morphisms in $\widetilde{Hom}_A$

$$i : A \longrightarrow T \otimes_A^{\mathbb{L}} T^{op}, \qquad ev : T \otimes_A^{\mathbb{L}} T^{op} \longrightarrow A,$$

that can be checked to be a duality datum in $\widetilde{Hom}_A$. Now, Hom_A is a symmetric monoïdal sub-category $\widetilde{Hom}_A$, containing all isomorphisms. Therefore, the unicity of a duality datum implies that $T \in Hom_A$ is dualizable if and only if the two morphisms i and ev as defined above in $\widetilde{Hom}_A$ lies in the sub-category Hom_A. That i belongs to Hom_A is equivalent to state that T is a compact object in $D(T \otimes_A^{\mathbb{L}} T^{op})$. In the same way, that ev belongs to Hom_A implies that $T(b,a)$ is compact in $D(A)$ for any $a, b \in T$. Therefore, in order to prove our point (1) it only remains to show that T is moreover isomorphic, in Hom_A, to a A-dg-algebra. But this follows from the following lemma.

Lemma 2.6 *If $T \in Hom_A$ is such that T is a compact object in $D(T \otimes_A^{\mathbb{L}} T^{op})$, then T is isomorphic to an A-dg-algebra.*

Proof of the Lemma We have a natural triangulated functor

$$D(T \otimes_A^{\mathbb{L}} T^{op}) \times D(T) \longrightarrow D(T),$$

induced by the composition in $\widetilde{Hom}_A$. The object T acts by the identity on $D(T)$, and by assumption T lies in the thick triangulated sub-category generated by representable objects on $D(T \otimes_A^{\mathbb{L}} T^{op})$. These representable objects are given by two objects $a, b \in T$, and they act on $D(T)$ by the functor

$$E \mapsto E(a) \otimes_A^{\mathbb{L}} T(b, -).$$

This shows that $E = id(E)$ lies in the thick triangulated sub-category of $D(T)$ generated by the representable objects $T(b, -)$. As only a finite number of b's are necessary to write the object T by retracts, sums, cones and shifts in $D(T \otimes_A^{\mathbb{L}} T^{op})$, we see that $D(T)$ is generated, as a triangulated category, by a finite number of representable objects. The direct sums of these representable modules provides a compact generator in $E \in D(T)$, and thus T becomes isomorphic in Hom_A to $\mathbb{R}\underline{Hom}_T(E, E)$, the derived endomorphism A-dg-algebra of the object E. □

(2) An invertible object is a dualizable object. From (1) we see that invertible objects are all isomorphic to smooth and proper A-dg-algebras. Moreover, the uniqueness of duality data implies that such a smooth and proper A-dg-algebra B is invertible in Hom_A if and only if the morphism

$$ev : B \otimes_A^{\mathbb{L}} B^{op} \longrightarrow A,$$

is an isomorphism. This morphism is itself determined by the diagonal bi-dg-module $B \in D(B \otimes_A^{\mathbb{L}} B^{op})$, and it is an isomorphism if and only the induced functor

$$\phi : D(B \otimes_A^{\mathbb{L}} B^{op}) \longrightarrow D(A)$$

is an equivalence of categories. By construction, this functor sends $B \otimes^{\mathbb{L}}_A B^{op}$ to $B \in D(A)$, and always preserves compact objects. Therefore, ϕ is fullyfaithful if and only if the morphism

$$B \otimes^{\mathbb{L}}_A B^{op} \simeq \mathbb{R}\underline{End}_{B\otimes^{\mathbb{L}}_A B^{op}}(B \otimes^{\mathbb{L}}_A B^{op}) \longrightarrow \mathbb{R}\underline{End}_A(B, B)$$

is an isomorphism in $D(A)$. In the same way, the essential image of ϕ generates $D(A)$ in the sense of triangulated categories, if and only if B is a compact generator of $D(A)$. We thus see that ϕ is an equivalence of categories if and only if B is a deraz A-algebra. □

Remark 2.7 The Proposition 2.5 also has an interpretation purely in terms of the category $\widetilde{Hom}_A$ we have introduced during its proof, at least if this category is enhanced into a symmetric monoïdal 2-category in the usual fashion (the category of morphisms between T and T' is the category $D(T \otimes^{\mathbb{L}}_A (T')^{op})$). Then, it can be proved that the an object T in this 2-category is equivalent to a smooth and proper A-*dg*-algebra if and only if it is a fully dualizable object in the sense of [12]. This explains the importance of smooth and proper *dg*-categories in the context of two dimensional field theories. Invertible objects are the same if one considers $\widetilde{Hom}_A$ as a monoïdal category or as a monoïdal 2-category.

An immediate corollary of the last proposition is the following result (point (5) is more a corollary of the proof).

Corollary 2.8 *Let A be a simplicial commutative ring.*

1. *An A-dg-algebra Morita equivalent to a deraz A-algebra is itself a deraz A-algebra.*
2. *If B and B' are two deraz A-algebras, then so is $B \otimes^{\mathbb{L}}_A B'$.*
3. *Any deraz A-algebra is a smooth and proper A-dg-algebra.*
4. *An A-dg-algebra B is a deraz A-algebra if and only if there is another A-dg-algebra B' such that $B \otimes^{\mathbb{L}}_A B'$ is Morita equivalent to A.*
5. *An A-dg-algebra B is a deraz A-algebra if and only if the A-dg-algebra $B \otimes^{\mathbb{L}}_A B^{op}$ is Morita equivalent to A.*

2.2 Derived Azumaya algebras over a field

In this paragraph we fix a base field k, and will show that all deraz k-algebras are Morita equivalent to underived Azumaya k-algebras. We start by the algebraically closed case.

Proposition 2.9 *If k is algebraically closed, then any deraz k-algebra is Morita equivalent to the trivial k-dg-algebra k.*

Proof Let B be a deraz k-algebra. We need to prove that B is isomorphic, in Hom_k, to the trivial k-algebra k. For this, we let $H^*(B)$ be the graded cohomology k-algebra of B. As we are working over a field, we have natural identifications of graded k-vector spaces.

$$H^*(B \otimes_k B^{op}) \simeq H^*(B) \otimes_k H^*(B)^{op},$$
$$H^*(\mathbb{R}\underline{Hom}_k(B, B)) \simeq Hom_k^*(H^*(B), H^*(B)).$$

These identifications, and conditions (Az-1) and (Az-2) imply that $H^*(B)$ is itself an Azumaya k-algebra, and more precisely that $H^*(B)$, considered as a graded bi-$H^*(B)$-module, induces an equivalence between the categories of graded modules

$$H^*(B) \otimes_k H^*(B)^{op}\text{-}Mod^{gr} \simeq k\text{-}Mod^{gr}.$$

This implies that in particular that $H^*(B)$ is a graded projective $H^*(B) \otimes_k H^*(B)^{op}$-module, and therefore that $H^*(B)$ is a semi-simple graded k-algebra (i.e. that $H^*(B)$-Mod^{gr} is semi-simple k-linear abelian category), and thus that any graded $H^*(B)$-module is graded projective.

We have a graded functor

$$\phi : D(B) \longrightarrow H^*(B)\text{-}Mod^{gr}$$

sending a B-dg-module E to its cohomology $H^*(E)$.

Lemma 2.10 *The functor above ϕ is an equivalence of* (*graded*) *categories.*

Proof Let call an object $E \in D(B)$ *graded free* if it is a (possibly infinite) sum of shifts of B. An object of $D(B)$ will then be called *graded projective* if it is a retract of a graded free object. It is clear by definition that the functor ϕ is fully faithful when restricted to graded free objects, and thus also when restricted to graded projective objects. In order to show that ϕ is fully faithful it is then enough to show that any object in $D(B)$ is graded projective.

Let $E \in D(B)$, and let $H^*(E)$ be the corresponding graded $H^*(B)$-module, which is graded projective as we saw. Let p be a projector on some graded free graded $H^*(B)$-module P with image $H^*(E)$. If we write

$$P = \bigoplus_d (H^*(B)[p]^{(I_p)}),$$

then p defines a projector q in $D(B)$ on the graded free object

$$Q := \bigoplus_d (B[p]^{(I_p)}) \in D(B).$$

As $D(B)$ is Karoubian complete (see for instance [27, Sous-lemme 3]), we now that this projector splits

$$q : Q \xrightarrow{k} E' \xrightarrow{j} Q,$$

with $E' \in D(B)$. We claim that E and E' are isomorphic objects in $D(B)$. Indeed, by the choice of p, there exists a factorisation in $H^*(B)-Mod^{gr}$

$$p:\ P \xrightarrow{r} H^*(E) \xrightarrow{i} P.$$

The images of k and j by ϕ provides another such factorisation

$$\phi(q) = p:\ P \xrightarrow{\phi(k)} H^*(E') \xrightarrow{\phi(l)} P.$$

By the uniqueness of such factorisation, we have that the composite morphism

$$H^*(E') \xrightarrow{\phi(l)} P \xrightarrow{r} H^*(E)$$

is an isomorphism. Now, from the definition of ϕ, the morphism r lifts uniquily to a morphism $r' : Q \longrightarrow E$ with $\phi(r') = r$. We thus get a morphism in $D(B)$

$$E' \xrightarrow{j} Q' \xrightarrow{r} E,$$

whose image by ϕ is $r \circ \phi(l)$, which we have seen to be an isomorphism. As the functor ϕ is conservative we have $E \simeq E'$ in $D(B)$.

We thus have that all objects in $D(B)$ are graded projective, and therefore that ϕ is fully faithful. All the objects $H^*(B)[p]$ belong to the essential image of ϕ, and therefore so do all graded free $H^*(B)$-module. As ϕ is fully faithful and as $D(B)$ is Karoubian closed, we deduce that any graded projective $H^*(B)$-module also belongs to the essential image of ϕ. But, as all graded $H^*(B)$-module are graded projective we have that ϕ is essentially surjective and thus an equivalence of categories. □

We use the previous lemma as follows. Let $E \in H^*(B)\text{-}Mod^{gr}$ be a simple object, which is automatically finite dimensional over k, as $H^*(B)$ is so. Because k is algebraically closed we have

$$Hom^0_{H^*(B)\text{-}Mod^{gr}}(E, E) \simeq k.$$

Moreover, for any $n \neq 0$, we have

$$Hom^n_{H^*(B)\text{-}Mod^{gr}}(E, E) = Hom^0_{H^*(B)\text{-}Mod^{gr}}(E, E[n]) = 0,$$

unless E would be isomorphic to $E[n]$ which would a condradiction with the finite dimensionality of E. Let now $E' \in D(B)$ be an object such that $\phi(E') \simeq E$. As the graded functor ϕ is an equivalence, we have

$$[E', E'] \simeq k, \qquad [E', E'[n]] = 0 \quad \text{if } n \neq 0.$$

This implies that, the k-dg-algebra $\mathbb{R}\underline{Hom}_k(E', E')$ is quasi-isomorphic to k, and thus that the pair of adjoint functors

$$E \otimes_k - : D(k) \longrightarrow D(B), \qquad \mathbb{R}\underline{Hom}_k(E, -) : D(B) \longrightarrow D(k)$$

defines a full embedding $D(k) \hookrightarrow D(B)$. Therefore, when considered as a morphism $k \to B$ in Hom_k, the object E exhibits k as a retract of B. By dualizing B we get that B^{op} is a retract of k in Hom_k. The object B^{op} is then the image of a projector p on k in Hom_k, which corresponds to a compact object $P \in D(k)$ with the property that $P \otimes_k P \simeq P$. The only possibility is $P \simeq k$, giving that the projector p is in fact the identity. This implies that B^{op} is isomorphic to k in Hom_k, and by duality that B is isomorphic to k. □

A traduction of the last proposition is the following more explicit statement.

Corollary 2.11 *Any derived Azumaya algebra over an algebraically closed field k is quasi-isomorphic to a finite dimensional graded matrix algebra $\underline{End}_k^*(V)$ (for V a graded finite dimensional k-vector space). In particular, a deraz k-algebra is always formal (i.e. quasi-isomorphic to its cohomology algebra).*

We have seen that any deraz k-algebra is trivial when k is an algebraically closed field, we will now show that for any field k the deraz k-algebra, up to Morita equivalence, are in one-to-one correspondence with equivalence classes of nonderived Azumaya algebras.

For this we introduce underived analog of our symmetric monoïdal category Hom_k. We let Cm_k be the category whose objects are k-linear categories, and for which the set of morphisms from C to C' in Cm_k is the set of isomorphism classes of $(C \otimes_k (C')^{op})$-bimodules P, such that $P(c,-)$ is a projective $(C')^{op}$-module for any $c \in C$. The category Cm_k is made into a symmetric monoïdal category by means of the tensor product of linear categories over k. Any k-algebra B defines an object $B \in Cm_k$, and it is easy to check (by following the same argument as for our Proposition 2.3), that such an algebra is Azumaya over k (in the sense of [7]), if and only if it defines an invertible object in Cm_k. Moreover, equivalence classes of Azumaya algebras over k are in one-to-one correspondence with isomorphism classes of invertible objects in Cm_k.

There is a natural functor

$$Cm_k \longrightarrow Hom_k$$

which consists of considering a k-linear category as a k-dg-category is an obvious way (i.e. concentrated in degree 0). This functor is a symmetric monoïdal functor (because k is a field here), and thus induces a map on isomorphism classes of invertible objects.

Proposition 2.12 *The above functor*

$$Cm_k \longrightarrow Hom_k$$

induces a bijection between isomorphism classes of invertible objects. In other words, the natural inclusion from k-algebras to k-dg-algebras induces a one-to-one correspondence between the set of equivalent classes of Azumaya k-algebras and the set of deraz k-algebras up to Morita equivalence.

Proof We need to prove two statements:

1. If B and B' are two Azumaya k-algebras which are isomorphic in Hom_k, then there also are isomorphic in Cm_k.
2. Any deraz k-algebra is isomorphic in Hom_k to an Azumaya k-algebra.

We start by proving (1). Let B and B' two Azumaya k-algebras which are isomorphic in Hom_k. We set $C := B \otimes_k (B')^{op}$. We define two stacks $Eq_0(B, B')$ and $Eq(B, B')$ over $Spec\, k$ (we will work with the fppf topology, see however [30]). The stack $Eq_0(B, B')$ sends a commutative k-algebra A to the groupoïd of C_A-modules M whose corresponding functor

$$B_A^{op}\text{-}Mod \longrightarrow (B'_A)^{op}\text{-}Mod,$$
$$N \mapsto N \otimes_B M$$

is an equivalence of categories. In the same way, the stack $Eq(B, B')$ sends a commutative k-algebra A to the groupoïd of objects $M \in D(C_A)$ whose corresponding functor

$$D(B_A^{op}) \longrightarrow D((B'_A)^{op}),$$
$$N \mapsto N \otimes_B^{\mathbb{L}} M$$

is an equivalence of categories (see [31, Corollary 3.21] for a justification that this is indeed a stack). The natural inclusion $C_A\text{-}Mod \longrightarrow D(C_A)$ defines a morphism of stacks

$$f : Eq_0(B, B') \longrightarrow Eq(B, B').$$

These stacks have natural actions by the group stacks

$$G_0 := Eq_0(B, B), \qquad G := Eq(B, B),$$

and the fact that everything becomes trivial over $\bar{k}$ implies that $Eq_0(B, B')$ and $Eq(B, B')$ are torsors, for the fppf topology, over $Eq_0(B, B)$ and $Eq(B, B)$ respectively. Now, as B is an invertible object in Cm_k, we have natural equivalences of group stacks

$$G_0 \simeq Eq_0(k, k), \qquad G \simeq Eq(k, k).$$

These group stacks can be computed explicitely, and we have

$$G_0 \simeq K(\mathbb{G}_m, 1), \qquad G \simeq \mathbb{Z} \times K(\mathbb{G}_m, 1).$$

The two torsors $Eq_0(B, B')$ and $Eq(B, B')$ are thus represented by cohomology classes

$$x_0 \in H^1_{fppf}(Spec\, k, G_0) \simeq H^2_{fppf}(Spec\, k, \mathbb{G}_m),$$
$$x \in H^1_{fppf}(Spec\, k, G) \simeq H^1_{fppf}(Spec\, k, \mathbb{Z}) \times H^2_{et}(Spec\, k, \mathbb{G}_m).$$

Moreover, the natural inclusion $G_0 \longrightarrow G$ sends the class x_0 to the class x. Under the above identification the morphism induced by the inclusion becomes the inclusion of the second factor

$$H^2_{fppf}(Spec\, k, \mathbb{G}_m) \hookrightarrow H^1_{fppf}(Spec\, k, \mathbb{Z}) \times H^2_{fppf}(Spec\, k, \mathbb{G}_m)$$

and is thus injective. By assumption $Eq(B, B')$ has a k-rational point, and thus $x = 0$. We therefore have $x_0 = 0$, or equivalently $Eq_0(B, B')(k) \neq \emptyset$. This implies that B and B' are by definition isomorphic in Cm_k, and thus are equivalent as Azumaya k-algebras.

We now prove the second point (2). It is enough to show that a given deraz k-algebra B is isomorphic in Hom_k to the image of a non-dg k-algebra, as this k-algebra will then automatically satisfies assumptions (Az-1) and (Az-2), which will imply that it is an Azumaya k-algebra (we use here that k is a field).

Let B be a fixed deraz k-algebra. If $\overline{k}$ is a given algebraic closure of k, we now from Proposition 2.9 that $B_{\overline{k}}$ is Morita equivalent to $\overline{k}$. This is equivalent to say that $D(B_{\overline{k}})$ possesses a compact generator whose endomorphism dg-algebra is quasi-isomorphic to $\overline{k}$. From our Corollary 2.8 we know that B is a smooth and proper k-dg-algebra, and therefore, as it is shown in [31, Theorem 3.6], the (∞)-stack of perfect B-dg-modules is locally of finite type over k. A consequence of this is the existence of a finite field extension k'/k, and of a compact generator $E' \in D(B_{k'})$, whose endomorphism dg-algebra is quasi-isomorphic to k'. In other words, $B_{k'}$ becomes isomorphic in $Hom_{k'}$ to k'. We consider the natural adjunction morphism of k-dg-algebras $B \longrightarrow B_{k'}$, and the corresponding forgetful functor

$$D(B_{k'}) \longrightarrow D(B).$$

We let E be the image of E' by this functor. The underlying complex of E is perfect over k, and as B is smooth this implies that E is a compact object in $D(B)$. Moreover B is a direct factor of $B_{k'} \simeq B^d$ in $D(B)$ (here d is the degree of the extension k'/k) and thus belongs to the thick triangulate subcategory generated by E. The object E is thus a compact generator of $D(B)$. We now show that the k-dg-algebra $\mathbb{R}\underline{End}_B(E)$ is quasi-isomorphic to a k-algebra. Indeed, for this it is enough to show that the natural morphism

$$k' \longrightarrow \mathbb{R}\underline{End}_B(E) \otimes_k k'$$

is a quasi-isomorphism, or equivalently that

$$k' \simeq \mathbb{R}\underline{End}_{B_{k'}}(E \otimes_k k').$$

But we have isomorphisms in $D(B_{k'})$

$$E \otimes_k k' \simeq E' \otimes_{k'} k' \otimes_k k' \simeq (E')^d,$$

where $d = deg(k'/k)$. By assumption on E' we have $\mathbb{R}\underline{End}_{B_{k'}}(E') \simeq k'$, showing that $\mathbb{R}\underline{End}_{B_{k'}}(E \otimes_k k')$ is quasi-isomorphic to the matrix k'-algebra $M_d(k')$. This implies that the derived endomorphism k-dg-algebra of the B-dg-module E is quasi-isomorphic to a non-dg algebra B_0 over k. As E is a compact generator we have that B_0 and B are isomorphic in Hom_k. □

Remark 2.13 The bijection on the set of invertible objects between the categories Cm_k and Hom_k does not extend to a stronger form of equivalence. For instance, the unit object in Hom_k has more automorphisms than in Cm_k (the one corresponding to the objects $k[n] \in D(k)$, for $n \in \mathbb{Z}$). The proposition is also wrong over more general rings, as we will see that there exist local rings over which there are deraz algebras non Morita equivalent to non-derived Azumaya algebras. We will however prove that the natural map from equivalent classes of Azumaya algebras to deraz algebras up to Morita equivalences stays injective over any scheme (or more generally any stack).

