DERIVED CATEGORIES OF COHERENT SHEAVES AND TRIANGULATED CATEGORIES OF SINGULARITIES

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Dedicated to Yuri Ivanovich Manin on the occasion of his 70th birthday

Abstract. In this paper we establish an equivalence between the category of graded D-branes of type B in Landau-Ginzburg models with homogeneous superpotential W and the triangulated category of singularities of the fiber of W over zero. The main result is the theorem which shows that the graded triangulated category of singularities of the cone over a projective variety is connected via a fully faithful functor to the bounded derived category of coherent sheaves on the base of the cone. This implies that the category of graded D-branes of type B in Landau-Ginzburg models with homogeneous superpotential W is connected via a fully faithful functor to the derived category of coherent sheaves on the projective variety defined by the equation $W=0$.

Introduction

With any algebraic variety X one can naturally associate two triangulated categories: the bounded derived category $\mathbf{D}^b(\mathrm{coh}(X))$ of coherent sheaves and the triangulated subcategory $\mathfrak{Perf}(X) \subset \mathbf{D}^b(\mathrm{coh}(X))$ of perfect complexes on X . If the variety X is smooth, then these two categories coincide. For singular varieties this is no longer true. In [22] we introduced a new invariant of a variety X - the triangulated category $\mathbf{D}_{\mathrm{Sg}}(X)$ of the singularities of X - as the quotient of $\mathbf{D}^b(\mathrm{coh}(X))$ by the full subcategory of perfect complexes $\mathfrak{Perf}(X)$. The category $\mathbf{D}_{\mathrm{Sg}}(X)$ captures many properties of the singularities of X .

Similarly we can define a triangulated category of singularities $\mathbf{D}_{\mathrm{Sg}}(A)$ for any noetherian algebra A . We set $\mathbf{D}_{\mathrm{Sg}}(A)=\mathbf{D}^b(\mathrm{mod}{-}A)/\mathfrak{Perf}(A)$, where $\mathbf{D}^b(\mathrm{mod}{-}A)$ is the bounded derived category of finitely generated right A-modules and $\mathfrak{Perf}(A)$ is its triangulated subcategory consisting of objects which are quasi-isomorphic to bounded complexes of projectives. We will again call $\mathfrak{Perf}(A)$ the subcategory of perfect complexes, but usually we will write $\mathbf{D}^b(\mathrm{proj}{-}A)$ instead of $\mathfrak{Perf}(A)$ since this category can also be identified with the derived category of the exact category $\mathrm{proj}{-}A$ of finitely generated right projective A-modules (see, e.g. [19]).

The investigation of triangulated categories of singularities is not only connected with a study of singularities but is mainly inspired by the Homological Mirror Symmetry Conjecture [20]. More precisely, the objects of these categories are directly related to D-branes of type B (B-branes) in Landau-Ginzburg models. Such models arise as a mirrors to Fano varieties [15]. For Fano varieties one has the derived categories of coherent sheaves (B-branes) and given a symplectic form one can

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propose a suitable Fukaya category (A-branes). Mirror symmetry should interchange these two classes of D-branes. Thus, to extend the Homological Mirror Symmetry Conjecture to the Fano case, one should describe D-branes in Landau-Ginzburg models.

To specify a Landau-Ginzburg model in general one needs to choose a target space X , and a holomorphic function W on X called a superpotential. The B-branes in the Landau-Ginzburg model are defined as W-twisted $\mathbb{Z}_2$-periodic complexes of coherent sheaves on X . These are chains $\{\cdots \stackrel{d}{\to} P_0 \stackrel{d}{\to} P_1 \stackrel{d}{\to} P_0 \stackrel{d}{\to} P_1 \stackrel{d}{\to} P_0 \cdots\}$ of coherent sheaves in which the composition of differentials is no longer zero, but is equal to multiplication by W (see, e.g. [17, 22, 23]). In the paper [22] we analysed the relationship between the categories of B-branes in Landau-Ginzburg models and triangulated categories of singularities. Specifically, we showed that for an affine X the product of the triangulated categories of singularities of the singular fibres of W is equivalent to the category of B-branes of (X, W) .

In this paper we consider the graded case. Let $A = \bigoplus_i A_i$ be a graded noetherian algebra over a field $\mathbf{k}$. We can define the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ of A as the quotient $\mathbf{D}^b(\mathrm{gr}{-}A)/\mathbf{D}^b(\mathrm{grproj}{-}A)$, where $\mathbf{D}^b(\mathrm{gr}{-}A)$ is the bounded derived category of finitely generated graded right A-modules and $\mathbf{D}^b(\mathrm{grproj}{-}A)$ is its triangulated subcategory consisting of objects which are isomorphic to bounded complexes of projectives.

The graded version of the triangulated category of singularities is closely related to the category of B-branes in Landau-Ginzburg models (X, W) equipped with an action of the multiplicative group $\mathbf{k}^*$ for which W is semi-invariant. The notion of grading on D-branes of type B was defined in the papers [16, 28]. In the presence of a $\mathbf{k}^*$-action one can construct a category of graded B-branes in the Landau-Ginzburg model (X, W) (Definition 3.1 and Subsection 3.3). Now our Theorem 3.10 gives an equivalence between the category of graded B-branes and the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$, where A is such that $\mathbf{Spec}(A)$ is the fiber of W over 0 .

This equivalence allows us to compare the category of graded B-branes and the derived category of coherent sheaves on the projective variety which is defined by the superpotential W . For example, suppose X is the affine space $\mathbb{A}^N$ and W is a homogeneous polynomial of degree d . Denote by $Y \subset \mathbb{P}^{N-1}$ the projective hypersurface of degree d which is given by the equation $W = 0$. If $d = N$, then the triangulated category of graded B-branes $\mathrm{DGrB}(W)$ is equivalent to the derived category of coherent sheaves on the Calabi-Yau variety Y . Furthermore, if $d < N$ (i.e. Y is a Fano variety), we construct a fully faithful functor from $\mathrm{DGrB}(W)$ to $\mathbf{D}^b(\mathrm{coh}(Y))$, and, if $d > N$ (i.e. Y is a variety of general type), we construct a fully faithful functor from $\mathbf{D}^b(\mathrm{coh}(Y))$ to $\mathrm{DGrB}(W)$ (see Theorem 3.11).