2.3 Local triviality for the étale topology

To finish this first section we prove the local triviality of deraz A-algebras for the étale topology. The proof starts with the case of a base field, and proceeds with a deformation theory argument. For the statement, and the proof, of the next proposition we use the notion of derived stacks, and we refer to [33, Sect. 2.2] or [28] for references.

Proposition 2.14 *Let A be a commutative simplicial ring and B be a deraz A-algebra. Then, there exists an étale covering $A \longrightarrow A'$ such that $B_{A'}$ is Morita equivalent to A'.*

Proof Let A and B as in the statement of the proposition. We work in the category of derived stacks over A, for the fppf topology, which we simply denote by $dSt(A)$ (see [30] for considerations on the fppf topology). We consider the derived stack $Eq(A, B)$, of Morita equivalences from A to B. By definition, its values over a simplicial A-algebra A' is the nerve of the category of quasi-isomorphisms between all compact $B_{A'}^{op}$-dg-modules E whose corresponding morphism $A' \to B_{A'}$ is an isomorphism in $Hom_{A'}$. This is an open sub-stack of $\mathcal{M}_B$, of all compact B^{op}-dg-modules, constructed and studied in [31] (we use here Corollary 2.8 to insure that B is smooth and proper in order to apply Theorem 3.6 of [31]). In particular, $Eq(A, B)$ is a geometric derived stack which is locally of finite presentation over A. Let A' be any simplicial A-algebra, and $E : \mathbb{R}\underline{Spec}\, A' \longrightarrow Eq(A, B)$ a given A'-point. Then, according to [31, Corollary 3.17], the tangent complex of $Eq(A, B)$ at the point E is given by

$$\mathbb{T}_{Eq(A,B),E} \simeq \mathbb{R}\underline{Hom}_{B_{A'}^{op}}(E, E)[1],$$

where the morphism E is considered as an object in $D(B^{op}_{A'})$. But, as E induces an isomorphism $A' \simeq B_{A'}$ in $Hom_{A'}$, the functor

$$\phi_E : D(A') \longrightarrow D(B^{op}_{A'}),$$

sending M to $M \otimes^{\mathbb{L}}_{A'} E$ is an equivalence of triangulated categories. Therefore we have

$$\mathbb{R}\underline{Hom}_{B^{op}_{A'}}(E, E) \simeq A'.$$

This implies that $\mathbb{T}_{Eq(B,A),E} \simeq A'[1]$ is of negative Tor amplitude, and thus that the derived stack $Eq(B, A)$ is smooth over A (see [33, Sect. 2.2.5]). Moreover, Proposition 2.9 shows that $Eq(B, A) \longrightarrow \mathbb{R}\underline{Spec}\, A$ is surjective over all geometric points, and it is therefore a smooth covering. This map possesses then local sections for the étale topology (see [33, Lemma 2.2.3.1]), or equivalently, B is locally, for the étale topology on A, Morita équivalent to A'.

The previous proposition implies the following refinement of Proposition 2.9.

Corollary 2.15 *Let k be a separably closed field. Then, any deraz k-algebra is quasi-isomorphic to a graded finite dimensional matrix algebra $\underline{End}^*_k(V)$.*

3 Derived Azumaya algebras over derived stacks

This section uses the notion of ∞-categories (truly only $(1,\infty)$-categories are considered), and we will use the approach based on Segal categories (see [21]) and simplicially enriched categories recalled in [35, Sect. 1]. Therefore, for us ∞-*category* is synonimus to *Segal category*, and *strict* ∞-*category* to $\mathbb{S}$-*category*. Needless to say that any other, reasonable enough, theory of ∞-categories could be used equivalently (for instance any of the one presented in [2]). We leave the reader to chose is favorite and to make its own translations of the constructions and results presented in the sequel.

3.1 The derived stack of locally presentable *dg*-categories

We have, up to now, ignored set theoritical issues. In this paragraph we will make some constructions involving non-small objects, that will be called *big*, but also bigger object that will be called *very big*, in a way that big objects are small with respect to very big one. It is unclear to me how to avoid using universes in order to make these constructions meaningful without going into messy cardinality bound issues. We will therefore assume here that we have added the universe axioms to the axiom of sets theory. I do not think that the results we prove in the sequel really depend on axioms of universes, but rather that it is an easy way to write things down. Once again, at the very end, when these constructions will be applied to prove our main theorems about existence of derived Azumaya algebras, only a very small part of the objects that we will describe will be truly used, a part which can be check to exits in ZFC without any universe axioms.

In this paragraph we use the terminology and notations from the homotopy theory of *dg*-categories. We refer the reader to [22, 26] for more details. In particular, for a *dg*-category T we note $[T]$ the associated homotopy category, obtained by replacing all complexes of morphisms by their H^0. When T is a triangulated *dg*-category (for instance when T has small limits and colimits), then $[T]$ is naturally endowed with a triangulated structure.

We let $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$ be three universes. Sets elements of $\mathbb{U}$ will be called *small* (which will be implicit if no other adjective is specified), elements of $\mathbb{V}$ *big*, and the ones of $\mathbb{W}$ *very big*. We let $s\mathbb{Z}\text{-}CAlg$ be the category of (small) commutative rings, endowed its usual model structure. For any $A \in s\mathbb{Z}\text{-}CAlg$ we $Dg(A)$ be the category of big A-*dg*-categories. We note that $Dg(A)$ possesses all limits and colimits along big sets, and is a $\mathbb{V}$-combinatorial model category in the sense of [32, Appendix] (the underlying category of $Dg(A)$ is itself is very big). For $u : A \to B$ a morphism of simplicial commutative rings, we have a direct image functor between very big categories

$$u_* : Dg(B) \longrightarrow Dg(A)$$

which consists of viewing a B-*dg*-category as a A-*dg*-category throught the morphism $A \to B$. This defines a functor

$$Dg : s\mathbb{Z}\text{-}CAlg^{op} \longrightarrow Cat_{\mathbb{W}},$$

from $s\mathbb{Z}\text{-}CAlg$ to the category of very big categories. For $A \in s\mathbb{Z}\text{-}CAlg$, the category $Dg(A)$ is equipped with a model structure and in particular with a notion of equivalences (here these are the quasi-equivalences of big A-*dg*-categories). Localizing along these equivalences we get a new functor

$$LDg : s\mathbb{Z}\text{-}CAlg^{op} \longrightarrow \infty\text{-}Cat_{\mathbb{W}},$$

defined by the obvious formula $LDg(A) := L(Dg(A))$. We perform the Grothendieck's construction of [35, Sect. 1.3] to turn the functor LDg into a fibered very big ∞-category

$$\pi : \int LDg \longrightarrow s\mathbb{Z}\text{-}CAlg.$$

Objects of the total space $\int LDg$ are pairs (A, T), where A is a simplicial commutative ring and T is a big A-*dg*-category. The simplicial set of morphisms in $\int LDg$, between (A, T) and (B, U) are given by (see [35, Sect. 1.3] for more details)

$$Map_{\int LDg}((A, T), (B, U)) = \coprod_{u:A \to B} L(Dg(A))(T, u_*(U)).$$

Before going further we need to remind here the notion of homotopy colimits in *dg*-categories. For T a big *dg*-category (over some commutative simplicial ring A), we consider the model category T-*Mod*, of big T-*dg*-modules. The T-*dg*-modules quasi-isomorphic to *dg*-modules of the form $T(a,-)$, for some object $a \in T$, are called *co-representable*. We then say that T *possesses small (homotopy) colimits*, if the co-representable T-*dg*-modules are stable by small homotopy limits in the model category T-*Mod*, as well as by the operations $E \otimes^{\mathbb{L}}_A$, for some small A-*dg*-module E. Dually, we can make sense of the expression T *possesses small (homotopy) limits*. As we will never consider limits and colimits in *dg*-categories in the strict sense, we will allow ourselves not to mention the word "homotopy".

We will also use the notion of adjunction between *dg*-categories. As for limits and colimits we will never mean a strict adjunction between *dg*-categories, but only a weak form of it, defined as follows. Let $f : T \longrightarrow T'$ be a *dg*-functor between *dg*-categories (over some base commutative simplicial ring A). We say that f *possesses a left adjoint* if there is a morphism $g : T' \longrightarrow T$, in the homotopy category of *dg*-categories, such that the two $T \otimes^{\mathbb{L}}_A (T')^{op}$-*dg*-modules

$$T(g(-),-), \qquad T(-,f(-))$$

are isomorphic in the derived category $D(T \otimes^{\mathbb{L}}_A (T')^{op})$. This is equivalent to say that there is a morphism

$$u : 1 \Rightarrow f \circ g,$$

inside the category $[\mathbb{R}\underline{Hom}(T',T')]$, such that for any $x \in T$ and $y \in T'$, the induced morphisms (well defined in the derived category $D(A)$ of A-*dg*-modules)

$$T(g(x),y) \to T'(fg(x),f(y)) \longrightarrow T'(x,f(y))$$

is an isomorphism (here, $\mathbb{R}\underline{Hom}(T',T')$ is the derived internal Hom of *dg*-categories of [26, Theorem 6.1]).

Now, recall that a big A-*dg*-category T is *locally presentable* if it possesses a small set of λ-small objects, for λ a regular cardinal elements of $\mathbb{U}$, and if it is closed by small colimits (in the sense above) (this is the *dg*-version of locally presentable ∞-categories of [11]). Equivalently, there exists a small A-*dg*-category T_0, and a fully faithful A-*dg*-functor (recall that $\widehat{T_0}$ is by definition the A-*dg*-category of all cofibrant T_0^{op}-*dg*-modules, see [26, Sect. 7])

$$T \hookrightarrow \widehat{T_0},$$

having a right adjoint (in the sense of the homotopy theory of A-*dg*-categories) which commutes furthermore with λ-filtered colimits, for λ a (small) regular cardinal. Another equivalent definition is to say that there exists a small *dg*-category T_0 over A, and a small set of morphisms S in $\widehat{T_0}$, such that T is a localization of $\widehat{T_0}$ along S, *inside the homotopy theory of dg-categories with colimits and continuous dg-functors* (sometimes called *Bousfield localizations*, to avoid possible confusions with the usual localizations inside the homotopy theory of dg-categories and all dg-functors). This, in turn, is equivalent to say that T is equivalent to the full sub-dg-category of $\widehat{T_0}$ consisting of all S-local objects, that is objects E such that for any $x \to y$ in S, the induced morphism on complexes of morphisms

$$\widehat{T_0}(y, E) \longrightarrow \widehat{T_0}(x, E)$$

is a quasi-isomorphism. Note that the inclusion dg-functor of S-local objects in $\widehat{T_0}$ always has a left adjoint, constructed by the so-called small objects argument.

By definition, locally presentable dg-categories are big dg-categories, even if we do not mention the adjective *big* explicitely. A A-dg-functor $T \longrightarrow T'$ between locally presentable A-dg-categories is *continuous* if it commutes with small direct sums, and thus will all small colimits (recall that a A-dg-functor always commutes with finite limits and colimits when they exist).

We let $Dg^{lp}(A)$ be the (non full !) sub-category of $Dg(A)$ consisting of locally presentable A-dg-categories and continuous morphisms between them. For $A \to B$, the direct image functor $Dg(B) \longrightarrow Dg(A)$ obviously preserves locally presentable objects as well as continuous dg-functors. We consider the sub-category $\int LDg^{lp} \subset \int LDg$ consisting of all pairs (A, T), with T a locally presentable A-dg-category, and continuous morphisms between those pairs. The natural projection

$$\pi : \int LDg^{lp} \longrightarrow s\mathbb{Z}\text{-}CAlg$$

is still a fibered very big ∞-category.

Lemma 3.1 *The ∞-functor $\int LDg^{lp} \longrightarrow s\mathbb{Z}\text{-}CAlg$ is also a cofibered ∞-category, so is a bifibered ∞-category.*

Proof of the Lemma By definition, the results states that for $u : A \to B$ in $s\mathbb{Z}\text{-}CAlg$, the ∞-functor

$$u_* : LDg^{lp}(B) \longrightarrow LDg^{lp}(A)$$

possesses a left adjoint u^* (in the sense of ∞-categories). This is equivalent to the existence, for an arbitrary $T \in LDg^{lp}(A)$, of an object $T \otimes_A^{ct} B \in LDg^{lp}(B)$, together with a morphism $T \longrightarrow u_*(T \otimes_A^{ct} B)$, such that for any $U \in LDg^{lp}(B)$, the composite morphism

$$\begin{aligned} Map_{LDg^{lp}(B)}(T \otimes_A^{ct} B, U) &\longrightarrow Map_{LDg^{lp}(A)}(u_*(T \otimes_A^{ct} B), u_*(U)) \\ &\longrightarrow Map_{LDg^{lp}(A)}(T, u_*(U)) \end{aligned}$$

is an equivalence (see [35, Sect. 1.2] for the notion of adjunction between ∞-categories). The object $T \otimes_A^{ct} B$ can be explicitly constructed as follows. We write T as the localization, in the sense of locally presentable B-dg-categories, of $\widehat{T_0}$, for T_0 a small A-dg-category, along a small set of morphisms S in $\widehat{T_0}$. That is we assume that T is the full sub-dg-category of $\widehat{T_0}$ consisting of all S-local objects. We consider the small B-dg-category $T_0 \otimes_A^{\mathbb{L}} B$, which comes equiped with a small set of morphisms $S \otimes_A^{\mathbb{L}} B$ in $\widehat{T_0 \otimes_A^{\mathbb{L}} B}$, consisting of all morphisms of $T_0^{op} \otimes_A^{\mathbb{L}} B$-$dg$-modules obtained by tensoring a morphism in S by B over A, or in other words the image of S by the base change dg-functor

$$- \otimes_A^{\mathbb{L}} B : \widehat{T_0} \longrightarrow \widehat{T_0 \otimes_A^{\mathbb{L}} B}.$$

We define $T \otimes_A^{ct} B$ to be the localization of the locally presentable B-dg-category $\widehat{T_0 \otimes_A^{\mathbb{L}} B}$ along the set $S \otimes_A^{\mathbb{L}} B$, that is the full sub-$dg$-category of all $S \otimes_A^{\mathbb{L}} B$-local objects in $\widehat{T_0 \otimes_A^{\mathbb{L}} B}$. Note that these are precisely the $T_0^{op} \otimes_A^{\mathbb{L}} B$-$dg$-modules E which are S-local when simply considered as T_0^{op}-dg-modules. Therefore, the natural A-dg-functor $\widehat{T_0} \longrightarrow u_*(\widehat{T_0 \otimes_A^{\mathbb{L}} B})$ induces, by localization, a natural A-dg-functor

$$T \longrightarrow u_*(T \otimes_A^{ct} B),$$

and [26, Theorem 7.1, Corollary 7.6] imply that the required universal property is satisfied. □

We consider the bifibered ∞-category of locally presentable dg-categories

$$\int LDg^{lp} \longrightarrow s\mathbb{Z}\text{-}CAlg.$$

Let $A \longrightarrow B_*$ be a fpqc hyper-coverings in $s\mathbb{Z}$-$CAlg$. It defines a co-augmented cosimplicial object

$$H : \Delta_+ \longrightarrow s\mathbb{Z}\text{-}CAlg.$$

Because of Lemma 3.1 above, given an object $T \in LDg^{lp}(A)$, there is a unique, up to equivalence, ∞-functor over $s\mathbb{Z}$-$CAlg$

$$F : \Delta_+ \longrightarrow \int LDg^{lp},$$

such that $F([-1]) \simeq T$, and such that F sends every morphism in Δ_+ to a cocartesian morphism in $\int LDg^{lp}$. As $\mathbb{Z}$ is the initial object in $s\mathbb{Z}$-$CAlg$, we have a natural ∞-functor over $s\mathbb{Z}$-$CAlg$

$$\int LDg^{lp} \longrightarrow LDg^{lp}(\mathbb{Z}) \times s\mathbb{Z}\text{-}CAlg,$$

which simply sends every fiber $LDg^{lp}(A')$ to $LDg^{lp}(\mathbb{Z})$ using the direct image ∞-functor along $\mathbb{Z} \longrightarrow A'$. Precomposing with F we get a new ∞-functor

$$\Delta^{+} \longrightarrow LDg^{lp}(\mathbb{Z}).$$

We have thus seen that the consequence of Lemma 3.1 is the existence, for any locally presentable A-dg-category T, and any fpqc hyper-coverings $A \to B_*$, of a co-augmented cosimplicial locally presentable dg-category T^*, with $T^{-1} = T$. This produces a natural morphism

$$T = T^{-1} \longrightarrow \underset{n\in\Delta}{Holim}\, T^{n}.$$

Intuitively, this cosimplicial object is given by

$$T^{n} = T \otimes_{A}^{ct} B_n$$

(which is only an intuitive picture as many homotopy coherences must also be given to describe the object T^*), and the previous natural morphism will therefore be denoted symbolically by

$$T \longrightarrow \underset{n\in\Delta}{Holim}\, T \otimes_{A}^{ct} B_n.$$

By construction this morphism actually exists not only in dg-categories over $\mathbb{Z}$ but in dg-categories over A (simply replace $s\mathbb{Z}$-$CAlg$ by sA-$CAlg$ in the construction above).

Definition 3.2 Let A be a commutative simplicial ring and T be a locally presentable A-dg-category. We say that T *has (fpqc) descent* if for any fpqc hyper-coverings $B \to B_*$ in sA-$CAlg$, the natural morphism

$$T \otimes_{A}^{ct} B \longrightarrow \underset{n\in\Delta}{Holim}\, T \otimes_{A}^{ct} B_n$$

is an equivalence of dg-categories.

The locally presentable dg-categories with descent are clearly stable by the base change $- \otimes_{A}^{ct} B$, and thus form a full sub ∞-category $\int LDg^{lp,desc} \subset \int LDg^{lp}$, which is cofibered over $s\mathbb{Z}$-$CAlg$. Using [35, Proposition 1.4] we turn this last cofibered ∞-category into a functor

$$\mathbb{D}g^{lp,desc} : s\mathbb{Z}\text{-}CAlg \longrightarrow \infty\text{-}Cat_{\mathbb{W}},$$

that is a *derived prestack.*

We will need the following behaviour of dg-categories with descent with respect to homotopy limits.

Lemma 3.3

1. *For any commutative simplicial ring A the very big ∞-category $\mathbb{D}g^{lp,desc}(A)$ has all small limits.*

2. *For any morphism* $u : A \to B$ *in* $s\mathbb{Z}\text{-}CAlg$, *the two* ∞*-functors*

$$- \otimes_A^{ct} B : \mathbb{D}g^{lp,desc}(A) \longrightarrow \mathbb{D}g^{lp,desc}(B),$$
$$u_* : \mathbb{D}g^{lp,desc}(B) \longrightarrow \mathbb{D}g^{lp,desc}(A)$$

commute with small limits.

Proof We start to observe that, for any $A \in s\mathbb{Z}\text{-}CAlg$, the ∞-category $LDg^{lp}(A)$ has small limits. This follows from the dg-analog of the fact that locally presentable ∞-categories are stable by limits (see [11]), and can actually be deduced from it by considering underlying ∞-categories of dg-categories as follows.

For any dg-category T over A, we can consider its complexes of morphisms and apply the Dold-Kan correspondence to get simplicial sets. As the Dold-Kan correspondence is a weak monoïdal functor, we can use the composition morphisms in T in order to define the composition morphisms on the corresponding simplicial sets (see [24, Theorem 4.16] for more details). We obtain this way a simplicial category sT, functoriality associated to T, that we consider as a strict ∞-category. This defines an ∞-functor

$$s : LDg(A) \longrightarrow \infty\text{-}Cat,$$

from the ∞-category of dg-categories over A to the ∞-category of ∞-categories $\infty\text{-}Cat$. When T has small colimits the ∞-category sT has small colimits. Moreover, if T has small colimits, then T is a locally presentable dg-category over A if and only if sT is a locally presentable ∞-category. As the ∞-functor $T \mapsto sT$ commutes with limits (as it is obtained by localizing a right Quillen functor, see [24]), we deduce from [11, Proposition 5.5.3.12] that $LDg^{lp}(A)$ is stable by small limits in $LDg(A)$, and thus itself possesses all small limits.

The next step is to see that, for any morphism of commutative simplicial rings $A \longrightarrow B$, the base change ∞-functor

$$- \otimes_A^{ct} B : LDg^{lp}(A) \longrightarrow LDg^{lp}(B)$$

commutes with small limits. Using [26, Theorem 7.2, Corollary 7.6], as well as the explicit construction of $T \otimes_A^{ct} B$ in terms of localizations (see the proof of Lemma 3.1), we see that there is a natural equivalence

$$T \otimes_A^{ct} B \simeq \mathbb{R}\underline{Hom}(B, T),$$

where the right hand side denotes the derived internal *Hom* object in $LDg(A)$. As it is clear that $T \mapsto \mathbb{R}\underline{Hom}(B, T)$ commutes with small limits (because it is a right adjoint), we deduce that $- \otimes_A^{ct} B$ commutes with small limits.

We are now ready to prove (1) and (2) of Lemma 3.3. Let

$$T_* : I \longrightarrow \mathbb{D}g^{lp,desc}$$

be a small diagram of locally presentable *dg*-category with descent over A, $T = Holim\, T_i$, and $A \to B_*$ be a fpqc hyper-covering of commutative simplicial rings. Now that we know that base change of locally presentable *dg*-categories commutes with limits, we have

$$T = Holim\, T_i \simeq Holim_i \big(Holim_n T_i \otimes_A^{ct} B_n\big) \simeq Holim_n \big(T \otimes_A^{ct} B_n\big),$$

showing that T is a locally presentable *dg*-category with descent over A. This shows (1), and (2) follows from what we have already seen, that $- \otimes_A^{ct} B$ commutes with small limits. □

The main properties of locally presentable *dg*-categories with descent is the following gluing property.

Theorem 3.4 *The derived prestack $\mathbb{D}g^{lp,desc}$ defined above is a stack for the fpqc topology on the model category $(s\mathbb{Z}\text{-}CAlg)^{op}$.*

Proof This follows easily from our descent criterion Corollary A.7, proved in the Appendix. The criterion is given in the language of ∞-sites, but we can simply localize our model site $(s\mathbb{Z}\text{-}CAlg)^{op}$ to consider $\mathbb{D}g^{lp,desc}$ as a prestack over the ∞-site $L(s\mathbb{Z}\text{-}CAlg)^{op}$. The descent condition over the model site $(s\mathbb{Z}\text{-}CAlg)^{op}$ and over $L(s\mathbb{Z}\text{-}CAlg)^{op}$ being equivalent, we can simply apply Corollary A.7 over the ∞-site $L(s\mathbb{Z}\text{-}CAlg)^{op}$.

The condition (1) is our Lemma 3.3, and the right adjoints of (2) are simply the forgetful ∞-functors u_*, which are clearly conservative. Condition (3) follows from the compatibility of the completed tensor product $- \otimes_A^{ct} B$ with the homotopy base change of rings

$$- \otimes_A^{ct} (B \otimes_A^{\mathbb{L}} B') \simeq - \otimes_B^{ct} B',$$

which itself follows from the explicit construction of $- \otimes_A^{ct} B$, and the usual associative property of the derived tensor product of simplicial rings. Finally, condition (4) follows easily from the fact that we restrict to *dg*-categories with descent: we have for any fpqc hyper-coverings $A \to B_*$, any $T \in \mathbb{D}g^{lp,desc}$, and any $K \in \mathbb{D}g^{lp,desc}(\mathbb{Z})$

$$\begin{aligned} Map_A(K \otimes^{ct} A, T) &\simeq Map_A(K \otimes^{ct} A, Holim_n(T \otimes_A^{ct} B_n)) \\ &\simeq Holim_n Map_A(K \otimes^{ct} A, (T \otimes_A^{ct} B_n)). \end{aligned}$$

We conclude that $\mathbb{D}g^{lp,desc}$ is a stack for the fpqc topology. □

Remark 3.5 We could have considered a weaker descent property, simply requiring descent for fppf hypercoverings, and even more restrictively fppf Čech descent (i.e. descent with respect of nerves of fppf coverings). It can be proven that every locally presentable *dg*-category has the fppf Čech descent (Jacob Lurie, private communication). The argument follows the same lines as in the proof of our Theorem 4.7, and proceeds by breaking the fppf descent into two independant steps: descent for finite flat maps and Nisnevich descent. As a consequence, the derived prestack $A \mapsto \mathbb{D}g^{lp}$ is a Čech stack for the fppf topology.

As we will see in the sequel there many examples of *dg*-categories with descent, and the two classes of all locally presentable *dg*-categories and of *dg*-categories with descent are rather closed.

Definition 3.6 Let F be a derived stack. A *locally presentable dg-category over F* is a morphism of derived stacks

$$F \longrightarrow \mathbb{D}g^{lp,desc}.$$

In intuitive terms, a locally *dg*-category over F consists of the following data.

1. For any commutative simplicial ring A and any morphism
$$u : \mathbb{R}\underline{Spec}\, A \longrightarrow F,$$
a locally presentable *dg*-category T_u with descent over A.
2. For any commutative simplicial rings A and B, and any commutative diagram
$$\begin{array}{ccc} \mathbb{R}\underline{Spec}\, A & \xrightarrow{f} & \mathbb{R}\underline{Spec}\, B \\ & \searrow^{u} & \downarrow^{v} \\ & & F, \end{array}$$
a *dg*-functor $f^* : T_v \longrightarrow T_u$, of *dg*-categories over A, such that the induced morphism $T_v \otimes_B^{ct} A \longrightarrow T_u$ is an equivalence.
3. Homotopy coherences conditions on f^* with respect to compositions.

The first two points above give a rather clear picture of what *dg*-categories over a derived stack F are, and the technical part of this notion is really in point (3). There are many ways of writing these homotopy coherences, which are encoded in the fact that the morphism $F \longrightarrow \mathbb{D}g^{lp,desc}$ is only a morphism in the ∞-category of derived stacks.

There are two trivial *dg*-categories over any given derived stack, the null *dg*-category and the unit *dg*-category. They both are global sections

$$* = Spec\, \mathbb{Z} \longrightarrow \mathbb{D}g^{lp,desc},$$

corresponding to the 0 *dg*-category, and to the unit *dg*-category $\widehat{\mathbb{Z}}$. Pulling back these two morphisms along the morphism $F \longrightarrow *$ provides two *dg*-categories over F. Using the intuitive description of *dg*-categories over F given above, the null *dg*-category is the one given by $T_u = 0$ for all u, and the unit *dg*-category is given by $T_u = \widehat{A}$ for all u (with the obvious transitions *dg*-functors f^*). The unit *dg*-category over F will be denoted by $\mathbf{1}_F$, or $\mathbf{1}$ if F is clear from the context, and should be understood as some kind of *categorical structure sheaf of F*.

The previous theorem shows that *dg*-categories with descent possess nice local properties. The following proposition implies that all compactly generated *dg*-categories are with descent.

Proposition 3.7 *Let A be a commutative simplicial ring and T_0 be a small dg-category over A. Then, the big dg-category $\widehat{T_0}$, of small T_0-dg-modules, is a locally presentable dg-category with descent.*

Proof The *dg*-categories of the form $\widehat{T_0}$ are clearly locally presentable. Moreover, for any simplicial commutative A-algebra B, we have

$$\widehat{T_0} \otimes_A^{ct} B \simeq \widehat{T_0 \otimes_A^{\mathbb{L}} B}.$$

Therefore, to show that $\widehat{T_0}$ is with descent it is enough to show that for any fpqc hyper-coverings $A \to B_*$ we have

$$\widehat{T_0} \simeq \operatorname*{Holim}_{n \in \Delta} \widehat{T_0 \otimes_A^{\mathbb{L}} B_n}.$$

But, from the results of [26] we have, for an arbitrary small A-*dg*-category T_0', and any $A' \in sA\text{-}CAlg$

$$\widehat{T_0' \otimes_A^{\mathbb{L}} A'} \simeq \mathbb{R}\underline{Hom}((T_0')^{op}, \widehat{A'}),$$

where $\mathbb{R}\underline{Hom}$ is the derived internal Hom between A-*dg*-categories. Therefore, as $\mathbb{R}\underline{Hom}$ commutes with homotopy limits (because it is a right adjoint in the sense of ∞-category), it is enough to treat the case $T_0 = A$. We thus have to show that

$$\widehat{A} \simeq \operatorname*{Holim}_{n \in \Delta} \widehat{B_n},$$

which is nothing else than fpqc descent for quasi-coherent *dg*-modules, as proved in [31, Lemma 3.1], but directly at the level of *dg*-categories rather than only at the level of nerves of quasi-isomorphisms. □

An interesting corollary of the previous proposition together with Lemma 3.3 is that any homotopy limit of compactly generated A-*dg*-categories is with descent (thought it might be non-compactly generated).

Corollary 3.8 *Any dg-category over a commutative simplicial ring A, which is equivalent to a homotopy limit of dg-categories of the form $\widehat{T_0}$ (for T_0 a small A-dg-category) is a locally presentable A-dg-category with descent.*

Remark 3.9

1. It should be noted that Proposition 3.7 together with Theorem 3.4 provides an explicit description of the derived stack $\widetilde{\mathbb{D}g^c}$, defined in [35, Sects. 4.3, 4.4] by brutal stackyfication of the prestack of compactly generated *dg*-categories. Indeed, $\widetilde{\mathbb{D}g^c}$ can be identified with the full sub-stack of $\mathbb{D}g^{lp,desc}$ consisting of all A-*dg*-categories T such that there is a fppf covering $A \to B$ with $T \otimes_A^{ct} B$ equivalent to $\widehat{T_0}$ for some small *b*-*dg*-category T_0. In other words, $\widetilde{\mathbb{D}g^c}$ is equivalent to the derived stack of locally presentable *dg*-categories with descent and with local (for the fppf topology) compact generators. Objects in $\widetilde{\mathbb{D}g^c}$ are then twisted forms of compactly generated *dg*-categories, *and do form a rigid symmetric monoïdal ∞-category.* A consequence of our Theorem 4.7 (more precisely of techniques of proof of Propositions 4.9, 4.12 and 4.13) is that $\mathbb{D}g^c$ is in fact a stack, and that any twisted form of a compactly generated *dg*-category (for the fppf topology) is itself compactly generated.
2. There are examples of locally presentable *dg*-categories which do not have descent. By Corollary 3.8 these are not compactly generated. For instance, for a general topological space X, the *dg*-category $L(X)$, of complexes of sheaves on X, is a locally presentable *dg*-category which does not satisfies descent in general.

We finish this paragraph with the description of the diagonal of the derived stack $\mathbb{D}g^{lp,desc}$. For this we let A a commutative simplicial ring and

$$T_1, T_2 : \mathbb{R}\underline{Spec}\, A \longrightarrow \mathbb{D}g^{lp,desc}$$

be two locally presentable *dg*-categories with descent over A. As $\mathbb{D}g^{lp,desc}$ is a stack in ∞-categories, we have a derived stack of simplicial sets $Map(T_1, T_2)$ of morphisms between the two objects T_1 and T_2. We will assume for simplicity that T_1 is compactly generated, that is of the form $\widehat{T_0}$ for some small *dg*-category T_0 over A. Then, using [26, Corollary 7.6] it is possible to prove that this derived stack can be described as follows. We let

$$T := \widehat{T_0^{op}} \otimes_A^{ct} T_2,$$

which is another *dg*-category with descent over A. Then, for any A-algebra A', we have a natural equivalence

$$Map(T_1, T_2)(A') \simeq Map_{A\text{-}dg\text{-}cat}(A, T \otimes_A^{ct} A').$$

In other words, the simplicial set $Map(T_1, T_2)(A')$ is a classifying space for objects in the *dg*-category $T \otimes_A^{ct} A'$. As a consequence, we have the following description for the homotopy groups of $Map(T_1, T_2)(A')$ (see [26, Corollary 7.6])

$$\pi_0(Map(T_1,T_2)(A')) \simeq [T \otimes_A^{ct} A']/iso,$$
$$\pi_1(Map(T_1,T_2)(A'),x) \simeq aut(x),$$
$$\pi_i(Map(T_1,T_2)(A'),x) \simeq Ext^{1-i}(x,x),$$

where the automorphisms and ext groups are computed in the triangulated category $[T \otimes_A^{ct} A']$, the homotopy category associated to $T \otimes_A^{ct} A'$.

3.2 The derived prestack of derived Azumaya algebras

In the previous paragraph we have defined the derived stack $\mathbb{D}g^{lp,desc}$, of locally presentable *dg*-categories with descent. We will now define the derived stack $\mathbb{D}g\mathbb{A}lg$, of *dg*-algebras, and relate these two derived stacks by means of a morphism

$$\phi : \mathbb{D}g\mathbb{A}lg \longrightarrow \mathbb{D}g^{lp,desc},$$

sending a *dg*-algebra to its *dg*-category of *dg*-modules. The image of ϕ, restricted to deraz algebras, is a derived subprestack of $\mathbb{D}g^{lp,desc}$, which by definition, will be the derived prestack of deraz algebras.

For $A \in s\mathbb{Z}\text{-}CAlg$, we set $A\text{-}dg\text{-}alg$, the category of (small) A-*dg*-algebras. Localizing along the quasi-isomorphisms we obtain an ∞-category

$$LDgAlg(A) := L(A\text{-}dg\text{-}alg).$$

For $u : A \to A'$ a morphism of simplicial commutative rings, we have the direct image functor $A'\text{-}dg\text{-}alg \longrightarrow A\text{-}dg\text{-}alg$, which after localization induces an ∞-functor

$$u_* : LDgAlg(A') \longrightarrow LDgAlg(A).$$

This defines a functor $DgAlg : s\mathbb{Z}\text{-}CAlg^{op} \longrightarrow \infty\text{-}Cat$, of which we take the Grothendieck's construction to obtain a fibered ∞-category

$$\int LDgAlg \longrightarrow s\mathbb{Z}\text{-}CAlg.$$

The existence of left adjoints to the ∞-functors u_*,

$$- \otimes_A^{\mathbb{L}} A' : LDgAlg(A) \longrightarrow LDgAlg(A'),$$

implies that $\int LDgAlg$ is a bifibered ∞-category over $s\mathbb{Z}\text{-}CAlg$. By [35, Sect. 1.3] it thus corresponds to a functor

$$\mathbb{D}g\mathbb{A}lg : s\mathbb{Z}\text{-}CAlg \longrightarrow \infty\text{-}Cat,$$

which is a derived prestack of ∞-categories.

Proposition 3.10 *The derived prestack $\mathbb{D}g\mathbb{A}lg$ is a stack for the fpqc topology.*

Proof The proposition is proven using the same argument as for the proof that quasi-coherent complexes form a derived stack (e.g. as in [31, Lemma 3.1]), but replacing the model categories of *dg*-modules by model categories of *dg*-algebras. We leave the details to the reader. □

We now define a morphism of derived stacks in ∞-categories

$$\phi : \mathbb{D}g\mathbb{A}lg \longrightarrow \mathbb{D}g^{lp,desc}.$$

For this, it is enough to construct an ∞-functor

$$\widetilde{\phi} : \int LDgAlg \longrightarrow \int LDg^{lp,desc},$$

covering the natural projections to $s\mathbb{Z}\text{-}CAlg$, and with the property that $\widetilde{\phi}$ takes cocartesian morphisms to cocartesian morphisms. As $\int LDgAlg$ and $\int LDg^{lp,desc}$ are obtained by the Grothendieck construction applied to natural functors

$$LDgAlg, LDg^{lp,desc} : s\mathbb{Z}\text{-}CAlg^{op} \longrightarrow \infty\text{-}Cat,$$

we start by defining a natural transformation $h : LDgAlg \Rightarrow LDg^{lp,desc}$, and we will check that the induced ∞-functor

$$\int h : \int LDgAlg \longrightarrow \int LDg^{lp,desc}$$

possesses the required properties. For this, let A be a simplicial commutative ring, and denote by $Dg_*(A)$ the category of big A-*dg*-category T endowed with a distinguished object $x \in T$ and *dg*-functor preserving the distinguished objects. We have

$$Dg_*(A) \simeq A/Dg(A),$$

where A is considered as an A-*dg*-category with a unique object. By localization we get a functor

$$LDg_* : s\mathbb{Z}\text{-}CAlg^{op} \longrightarrow \infty\text{-}Cat,$$

which is the pointed version of LDg. We consider $LDg_*^{lp,desc}$ the sub-functor of LDg_* consisting of pointed *dg*-categories (T, x) with T a locally presentable *dg*-category with descent and continuous *dg*-functors. We have a diagram of functors

$$LDgAlg \xleftarrow{\ a\ } LDg_*^{lp,desc} \xrightarrow{\ p\ } LDg^{lp,desc},$$

where p forgets the distinguished objects and a sends a pair (T, x) to the *dg*-algebra $T(x, x)$. We now consider $\mathcal{D}$ the sub-functor of $LDg_*^{lp,desc}$ consisting of pairs equivalent to $(B\text{-}dg\text{-}mod^c, B)$, for B a *dg*-algebra (where

B-dg-mod^c is the dg-category of cofibrant B-dg-modules and B is considered as a dg-module over itself). We have another diagram

$$LDgAlg \xleftarrow{a} \mathcal{D} \xrightarrow{p} LDg^{lp,desc},$$

and it is easy to check that a is now an equivalence. Therefore, we obtain a well defined morphism of ∞-functors, well defined up to equivalence

$$h : LDgAlg \longrightarrow LDg^{lp,desc},$$

from which we obtain an ∞-functor of fibered ∞-categories

$$\int h : \int LDgAlg \longrightarrow \int LDg^{lp,desc},$$

which by construction preserves the Cartesian morphisms. This morphism sends a dg-algebra B to the dg-category with descent $\widehat{B}$. We claim that it does also preserve the cocartesian morphisms. Coming back to the definitions we see that this is equivalent to state that for $A \to A'$ a morphism in $s\mathbb{Z}$-$CAlg$, and for B a A-dg-algebra, the natural adjunction morphism

$$\widehat{B^{op}} \otimes_A^{ct} A' \longrightarrow \widehat{B^{op} \otimes_A^{\mathbb{L}} A'},$$

is a quasi-equivalence. But, this is true by the definition of the base change $\otimes_A^{ct} A'$, and by the results of [26].