This result follows from a more general statement for graded Gorenstein algebras (Theorem 2.5). It gives a relation between the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and the bounded derived category $\mathbf{D}^b(\mathrm{qgr}\, A)$, where $\mathrm{qgr}\, A$ is the quotient of the abelian category of graded finitely generated A-modules by the subcategory of torsion modules. More precisely, for Gorenstein algebras we obtain a fully faithful functor between $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and $\mathbf{D}^b(\mathrm{qgr}\, A)$, and the direction of this functor depends on the Gorenstein parameter a of A. In particular, when the Gorenstein parameter a is equal to zero, we obtain an equivalence between these categories. Finally, the famous theorem of Serre, which identifies $\mathbf{D}^b(\operatorname{qgr} A)$ with $\mathbf{D}^b(\operatorname{coh}(\mathbf{Proj}\,(A)))$ when A is generated by its first component, allows to apply this result to geometry.

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1. Triangulated categories of singularities for graded algebras.

1.1. **Localization in triangulated categories and semiorthogonal decomposition.** Recall that a triangulated category $\mathcal{D}$ is an additive category equipped with the additional data:

a) an additive autoequivalence $[1] : \mathcal{D} \longrightarrow \mathcal{D},$ which is called a translation functor,

b) a class of exact (or distinguished) triangles:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1],$$

which must satisfy a certain set of axioms (see [27], also [12, 19, 21]).

A functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ between two triangulated categories is called exact if it commutes with the translation functors, i.e. $F \circ [1] \cong [1] \circ F,$ and transforms exact triangles into exact triangles.

With any pair $\mathcal{N} \subset \mathcal{D},$ where $\mathcal{N}$ is a full triangulated subcategory, in a triangulated category $\mathcal{D}$, we can associate the quotient category $\mathcal{D}/\mathcal{N}$. To construct it let us denote by $\Sigma(\mathcal{N})$ a class of morphisms s in $\mathcal{D}$ fitting into an exact triangle

$$X \xrightarrow{s} Y \longrightarrow N \longrightarrow X[1]$$

with $N \in \mathcal{N}$. It can be checked that $\Sigma(\mathcal{N})$ is a multiplicative system. Define the quotient $\mathcal{D}/\mathcal{N}$ as the localization $\mathcal{D}[\Sigma(\mathcal{N})^{-1}]$ (see [10, 12, 27]). It is a triangulated category. The translation functor on $\mathcal{D}/\mathcal{N}$ is induced from the translation functor in the category $\mathcal{D}$, and the exact triangles in $\mathcal{D}/\mathcal{N}$ are the triangles isomorphic to the images of exact triangles in $\mathcal{D}$. The quotient functor $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{N}$ annihilates $\mathcal{N}$. Moreover, any exact functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ between triangulated categories, for which $F(X) \simeq 0$ when $X \in \mathcal{N},$ factors uniquely through Q. The following lemma is obvious.

Lemma 1.1. *Let $\mathcal{N}$ and $\mathcal{N}'$ be full triangulated subcategories of triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ respectively. Let $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ be an adjoint pair of exact functors such that $F(\mathcal{N}) \subset \mathcal{N}'$ and $G(\mathcal{N}') \subset \mathcal{N}$. Then they induce functors*

$$\overline{F} : \mathcal{D}/\mathcal{N} \longrightarrow \mathcal{D}'/\mathcal{N}', \qquad \overline{G} : \mathcal{D}'/\mathcal{N}' \longrightarrow \mathcal{D}/\mathcal{N}$$

which are adjoint as well. Moreover, if the functor $F : \mathcal{D} \to \mathcal{D}'$ is fully faithful, then the functor $\overline{F} : \mathcal{D}/\mathcal{N} \longrightarrow \mathcal{D}'/\mathcal{N}'$ is also fully faithful.

Now recall some definitions and facts concerning admissible subcategories and semiorthogonal decompositions (see [7, 8]). Let $\mathcal{N} \subset \mathcal{D}$ be a full triangulated subcategory. The right orthogonal to $\mathcal{N}$ is the full subcategory $\mathcal{N}^{\perp} \subset \mathcal{D}$ consisting of all objects M such that $\operatorname{Hom}(N, M) = 0$

for any $N \in \mathcal{N}$. The left orthogonal ${}^{\perp}\mathcal{N}$ is defined analogously. The orthogonals are also triangulated subcategories.

Definition 1.2. *Let $I : \mathcal{N} \hookrightarrow \mathcal{D}$ be an embedding of a full triangulated subcategory $\mathcal{N}$ in a triangulated category $\mathcal{D}$. We say that $\mathcal{N}$ is* right admissible *(respectively* left admissible*) if there is a right (respectively left) adjoint functor $Q : \mathcal{D} \to \mathcal{N}$. The subcategory $\mathcal{N}$ will be called admissible if it is right and left admissible.*

Remark 1.3. For the subcategory $\mathcal{N}$ the property of being right admissible is equivalent to requiring that for each $X \in \mathcal{D}$ there is an exact triangle $N \to X \to M$, with $N \in \mathcal{N}, M \in \mathcal{N}^{\perp}$.

Lemma 1.4. *Let $\mathcal{N}$ be a full triangulated subcategory in a triangulated category $\mathcal{D}$. If $\mathcal{N}$ is right (respectively left) admissible, then the quotient category $\mathcal{D}/\mathcal{N}$ is equivalent to $\mathcal{N}^{\perp}$ (respectively ${}^{\perp}\mathcal{N}$). Conversely, if the quotient functor $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{N}$ has a left (respectively right) adjoint then $\mathcal{D}/\mathcal{N}$ is equivalent to $\mathcal{N}^{\perp}$ (respectively ${}^{\perp}\mathcal{N}$).*

If $\mathcal{N} \subset \mathcal{D}$ is a right admissible subcategory, then we say that the category $\mathcal{D}$ has a weak semiorthogonal decomposition $\langle \mathcal{N}^{\perp}, \mathcal{N} \rangle$. Similarly if $\mathcal{N} \subset \mathcal{D}$ is a left admissible subcategory, we say that $\mathcal{D}$ has a weak semiorthogonal decomposition $\langle \mathcal{N}, {}^{\perp}\mathcal{N} \rangle$.