We have thus defined

$$\int h : \int LDgAlg \longrightarrow \int LDg^{lp,desc},$$

which preserves cocartesian morphisms, and therefore induces a morphism of derived stacks of ∞-categories

$$\phi : \mathbb{D}g\mathbb{A}alg \longrightarrow \mathbb{D}g^{lp,desc}.$$

We now work in the ∞-category dSt_{fppf}, of derived stacks (of simplicial sets) over the model category $dAff := s\mathbb{Z}\text{-}CAlg^{op}$, endowed with the fppf topology (see [30]). The derived stacks of ∞-categories $\mathbb{D}g\mathbb{A}alg$ and $\mathbb{D}g^{lp,desc}$ give rise to derived stacks of simplicial sets by considering their *underlying spaces* as in [35, Sect. 1][1]

$$\mathcal{I}(\mathbb{D}g\mathbb{A}alg), \qquad \mathcal{I}(\mathbb{D}g^{lp,desc}).$$

These derived stacks can also be considered as stacks in ∞-groupoïds (using the equivalence between simplicial sets and ∞-groupoïds), and they then correspond to the maximal substacks in groupoïds of $\mathbb{D}g\mathbb{A}alg$ and $\mathbb{D}g^{lp,desc}$ respectively (in other words the symbol $\mathcal{I}$ simply means that we restrict to

[1] In the sequel we will often neglect writing "$\mathcal{I}$", and assume implicitely that the underlying space functor has been applied if necessary.

equivalences between objects, and throw away any non invertible morphism). The morphism ϕ induces a morphism of derived stacks

$$\phi : \mathcal{I}(\mathbb{D}g\mathbb{A}alg) \longrightarrow \mathcal{I}(\mathbb{D}g^{lp,desc}).$$

We restrict this morphism to the derived substack $\mathbb{D}g\mathbb{A}alg^{Az} \subset \mathcal{I}(\mathbb{D}g\mathbb{A}alg)$, consisting of all deraz algebras (this is a substack by Proposition 2.3(2)), and its stacky image $\mathbb{D}g^{Az} \subset \mathcal{I}(\mathbb{D}g^{lp,desc})$. By definition, for any $A \in s\mathbb{Z}\text{-}CAlg$, $\mathbb{D}g\mathbb{A}alg^{Az}(A)$ is equivalent to the nerve of the category of quasi-isomorphisms between deraz A-algebras, whereas $\mathbb{D}g^{Az}$ is equivalent to the nerve of the category of quasi-equivalences between locally presentable big A-*dg*-categories with descent, locally equivalent (for the fppf topology) to $\widehat{B}$ for B a deraz algebra. The morphism ϕ simply sends a deraz algebra B to $\widehat{B}$.

We are now ready to define derived Azumaya algebras over an arbitrary derived stack.

Definition 3.11 Let $F \in dSt_{fppf}$ be a derived stack.

1. The *classifying space of derived Azumaya algebras over F* is the full sub-simplicial set $\mathbb{D}eraz(F)$ of $Map_{dSt_{fppf}}(F, \mathbb{D}g^{Az})$, consisting of all morphisms $F \to \mathbb{D}g^{Az}$ that factors throught $\phi : \mathbb{D}g\mathbb{A}lg^{Az} \longrightarrow \mathbb{D}g^{Az}$.
2. The *classifying space of derived Azumaya dg-categories over F* is

$$\mathbb{D}eraz^{dg}(F) := Map_{dSt_{fppf}}(F, \mathbb{D}g^{Az}).$$

We now finish by the following description of the derived stack $\mathbb{D}g^{Az}$, which is a simple corollary of the results of our first section.

Corollary 3.12 *There exist a natural equivalence of derived stacks*

$$\psi : K(\mathbb{Z}, 1) \times K(\mathbb{G}_m, 2) \simeq \mathbb{D}g^{Az}.$$

In particular, for any $F \in dSt_{fppf}$, there is a natural monomorphism of simplicial sets

$$\begin{aligned}\psi : \mathbb{D}eraz(F) &\hookrightarrow \mathbb{H}^1_{fppf}(F, \mathbb{Z}) \times \mathbb{H}^2_{fppf}(F, \mathbb{G}_m)\\ &:= Map_{dSt_{fppf}}(F, K(\mathbb{Z}, 1)) \times Map_{dSt_{fppf}}(F, K(\mathbb{G}_m, 2)),\end{aligned}$$

and thus an injective map

$$\pi_0(\mathbb{D}eraz(F)) \hookrightarrow H^1_{fppf}(F, \mathbb{Z}) \times H^2_{fppf}(F, \mathbb{G}_m).$$

Proof By definition, any object in $\mathbb{D}g^{Az}(A)$ is locally, for the étale topology, equivalent to some object $\widehat{B}$, for B a deraz A-algebra. By Proposition 2.14, B is itself locally Morita equivalent, for the étale topology, to the trivial deraz algebra A. Therefore, any object in $\mathbb{D}g^{Az}(A)$ is locally equivalent to the trivial object $\widehat{A}$. This implies that there exists a natural equivalence of derived stacks

$$K(G,1) \simeq \mathbb{D}g^{Az},$$

where G is the derived group stack of auto-equivalences of the trivial object. By [26, Theorem 8.15], the derived group stack G is simply the group of invertible perfect dg-modules, which are all of the form $\mathcal{L}[n]$, for $\mathcal{L}$ a line bundle and n some integer (the group law being given by the tensor product). We thus have a natural equivalence of derived group stacks

$$G \simeq \mathbb{Z} \times K(\mathbb{G}_m, 1).$$

We get this way the required equivalence

$$\psi : K(G,1) \simeq K(\mathbb{Z},1) \times K(\mathbb{G}_m, 2) \simeq \mathbb{D}g^{Az}.$$ □

In [35, Sect. 4.3], we have defined $\mathbb{D}g^c$, the derived prestack of compactly generated dg-categories, as a stack in symmetric monoïdal ∞-categories. The exact same construction provides a refined versions of the derived stacks $\mathbb{D}g\mathbb{A}lg$ and $\mathbb{D}g^{lp,desc}$ as derived stacks of symmetric monoïdal ∞-categories. The morphism

$$\phi : \mathbb{D}g\mathbb{A}lg \longrightarrow \mathbb{D}g^{lp,desc}$$

can also be refined as a symmetric monoïdal morphism. For any derived stack F, these symmetric monoïdal structures induce symmetric monoïdal group-like structures on the classifying spaces $\mathbb{D}eraz(F)$ and $\mathbb{D}g^{Az}(F)$, and ϕ induces a symmetric monoïdal morphism between these two objects. By the equivalence between symmetric monoïdal group-like ∞-groupoïds and connective spectra (see [35, Sect. 2.1]), another equivalent way to state the existence of these group structures is simply by stating that $\mathbb{D}eraz(F)$ and $\mathbb{D}g^{Az}(F)$ are the 0th space of natural spectra, and that ϕ is itself the restriction to the 0th spaces of a morphism of spectra. As a consequence, we see that the sets $\pi_0(\mathbb{D}eraz(F))$ and $\pi_0(\mathbb{D}g^{Az}(F))$ come equiped with natural abelian group structures, both induced by the tensor products of locally presentable dg-categories.

Remark 3.13 The equivalence of Corollary 3.12 can be improved as an equivalence between symmetric monoïdal derived stacks, where the monoïdal structures is the standard one on $K(\mathbb{Z},1) \times K(\mathbb{G}_m, 2)$, and induced by the tensor product of locally presentable dg-categories on the right hand side, as explained above.

We now make the following definitions, generalizing the two well known definitions of Brauer group of a scheme.

Definition 3.14 Let $F \in dSt_{fppf}$ be a derived stack.

1. *The derived algebraic Brauer group of* F is defined by

$$dBr(F) := \pi_0(\mathbb{D}eraz(F)),$$

where the group structure on the right hand side is induced by the tensor product of deraz algebras.
2. *The derived categorical Brauer group of* F is defined by

$$dBr_{cat}(F) := \pi_0(\mathbb{D}g^{Az}(F)),$$

where the group structure on the right hand side is induced by the tensor product of locally presentable dg-categories.
3. *The big derived categorical Brauer group of* F is defined by

$$dBr_{cat,big}(F) := \pi_0(\mathbb{D}g^{lp,desc,inv}(F)),$$

where the group structure on the right hand side is induced by the tensor product of locally presentable dg-categories, and where $\mathbb{D}g^{lp,desc,inv}$ denotes the monoïdal sub-stack of $\mathbb{D}g^{lp,desc}$ consisting of all invertible objects and equivalences between them.
4. *The big derived cohomological Brauer group* of F is defined by

$$dBr'_{big}(F) := H^1_{fppf}(F, \mathbb{Z}) \times H^2_{fppf}(F, \mathbb{G}_m).$$

We obviously have natural injective morphisms of groups

$$dBr(F) \subset dBr_{cat}(F) \simeq dBr'_{big}(F) \subset dBr_{cat,big}(F),$$

where the isomorphism in the middle follows from Corollary 3.12, the first inclusion is induced by the morphism ϕ, and the last inclusion follows from the fact the inclusion of symmetric monoïdal substacks (due to Proposition 2.5(2))

$$\mathbb{D}g^{Az} \subset \mathbb{D}g^{lp,desc,inv}.$$

We consider now two copies of the morphism ϕ

$$\mathbb{D}g\mathbb{A}lg^{Az} \times \mathbb{D}g\mathbb{A}lg^{Az} \longrightarrow \mathbb{D}g^{Az} \times \mathbb{D}g^{Az}.$$

Pulling-back the diagonal along this morphism provides a natural morphism of derived stacks

$$\pi : \mathcal{M} \longrightarrow \mathbb{D}g\mathbb{A}lg^{Az} \times \mathbb{D}g\mathbb{A}lg^{Az}.$$

It is easy to describe the projection π. Indeed, if $X = \mathbb{R}\underline{Spec}\, A$ is an affine derived scheme, and $X \longrightarrow \mathbb{D}g\mathbb{A}lg^{Az} \times \mathbb{D}g\mathbb{A}lg^{Az}$, is a pair of deraz A-algebras B_1 and B_2, the fiber product

$$\mathcal{M}_{B_1,B_2} := \mathcal{M} \times^h_{\mathbb{D}g\mathbb{A}lg^{Az} \times \mathbb{D}g\mathbb{A}lg^{Az}} X,$$

is the derived stack over $Spec\, A$ of Morita equivalences between B_1 and B_2, that we have denoted by $Eq(B_1, B_2)$ during our proof of Proposition 2.14.

This description of the correspondence π implies the following description of the group $dBr(F)$, for a derived stack F. Its elements are simply morphisms of derived stacks $F \longrightarrow \mathbb{D}g\mathbb{A}lg^{Az}$, up to equivalences, which by definition are called *deraz algebras over* F. Two such elements are declared to be *Morita equivalent* if there exists a commutative diagram of derived stacks

$$\begin{array}{ccc} & & \mathcal{M} \\ & \nearrow & \downarrow \pi \\ F & \xrightarrow[B_1, B_2]{} & \mathbb{D}g\mathbb{A}lg^{Az} \times \mathbb{D}g\mathbb{A}lg^{Az}, \end{array}$$

a terminology justified by the description of π above. We then have a natural identification between the group $dBr(F)$ and the Morita equivalent classes of deraz algebras over F. This last remark justifies the fact that $\pi_0(\mathbb{D}eraz(F))$ is called the derived algebraic Brauer group of F. When F is moreover a scheme, then $dBr(F)$ can even be described more concretely in Termes of sheaves of quasi-coherent dg-algebras, locally satisfying the two conditions (Az-1) and (Az-2) of Definition 2.1. These sheaves of dg-algebras are then taken up to Morita equivalences, a Morita equivalence being here a sheaf of quasi-coherent bi-dg-modules inducing local equivalences on derived categories of dg-modules, as this is done in the concrete treatment of categorical sheaves of [34].

4 Existence of derived Azumaya algebras

We now arrive at the question of existence of derived Azumaya algebras, which is the problem of surjectity of the natural embeddings (see Definition 3.14)

$$dBr(F) \subset dBr_{cat}(F) \subset dBr_{cat,big}(F),$$

for a given derived stack F. We remind here that we have proved that the derived stack $\mathbb{D}g^{Az}$ is naturally equivalent to $K(\mathbb{Z}, 1) \times K(\mathbb{G}_m, 2)$, and therefore that we have a natural isomorphism

$$dBr_{cat}(F) \simeq H^1_{fppf}(F, \mathbb{Z}) \times H^2_{fppf}(F, \mathbb{G}_m).$$

The question of surjectivity of the first map is therefore the question of representability of cohomology classes in terms of derived Azumaya algebras, and is a derived generalization of the original question of Grothendieck of representability of torsion classes in $H^2(X, \mathbb{G}_m)$ by Azumaya algebras over X (see [7]). The new cohomological term $H^1(X, \mathbb{Z})$ is somehow anecdotal, but the fact that non torsion elements are taken into account is the true new feature of the derived setting. The surjectivity of the second map seems to us a more exotic question, a bit outside of the scope of the present paper. In this section we will show our main theorem, which implies that the morphism ϕ is surjective for quasi-compact and quasi-separated schemes (and more generally for a large class of Deligne-Mumford stacks).

4.1 Derived Azumaya algebras and compact generators

Before starting to study the surjectivity of the inclusion $dBr(F) \subset dBr_{cat}(F)$, we will give a simple and purely categorical criterion for a class $\alpha \in dBr_{cat}(F)$ to belongs to $dBr(F)$. For this we will first study a slightly more general problem. Let F be a fixed derived stack and

$$\alpha : F \longrightarrow \mathbb{D}g^{lp,desc}$$

be a dg-category with descent on F (here we really mean a morphism of derived stacks from F to $\mathcal{I}(\mathbb{D}g^{lp,desc})$, the underlying derived stacks of simplicial sets of the stacks of ∞-categories $\mathbb{D}g^{lp,desc}$). We first study the existence of a dg-algebra B_α over F, with $\phi(B) = \alpha$. In other words, we ask the question of existence of a commutative diagram of derived stacks

$$\begin{array}{ccc} & & \mathbb{D}g\mathbb{A}lg \\ & \overset{B_\alpha}{\nearrow} & \downarrow \phi \\ F & \xrightarrow[\alpha]{} & \mathbb{D}g^{lp,desc}. \end{array}$$

When B_α exists we will say that B_α *realizes* α. For this, we let $L_\alpha(F)$ be the dg-category of global sections of α

$$L_\alpha(F) := \Gamma(F, \alpha) \in \mathbb{D}g^{lp,desc}(\mathbb{Z}),$$

defined as follows. The morphism α corresponds, by [35, Proposition 1.4], to a Cartesian ∞-functor

$$u_\alpha : \int F \longrightarrow \int LDg^{lp,desc}.$$

We then use the natural ∞-functor

$$\int LDg^{lp,desc} \longrightarrow LDg(\mathbb{Z}),$$

obtained by functoriality because $\mathbb{Z}$ is initial in $s\mathbb{Z}\text{-}CAlg$. By definition we have

$$L_\alpha(F) := \underset{\int F}{Holim}\, u_\alpha \in LDg(\mathbb{Z}).$$

The homotopy category of $L_\alpha(F)$ is a triangulated category denoted by $D_\alpha(F)$.

Definition 4.1 With the same notations as above, the triangulated category $D_\alpha(F)$ is called the *α-twisted derived category of F*. The dg-category $L_\alpha(F)$ is it self called the *α-twisted derived dg-category of F*.

Note that by definition $L_\alpha(F)$ is a priori only a big *dg*-category. When F is not too big, then $L_\alpha(F)$ can be proven to be a locally presentable *dg*-category (see the Corollary 4.3 below).

The construction $L_\alpha(F)$ is functorial in F as follows. If $F' \to F$ is a morphism of derived stacks, and α is a locally presentable *dg*-category over F. There is a natural ∞-functor

$$\int F' \longrightarrow \int F,$$

and thus an induced morphism on the corresponding limits

$$L_\alpha(F') \longrightarrow L_\alpha(F),$$

where we still denote by α the *dg*-category α pulled back on F'. With a bit more work we can enhanced this construction to an ∞-functor

$$L_\alpha : (dSt/F)^{op} \longrightarrow \mathbb{D}g(\mathbb{Z}),$$

from the ∞-category of derived stacks over F to the ∞-category of big $\mathbb{Z}$-*dg*-categories.

Lemma 4.2 *Let F be a derived stack and F_i be a small diagram of objects in dSt/F. Then, for any locally presentable dg-category α over F the natural dg-functor*

$$L_\alpha(Hocolim_i F) \longrightarrow Holim_i L_\alpha(F_i)$$

is a quasi-equivalence.

Proof The ∞-category dSt/F is an ∞-topos having the ∞-site $dAff/F$, of derived affine schemes over F, as a generating site. Therefore, there is an equivalence between the ∞-category of ∞-functors

$$(dSt/F)^{op} \longrightarrow \mathbb{D}g(\mathbb{Z})$$

sending colimits to limits, and ∞-functors $(dAff/F)^{op} \longrightarrow \mathbb{D}g(\mathbb{Z})$ satisfying the descent condition for fppf hypercoverings. The lemma follows from the fact that

$$L_\alpha(-) : (dSt/F)^{op} \longrightarrow \mathbb{D}g(\mathbb{Z})$$

is the left Kan extension of the ∞-functor

$$u_\alpha : \int F \simeq (dAff/F)^{op} \longrightarrow \mathbb{D}g(\mathbb{Z}),$$

and that this last ∞-functor does have descent for the fppf topology (because it has descent for the fpqc topology by definition of *dg*-categories with descent). □

Corollary 4.3 *Let F be a small derived stacks (i.e. a small colimit of representable derived stacks), and α a locally presentable dg-category over F. Then the dg-category $L_\alpha(F)$ is locally presentable with descent.*

Proof Follows from the previous lemma and from Lemma 3.3. □

Note that when $\alpha = \mathbf{1}$ is the trivial *dg*-category over F, $D_\alpha(F)$ is simply the derived category of quasi-coherent complexes on F (this can be taken as a definition of the derived category of quasi-coherent complexes on F, or can be proven to be equivalent to other definitions, see e.g. [14, 16, 28]).

We come back to a small derived stack F together with a locally presentable *dg*-category α over F. By adjunction, any object $E \in D_\alpha(F)$ gives rise to a unique *dg*-functor $E : \widehat{\mathbb{Z}} \longrightarrow L_\alpha(F)$ pointing E, and thus to a morphism

$$\mathbf{1}_F \longrightarrow \alpha,$$

of locally presentable *dg*-categories over F (here $\mathbf{1}_F$ is the unit *dg*-category over F). Evaluating this morphism at a morphism $u : \mathbb{R}\underline{Spec}\, A \longrightarrow F$, we get a *dg*-functor $A \longrightarrow u^*(\alpha)$, and therefore an object $E_u \in \overline{u^*}(\alpha)$. Keeeping these notations we have the following definition.

Definition 4.4 Let F be a small derived stack together with a locally presentable *dg*-category α over F. An object $E \in D_\alpha(F)$ is a *a compact local generator* if for any commutative simplicial ring A, and any morphism

$$u : \mathbb{R}\underline{Spec}\, A \to F$$

the object E_u is compact generator for the triangulated *dg*-categor $[u^*(\alpha)]$.

Note that, despite the terminology, a compact local generator is not necessarily a compact object in $D_\alpha(F)$. Note also that a important point in the above definition is that E is globally defined, and that a compact local generator should not be confused with the notion of *local compact generators*, that would rather be compact generators defined locally on F. The question to know if local data of compact generators gives rise to a global compact local generator is the main question concerning the existence of derived Azumaya algebras, as we will see later.

The following result shows that, over an affine object, a compact generator is precisely the same thing as a compact local generator. Moreover, for a given global object, being a compact generator is a local property for the fpqc topology.

Lemma 4.5 *Let T be a locally presentable dg-category with descent over a commutative simplicial ring A.*

1. *An object $E \in T$ is a compact local generator if and only if it is a compact generator of the triangulated category $[T]$.*
2. *Let $A \longrightarrow B$ be a fpqc covering of commutative simplicial rings. Then, a compact object E of T is a compact generator for $[T]$ if and only if its base change $E \otimes_A^{\mathbb{L}} B$ is a compact generator for $[T \otimes_A^{ct} B]$.*

Proof (1) Assume first that E is a compact generator of $[T]$. We need to prove that for any morphism of simplicial commutative rings $A \to A'$, the object $E \otimes_A^{\mathbb{L}} A' \in T \otimes_A^{ct} A'$, obtained by base change, is a compact generator. For this we use the triangulated adjunction

$$- \otimes_A^{\mathbb{L}} A' : [T] \longleftrightarrow [T \otimes_A^{ct} A'] : f,$$

given on the one side from the adjunction dg-functor $T \longrightarrow T \otimes_A^{ct} A'$, and on the other side from the forgetful dg-functor $T \otimes_A^{ct} A' \longrightarrow T \otimes_A^{ct} A \simeq T$. The fact that these dg-functors are adjoints to each others simply follows from the usual adjunction of locally presentable A-dg-categories

$$- \otimes_A^{\mathbb{L}} A' : \widehat{A} \longleftrightarrow \widehat{A'} : f,$$

tensored over A by T (or it also follows from the explicit construction of $\otimes_A^{ct} A'$ obtained by writing T as a localisation of some $\widehat{T_0}$). We claim moreover that the right adjoint f is a conservative dg-functor. Indeed, write T as a localisation of some dg-categories $\widehat{T_0}$, for T_0 a small dg-category over A, and along a small set of maps S in $\widehat{T_0}$. Then the forgetful dg-functor f is simply the forgetful dg-functor

$$\widehat{T_0 \otimes_A^{\mathbb{L}} B} \longrightarrow \widehat{T_0}$$

restricted to S-local objects, which is clearly conservative. It is then a formal consequence of this adjunction, and of the fact that f is continuous and conservative, that $E \otimes_A^{\mathbb{L}} A'$ is a compact generator for $[T \otimes_A^{ct} A']$.

Conversely, is E is a compact local generator then, taking the identity $A \to A$ gives that E is a compact generator of $[T]$.

(2) We again use the adjunction

$$- \otimes_A^{\mathbb{L}} B : [T] \longleftrightarrow [T \otimes_A^{ct} B] : f.$$

We let T_E be the smallest sub-dg-category of T containing E and which is stable by colimits. By assumption on E the inclusion dg-functor $T_E \subset T$ induces an equivalence after base change

$$T_E \otimes_A^{ct} B \simeq T \otimes_A^{ct} B.$$

But both dg-categories T_E and T are locally presentable with descent over A, as this follows from Proposition 3.7 for the case of T_E and by assumption for T. Theorem 3.4 implies that the base change ∞-functor

$$- \otimes_A^{ct} B : \mathbb{D}g^{lp,desc}(A) \longrightarrow \mathbb{D}g^{lp,desc}(B)$$

is conservative. We thus have that the inclusion $T_E \subset T$ is an equivalence of *dg*-categories, or equivalently that E is a compact generator of $[T]$. □

The following proposition is a simple, but useful, observation.

Proposition 4.6 *Let F be any derived stack and α be a locally presentable dg-category over F. Then, α is realizable by a dg-algebra B_α over F, if and only if the triangulated category $D_\alpha(F)$ admits a compact local generator.*

Proof As we have defined the derived stack $\mathbb{D}g^{lp,desc}$, we can define the derived stack $\mathbb{D}g_*^{lp,desc}$, of *pointed locally presentable dg-categories with descent*. This derived stack simply classifies locally presentable *dg*-categories with descent T, together with an object $E \in T$, as well as continuous *dg*-functors that preserves this object. We have a sub derived prestack $\mathbb{D}g^{lp,desc,cp} \subset \mathbb{D}g_*^{lp,desc}$ of pointed *dg*-categories for which the distinguished object is a compact generator. We observe that the morphism ϕ factorizes as

$$\mathbb{D}g\mathbb{A}lg \longrightarrow \mathbb{D}g^{lp,desc,cp} \longrightarrow \mathbb{D}g^{lp,desc},$$

because for a *dg*-algebra B, the *dg*-category $\widehat{B^{op}}$ possesses a canonical compact generator given by B itself. Moreover, the induced morphism

$$\mathbb{D}g\mathbb{A}lg \longrightarrow \mathbb{D}g^{lp,desc,cp}$$

is an equivalence of derived prestacks (so in particular the right hand side is itself a stack).

Now, given $\alpha : F \longrightarrow \mathbb{D}g^{lp,desc}$, a factorization of α throught $\mathbb{D}g_*^{lp,desc}$ is precisely equivalent to the data of an object $E \in L_\alpha(F)$. Moreover, this object E is a compact local generator exactly when this factorization lands in the substack $\mathbb{D}g^{lp,desc,cp} \subset \mathbb{D}g_*^{lp,desc}$. □

The main existence statement concerning compact local generators is the following theorem.

Theorem 4.7 *Let X be quasi-compact and quasi-separated derived scheme, and α be a locally presentable dg-category with descent over X. We assume that there is a fppf covering $X' \longrightarrow X$, such that $D_\alpha(X')$ possesses a compact local generator. Then the triangulated category $D_\alpha(X)$ possesses a compact generator which is also a compact local generator.*

The proof of the previous theorem will take us some time and will be given is the next paragraphes: it will be a direct consequences of Propositions 4.9, 4.12 and 4.13 below. We however already state the following main corollary.

Corollary 4.8 *Let X be a quasi-compact and quasi-separated derived scheme, and α be a locally presentable dg-category with descent over X.*

Then, α possesses a compact local generator locally for the fppf topology on X if and only if it is realized by a dg-algebra B_α on X. In particular we have natural isomorphisms

$$dBr(X) \simeq dBr_{cat}(X) \simeq H^1_{fppf}(X, \mathbb{Z}) \times H^2_{fppf}(X, \mathbb{G}_m),$$

so any class in $H^2_{fppf}(X, \mathbb{G}_m)$ is realizable by a derived Azumaya algebra over X.

Proof Assume first that α possesses a compact local generator locally for the fppf topology on X, that is there exist a fppf covering $u : X' \longrightarrow X$, and an object $E \in D_{u^*(\alpha)}(X')$, which is a compact generator when restricted to any affine derived scheme over X'. Then Theorem 4.7 implies that $D_\alpha(X)$ possesses a compact local generator, which by Proposition 4.6 implies that α is realized by a dg-algebra B_α over X. Conversely, if α is realized by B_α then Proposition 4.6 tells us that $D_\alpha(X)$ admits a compact local generator.

The second part of the corollary, concerning deraz algebras, is a formal consequence of the first part. Indeed, we already know that the natural morphism

$$\phi : dBr(X) \longrightarrow dBr_{cat}(X),$$

is injective. The surjectivity of this morphism is equivalent to state that any locally dg-category α on X, which is locally (for the fppf topology) equivalent to the unit dg-category, is realized by a deraz algebra over X. By the corollary, such a dg-category α is realized by some dg-algebra B_α. This dg-algebra, is then locally on X_{fppf}, Morita equivalent to the unit dg-algebra, and thus is locally a deraz dg-algebra. Proposition 2.3(2) therefore implies that B_α is a deraz dg-algebra over X, which is an antecedant of α by the morphism ϕ.

Finally, the isomorphism $dBr_{cat}(X) \simeq H^1_{fppf}(X, \mathbb{Z}) \times H^2_{fppf}(X, \mathbb{G}_m)$, has been already proved in Corollary 3.12. □

4.2 Gluing compact generators for the Zariski topology

In this paragraph we start proving our Theorem 4.7 by first proving the following special case.

Proposition 4.9 *Let X be quasi-compact and quasi-separated derived scheme, and α be a locally presentable dg-category with descent over X. We assume that there is a Zariski covering $\{U_i\}$ of X, such that each $D_{\alpha_{U_i}}(U_i)$ possesses a compact local generator. Then the triangulated category $D_\alpha(X)$ possesses a compact generator which is also a compact local generator.*

Proof This is essentially the same proof as the proof of the existence of a compact generator for the quasi-coherent derived categories of quasi-compact and quasi-separated schemes, as this is done in [3].

We start by some local construction over affine derived schemes. Assume that $X := \mathbb{R}\underline{Spec}\, A$, for some commutative simplicial ring A, and that α is realized by a dg-algebra B over A. Let $U \subset X$ be a quasi-compact open affine derived sub-scheme. The sub-scheme U is given by certain elements $f_1, \ldots, f_n \in A_0$, as the union of the elementary opens

$$U_i := X_{f_i} = \mathbb{R}\underline{Spec}\, A[f_i^{-1}] \subset X.$$

We let $K = K(A, f)$ be the simplicial A-module obtained by freely adding 1-simplices h_i to A with the property

$$d_0(h_i) = f_i, \qquad d_1(h_i) = 0 \quad \forall i.$$

Considered as an A-dg-module, K is perfect, and thus a compact object in $D(A)$. It is moreover a compact generator for the full triangulated sub-category of $D(A)$ consisting of all A-dg-modules set-theorically supported on the closed sub-scheme $X - U$ (i.e. A-dg-modules E such that $E \otimes_A^{\mathbb{L}} A[f_i^{-1}] \simeq 0$ for all i), as shown by the following lemma.

Lemma 4.10 *The right orthogonal complement of K in $D(A)$ is generated, by homotopy colimits and limits by the objects $A[f_i^{-1}]$.*

Proof of the Lemma We denote by $D_U(A)$ the full sub-category of $D(A)$ generated by the $A[f_i^{-1}]$ and homotopy colimits and limits. It is the essential image of the direct image (fully faithful) functor

$$D(U) \longrightarrow D(X) \simeq D(A),$$

induced by the inclusion $U \hookrightarrow X$.

The A-dg-module K can be written as a tensor product

$$K \simeq \otimes_i^{\mathbb{L}} K_i,$$

where, for each i, K_i is the cone of the morphism $\times f_i : A \longrightarrow A$. We write

$$K_{\neq j} := \otimes_{i \neq j}^{\mathbb{L}} K_i.$$

The object $A[f_i^{-1}]$ is right orthogonal to K_i, and thus to K because of the formula for any $M \in D(A)$

$$\mathbb{R}\underline{Hom}_A(K, M) \simeq \mathbb{R}\underline{Hom}_A(K_{\neq i}, \mathbb{R}\underline{Hom}_A(K_i, M)).$$

As K is compact this implies that $D_U(A)$ is contained in the right orthogonal complement of K.

Conversely, assume that M is right orthogonal to K. There is a morphism

$$u : M \longrightarrow M',$$

with the property that M' belongs to $D_U(A)$, and that the cone C of u restricts to zero over the open U (take $M' = u_* u^*(M)$, where u^* and u_* are the inverse and direct image functors between $D(U)$ and $D(X)$).

As M and M' are both right orthogonal to K, so is C. We thus have

$$\mathbb{R}\underline{Hom}_A(K,C)\simeq \mathbb{R}\underline{Hom}_A(K_n,\mathbb{R}\underline{Hom}_A(K_{\neq n},C))\simeq 0.$$

This implies that $\mathbb{R}\underline{Hom}_A(K_{\neq n},C)$ is naturally a dg-module over $A[f_n^{-1}]$, and that we have

$$\begin{aligned}\mathbb{R}\underline{Hom}_A(K_{\neq n},C)&\simeq \mathbb{R}\underline{Hom}_A(K_{\neq n},C)\otimes^{\mathbb{L}}_A A[f_n^{-1}]\\&\simeq \mathbb{R}\underline{Hom}_A(K_{\neq n},C\otimes^{\mathbb{L}}_A A[f_n^{-1}])\simeq 0.\end{aligned}$$

By a descending induction on n we thus see that

$$\mathbb{R}\underline{Hom}_A(K_1,C)\simeq 0.$$

Therefore, we have

$$C\simeq C\otimes^{\mathbb{L}}_A A[f_1^{-1}]\simeq 0,$$

and thus $M=M'$, and $M\in D_U(A)$. This finishes the proof of the lemma. □

We now consider $B_{-K}:=B\otimes^{\mathbb{L}}_A K\in D(B)$, the free B-dg-module over K. By construction B_{-K} is a compact object in $D(B)$, which is a generator for the full sub-category $L_\alpha(X,K)$, of B-dg-modules which are set theoretically supported on $X-U$ (that is restrict to zero over U). We denote by $L_\alpha(X,U)$ the full sub-category of $L_\alpha(X)=\widehat{B^{op}}$, consisting of B-dg-modules $E\in L_\alpha(X)$ with $K\otimes^{\mathbb{L}}_A E\simeq 0$. As K is a perfect A-dg-module this is equivalent to say that $\mathbb{R}\underline{Hom}_{A\text{-}dg\text{-}mod}(K,E)\simeq 0$, that is that E is right orthogonal to B_{-K} in $L_\alpha(X)$. The sub-category $L_\alpha(X,U)$ therefore consists of all B-dg-modules whose underlying A-dg-module belongs to $D_U(A)\subset D(A)$.

We now consider the natural inverse image dg-functor

$$L_\alpha(X)\longrightarrow L_\alpha(U),$$

precomposed by the natural inclusion $L_\alpha(X,U)\subset L_\alpha(X)$. This provides a natural dg-functor

$$L_\alpha(X,U)\longrightarrow L_\alpha(U).$$

By definition of $L_\alpha(U)$, and by the gluing Lemma 4.2 we have

$$L_\alpha(U)\simeq Holim L_\alpha(X_f),$$

where the limit is taken over all elementary open affines

$$X_f=\mathbb{R}\underline{Spec}\,A[f^{-1}]\subset U\subset X.$$

We claim that the induced dg-functor

$$L_\alpha(X,U)\longrightarrow L_\alpha(U)\simeq Holim L_\alpha(X_f)$$

is an equivalence of dg-categories. This simply is the descent for quasi-coherent B-dg-modules for the Zariski topology, and can be either proven directly or deduced from the descent of quasi-coherent complexes as follows. We have natural equivalence of *dg*-categories

$$L_\alpha(X_f) \simeq \mathbb{R}\underline{Hom}(B, L(X_f)), \qquad L_\alpha(X,U) \simeq \mathbb{R}\underline{Hom}(B, L(U)),$$

where the internal Hom's are relative to the ∞-category of locally presentable *dg*-categories over A, and $L(-)$ stands for $L_{\mathbf{1}}(-)$, the *dg*-category of quasi-coherent complexes. We therefore have a commutative square of A-*dg*-categories

$$\begin{array}{ccc} L_\alpha(X,U) & \longrightarrow & Holim L_\alpha(X_f) \\ \sim\downarrow & & \downarrow\sim \\ \mathbb{R}\underline{Hom}(B, L(U)) & \longrightarrow & Holim\mathbb{R}\underline{Hom}(B, L(X_f)). \end{array}$$

The vertical morphisms are equivalences. The bottom horizontal morphism is also an equivalence as we have

$$L(U) \simeq Holim L(X_f),$$

because quasi-coherent complexes form a stack for the fpqc topology, and thus for the Zariski topology (see e.g. [31, Lemma 3.1]).

The consequence of this first local discussion is the existence of a semi-orthogonal decomposition of compactly generated locally presentable *dg*-categories

$$\langle B_{-K} \rangle \subset L_\alpha(X) \longrightarrow L_\alpha(U),$$

where $\langle B_{-K} \rangle \subset L_\alpha(X)$ is the full sub-*dg*-category generated by B_{-K}, colimits and shifts. In particular, the localisation theorem of Thomason-Neeman [15, Theorem 2.1], and [3, Corollary 3.2.3], applies. We therefore have that for any compact object $E \in L_\alpha(U)$, there exists a compact object $E' \in L_\alpha(X)$, whose image by

$$L_\alpha(X) \longrightarrow L_\alpha(U)$$

is $E \oplus E[1]$.

We now prove Proposition 4.9, by induction on the number of open affines needed to cover the derived scheme X. First of all if $X = \mathbb{R}\underline{Spec}\, A$ is affine, we have $L_\alpha(X) \simeq \widehat{B^{op}}$, for some A-*dg*-algebra B realizing α. Then B, as a *dg*-module over itself is both, a compact generator and a compact local generator thanks to Lemma 4.5.

Assume now that the proposition ((1) + (2)) holds for all quasi-compact and quasi-separated derived schemes which are union of m derived affine schemes, for $m > 0$. Let X be a quasi-compact derived scheme, covering by two open sub-schemes U and V, with V affine and U covered by m affine open sub-schemes. We let $W = U \cap V$. Lemma 4.2 tells us that we have a homotopy Cartesian square of *dg*-categories

$$\begin{array}{ccc} L_\alpha(X) & \longrightarrow & L_\alpha(U) \\ \downarrow & & \downarrow \\ L_\alpha(V) & \longrightarrow & L_\alpha(W). \end{array}$$

By the induction hypothesis, let E_U a compact generator of $L_\alpha(U)$ which is also a compact local generator. By what we have seen in the affine case, the image of $E_U \oplus E_U[1]$ in $L_\alpha(W)$ lifts to a compact object in $L_\alpha(V)$. Replacing E_U by $E_U \oplus E_U[1]$ we simply assume that there is a compact object in $L_\alpha(V)$ whose image in $L_\alpha(W)$ is equivalent to the image of E_U. By the homotopy Cartesian square above there is an object $E_X \in L_\alpha(X)$ whose image is equivalent to E_U in $L_\alpha(U)$. Moreover, because of the same homotopy Cartesian of *dg*-categories the object E_X has compact components in $L_\alpha(V)$, $L_\alpha(U)$ and $L_\alpha(W)$ and thus is easily seen to be itself compact.

The derived scheme V is affine, and W is a quasi-compact open in V. By what we have seen in the beginning of the proof the *dg*-functor

$$L_\alpha(V) \longrightarrow L_\alpha(W)$$

is a localization obtained by killing a compact object E_0 of $L_\alpha(V)$ (denoted by B_{-K} in our discussion). By construction, this compact object goes to zero in $L_\alpha(W)$, and is a compact generator for the full sub-*dg*-category of $L_\alpha(V)$ of objects sent to zero in $L_\alpha(W)$. The homotopy Cartesian square above implies that E_0 naturally lifts to a compact object $F_X \in L_\alpha(X)$, whose restriction to U is zero. We let

$$E := E_X \oplus F_X.$$

We claim that E is a compact generator of $L_\alpha(X)$, as well as a compact local generator. Indeed, an object E' is right orthogonal to E in $L_\alpha(X)$, if and only if it is right orthogonal to E_X and F_X. But, by construction, we have

$$L_\alpha(X)(F_X, E') \simeq 0 \times_0^h L_\alpha(V)(E_0, E'_{|V}) \simeq L_\alpha(V)(E_0, E'_{|V}).$$

Therefore, E' is right orthogonal to F_X if and only if its restriction $E'_{|V}$ to V is right orthogonal to E_0. By definition of E_0 this is in turn equivalent to the fact that $E'_{|V}$ is supported on the open $W \subset V$, or equivalently that for any object $F \in L_\alpha(V)$, the restriction morphism

$$L_\alpha(V)(F, E'_{|V}) \longrightarrow L_\alpha(W)(F_{|W}, E'_{|W})$$

is a quasi-isomorphism. In particular, we must have

$$\begin{aligned} L_\alpha(X)(E_X, E') &\simeq L_\alpha(U)(E_U, E'_{|U}) \times^h_{L_\alpha(W)(E_{|W}, E'_{|W})} L_\alpha(V)(E_{|V}, E'_{|V}) \\ &\simeq L_\alpha(U)(E_U, E'_{|U}). \end{aligned}$$

This implies that if E' is orthogonal to both, E_X and F_X, then we have $E'_{|U} = 0$, thus $E'_{|W} = 0$ and therefore $E'_{|V} = 0$ because $E'_{|V}$ is supported on W. This shows that E' must be zero, and that E is indeed a compact generator for $L_\alpha(X)$.

By construction, E is a compact generator such that $E_{|U} \simeq E_U$ is a compact local generator of $L_\alpha(U)$. Moreover, $E_{|V}$ is a compact object in $L_\alpha(V)$ with the following two properties:

1. its restriction to each open affine $V_f \subset W$ is a compact generator of $L_\alpha(V_f)$,
2. $E_{|V}$ contains E_0 as a direct factor.