Definition 1.5. *A sequence of full triangulated subcategories $(\mathcal{N}_1, \ldots, \mathcal{N}_n)$ in a triangulated category $\mathcal{D}$ will be called a* weak semiorthogonal decomposition *of $\mathcal{D}$ if there is a sequence of left admissible subcategories $\mathcal{D}_1 = \mathcal{N}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_n = \mathcal{D}$ such that $\mathcal{N}_p$ is left orthogonal to $\mathcal{D}_{p-1}$ in $\mathcal{D}_p$. We will write $\mathcal{D} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$. If all N_p are admissible in $\mathcal{D}$ then the decomposition $\mathcal{D} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$ is called* semiorthogonal.

The existence of a semiorthogonal decomposition on a triangulated category $\mathcal{D}$ clarifies the structure of $\mathcal{D}$. In the best scenario, one can hope that $\mathcal{D}$ has a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$ in which each elementary constituent $\mathcal{N}_p$ is as simple as possible, i.e. is equivalent to the bounded derived category of finite dimensional vector spaces.

Definition 1.6. *An object E of a $\mathbf{k}$-linear triangulated category $\mathcal{T}$ is called* exceptional *if $\mathrm{Hom}(E, E[p]) = 0$ when $p \neq 0$, and $\mathrm{Hom}(E, E) = \mathbf{k}$. An* exceptional collection *in $\mathcal{T}$ is a sequence of exceptional objects $(E_0, \ldots, E_n)$ satisfying the semiorthogonality condition $\mathrm{Hom}(E_i, E_j[p]) = 0$ for all p when $i > j$.*

If a triangulated category $\mathcal{D}$ has an exceptional collection $(E_0, \ldots, E_n)$ which generates the whole $\mathcal{D}$ then we say that the collection is full. In this case $\mathcal{D}$ has a semiorhtogonal decomposition with $\mathcal{N}_p = \langle E_p \rangle$. Since E_p is exceptional each of these categories is equivalent to the bounded derived category of finite dimensional vector spaces. In this case we write $\mathcal{D} = \langle E_0, \ldots, E_n \rangle$.

Definition 1.7. *An exceptional collection $(E_0, \ldots, E_n)$ is called strong if, in addition, $\mathrm{Hom}(E_i, E_j[p]) = 0$ for all i and j when $p \neq 0$.*

1.2. **Triangulated categories of singularities for algebras.** Let $A=\bigoplus_{i\ge 0} A_i$ be a noetherian graded algebra over a field $\mathbf{k}$. Denote by $\mathrm{mod}{-}A$ and $\mathrm{gr}{-}A$ the category of finitely generated right modules and the category of finitely generated graded right modules respectively. Note that morphisms in $\mathrm{gr}{-}A$ are homomorphisms of degree zero. These categories are abelian. We will also use the notation $\mathrm{Mod}{-}A$ and $\mathrm{Gr}{-}A$ for the abelian categories of all right modules and all graded right modules and we will often omit the prefix "right". Left A-modules are will be viewed as right A°-modules and $A-B$ bimodules as right $A^\circ - B$-modules, where A° is the opposite algebra.

The twist functor (p) on the category $\mathrm{gr}{-}A$ is defined as follows: it takes a graded module $M=\oplus_i M_i$ to the module $M(p)$ for which $M(p)_i = M_{p+i}$ and takes a morphism $f: M\longrightarrow N$ to the same morphism viewed as a morphism between the twisted modules $f(p): M(p)\longrightarrow N(p)$.

Consider the bounded derived categories $\mathbf{D}^b(\mathrm{gr}{-}A)$ and $\mathbf{D}^b(\mathrm{mod}{-}A)$. They can be endowed with natural structures of triangulated categories. The categories $\mathbf{D}^b(\mathrm{gr}{-}A)$ and $\mathbf{D}^b(\mathrm{mod}{-}A)$ have full triangulated subcategories consisting of objects which are isomorphic to bounded complexes of projectives. These subcategories can also be considered as the derived categories of the exact categories of projective modules $\mathbf{D}^b(\mathrm{grproj}{-}A)$ and $\mathbf{D}^b(\mathrm{proj}{-}A)$ respectively (see, e.g. [19]). They will be called the subcategories of perfect complexes. Observe also that the category $\mathbf{D}^b(\mathrm{gr}{-}A)$ (respectively $\mathbf{D}^b(\mathrm{mod}{-}A)$) is equivalent to the category $\mathbf{D}^b_{\mathrm{gr}-A}(\mathrm{Gr}{-}A)$ (respectively $\mathbf{D}^b_{\mathrm{mod}-A}(\mathrm{Mod}{-}A)$) of complexes of arbitrary modules with finitely generated cohomologies (see [5]). We will tacitly use this equivalence throughout our considerations.

Definition 1.8. *We define triangulated categories of singularities* $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *and* $\mathbf{D}_{\mathrm{Sg}}(A)$ *as the quotient* $\mathbf{D}^b(\mathrm{gr}{-}A)/\mathbf{D}^b(\mathrm{grproj}{-}A)$ *and* $\mathbf{D}^b(\mathrm{mod}{-}A)/\mathbf{D}^b(\mathrm{proj}{-}A)$ *respectively.*

Remark 1.9. As in the commutative case [22, 23], the triangulated categories of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and $\mathbf{D}_{\mathrm{Sg}}(A)$ will be trivial if A has finite homological dimension. Indeed, in this case any A-module has a finite projective resolution, i.e. the subcategories of perfect complexes coincide with the full bounded derived categories of finitely generated modules.