We claim that these two properties, together with the fact that V is affine implies that $E_{|V}$ is a compact generator of $L_\alpha(V)$. Indeed, let us write $V \simeq \mathbb{R}\underline{Spec}\, A$, and $W = \bigcup V_{f_i}$, for a finite number of elements $f_i \in A_0$. Then, if an object F is right orthogonal to $E_{|V}$ in $L_\alpha(V)$, by the property (2) above the object F must be supported on W. Moreover, we have

$$L_\alpha(V)(E_{|V}, F) \otimes^{\mathbb{L}}_A A[f^{-1}] \simeq L_\alpha(V_f)(E_{|V_f}, F_{|V_f}).$$

Therefore, if F is right orthogonal to $E_{|V}$ we must have $F_{|V_f} = 0$, and as F is supported on W this implies that $F \simeq 0$. □

4.3 Gluing compact generators for the fppf topology

We will proceed in two steps, using our previous result about gluing compact generators for the Zariski topology Proposition 4.9. We start by proving that existence of compact generators locally for the étale topology implies the existence of compact generators locally for the Zariski topology. Then, we prove that existence of compact generators locally for the fppf topology implies the existence of compact generators locally for the étale topology. In both cases we will use the following lemma.

Lemma 4.11 *Let $A \longrightarrow B$ be a finite and faithfully flat morphism between commutative simplicial rings, and T be a locally presentable dg-category with descent over A. Then $[T]$ possesses a compact generator if and only if $[T \otimes^{ct}_A B]$ does so.*

Proof We use the adjunction

$$- \otimes^{\mathbb{L}}_A B : [T] \longrightarrow [T \otimes^{ct}_A B] : f.$$

The right adjoint f is easily seen to be conservative, as we already have seen this during the proof of Lemma 4.5(1). During the same proof we also have seen that $E \otimes^{\mathbb{L}}_A B$ is a compact generator of $[T \otimes^{ct}_A B]$ when E is a compact generator of $[T]$.

Conversely, assume that E is a compact generator of $[T \otimes_A^{ct} B]$. We claim that $f(E)$ is a compact generator of $[T]$. For this we use that f does itself have a right adjoint

$$f^! : [T] \longrightarrow [T \otimes_A^{ct} B].$$

This right adjoint is a priori not a continuous functor, but B being finite flat over A implies that it is continuous. Indeed, the direct image dg-functor

$$p : \widehat{B} \longrightarrow \widehat{A},$$

possesses a right adjoint

$$p^! : \mathbb{R}\underline{Hom}_A(B, -) \otimes^{\mathbb{L}} - : \widehat{A} \longrightarrow \widehat{B}.$$

This right adjoint commutes with direct sums because B is finite flat over A, thus projective and of finite type (see [33, Lemma 2.2.2.2]), and thus compact as a A-dg-module. In particular, if $B^\vee := \mathbb{R}\underline{Hom}_A(B, A)$ denotes the A-dg-module dual of B then we have

$$p(p^!(M)) \simeq B^\vee \otimes^{\mathbb{L}} M,$$

for all $M \in D(A)$. Tensoring the continuous dg-functor $p^!$ with T gives a continuous dg-functor

$$f^! : [T] \longrightarrow [T \otimes_A^{ct} B],$$

which is right adjoint to the direct image dg-functor f. It follows formally from the existence of $f^!$ that f preserves compact objects, and thus that $f(E)$ is compact in $[T]$. Moreover, $B^\vee$ is a faithfully flat A-dg-module, and thus $f \circ f^! \simeq B^\vee \otimes^{\mathbb{L}}$—is a conservative functor. As f is conservative, it follows that so is $f^!$. This, again formally, implies that $f(E)$ is a generator of $[T]$. □

We now show that local compact generators for the étale topology can be glued to get compact local generators for the Zariski topology. The proof of the proposition we give below is an adaptation of Gabber's proof of the existence of Azumaya algebras over affine schemes (see [6]).

Proposition 4.12 *Let $A \longrightarrow B$ be a étale covering of commutative simplicial rings, and T be a locally presentable dg-category over A. Then $[T]$ possesses a compact generator if and only if $[T \otimes_A^{ct} B]$ does so.*

Proof By Lemma 4.5 the base change of a compact generator for $[T]$ is a compact generator for $[T \otimes_A^{ct} B]$. So let E be a compact generator for $[T \otimes_A^{ct} B]$. We are allowed to refine our étale covering $A \to B$. Moreover, using Proposition 4.9 we are also allowed to work locally for the Zariski topology on A. We can therefore assume that our étale morphism $A \longrightarrow B$ is a standard étale morphism: there exists a commutative diagram of commutative simplicial rings

$$\begin{array}{ccc} A & \xrightarrow{p} & C \\ & \searrow & \downarrow{\scriptstyle j} \\ & & B, \end{array}$$

where j is a Zariski open immersion (that can be taken to be principal, that is of the form $C[c^{-1}]$ for some $c \in \pi_0(C)$), and where C is equivalent to $A[X]/p$, for p a monic polynomial in $\pi_0(A)$. Before going further let us give more details about $C = A[X]/p$. Here $A[X]$ is simply the free commutative simplicial A-algebra, and p is a monic polynomial over the ring $\pi_0(A)$

$$p = x^d + \sum_i x^i a_i \in \pi_0(A)[X].$$

As homotopy classes of morphisms $A[X] \to A'$ are in one-to-one correspondence with $\pi_0(A')$, we can represent p by a morphism of commutative simplicial rings

$$p : A[X] \longrightarrow A[X]$$

well defined in the homotopy category of commutative simplicial rings. By definition $A[X]/p$ is the commutative simplicial ring fitting in the following homotopy cocartesian square

$$\begin{array}{ccc} A[X] & \xrightarrow{p} & A[X] \\ \downarrow & & \downarrow \\ A & \longrightarrow & A[X]/p. \end{array}$$

We now consider the following morphism of commutative rings

$$\mathbb{Z}[X_1, \ldots, X_d]^{\Sigma_d} \hookrightarrow \mathbb{Z}[X_1, \ldots, X_d].$$

The left hand side is isomorphic to a polynomial ring $\mathbb{Z}[T_1, \ldots, T_d]$, where the identification is by sending T_i to the ith elementary function on the set of variables X_j

$$T_i \mapsto \phi_i(X) = (-1)^{d-i} \sum_{n_1+\cdots+n_d=i} X_1^{n_1} \cdots X_d^{n_d}.$$

The polynomial p is determined by a morphism of commutative simplicial rings (again, well defined in the homotopy category $Ho(s\mathbb{Z}\text{-}CAlg)$)

$$\mathbb{Z}[T_1, \ldots, T_d] \longrightarrow A,$$

sending T_i to the element $a_{i-1} \in \pi_0(A)$. We form the homotopy push-out

$$\begin{array}{ccc} \mathbb{Z}[T_1, \dots, T_d] & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathbb{Z}[X_1, \dots, X_d] & \longrightarrow & C'. \end{array}$$

As $\mathbb{Z}[X_1, \dots, X_d]$ is finite and free as a $\mathbb{Z}[T_1, \dots, T_d]$-module, $A \to C'$ is a finite and flat morphism of commutative simplicial rings, coming equiped with a natural action of Σ_d. There exists a natural commutative diagram, in $Ho(s\mathbb{Z}\text{-}CAlg)$

$$\begin{array}{ccc} A & \longrightarrow & C \\ & \searrow & \downarrow \\ & & C'. \end{array}$$

To see this, let $q \in A_0[X]$ be a lift of $p \in \pi_0(A)[X]$. Then, the datum of a morphism in $Ho(s\mathbb{Z}\text{-}CAlg)$

$$C \longrightarrow C',$$

is equivalent to the data of a pair (c, h), where $c \in C'_0$, $h \in C'_1$ with

$$d_0(h) = q(c), \qquad d_1(h) = 0,$$

two such data (x, h) and (y, k) considered as being equivalent if they are homotopic (in a rather obvious meaning). The morphism of commutative simplicial A-algebras $C \longrightarrow C'$ is then obtained by chosing, say, the image x_1, of X_1 in C'_0, which satisfies by construction that $q(x_1)$ is natural homotopic to zero. In fact, by the construction of C', the image of the polynomial q in C'_0 comes equiped with a natural homotopy to the polynomial $\prod_i (X - x_i)$.

We now base change the Zariski open $j : C \longrightarrow B$ to get another Zariski open

$$C' \longrightarrow B' := C' \otimes^{\mathbb{L}}_C B.$$

By hypothesis $[T \otimes^{ct}_A B]$ has a compact generator, which by Lemma 4.5 implies that $[T \otimes^{ct}_A B']$ also possesses a compact generator. By using the action of Σ_d on C', by autormophisms over A, we obtain a finite family of Zariski open

$$C' \longrightarrow B'_\sigma, \quad \sigma \in \Sigma_d.$$

This family of Zariski open has the property that each triangulated category $[T \otimes^{ct}_A B'_\sigma]$ possesses a compact generator.

We now let $T' := T \otimes^{ct}_A C'$, which is a locally presentable dg-category with descent over C'. It is such that each $T' \otimes^{ct}_{C'} B'_\sigma$ has a compact generator. Moreover, the Zariski open B'_σ form a Zariski open covering of C'. This can be checked on the level of geometric points, and thus becomes a non simplicial statement. Indeed, let $u : C' \longrightarrow L$ be morphism with L an algebraically closed field. By definition of C' the morphism u is determined by the data of elements $\alpha_i \in L$ such that $p(X) = \prod_i (X - \alpha_i)$, where p is considered as a polynomial with coefficients in L using the morphism $A \to C' \to L$. The point $A \to L$ lifts to a point $B \to L$ (because $A \to B$ is an étale covering), and thus to a point $v : C \to L$. The morphism v is determined by $\alpha \in L$ which is a root of $p(X)$, which must be equal to α_i for some i. If we chose $\sigma \in \Sigma_d$ with $\sigma^{-1}(i) = 1$, then we have a commutative diagram

$$\begin{array}{ccccc} C & \longrightarrow & B & \longrightarrow & L \\ \downarrow & & & \nearrow & \\ C' & & u & & \\ \downarrow{\scriptstyle \sigma} & \nearrow & & & \\ C', & & & & \end{array}$$

where the morphism at the top is the point v. This implies that σ^{-1} sends the Zariski open B' of C' to a new Zariski open containing the point u. This finishes to show that B'_σ form a Zariski covering of C' such that $[T' \otimes_A^{ct} B'_\sigma]$ possesses a compact generator. We then conclude using Proposition 4.9 that T' has a compact generator, which by Lemma 4.11 implies that T also has a compact generator. □

We now finish the last step of the proof of our Theorem 4.7 by proving that the local compact generators for the fppf topology can be glued to a local compact generator for the Zariski topology, and thus to a global compact generator by our previous Proposition 4.9.

Proposition 4.13 *Let $A \longrightarrow B$ be a fppf covering of commutative simplicial rings, and T be a locally presentable dg-category over A. Then $[T]$ possesses a compact generator if and only if $[T \otimes_A^{ct} B]$ does.*

Proof As for the proofs of Propositions 4.9 and 4.12 one implication is already known. So let E be a compact generator for $[T \otimes_A^{ct} B]$. From [30, Lemma 2.14] (only the easiest affine case is needed here) we know the existence of a smooth covering $p : X \to \mathbb{R}\underline{Spec}\, A$, classifying quasi-smooth quasi-sections of $A \to B$. Taking local sections of p for the étale topology, we do get the existence of a commutative diagram of commutative simplicial rings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B', \end{array}$$

where $A \to A'$ is an étale covering, and $A' \to B'$ is a finite flat morphism of a certain rank m (and we have moreover that $B \to B'$ is a quasi-smooth morphism, but we will not use this additional property). Therefore, using Lemma 4.5 and the hypothesis, $[T \otimes_A^{ct} B']$ possesses a compact generator. By the Lemma 4.11 applied to the finite flat map $A' \to B'$, we have that $[T \otimes_A^{ct} A']$ also have a compact generator, which by Proposition 4.12 implies that T has a compact generator. This finishes the proof of the proposition. □

5 Applications and complements

5.1 Existence of derived Azumaya algebras

In this subsection we refine a bit the existence statement for derived Azumaya algebras Corollary 4.8, by also considering certain algebraic spaces and stacks.

Corollary 5.1 *Let X be a derived stack. Then, the natural map*

$$dBr(X) \longrightarrow H^1_{et}(X, \mathbb{Z}) \times H^2_{et}(X, \mathbb{G}_m)$$

is bijective in each of the following cases.

1. *X is a quasi-compact and quasi-separated derived scheme.*
2. *X is a smooth and separated algebraic space of finite type over a field of characteristic zero k.*
3. *X is a quasi-compact, separated and derived Deligne-Mumford stack whose coarse moduli space is a derived scheme.*

Proof (1) This is our Corollary 4.8. For (2), let $p : X' \longrightarrow X$ be a proper birational morphism with X' smooth. As X is smooth we have $H^1_{et}(X, \mathbb{Z}) = 0$. Let $\alpha \in H^2_{et}(X, \mathbb{G}_m)$, that we represent by a locally presentable dg-category over X. We have a pull-back functor

$$p^* : L_\alpha(X) \longrightarrow L_\alpha(X').$$

This functor possesses a right adjoint

$$p_* : L_\alpha(X') \longrightarrow L_\alpha(X),$$

which is fully faithful, as this easily follows from the fact that X has rational singularities (and thus $\mathbb{R}p_*(\mathcal{O}_{X'}) \simeq \mathcal{O}_X$). Therefore, $L_\alpha(X)$ becomes a direct summand (in the sense of orthogonal decomposition) of $L_\alpha(X')$. It is easy to deduce from this that the image by p_* of a compact generator of $L_\alpha(X')$ is a compact generator of $L_\alpha(X)$. In the same way, the image by p_* of a compact local generator of $L_\alpha(X')$ is a compact local generator of $L_\alpha(X)$, by using a base change to a affine scheme U étale over X. As we know that such compact local generator exists for $L_\alpha(X')$ we see that $L_\alpha(X)$ has a compact local generator, and thus that α is represented by a derived Azumaya algebra over X.

We now consider the case (3). We let $\pi : X \longrightarrow M$ be the projection of X to its coarse moduli space M, which by assumption is a quasi-compact and separated derived scheme. We let $\alpha \in H^1_{et}(X, \mathbb{Z}) \times H^2_{et}(X, \mathbb{G}_m)$, that we represent by a locally presentable dg-category α over X. We then consider $\pi_*(\alpha)$, which is a locally presentable dg-category over M defined by

$$(U \mapsto M) \mapsto L_\alpha(\pi^{-1}(U)).$$

We have

$$L_\alpha(X) \simeq L_{\pi_*(\alpha)}(M).$$

Now, locally for the étale topology on M, the derived stack X is equivalent to a quotient stack $[U/G]$, for G a finite group acting on $U = \mathbb{R}\underline{Spec}\, A$ a derived affine scheme. More precisely, there is an étale covering $\{V_i \longrightarrow M\}$, and pull-back diagramms

$$\begin{array}{ccc} [U_i/G_i] & \longrightarrow & X \\ \downarrow & & \downarrow \\ V_i & \longrightarrow & M, \end{array}$$

with $[U_i/G_i]$ as above ($U_i \simeq \mathbb{R}\underline{Spec}\, A_i$ and G_i a finite group acting on U_i). If we let $B_i := A_i[G_i]$, the twisted group simplicial algebra of A_i by G_i, then we have

$$L_\alpha([U_i/G_i]) \simeq \widehat{B_i^{op}}.$$

This shows that the locally presentable dg-category $\pi_*(\alpha)$ on M possesses compact generators locally for the étale topology on M. Theorem 4.7 tells us that $L_{\pi_*(\alpha)}(M) \simeq L_\alpha(X)$ possesses a compact local generator, and thus that α is represented by a deraz algebra over X. □

The previous corollary implies the existence of derived Azumaya algebras in the two following concrete examples.

The first situation is the famous example of Mumford, of a normal local $\mathbb{C}$-algebra A of dimension 2, with a non-torsion class $\alpha \in H^2_{fppf}(Spec\, A, \mathbb{G}_m)$ (see [7]). This class is then realized by an A-dg-algebra B, which, as α is non-torsion, is not Morita equivalent to an Azumaya non-derived algebra. It seems an interesting question to write down explicit derived Azumaya algebras, say by generators and relations, representing the class α.

For our second example, let X be any smooth and proper scheme over some ring of characteristic zero k. We assume that there is a class $\alpha \in H^2(X, \mathcal{O})$ which is non-zero. We see the class α as a first order deformation of the trivial class in $H^2_{fppf}(X, \mathbb{G}_m)$, and thus consider it as a class in $H^2_{fppf}(X[\epsilon], \mathbb{G}_m)$ that restricts to zero in $H^2_{fppf}(X, \mathbb{G}_m)$. Corollary 5.1 tells us that the class α is then realized by a derived Azumaya algebra B over $X[\epsilon]$, whose restriction to X is Morita equivalent to $\mathcal{O}$. Again, as the class α is non-zero it is non-torsion, and the deraz algebra B is thus not Morita equivalent to an Azumaya algebra.

An important corollary of Corollary 5.1, or rather of the proof of point (2), is the following existence statement of compact generator for derived categories of Deligne-Mumford stacks.

Corollary 5.2 *Let X be a quasi-compact and separated Deligne-Mumford stack whose coarse moduli space is a scheme. Then the derived category $D(X)$ of quasi-coherent sheaves on X admits a compact generator.*

5.2 Localization theorem for twisted K-theory

We present here an application of the existence of derived Azumaya algebras to twisted K-theory, by deducing the existence of a localization exact triangle for the K-theory of twisted perfect complexes. For this we need to introduce some notations. Let α be a locally presentable dg-category over some derived stack X. The compact object in $L_\alpha(X)$ form a small triangulated dg-category $L_\alpha(X)_c$, of which we can construct its non-connective K-theory spectrum (see [18])

$$K_\alpha(X) := K(L_\alpha(X)_c).$$

For instance, when $\alpha = \mathbf{1}_X$ is the unit dg-category over X, then $K_\alpha(X)$ is nothing else than the K-theory spectrum of perfect complexes over the derived stack X.

We will also use the following notation: if Y is a closed sub-stack of X we note $L_\alpha(X, Y)$ the full sub-dg-category of $L_\alpha(X)$ of all objects that restricts to zero over the open complement $X - Y$. We thus have a sequence of dg-categories

$$L_\alpha(X, Y)_c \longrightarrow L_\alpha(X)_c \longrightarrow L_\alpha(X - Y)_c,$$

which in general is only exact on the left. Exactness on the right (i.e. the fact that any object in $L_\alpha(X - Y)_c$ is a retract of a the image of an object in $L_\alpha(X)_c$) is precisely what is needed in order to apply Thomason's localization theorem to obtain an exact triangle on the level of K-theory spectra.

Corollary 5.3 *Let X be a separated and quasi-compact derived Deligne-Mumford stack whose coarse moduli space is a derived scheme. Let α be a locally presentable dg-category over X, which locally for the étale topology* *on X possesses a compact generator. Let Y be a finitely presented closed sub-stack in X, then the sequence of dg-categories*

$$L_\alpha(X,Y)_c \longrightarrow L_\alpha(X)_c \longrightarrow L_\alpha(X-Y)_c$$

induces an exact triangle on the corresponding (non-connective) K-theory spectra

$$K_\alpha(X,Y) \longrightarrow K_\alpha(X) \longrightarrow K_\alpha(X-Y).$$

Proof We have to prove that the sequence of *dg*-categories

$$L_\alpha(X,Y)_c \longrightarrow L_\alpha(X)_c \longrightarrow L_\alpha(X-Y)_c$$

is exact, that is the right hand side identifies itself naturally with the quotient *dg*-category $L_\alpha(X)_c/L_\alpha(X,Y)_c$, where this quotient has to be understood in the homotopy theory of small *dg*-categories up to Morita equivalences (see [23]).

Let $\pi : X \longrightarrow M$ be the projection of X to its coarse moduli space, which by assumption is a quasi-compact and separated derived scheme. By pushing α down to M, and noticing that $L_\alpha(X) \simeq L_{\pi_*(\alpha)}(M)$ (as well as with versions with supports), we see that we can in fact assume that $X = M$ and that X is in fact a derived scheme. By Theorem 4.7 α is realized by a sheaf of *dg*-algebra B over X with quasi-coherent cohomology. The *dg*-categories $L_\alpha(X)$ and $L_\alpha(X-Y)$ are then naturally equivalent to the *dg*-categories of B-*dg*-modules over X with quasi-coherent cohomology, and B-*dg*-modules over $X-Y$ with quasi-coherent cohomology. The pull-back functor

$$j^* : L_\alpha(X) \longrightarrow L_\alpha(X-Y)$$

then possesses a fully faithful right adjoint

$$j_* : L_\alpha(X-Y) \longrightarrow L_\alpha(X),$$

simply induced by push-forward on the level of complexes with quasi-coherent cohomology. This implies that we have a semi-orthogonal decomposition of locally presentable *dg*-categories

$$L_\alpha(X,Y) \longrightarrow L_\alpha(X) \longrightarrow L_\alpha(X-Y).$$

We know by our Theorem 4.7 that $L_\alpha(X)$ and $L_\alpha(X-Y)$ are compactly generated, and it is easy to see that the *dg*-functor j^* preserves compact objects. The kernel $L_\alpha(X,Y)$ is also compactly generated as another application of Theorem 4.7 to α_Y, the kernel of the natural morphism

$$\alpha \longrightarrow j_*j^*(\alpha),$$

of locally presentable *dg*-categories over X, which we have seen to have compact generators locally for the Zariski topology on X (it was denoted B_{-K} in the proof of Proposition 4.9). It follows formally from this that the sequence induced on compact objects

$$L_\alpha(X,Y)_c \longrightarrow L_\alpha(X)_c \longrightarrow L_\alpha(X-Y)_c$$

is an exact sequence in the Morita theory of small *dg*-categories. □

5.3 Direct images of smooth and proper categorical sheaves

In [34] we have introduced a certain class of categorical sheaves, as categorified versions of vector bundles. By definition these categorical sheaves are locally presentable *dg*-categories over schemes, which locally for the Zariski topology are smooth and proper in the sense of our Proposition 2.5. The result of the present paper have several important consequences concerning functoriality properties for these categorical sheaves. The following result states various properties of direct images of categorical sheaves along certain kind of morphisms. Probably the most important case is that smooth and proper locally presentable *dg*-categories are stable by direct images along smooth and proper morphisms. This gives a general tool in order to construct smooth and/or proper *dg*-categories by integration of smooth and/or proper *dg*-categories over smooth and/or proper schemes.

Corollary 5.4 *Let X be a quasi-separated and quasi-compact derived scheme, equiped with a morphism $X \longrightarrow \mathbb{R}\underline{Spec}\,k$, for some commutative simplicial ring k. Let α be a locally presentable dg-category over X, and assume that there is an étale covering $U = \mathbb{R}\underline{Spec}\,A \longrightarrow X$, such that $L_\alpha(U)$ possesses a compact generator, and therefore is equivalent to $\widehat{B^{op}}$ for some A-dg-algebra B.*

1. *The dg-category $L_\alpha(X)$ possesses a compact generator which is also a compact local generator.*
2. *If B is smooth as an A-dg-algebra, and if X is smooth over k, then the compactly generated dg-category $L_\alpha(X)$ is smooth over k.*
3. *If B is a proper A-dg-algebra (in the sense of Proposition 2.5), and if X is proper and flat over k, then the compactly generated dg-category $L_\alpha(X)$ is proper over k.*
4. *If B is saturated as an A-dg-algebra (i.e. smooth and proper), and if X is smooth and proper over k, then $L_\alpha(X)$ is saturated as a k-dg-category.*

Proof (1) This is Theorem 3.4.

(2) By (1) there is a compact local generator in $L_\alpha(X)$, and thus $L_\alpha(X)$ can be written as the *dg*-category of *dg*-modules with quasi-coherent cohomology over a *dg*-algebra B_α over X. We thus have

$$L_\alpha(X) \simeq \underset{\mathbb{R}\underline{Spec}\,A' \to X}{Holim} \widehat{B^{op}_{A'}},$$

where $B_{A'}$ is the A'-dg-algebra which is the restriction of B_α over $\mathbb{R}\underline{Spec}\, A'$. As X is quasi-compact and quasi-separated this limit can be taken to be a finite limit, corresponding to a finite hypercoverings of X by affine open subschemes.

As the compact local generator is also a compact generator, we have

$$L_\alpha(X) \simeq \widehat{B_X^{op}},$$

where $B_X := Holim B_{A'}$ is the dg-algebra of global sections of B over X. We need to prove that B_X is a smooth dg-algebra over k.

Lemma 5.5 *With the notation as above, the natural dg-functor*

$$\widehat{B_X} \longrightarrow \underset{\mathbb{R}\underline{Spec}\, A' \to X}{Holim} \widehat{B_{A'}}$$

is a quasi-equivalence.

Proof of the Lemma We start by the quasi-equivalence

$$\widehat{B_X^{op}} \longrightarrow \underset{\mathbb{R}\underline{Spec}\, A' \to X}{Holim} \widehat{B_{A'}^{op}}.$$

All the dg-categories envolved in this equivalence are compactly generated dg-categories. Moreover, all the dg-functors envolved in the above diagram are compact dg-functors, that is dg-functors preserving compact objects. As the homotopy limit on the right hand side is finite, the compact objects of $Holim_{\mathbb{R}\underline{A'}\to X} \widehat{B_{A'}^{op}}$ form a dg-category equivalent to $Holim_{\mathbb{R}\underline{Spec}\, A' \to X}(\widehat{B_{A'}^{op}})_c$, where $(\widehat{B_{A'}^{op}})_c$ is the full sub-dg-category of compact objects in $\widehat{B_{A'}^{op}}$. We thus have a quasi-equivalence

$$(\widehat{B_X^{op}})_c \simeq \underset{\mathbb{R}\underline{Spec}\, A' \to X}{Holim} (\widehat{B_{A'}^{op}})_c.$$

Passing to opposite dg-categories we find a new quasi-equivalence of dg-categories

$$(\widehat{B_X})_c \simeq (\widehat{B_X^{op}})_c^{op} \simeq \left(\underset{\mathbb{R}\underline{Spec}\, A' \to X}{Holim} (\widehat{B_{A'}^{op}})_c\right)^{op} \simeq \underset{\mathbb{R}\underline{Spec}\, \underline{A'} \to X}{Holim} (\widehat{B_{A'}})_c.$$

Now, again because the homotopy limit is finite we have

$$\left(\underset{\mathbb{R}\underline{Spec}\, \underline{A'} \to X}{Holim} (\widehat{B_{A'}})\right)_c \simeq \underset{\mathbb{R}\underline{Spec}\, \underline{A'} \to X}{Holim} (\widehat{B_{A'}})_c.$$

The conclusion is then that the natural dg-functor

$$\phi : \widehat{B_X} \longrightarrow \underset{\mathbb{R}\underline{Spec}\, A' \to X}{Holim} \widehat{B_{A'}},$$

induces a quasi-equivalence on the full sub-*dg*-categories of compact objects. By Theorem 3.4, applied to the locally presentable *dg*-category over X realized by the *dg*-algebra B_α^{op}, the *dg*-category $Holim_{\mathbb{R}\underline{Spec}\, A'\to X}\widehat{B_{A'}}$ is compactly generated. This implies that ϕ is thus a quasi-equivalence. □

According to Lemma 5.5, we have

$$B_X \widehat{\otimes_k^{\mathbb{L}}} B_X^{op} \simeq \underset{\mathbb{R}\underline{Spec}\, A'\to X}{Holim} B_{A'} \widehat{\otimes_k^{\mathbb{L}}} B_{A'}^{op} \simeq L_{\alpha^\vee \otimes_k^{ct} \alpha}(X \times_k X),$$

where $\alpha^\vee \otimes_k^{ct} \alpha$ is the external product of α by its dual $\alpha^\vee$ (realized by B_α^{op}) on the derived stack $X \times_k X$. In other words, $B_X \widehat{\otimes_k^{\mathbb{L}}} B_X^{op}$ is naturally equivalent to the *dg*-category of *dg*-modules over $B_\alpha \otimes_k^{\mathbb{L}} B_\alpha^{op}$, which is itself a *dg*-algebra over $X \times_k X$. The diagonal bi-*dg*-module B_X in $B_X \widehat{\otimes_k^{\mathbb{L}}} B_X^{op}$ corresponds by these equivalences to B_α, considered as a *dg*-module over $B_\alpha \otimes_k^{\mathbb{L}} B_\alpha^{op}$, which is itself a *dg*-algebra over $X \times_k X$. As X is strongly of finite presentation over k, its quasi-coherent cohomology commutes with direct sums. As a consequence, to show that B_α is a compact object in $L_{\alpha^\vee \otimes_k^{ct} \alpha}(X \times_k X)$, it is enough to show that it is locally, for the étale topology on $X \times_k X$, a compact object. We can therefore replace X by the étale covering $U = \mathbb{R}\underline{Spec}\, A \to X$, which is affine and smooth over k.

We thus assume that $X = \mathbb{R}\underline{Spec}\, A$ is affine and smooth over k, and that α is realized by a smooth A-*dg*-algebra B. In order to finish the proof of (2) in this special case we need to show that B is moreover smooth over k. For this, we write the following base change formula

$$B \otimes_A^{\mathbb{L}} B^{op} \simeq (B \otimes_k^{\mathbb{L}} B^{op}) \otimes^{\mathbb{L}}_{A \otimes_k^{\mathbb{L}} A} A.$$

As A is smooth over k, it is a compact object in $D(A \otimes_k^{\mathbb{L}} A)$, and the previous formula shows then that $B \otimes_A^{\mathbb{L}} B^{op}$ is compact in $D(B \otimes_k^{\mathbb{L}} B^{op})$, or in other words that the direct image functor

$$D(B \otimes_A^{\mathbb{L}} B^{op}) \longrightarrow D(B \otimes_k^{\mathbb{L}} B^{op})$$

preserves compact objects. In particular it does send B itself to a compact object, showing that B is compact in $D(B \otimes_k^{\mathbb{L}} B^{op})$.

To finish the proof of the corollary we need to prove (3), (4) being a direct consequence of (3) and (2). As we saw, we can write $L_\alpha(X) \simeq \widehat{B_X^{op}}$, for B_X the *dg*-algebra of global section of B_α. By assumption B_α is a perfect complex on X, and thus its global sections form a perfect k-*dg*-module because X is proper and flat over k. □

Remark 5.6 With a bit of more work, but using essentially the same arguments, we can extend Corollary 5.4 to the case where X is more generally a seperated quasi-compact tame Deligne-Mumford derived stack, assuming that its coarse moduli space is a scheme. We get in particular that the *dg*-category $L(X)$, of quasi-coherent complexes on X, is smooth (resp. proper) over k when X is so. In particular, when X is a smooth and proper Deligne-Mumford derived k-stack, whose coarse moduli space is a derived scheme, the *dg*-category $L(X)$ is saturated, and therefore [31, Theorem 3.6] can be applied in order to prove the existence of a locally algebraic derived stacks, locally of finite presentation over k, $\mathcal{M}_X$, classifying perfect complexes on X. For instance, when X is a μ_n-gerbe over a smooth and proper k-scheme M, we deduce the existence of a moduli stack of twisted sheaves and twisted perfect complexes on M.

Appendix: A descent criterion

A.1 Relative limits

We let I be any big category (see our conventions for universes in our section on locally presentable *dg*-categories, Sect. 3.1) and we consider ∞-Cat/I, the category of (big) ∞-categories over I. Given $\mathcal{C} \to I$ and $\mathcal{D} \to I$ two ∞-categories over I, we can form the ∞-category of relative ∞-functors

$$\mathbb{R}\underline{Hom}_I(\mathcal{C}, \mathcal{D}).$$

This ∞-category is characterized, up to equivalence, by the following universal property: for any ∞-category A, we have functorial bijections

$$[A, \mathbb{R}\underline{Hom}_I(\mathcal{C}, \mathcal{D})] \simeq [A \times \mathcal{C}, \mathcal{D}]_I,$$

where $[-,-]$ denotes the morphisms sets in the homotopy category of ∞-categories, and $[-,-]$ the morphism sets in the homotopy category of ∞-categories over I. Using functorial fibrant replacement of Segal categories it is possible to define a strictly functorial model for $\mathbb{R}\underline{Hom}_I(\mathcal{C}, \mathcal{D})$, by considering the Segal category of morphisms over I,

$$\underline{Hom}_I(\mathcal{C}, \mathcal{D}'),$$

where $\mathcal{D}' \to I$ is a functorial fibrant replacement of the projection $\mathcal{D} \to I$.

We now fix $\mathcal{C} \longrightarrow I$, an ∞-category over I. Let J be a small category, and denote by J_+ the cone over J, obtained from J by adding a new initial object $+$ to J (with no morphisms from an object of J to $+$).

Definition A.1 An ∞-category over I, $\mathcal{C} \longrightarrow I$, *has small relative limits along* J the ∞-functor

$$\mathbb{R}\underline{Hom}(J_+, \mathcal{C}) \longrightarrow \mathbb{R}\underline{Hom}(J, \mathcal{C}) \times^h_{\mathbb{R}\underline{Hom}(J,I)} \mathbb{R}\underline{Hom}(J_+, I)$$

possess a fully faithful right adjoint.

The right adjoint of the previous definition will be denoted by

$$Lim_{J/I} : \mathbb{R}\underline{Hom}(J, \mathcal{C}) \times^h_{\mathbb{R}\underline{Hom}(J,I)} \mathbb{R}\underline{Hom}(J_+, I) \longrightarrow \mathbb{R}\underline{Hom}(J_+, \mathcal{C}),$$

and is called the *relative limit ∞-functor*.

Unfolding the previous definition gives the following equivalent definition: $\mathcal{C} \to I$ has relative limits along J if for any commutative diagram in the ∞-category of ∞-categories

$$\begin{array}{ccc} J & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ J_+ & \longrightarrow & I, \end{array}$$

there exists a lift $J_+ \longrightarrow \mathcal{C}$, in the ∞-category of ∞-categories, which is a final object in the ∞-category of all possible lifts.

Suppose that $I = *$, then an ∞-category $\mathcal{C}$, trivially considered over $*$, has relative limits along J if and only if it has limits along J. Another extreme case is when I is arbitrary but $J = *$. Then it is not hard to see that $\mathcal{C} \longrightarrow I$ has relative limits along $*$ if and only if it is a fibered ∞-category in the sense of [35, Sect. 1.3]. Finally, if $i \in I$, and if we consider the obvious functor $J_+ \to \{i\} \subset I$, we see that if $\mathcal{C} \to I$ has relative limits along J, then its fiber $\mathcal{C}_i$ at i is an ∞-category which admits limits along J. These two cases essentially cover anything that can happen, as shown in the next lemma.

Lemma A.2 *Let $\mathcal{C} \to I$ be an ∞-category over I. Then $\mathcal{C}$ has relative limits along all small categories J if and only if it satisfies the following two conditions.*

1. *The ∞-category $\mathcal{C}$ is fibered over I.*
2. *For any object $i \in I$, the ∞-category $\mathcal{C}_i$, fiber at i, possesses all small limits.*

Proof We already have seen that the two conditions are necessary. Assume that they hold. Let J be a small category, and let

$$\begin{array}{ccc} J & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ J_+ & \longrightarrow & I, \end{array}$$

be a commutative diagram of ∞-categories. We must show that a final objects in the ∞-category of lifts $J_+ \longrightarrow \mathcal{C}$ exists. By pulling-back everything over J_+ we can in fact assume that $J_+ = I$. As $\mathcal{C}$ is fibered, it can be written as $\int F$, for some functor $F : I^{op} \longrightarrow \infty\text{-}Cat$. The initial object $+$ of I induces a natural transformation

$$F \longrightarrow ct(F(i_0)),$$

from F to the constant presheaf with values $F(i_0) \simeq \mathcal{C}_{i_0}$. By passing to the Grothendiek integral we get an ∞-functor over I

$$\phi : \mathcal{C} \longrightarrow \mathcal{C}_{i_0} \times I.$$

Precomposing with $J \to \mathcal{C}$, we get an ∞-functor

$$J \longrightarrow \mathcal{C}_{i_0},$$

and we let $x_0 \in \mathcal{C}_{i_0}$ its limit. By construction, x_0 comes equiped with a morphism towards the diagram $J \to \mathcal{C}_{i_0}$, and thus defines a lift

$$J_+ \longrightarrow \mathcal{C},$$

which, as x_0 is a limit, is final among all possible lifts. □

Remark A.3 There is a dual notion of relative colimits, that we leave to the reader. We note also that every we said so far about relative limits remain valid when the base category I is replaced by an ∞-category (see [35, Sect. 1.3]).

A.2 The stack of stacks

Let T be a (very big) ∞-topos in the sense of [35, Sect. 1.5]. We will typically apply the constructions and results of this part to the case where T is the ∞-topos of big stacks over the big model site $(s\mathbb{Z}\text{-}CAlg)^{op}$, of small commutative simplicial rings endowed with the fppf topology. According to our conventions about universes T will therefore be a very big ∞-topos.

We consider $Mor(T) := \mathbb{R}\underline{Hom}(\Delta^1, T)$, the ∞-category of morphisms in T. The projection $Mor(T) \longrightarrow T$, associating to every morphism its target, makes $Mor(T)$ into a fibered ∞-category over T. It corresponds to an ∞-functor

$$\mathbb{S} : T^{op} \longrightarrow \infty\text{-}\underline{Cat},$$

whose value at an object $x \in T$ is the comma ∞-category T/x. The ∞-functor $\mathbb{S}$ is called the *universal family* over T. Its main property is the following exactness condition.

Lemma A.4 *The ∞-functor $\mathbb{S}$ commutes with limits (i.e. sends colimits in T to limits in ∞-$\underline{Cat}$).*

Proof It is enough to show that the $\mathbb{S}$ commutes with (possibly infinite) products, as well as with fibered products. In other words, we need to show the following two statements.

1. Given a (small) family of objects $x_\alpha \in T$, $\alpha \in A$, with sum $x := \sum x_\alpha$, the base change ∞-functor

$$\prod_\alpha (- \times_x x_\alpha) : T/x \longrightarrow \prod_\alpha T/x_\alpha$$

is an equivalence of ∞-categories.

2. Given a cocartesian diagram of objects in T

$$\begin{array}{ccc} t & \longrightarrow & y \\ \downarrow & & \downarrow \\ x & \longrightarrow & s \end{array}$$

the induced diagram of ∞-categories

$$\begin{array}{ccc} T/s & \xrightarrow{-\times_s x} & T/x \\ {\scriptstyle -\times_s y}\downarrow & & \downarrow{\scriptstyle -\times_x z} \\ T/y & \xrightarrow[-\times_y z]{} & T/z \end{array}$$

is (homotopy) Cartesian.

Condition (1) above follows from the fact that sums are disjoints in T, and that colimits are stable by base change. Indeed, an inverse ∞-functor is given by sending a family of objects $y_\alpha \to x_\alpha$ to its sum $\coprod y_\alpha \to x$.