Homomorphisms of (graded) algebras $f: A\to B$ induce functors between the associated derived categories of singularities. Furthermore, if B has a finite Tor-dimension as an A-module then we get the functor $\overset{\mathbf{L}}{\otimes}_A B$ between the bounded derived categories of finitely generated modules which maps perfect complexes to perfect complexes. Therefore, we get functors between triangulated categories of singularities

$$\overset{\mathbf{L}}{\otimes}_A B: \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)\longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(B) \quad\text{and}\quad \overset{\mathbf{L}}{\otimes}_A B: \mathbf{D}_{\mathrm{Sg}}(A)\longrightarrow \mathbf{D}_{\mathrm{Sg}}(B).$$

If, in addition, B is finitely generated as an A-module, then these functors have right adjoints induced from the functor which sends a complex of B-modules to itself considered as a complex of A-modules.

More generally, suppose ${}_A\underline{M}^{\bullet}_B$ is a complex of graded $A-B$ bimodules which as a complex of graded B-modules is quasi-isomorphic to a perfect complex. Suppose that ${}_A\underline{M}^{\bullet}$ has a finite

Tor-amplitude as a left A-module. Then we can define the derived tensor product functor $\overset{\mathbf{L}}{\otimes}_A \underline{M}^{\bullet}_B : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^b(\mathrm{gr}{-}B)$. Moreover, since $\underline{M}^{\bullet}_B$ is perfect over B this functor sends perfect complexes to perfect complexes. Therefore, we get an exact functor

$$\overset{\mathbf{L}}{\otimes}_A \underline{M}^{\bullet}_B : \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(B).$$

In the ungraded case we also get the functor $\overset{\mathbf{L}}{\otimes}_A \underline{M}^{\bullet}_B : \mathbf{D}_{\mathrm{Sg}}(A) \longrightarrow \mathbf{D}_{\mathrm{Sg}}(B)$.

1.3. **Morphisms in categories of singularities.** In general, it is not easy to calculate spaces of morphisms between objects in a quotient category. The following lemma and proposition provide some information about the morphism spaces in triangulated categories of singularities.

Lemma 1.10. *For any object $T \in \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ (respectively $T \in \mathbf{D}_{\mathrm{Sg}}(A)$) and for any sufficiently large k, there is a module $M \in \mathrm{gr}{-}A$ (respectively $M \in \mathrm{mod}{-}A$), depending on T and k, and such that T is isomorphic to the image of $M[k]$ in the triangulated category of singularities. If, in addition, the algebra A has finite injective dimension, then for any sufficiently large k the corresponding module M satisfies $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$.*

Proof. The object T is represented by a bounded complex of modules $\underline{T}^{\bullet}$. Choose a bounded above projective resolution $\underline{P}^{\bullet} \xrightarrow{\sim} \underline{T}^{\bullet}$ and a sufficiently large $k \gg 0$. Consider the stupid truncation $\sigma^{\geq -k+1}\underline{P}^{\bullet}$ of $\underline{P}^{\bullet}$. Denote by M the cohomology module $H^{-k+1}(\sigma^{\geq -k+1}\underline{P}^{\bullet})$. Clearly $T \cong M[k]$ in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ (respectively $\mathbf{D}_{\mathrm{Sg}}(A)$).

If now A has finite injective dimension, then morphism spaces $\mathrm{Hom}(\underline{T}^{\cdot}, A[i])$ in $\mathbf{D}^b(\mathrm{gr}{-}A)$ (respectively $\mathbf{D}^b(\mathrm{mod}{-}A)$) are trivial for all but finitely many $i \in \mathbb{Z}$. So if M corresponds to T and a sufficiently large k, then we will have $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. □

Proposition 1.11. *Let M be an A-module such that $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. Then for any A-module N we have*

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{Sg}}(A)}(M, N) \cong \mathrm{Hom}_A(M, N)/\mathcal{R}$$

where $\mathcal{R}$ is the subspace of elements factoring through a projective, i.e. $e \in \mathcal{R}$ iff $e = \beta\alpha$ with $\alpha : M \to P$ and $\beta : P \to N$, where P is projective. If M is a graded module, then for any graded A-module N

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(M, N) \cong \mathrm{Hom}_{\mathrm{gr}-A}(M, N)/\mathcal{R}.$$

Proof. We will only discuss the graded case. By the definition of localization any morphism from M to N in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ can be represented by a pair

$$M \xrightarrow{a} \underline{T}^{\bullet} \xleftarrow{s} N, \tag{1}$$

of morphisms in $\mathbf{D}^b(\mathrm{gr}{-}A)$, such that the cone $\underline{C}^{\bullet}(s)$ is a perfect complex. Consider a bounded above projective resolution $\underline{Q}^{\bullet} \to N$ and its stupid truncation $\sigma^{\geq -k}\underline{Q}^{\bullet}$ for sufficiently large k.

There is an exact triangle

$$E[k] \longrightarrow \sigma^{\geq -k}\underline{Q}^{\bullet} \longrightarrow N \xrightarrow{s'} E[k+1],$$

where E denotes the module $H^{-k}(\sigma^{\geq -k}\underline{Q}^{\bullet})$. Choosing k to be sufficiently large we can guarantee that $\mathrm{Hom}(\underline{C}^{\bullet}(s), E[i]) = 0$ for all $i > k$. Using the triangle

$$\underline{C}^{\bullet}(s)[-1] \longrightarrow N \xrightarrow{s} \underline{T}^{\bullet} \longrightarrow \underline{C}^{\bullet}(s),$$

we find that the map $s' : N \to E[k+1]$ can be lifted to a map $\underline{T}^{\bullet} \to E[k+1]$. The map $\underline{T}^{\bullet} \to E[k+1]$ induces a pair of the form

(2) $$M \xrightarrow{a'} E[k+1] \xleftarrow{s'} N,$$

and this pair gives the same morphism in $\mathbf{D}_{\mathrm{Sg}}(A)$ as the pair (1). Since $\mathrm{Ext}^i(M, P) = 0$ for all $i > 0$ and any projective module P , we obtain

$$\mathrm{Hom}(M,\ (\sigma^{\geq -k}\underline{Q}^{\bullet})[1]) = 0.$$

Hence, the map $a' : M \to E[k+1]$ can be lifted to a map f which completes the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "a'"'] & & N \arrow[dl, "s'"] \\ & E[k+1] & \end{tikzcd}$$

Thus, the map f is equivalent to the map (2) and, as consequence, to the map (1). Hence, any morphism from M to N in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is represented by a morphism from M to N in the category $\mathbf{D}^b(\mathrm{gr}{-}A)$.

Now if f is the 0-morphism in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$, then without a loss of generality we can assume that the map a is the zero map. In this case we will have $a' = 0$ as well. This implies that f factors through a morphism $M \to \sigma^{\geq -k}\underline{Q}^{\bullet}$. By the assumption on M any such morphism can be lifted to a morphism $M \to Q^0$. Hence, if f is the 0-morphism in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ then it factors through Q^0. The same proof works in ungraded case (see [22]). □

Next we describe a useful construction utilizing the previous statements. Let $\underline{M}^{\bullet}$ and $\underline{N}^{\bullet}$ be two bounded complexes of (graded) A-modules. Assume that $\mathrm{Hom}(\underline{M}^{\bullet}, A[i])$ in the bounded derived categories of A-modules are trivial except for finite number of $i \in \mathbb{Z}$. By Lemma 1.10 for sufficiently large k there are modules $M, N \in \mathrm{gr}{-}A$ (resp. $M, N \in \mathrm{mod}{-}A$) such that $\underline{M}^{\bullet}$ and $\underline{N}^{\bullet}$ are isomorphic to the images of $M[k]$ and $N[k]$ in the triangulated category of singularities. Moreover it follows immediately from the assumption on $\underline{M}^{\bullet}$ and the construction of M , that for any sufficiently large k we have $\mathrm{Ext}^i_A(M, A) = 0$ whenever $i > 0$. Hence, by Proposition 1.11, we get

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(\underline{M}^{\bullet}, \underline{N}^{\bullet}) \cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(M, N) \cong \mathrm{Hom}_A(M, N)/\mathcal{R}$$

where $\mathcal{R}$ is the subspace of elements factoring through a projective module. This procedure works in the ungraded situation as well.

2. Categories of coherent sheaves and categories of singularities.

2.1. **Quotient categories of graded modules.** Let $A = \bigoplus_{i \ge 0} A_i$ be a noetherian graded algebra. We suppose that A is connected, i.e. $A_0 = \mathbf{k}$. Denote by $\operatorname{tors}-A$ the full subcategory of $\operatorname{gr}-A$ which consists of all graded A-modules which are finite dimensional over $\mathbf{k}$.

An important role will be played by the quotient abelian category $\operatorname{qgr} A = \operatorname{gr}-A/\operatorname{tors}-A$. It has the following explicit description. The objects of $\operatorname{qgr} A$ are the objects of $\operatorname{gr}-A$ (we denote by πM the object in $\operatorname{qgr} A$ which corresponds to a module M). The morphisms in $\operatorname{qgr} A$ are given by

$$\operatorname{Hom}_{\operatorname{qgr}}(\pi M, \pi N) := \varinjlim_{M'} \operatorname{Hom}_{\operatorname{gr}}(M', N) \tag{3}$$

where M' runs over submodules of M such that M/M' is finite dimensional.

Given a graded A-module M and an integer p, the graded A-submodule $\bigoplus_{i \ge p} M_i$ of M is denoted by $M_{\ge p}$ and is called the p-th tail of M. In the same way we can define the p-th tail $\underline{M}^{\bullet}_{\ge p}$ of any complex of modules $\underline{M}^{\bullet}$. Since A is noetherian, we have

$$\operatorname{Hom}_{\operatorname{qgr}}(\pi M, \pi N) = \lim_{p \to \infty} \operatorname{Hom}_{\operatorname{gr}}(M_{\ge p}, N).$$

We will also identify M_p with the quotient $M_{\ge p}/M_{\ge p+1}$.

Similarly, we can consider the subcategory $\operatorname{Tors}-A \subset \operatorname{Gr}-A$ of torsion modules. Recall that a module M is called torsion if for any element $x \in M$ one has $xA_{\ge p} = 0$ for some p. Denote by $\operatorname{QGr} A$ the quotient category $\operatorname{Gr}-A/\operatorname{Tors}-A$. The category $\operatorname{QGr} A$ contains $\operatorname{qgr} A$ as a full subcategory. Sometimes it is convenient to work in $\operatorname{QGr} A$ instead of $\operatorname{qgr} A$.

Denote by $\varPi$ and π the canonical projections of $\operatorname{Gr}-A$ to $\operatorname{QGr} A$ and of $\operatorname{gr}-A$ to $\operatorname{qgr} A$ respectively. The functor $\varPi$ has a right adjoint $\varOmega$ and, moreover, for any $N \in \operatorname{Gr}-A$

$$\varOmega \varPi N \cong \bigoplus_{n=-\infty}^{\infty} \operatorname{Hom}_{\operatorname{QGr}}(\varPi A, \varPi N(n)). \tag{4}$$

For any $i \in \mathbb{Z}$ we can consider the full abelian subcategories $\operatorname{Gr}-A_{\ge i} \subset \operatorname{Gr} A$ and $\operatorname{gr}-A_{\ge i} \subset \operatorname{gr} A$ which consist of all modules M such that $M_p = 0$ when $p < i$. The natural projection functor $\varPi_i : \operatorname{Gr}-A_{\ge i} \longrightarrow \operatorname{QGr}-A$ has a right adjoint $\varOmega_i$ satisfying

$$\varOmega_i \varPi_i N \cong \bigoplus_{n=i}^{\infty} \operatorname{Hom}_{\operatorname{QGr}}(\varPi A, \varPi_i N(n)).$$

Since the category $\operatorname{QGr} A$ is an abelian category with enough injectives there is a right derived functor

$$\mathbf{R}\varOmega_i : \mathbf{D}^+(\operatorname{QGr} A) \longrightarrow \mathbf{D}^+(\operatorname{Gr}-A_{\ge i})$$

defined as

$$\mathbf{R}\Omega_i M \cong \bigoplus_{k=i}^{\infty} \mathbf{R}\operatorname{Hom}_{\mathrm{QGr}}(\Pi A, M(k)). \tag{5}$$

Assume now that the algebra A satisfies condition "χ" from [1, Sec. 3]. We recall that by definition a connected noetherian graded algebra A satisfies condition "χ" if for every $M \in \mathrm{gr}{-}A$ the grading on the space $\operatorname{Ext}^i_A(\mathbf{k}, M)$ is right bounded for all i . In this case it was proved in [1, Prop. 3.14] that the restrictions of the functors Ω_i to the subcategory $\mathrm{qgr}\, A$ give functors $\omega_i : \mathrm{qgr}\, A \longrightarrow \mathrm{gr}{-}A_{\geq i}$ which are right adjoint to π_i. Moreover, it follows from [1, Th. 7.4] that the functor ω_i has a right derived

$$\mathbf{R}\omega_i : \mathbf{D}^+(\mathrm{qgr}\, A) \longrightarrow \mathbf{D}^+(\mathrm{gr}{-}A_{\geq i})$$

and all $\mathbf{R}^j\omega_i \in \mathrm{tors}{-}A$ for $j > 0$.