In order to prove condition (2), we consider the induced ∞-functor

$$\phi : T/s \longrightarrow T/x \times^h_{T/z} T/y.$$

This ∞-functor possesses a left adjoint, constructed as follows. We consider the diagram of ∞-categories

$$\begin{array}{ccc} T/s & \xrightarrow{-\times_s x} & T/x \\ {\scriptstyle -\times_s y}\downarrow & & \downarrow{\scriptstyle -\times_x z} \\ T/y & \xrightarrow[-\times_y z]{} & T/z, \end{array}$$

which we consider as a functor

$$F : I^{op} \longrightarrow \infty\text{-}Cat,$$

where, obviously $I = \Delta^1 \times \Delta^1$ is the category classifying commutative squares. We perform its Grothendieck's construction to get a fibered ∞-category

$$\mathcal{C} := \int F \longrightarrow I.$$

We note that I is also the cocone over the category $J = 1 \longleftarrow 0 \longrightarrow 2$

$$I = \begin{array}{ccc} 0 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & +. \end{array}$$

The ∞-category $\mathcal{C}$ is also fibered, as the base change ∞-functors possesses left adjoints given by the forgetful ∞-functors: for $u : x \to y$ in T, the left adjoint to $- \times_x y$ is $T/x \to T/y$ simply obtained by composition with u.

There is a natural equivalence of ∞-categories

$$T/x \times^h_{T/z} T/y \simeq \mathbb{R}\underline{Hom}^{cart}_I(J, \mathcal{C}),$$

where the right hand side is the ∞-category of Cartesian sections of $\mathcal{C} \to I$ over the sub-category $J \subset I$ (this formally follows from [35, Proposition 1.4]). Similarly, we have a natural equivalence of ∞-categories

$$T/s \simeq \mathbb{R}\underline{Hom}^{cart}_I(I, \mathcal{C}),$$

and thus a natural inclusion ∞-functor $T/s \subset \mathbb{R}\underline{Hom}_I(I, \mathcal{C})$. This inclusion ∞-functor is right adjoint to the evaluation at $+$

$$\mathbb{R}\underline{Hom}_I(I, \mathcal{C}) \longrightarrow T/s.$$

Using these identifications, the ∞-functor ϕ becomes the restriction ∞-functor

$$\mathbb{R}\underline{Hom}^{cart}_I(I, \mathcal{C}) \longrightarrow \mathbb{R}\underline{Hom}^{cart}_I(J, \mathcal{C}).$$

But, as $I = J_+$, and as $\mathcal{C}$ possesses relative colimits (see Lemma A.2), this restriction ∞-functor possesses a left adjoint

$$\mathbb{R}\underline{Hom}^{cart}_I(J, \mathcal{C}) \xrightarrow{Colim_{J/I}} \mathbb{R}\underline{Hom}_I(J, \mathcal{C}) \xrightarrow{ev_+} T/s.$$

This provides a left adjoint

$$\psi : T/x \times^h_{T/z} T/y \longrightarrow T/s$$

to the ∞-functor ϕ.

The unit and counit of the adjunction (ψ, ϕ) are explicitely given as follows. Let $u \to s$ be an object in T/s. Then the counit morphism in T/s is

$$(u \times_s x) \coprod_{(u \times_s z)} (u \times_s y) \longrightarrow u.$$

That this morphism is an equivalence in the ∞-category T/s is simply the compatibility of colimits with base change in T

$$(u \times_s x) \coprod_{(u \times_s z)} (u \times_s y) \simeq u \times_s \left(x \coprod_s y \right) \simeq u \times_s s \simeq u.$$

The unit of the adjunction is itself given as follows. An object in $T/x \times^h_{T/z} T/y$ is represented by a triple

$$(x' \to x),\ (y' \to y), \quad \alpha : x' \times_x z \simeq y' \times_y z,$$

consisting of two objects $x' \in T/x$, $y' \in T/y$, and an equivalence α between the two base changes in T/z. The image by ψ of this object is

$$x' \coprod_{x' \times_x z} y' \to s,$$

where α is used in order to define the morphism $x' \times_x z \simeq y' \times_y z \to y'$. Therefore, to prove that the unit of the adjunction is an equivalence we need to show that both morphisms

$$x' \longrightarrow \left(x' \coprod_{x' \times_x z} y' \right) \times_s x, \qquad y' \longrightarrow \left(x' \coprod_{x' \times_x z} y' \right) \times_s y$$

are equivalences in T. For this, we write T as an exact localisation of an ∞-category of the form $\widehat{T_0}$, of very big prestacks over a big ∞-category T_0. Then, to check that the above two morphisms are always equivalences in T it is enough to do this replacing T by $\widehat{T_0}$. Moreover, by evaluation at each object of T_0 we can further assume that $T_0 = *$. In other words, we can assume that T is simply the ∞-topos of (very big) Kan complexes. But in this case, that the above two morphisms are weak equivalences in simply a form of the Van Kampen theorem (see for instance [17, Proposition 8.1]). □

Remark A.5 Lemma A.4 provides a characterization of ∞-topos. Indeed, a locally presentable ∞-category T is an ∞-topos if and only if $x \mapsto T/x$ transforms colimits in T into limits in ∞-cat. This is known as the *distributive law* (see e.g. [11, Theorem 6.1.0.6]).

An interesting consequence of the previous lemma is the following construction. Suppose that $x_* : I \longrightarrow T$ is a (small) diagram of objects in T, together with a colimit $x_0 = colim_{i \in I} x_i$. Then, the pull-back ∞-functor

$$T/x_0 \longrightarrow \underset{i \in I}{Holim}\, T/x_i$$

is an equivalence of ∞-categories. There are two different ways of writing an inverse ∞-functor, one using left adjoints of the base changes functors, as we have done during the proof of the lemma, and a second one using the right adjoints to the base changes. Let us describe now this second construction. For a given morphism $p : x \to y$ in T, the base change

$$T/y \longrightarrow T/x$$

possesses a right adjoint

$$p_* : T/x \longrightarrow T/y,$$

which is given by the relative internal Hom object.

We consider $x \mapsto T/x$ and $p \mapsto p_*$ as a fibered ∞-category $\mathcal{C} \longrightarrow T^{op}$. This fibered ∞-category possesses relative limits according to Lemma A.2. We have

$$\underset{i \in I}{Holim}\, T/x_i \simeq \mathbb{R}\underline{Hom}^{cart}_{T^{op}}(I^{op}, \mathcal{C}).$$

Therefore, using existence of relative limits along I we produce an ∞-functor

$$Lim_{I/T} : \mathbb{R}\underline{Hom}^{cart}_{T}(I^{op}, \mathcal{C}) \longrightarrow \mathbb{R}\underline{Hom}_{T}(I^{op}_{+}, \mathcal{C}),$$

where $I_+ \longrightarrow T$ is fixed using the colimit object x_0 as the value at the initial point $+ \in I_+$. Composing with the evaluation at $+$ we get a well defined ∞-functor

$$\underset{i \in I}{Holim}\, T/x_i \longrightarrow T/x_0$$

which is a right adjoint to the equivalence $T/x_0 \simeq HolimT/x_i$. This right adjoint is therefore also an equivalence of ∞-categories. It can be described intuitively as follows. Consider an object x' in $HolimT/x_i$. Such an object is represented by a system of objects $x'_i \to x_i$, together with natural equivalences $x'_i \times_{x_i} x_j \simeq x'_j$, for all $i \to j$ a morphism in I. For each i we have the natural morphism $p : x_i \to x_0$, and we consider $p_*(x_i)$. By adjunction, we get that way an I-diagram of objects $i \mapsto p_*(x_i)$ inside the ∞-category T/x_0. The limit of this diagram is the image of x' in T/x_0. In other words, the ∞-functor $Holim_{i \in I} T/x_i \longrightarrow T/x_0$ sends x' to

$$\underset{i \in I}{lim}\, p_*(x_i) \in T/x_0.$$

As this ∞-functor is an equivalence we get that for any $y \to x_0$, the natural morphism

$$y \to limp_*(y \times_{x_0} x_i)$$

is an equivalence in T. In the same way, for any x' in $HolimT/x_i$, we the natural morphisms

$$x'_i \to x'_i \times_{x_0} (limp_*(x'_i)$$

are all equivalences in T.

A.3 The descent statement

We now use our last two paragraph to state and prove our descent statement. It consists of a criterion to show that a prestack in ∞-categories is a stack when it comes equiped with good forgetful functors to the stack of stacks $\mathbb{S}$ defined our last paragraph.

We let T be a very big ∞-topos, generated by a big ∞-site (C, τ). We let $j : C \longrightarrow T$ be the natural embedding. The universal family $\mathbb{S}$, restricted to C is still denoted by $\mathbb{S}$, and is called the *stack of stacks*. It can be described explicitely as follows. For each object $U \in C$, $\mathbb{S}(U)$ is the ∞-category of stacks over C/U, the comma ∞-category endowed with the induced topology. For $u : U \to V$ in C, the base change

$$u^* : \mathbb{S}(V) \longrightarrow \mathbb{S}(U)$$

is simply obtained by the restriction along $u\circ : C/U \longrightarrow C/V$. Recall that all the ∞-functors u^* possess right adjoint u_*.

Proposition A.6 *With the same notations as above, let*

$$F : C^{op} \longrightarrow \infty\text{-}\underline{Cat}_{\mathbb{W}}$$

be a prestack of ∞-categories. We assume that the following conditions hold.

1. *For each object $x \in T$ the ∞-category $F(x)$ has small limits.*
2. *For each morphism $u : x \to y$, the ∞-functor*

$$u^* : F(y) \longrightarrow F(x)$$

   has right adjoint u_*.
3. *There exists a family of morphism (in the ∞-category of prestacks of ∞-categories)*

$$\phi_\alpha : F \longrightarrow \mathbb{S},$$

   such that for each $x \in T$ the corresponding family of ∞-functors

$$\{\phi_\alpha(x) : F(x) \longrightarrow \mathbb{S}(x)\}_\alpha$$

   is conservative.
4. *For each $x \in T$, and all α, the ∞-functor*

$$\phi_\alpha(x) : F(x) \longrightarrow \mathbb{S}(x)$$

   commutes with limits.
5. *For each morphism $u : x \to y$ in T, and for all α, the natural transformation*

$$\phi_\alpha \circ u_* \Rightarrow u_* \phi_\alpha$$

   is an equivalence (in the ∞-category of ∞-functors from $F(x)$ to $\mathbb{S}(y)$).

Then F is a stack (i.e. has the descent property of all τ-hypercoverings).

Proof Few words of explanations concerning condition (5) first. We have a natural equivalence of ∞-functors $u^* \circ \phi_\alpha \simeq \phi_\alpha \circ^*$, coming from the fact that the ϕ_α are natural transformations. By adjunction these natural equivalences provide natural transformations

$$\phi_\alpha \circ u_* \Rightarrow u_* \phi_\alpha,$$

which by condition (5) must be equivalences.

Let $H_* \to X$ be a hyper-coverings over $X \in C$. We must show that the natural ∞-functor

$$f : F(X) \longrightarrow \underset{n \in \Delta}{Holim}\, F(H_n)$$

is an equivalence of ∞-categories. For this, we use the morphisms ϕ_α, in order to get commutative diagrams

$$\begin{array}{ccc} F(X) & \xrightarrow{f} & \underset{n\in\Delta}{Holim}\, F(H_n) \\ \downarrow{\scriptstyle \phi_\alpha} & & \downarrow{\scriptstyle \alpha} \\ \mathbb{S}(X)' & \xrightarrow[f]{} & Holim\, \mathbb{S}(H_n). \end{array}$$

Because of Lemma A.4 we know that $\mathbb{S}$ is a stack, and therefore that the bottom horizontal ∞-functor is an equivalence. Using relative limits, we can construct, for F and $\mathbb{S}$, right adjoints g and g' to the ∞-functors f and g (as we have done for $\mathbb{S}$ in our last paragraph), and because of conditions (4) and (5) we get adjoints commutative diagrams

$$\begin{array}{ccc} \underset{n\in\Delta}{Holim}\, F(H_n) & \xrightarrow{g} & F(X) \\ \downarrow{\scriptstyle \phi_\alpha} & & \downarrow{\scriptstyle \alpha} \\ Holim\, \mathbb{S}(H_n)' & \xrightarrow[g]{} & \mathbb{S}(X). \end{array}$$

Therefore, to prove that F is stack we simply need to prove that the unit and counit of the adjunction (f, g) are equivalences. But, as the ϕ_α form a conservative family this follows from the corresponding statement for $\mathbb{S}$, which we already know is true because f' is an equivalence by Lemma A.4. □

A nice consequence of the previous proposition is the following descent criterion. We state it using model sites as this is the version that we will use. It is however true for any ∞-site.

Corollary A.7 *Let (C, τ) be a big ∞-site and*

$$F : C^{op} \longrightarrow \infty\text{-}\underline{Cat}_{\mathbb{W}}$$

be a prestack in very big ∞-categories. We assume that the following conditions hold.

1. *For any object $U \in C$, the ∞-category $F(U)$ has all small limits, and for any $u : V \to U$ the base change ∞-functor*

$$u^* : F(U) \longrightarrow F(V)$$

commutes with all small limits.

2. *For any $u : V \to U$ in C, the base change ∞-functor u^* possesses a right adjoint*

$$u_* : F(U) \longrightarrow F(V),$$

which is moreover conservative.

3. *For each Cartesian square in the ∞-category C*

$$\begin{array}{ccc} V' & \xrightarrow{-u'} & U' \\ {\scriptstyle -q'}\downarrow & & \downarrow{\scriptstyle q} \\ V & \xrightarrow[u]{} & U, \end{array}$$

and any object $x \in F(V)$, the natural morphism

$$q^* u_*(x) \longrightarrow q'_*(u')^*(x)$$

is an equivalence in $F(U')$.

4. *For each object $U \in C$, and each two objects $x, y \in F(U)$, the simplicial presheaf*

$$Map(x, y) : (C/U)^{op} \longrightarrow SSet_{\mathbb{W}},$$

sending $u : V \to U$ to the simplicial set $Map_{F(V)}(u^*(x), u^*(y))$, is a stack for the induced model topology.

Then F is a stack.

Proof We let A be the set of equivalence classes of objects in the ∞-category $F(*)$. For any $x \in A$, we consider the morphism

$$\phi_x : F \longrightarrow \mathbb{S}$$

defines as follows. For each object $U \in C$, we let $x_U := p^*(x)$, where $p : U \longrightarrow *$ is the natural projection. For any $y \in F(U)$ we set

$$\phi_x(y) := Map(x_U, y),$$

where $Map(x_U, y)$ is the prestack over C/U defined by sending $u : V \to U$ to the simplicial set $Map_{F(V)}(x_V, u^*(y))$. By the condition (4), the prestack $\phi_x(y)$ is a stack over C/U. Moreover, we obviously have, for any $u : V \to U$ and $y \in F(U)$

$$\phi_x(u^*(y)) = u^*(\phi_x(y)),$$

showing that ϕ_x defines a morphism of prestacks

$$\phi_x : F \longrightarrow \mathbb{S}.$$

When x varies in A we get our family of morphisms

$$\{\phi_x : F \longrightarrow \mathbb{S}\}_{x \in A}.$$

It is then formal to check that this family satisfies the properties of Proposition A.6. □

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References

1. Baker, A., Richter, B., Szymik, M.: Brauer groups for commutative S-algebras. Preprint arXiv:1005.5370
2. Bergner, J.: Three models for the homotopy theory of homotopy theories. Topology **46**(4), 397–436 (2007)
3. Bondal, A., Van Den Bergh, M.: Generators and representability of functors in commutative and non-commutative geometry. Mosc. Math. J. **3**(1), 1–36 (2003)
4. de Jong, A.: A result of Gabber. Preprint available at http://www.math.columbia.edu/~dejong/
5. Edidin, D., Hassett, B., Kresch, A., Vistoli, A.: Brauer groups and quotient stacks. Am. J. Math. **123**(4), 761–777 (2001)
6. Gabber, O.: Some theorems on Azumaya algebras. In: The Brauer Group. Lecture Notes in Math., vol. 844, pp. 129–209. Springer, Berlin (1981)
7. Grothendieck, A.: Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses. In: Dix Exposés sur la Cohomologie des Schémas, pp. 46–66. North-Holland, Amsterdam (1968)
8. Heinloth, J., Schröer, S.: The bigger Brauer group and twisted sheaves. J. Algebra **322**(4), 1187–1195 (2009)
9. Johnson, N.: Azumaya objects in bicategories. Preprint arxiv:1005.4878
10. Lieblich, M.: Moduli of twisted sheaves and generalized Azumaya algebras. Oberwolfach Rep. **1**(3) (2004)
11. Lurie, J.: Higher Topos Theory. Annals of Mathematics Studies, vol. 170, pp. xviii+925. Princeton University Press, Princeton (2009).
12. Lurie, J.: On the Classification of Topological Field Theories. Current Developments in Mathematics, vol. 2008, pp. 129–280. Int. Press, Somerville (2009)
13. Lurie, J.: Derived algebraic geometry XII: proper morphisms, completions, and the Grothendieck existence theorem. Preprint available at http://www.math.harvard.edu/~lurie/
14. Lurie, J.: Quasi-coherent sheaves and Tannaka duality theorems. Preprint available at http://www.math.harvard.edu/~lurie/
15. Neeman, A.: Triangulated Categories. Annals of Mathematics Studies, vol. 148, pp. viii+449. Princeton University Press, Princeton (2001).
16. Pridham, J.: Presenting higher stacks as simplicial schemes. arXiv:0905.4044
17. Rezk, C.: Fibrations and homotopy colimits of simplicial sheaves. Preprint available at http://www.math.uiuc.edu/~rezk/papers.html
18. Schlichting, M.: Negative KK-theory of derived categories. Math. Z. **253**(1), 97–134 (2006)
19. Schwede, S., Shipley, B.: Algebras and modules in monoïdal model categories. Proc. Lond. Math. Soc. (3) **80**(2), 491–511 (2000)

20. Schwede, S., Shipley, B.: Equivalences of monoïdal model categories. Algebr. Geom. Topol. **3**, 287–334 (2003)
21. Simpson, C.: Homotopy Theory of Higher Categories: From Segal Categories to n-Categories and Beyond. New Mathematical Monographs, vol. 19. Cambridge University Press, Cambridge (2012).
22. Tabuada, G.: Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. C. R. Acad. Sci. Paris **340**, 15–19 (2005)
23. Tabuada, G.: Invariants additifs de DG-catégories. Int. Math. Res. Not. **53**, 3309–3339 (2005)
24. Tabuada, G.: Differential graded versus Simplicial categories. Topol. Appl. **157**(3), 563–593 (2010)
25. Thomason, R.W.: The classification of triangulated subcategories. Compos. Math. **105**(1), 1–27 (1997)
26. Toën, B.: The homotopy theory of dg-categories and derived Morita theory. Invent. Math. **167**(3), 615–667 (2007)
27. Toën, B.: Anneaux de définition des dg-algèbres propres et lisses. Bull. Lond. Math. Soc. **40**(4), 642–650 (2008)
28. Toën, B.: Higher and derived stacks: a global overview. In: Algebraic Geometry—Seattle 2005. Proc. Sympos. Pure Math., vol. 80, pp. 435–487. Amer. Math. Soc., Providence (2009), Part 1
29. Toën, B.: Lectures on dg-categories. In: Topics in Algebraic and Topological K-theory. Lecture Notes in Mathematics, vol. 2008. Springer, Berlin (2011)
30. Toën, B.: Descente fidèlement plate pour les n-champs d'Artin. Compos. Math. **147**(05), 1382–1412 (2011)
31. Toën, B., Vaquié, M.: Moduli of objects in dg-categories, Ann. Sci. ENS **40**(3) 387–444
32. Toën, B., Vezzosi, G.: Homotopical algebraic geometry I: topos theory. Adv. Math. **193**(2), 257–372 (2005)
33. Toën, B., Vezzosi, G.: Homotopical algebraic geometry II: geometric stacks and applications. Mem. Amer. Math. Soc. **193**(902), x+224 (2008)
34. Toën, B., Vezzosi, G.: Chern character, loop spaces and derived algebraic geometry. In: Algebraic Topology. Abel Symp., vol. 4, pp. 331–354. Springer, Berlin (2009)
35. Toën, B., Vezzosi, G.: ∞-Catégories monoïdales rigides, traces et caractère de Chern. Preprint arXiv:0903.3292