If, in addition, the algebra A is Gorenstein (i.e. if it has a finite injective dimension n and $D(\mathbf{k}) = \mathbf{R}\operatorname{Hom}_A(\mathbf{k}, A)$ is isomorphic to $\mathbf{k}(a)[-n]$) we obtain the right derived functor

$$\mathbf{R}\omega_i : \mathbf{D}^b(\mathrm{qgr}\, A) \longrightarrow \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$$

between bounded derived categories (see [30, Cor. 4.3]). It is important to note that the functor $\mathbf{R}\omega_i$ is fully faithful because $\pi_i\mathbf{R}\omega_i$ is isomorphic to the identity functor ([1, Prop. 7.2]).

2.2. **Triangulated categories of singularities for Gorenstein algebras.** The main goal of this section is to establish a connection between the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and the derived category $\mathbf{D}^b(\mathrm{qgr}\, A)$, in the case of a Gorenstein algebra A .

When the algebra A has finite injective dimension as right and as left module over itself (i.e. A is a dualizing complex for itself) we get two functors

$$D \;:= \mathbf{R}\operatorname{Hom}_A\,(-, A) : \mathbf{D}^b(\mathrm{gr}{-}A\;)^\circ \longrightarrow \mathbf{D}^b(\mathrm{gr}{-}A^\circ), \tag{6}$$

$$D^\circ := \mathbf{R}\operatorname{Hom}_{A^\circ}(-, A) : \mathbf{D}^b(\mathrm{gr}{-}A^\circ)^\circ \longrightarrow \mathbf{D}^b(\mathrm{gr}{-}A\;), \tag{7}$$

which are quasi-inverse triangulated equivalences (see [29, Prop. 3.5]).

Definition 2.1. *We say that a connected graded noetherian algebra A is Gorenstein if it has a finite injective dimension n and $D(\mathbf{k}) = \mathbf{R}\operatorname{Hom}_A(\mathbf{k}, A)$ is isomorphic to $\mathbf{k}(a)[-n]$ for some integer a , which is called the Gorenstein parameter of A . (Such algebra is also called AS-Gorenstein, where "AS" stands for "Artin-Schelter".)*

Remark 2.2. It is known (see [30, Cor. 4.3]) that any Gorenstein algebra satisfies condition "χ" and for any Gorenstein algebra A and for any $i \in \mathbb{Z}$ we have derived functors

$$\mathbf{R}\omega_i : \mathbf{D}^b(\mathrm{qgr}\, A) \longrightarrow \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$$

which are fully faithful.

Now we describe the images of the functors $\mathbf{R}\omega_i$. Denote by $\mathcal{D}_i$ the subcategories of $\mathbf{D}^b(\mathrm{gr}{-}A)$ which are the images of the composition of $\mathbf{R}\omega_i$ and the natural inclusion of $\mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$ to $\mathbf{D}^b(\mathrm{gr}{-}A)$. All $\mathcal{D}_i$ are equivalent to $\mathbf{D}^b(\mathrm{qgr}\,{-}A)$. Further, for any integer i denote by $\mathcal{S}_{<i}(A)$ (or simple $\mathcal{S}_{<i}$) the full triangulated subcategory of $\mathbf{D}^b(\mathrm{gr}{-}A)$ generated by the modules $\mathbf{k}(e)$ with $e > -i$. In other words, the objects of $\mathcal{S}_{<i}$ are complexes $\underline{M}^{\bullet}$ for which the tail $\underline{M}^{\bullet}_{\geq i}$ is isomorphic to zero. Analogously, we define $\mathcal{S}_{\geq i}$ as the triangulated subcategory which is generated by the objects $\mathbf{k}(e)$ with $e \leq -i$. In other words, the objects of $\mathcal{S}_{\geq i}$ are complexes of torsion modules from $\mathrm{gr}{-}A_{\geq i}$. It is clear that $\mathcal{S}_{<i} \cong \mathcal{S}_{<0}(-i)$ and $\mathcal{S}_{\geq i} \cong \mathcal{S}_{\geq 0}(-i)$.

Furthermore, denote by $\mathcal{P}_{<i}$ the full triangulated subcategory of $\mathbf{D}^b(\mathrm{gr}{-}A)$ generated by the free modules $A(e)$ with $e > -i$ and denote by $\mathcal{P}_{\geq i}$ the triangulated subcategory which is generated by the free modules $A(e)$ with $e \leq -i$. As above we have $\mathcal{P}_{<i} \cong \mathcal{P}_{<0}(-i)$ and $\mathcal{P}_{\geq i} \cong \mathcal{P}_{\geq 0}(-i)$.

Lemma 2.3. *Let* $A = \bigoplus\limits_{i\geq 0} A_i$ *be a connected graded noetherian algebra. Then the subcategories* $\mathcal{S}_{<i}$ *and* $\mathcal{P}_{<i}$ *are left and respectively right admissible for any* $i \in \mathbb{Z}$ *. Moreover, there are weak semiorthogonal decompositions*

$$\mathbf{D}^b(\mathrm{gr}{-}A) = \langle \mathcal{S}_{<i}, \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})\rangle, \qquad \mathbf{D}^b(\mathrm{tors}{-}A) = \langle \mathcal{S}_{<i}, \mathcal{S}_{\geq i}\rangle, \tag{8}$$

$$\mathbf{D}^b(\mathrm{gr}{-}A) = \langle \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i}), \mathcal{P}_{<i}\rangle, \qquad \mathbf{D}^b(\mathrm{grproj}{-}A) = \langle \mathcal{P}_{\geq i}, \mathcal{P}_{<i}\rangle. \tag{9}$$

Proof. For any complex $\underline{M}^{\bullet} \in \mathbf{D}^b(\mathrm{mod}{-}A)$ there is an exact triangle of the form

$$\underline{M}^{\bullet}_{\geq i} \longrightarrow \underline{M}^{\bullet} \longrightarrow \underline{M}^{\bullet}/\underline{M}^{\bullet}_{\geq i}.$$

By definition the object $\underline{M}^{\bullet}/\underline{M}^{\bullet}_{\geq i}$ belongs to $\mathcal{S}_{<i}$ and the object $\underline{M}^{\bullet}_{\geq i}$ is in the left orthogonal ${}^{\perp}\mathcal{S}_{<i}$. Hence, by Remark 1.3, $\mathcal{S}_{<i}$ is left admissible. Moreover, $\underline{M}^{\bullet}_{\geq i}$ also belongs to $\mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$, i.e. $\mathbf{D}^b(\mathrm{gr}{-}A_{\geq i}) \cong {}^{\perp}\mathcal{S}_{<i}$ in the category $\mathbf{D}^b(\mathrm{gr}{-}A)$. If $\underline{M}^{\bullet}$ is a complex of torsion modules then $\underline{M}^{\bullet}_{\geq i}$ belongs to $\mathcal{S}_{\geq i}$. Thus, we obtain both decompositions of (8).

To prove the existence of the decompositions (9) we first note that, due to the connectedness of A , any finitely generated graded projective A-module is free. Second, any finitely generated free module P has a canonical split decomposition of the form

$$0 \longrightarrow P_{<i} \longrightarrow P \longrightarrow P_{\geq i} \longrightarrow 0,$$

where $P_{<i} \in \mathcal{P}_{<i}$ and $P_{\geq i} \in \mathcal{P}_{\geq i}$. Third, any bounded complex of finitely generated A-modules $\underline{M}^{\bullet}$ has a bounded above free resolution $\underline{P}^{\bullet} \to \underline{M}^{\bullet}$ such that $P^{-k} \in \mathcal{P}_{\geq i}$ for all $k \gg 0$. This implies that the object $\underline{P}^{\bullet}_{<i} \in \mathcal{P}_{<i}$ from the exact sequence of complexes

$$0 \longrightarrow \underline{P}^{\bullet}_{<i} \longrightarrow \underline{P}^{\bullet} \longrightarrow \underline{P}^{\bullet}_{\geq i} \longrightarrow 0,$$

is a bounded complex. Since $\underline{P}^{\bullet}$ is quasi-isomorphic to a bounded complex, the complex $\underline{P}^{\bullet}_{\geq i}$ is also quasi-isomorphic to some bounded complex $\underline{K}^{\bullet}$ from $\mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$. Thus, any object $\underline{M}^{\bullet}$ has a decomposition

$$\underline{P}^{\bullet}_{<i} \longrightarrow \underline{M}^{\bullet} \longrightarrow \underline{K}^{\bullet},$$

where $\underline{P}^{\bullet}_{<i} \in \mathcal{P}_{<i}$ and $\underline{K}^{\bullet} \in \mathbf{D}^b(\mathrm{gr}{-}A_{\ge i})$. This proves the decompositions (9). $\square$

Lemma 2.4. *Let $A = \bigoplus_{i\ge 0} A_i$ be a connected graded noetherian algebra which is Gorenstein. Then the subcategories $\mathcal{S}_{\ge i}$ and $\mathcal{P}_{\ge i}$ are right and respectively left admissible. Moreover, for any $i \in \mathbb{Z}$ there are weak semiorthogonal decompositions*

$$\mathbf{D}^b(\mathrm{gr}{-}A_{\ge i}) = \langle \mathcal{D}_i, \mathcal{S}_{\ge i}\rangle, \quad \mathbf{D}^b(\mathrm{gr}{-}A_{\ge i}) = \langle \mathcal{P}_{\ge i}, \mathcal{T}_i\rangle, \tag{10}$$

where the subcategory $\mathcal{D}_i$ is equivalent to $\mathbf{D}^b(\mathrm{qgr}\, A)$ under the functor $\mathbf{R}\omega_i$, and $\mathcal{T}_i$ is equivalent to $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$.

Proof. The functor $\mathbf{R}\omega_i$ is fully faithful and has the left adjoint π_i. Thus, we obtain a semiorthogonal decomposition

$$\mathbf{D}^b(\mathrm{gr}{-}A_{\ge i}) = \langle \mathcal{D}_i, {}^{\perp}\mathcal{D}_i\rangle,$$

where $\mathcal{D}_i \cong \mathbf{D}^b(\mathrm{qgr}\, A)$. Furthermore, the orthogonal ${}^{\perp}\mathcal{D}_i$ consists of all objects $\underline{M}^{\bullet}$ satisfying $\pi_i(\underline{M}^{\bullet}) = 0$. Thus, ${}^{\perp}\mathcal{D}_i$ coincides with $\mathcal{S}_{\ge i}$. Hence, $\mathcal{S}_{\ge i}$ is right admissible in $\mathbf{D}^b(\mathrm{gr}{-}A_{\ge i})$ which is right admissible in whole $\mathbf{D}^b(\mathrm{gr}{-}A)$. This implies that $\mathcal{S}_{\ge i}$ is right admissible in $\mathbf{D}^b(\mathrm{gr}{-}A)$ as well.

The functor D from (6) establishes an equivalence of the subcategory $\mathcal{P}_{\ge i}(A)^{\circ}$ with the subcategory $\mathcal{P}_{<-i+1}(A^{\circ})$ which is right admissible by Lemma 2.3. Hence, $\mathcal{P}_{\ge i}(A)$ is left admissible and there is a decomposition of the form

$$\mathbf{D}^b(\mathrm{gr}{-}A_{\ge i}) = \langle \mathcal{P}_{\ge i}, \mathcal{T}_i\rangle$$

with some $\mathcal{T}_i$.

Now applying Lemma 1.1 to the full embedding of $\mathbf{D}^b(\mathrm{gr}{-}A_{\ge i})$ to $\mathbf{D}^b(\mathrm{gr}{-}A)$ and using Lemma 1.4 we get a fully faithful functor from $\mathcal{T}_i \cong \mathbf{D}^b(\mathrm{gr}{-}A_{\ge i})/\mathcal{P}_{\ge i}$ to $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) = \mathbf{D}^b(\mathrm{gr}{-}A)/\mathbf{D}^b(\mathrm{grproj}{-}A)$. Finally, since this functor is essentially surjective on objects it is actually an equivalence. $\square$

Theorem 2.5. *Let $A = \bigoplus_{i\ge 0} A_i$ be a connected graded noetherian algebra which is Gorenstein with Gorenstein parameter a. Then the triangulated categories $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and $\mathbf{D}^b(\mathrm{qgr}\, A)$ are related as follows:*

(i) *if $a > 0$, there are fully faithful functors $\Phi_i : \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \longrightarrow \mathbf{D}^b(\mathrm{qgr}\, A)$ and semiorthogonal decompositions*

$$\mathbf{D}^b(\mathrm{qgr}\, A) = \langle \pi A(-i-a+1), \dots, \pi A(-i), \Phi_i \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)\rangle,$$

where $\pi : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^b(\mathrm{qgr}\, A)$ is the natural projection;

(ii) *if $a < 0$, there are fully faithful functors $\Psi_i : \mathbf{D}^b(\mathrm{qgr}\, A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and semiorthogonal decompositions*

$$\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) = \langle q\mathbf{k}(-i), \dots, q\mathbf{k}(-i+a+1), \Psi_i \mathbf{D}^b(\mathrm{qgr}\, A)\rangle,$$

where $q : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is the natural projection;

(iii) *if* $a = 0$, *there is an equivalence* $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{qgr} A)$.

Proof. Lemmas 2.3 and 2.4 gives us that the subcategory $\mathcal{T}_i$ is admissible in $\mathbf{D}^b(\mathrm{gr}{-}A)$ and the right orthogonal $\mathcal{T}_i^{\perp}$ has a weak semiorthogonal decomposition of the form

$$\mathcal{T}_i^{\perp} = \langle \mathcal{S}_{<i}, \mathcal{P}_{\geq i} \rangle. \tag{11}$$

Now let us describe the right orthogonal to the subcategory $\mathcal{D}_i$. First, since A is Gorenstein the functor D takes the subcategory $\mathcal{S}_{\geq i}(A)$ to the subcategory $\mathcal{S}_{<-i-a+1}(A^{\circ})$. Hence, D sends the right orthogonal $\mathcal{S}^{\perp}_{\geq i}(A)$ to the left orthogonal ${}^{\perp}\mathcal{S}_{<-i-a+1}(A^{\circ})$ which coincides with the right orthogonal $\mathcal{P}^{\perp}_{<-i-a+1}(A^{\circ})$ by Lemma 2.3. Therefore, the subcategory $\mathcal{S}^{\perp}_{\geq i}$ coincides with ${}^{\perp}\mathcal{P}_{\geq i+a}$. On the other hand, by Lemmas 2.3 and 2.4 we have that

$${}^{\perp}\mathcal{P}_{\geq i+a} = \mathcal{S}^{\perp}_{\geq i} \cong \langle \mathcal{S}_{<i}, \mathcal{D}_i \rangle.$$

This implies that the right orthogonal $\mathcal{D}_i^{\perp}$ has the following decomposition

$$\mathcal{D}_i^{\perp} = \langle \mathcal{P}_{\geq i+a}, \mathcal{S}_{<i} \rangle. \tag{12}$$

Assume that $a \geq 0$. In this case, the decomposition (12) is not only semiorthogonal but is in fact mutually orthogonal, because $\mathcal{P}_{\geq i+a} \subset \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$. Hence, we can interchange $\mathcal{P}_{\geq i+a}$ and $\mathcal{S}_{<i}$, i.e.

$$\mathcal{D}_i^{\perp} = \langle \mathcal{S}_{<i}, \mathcal{P}_{\geq i+a} \rangle.$$

Thus, we obtain that $\mathcal{D}_i^{\perp} \subset \mathcal{T}_i^{\perp}$ and, consequently, $\mathcal{T}_i$ is a full subcategory of $\mathcal{D}_i$. Moreover, we can describe the right orthogonal to $\mathcal{T}_i$ in $\mathcal{D}_i$. Actually, there is a decomposition

$$\mathcal{P}_{\geq i} = \langle \mathcal{P}_{\geq i+a}, \mathcal{P}_i^a \rangle,$$

where $\mathcal{P}_i^a$ is the subcategory generated by the modules $A(-i-a+1), \dots, A(-i)$. Moreover, these modules form an exceptional collection. Thus, the category $\mathcal{D}_i$ has the semiorthogonal decomposition

$$\mathcal{D}_i = \langle A(-i-a+1), \dots, A(-i), \mathcal{T}_i \rangle,$$

Since $\mathcal{D}_i \cong \mathbf{D}^b(\operatorname{qgr} A)$ and $\mathcal{T}_i \cong \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ we obtain the decomposition

$$\mathbf{D}^b(\operatorname{qgr} A) \cong \langle \pi A(-i-a+1), \dots, \pi A(-i), \Phi_i \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \rangle,$$

where the fully faithful functor Φ_i is the composition $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \xrightarrow{\sim} \mathcal{T}_i \hookrightarrow \mathbf{D}^b(\mathrm{gr}{-}A) \xrightarrow{\pi} \mathbf{D}^b(\operatorname{qgr} A)$.

Assume now that $a \leq 0$. In this case, the decomposition (11) is not only semiorthogonal but is in fact mutually orthogonal, because the algebra A is Gorenstein and $\mathbf{R}\operatorname{Hom}_A(\mathbf{k}, A) = \mathbf{k}(a)[-n]$ with $a \leq 0$. Hence, we can interchange $\mathcal{P}_{\geq i}$ and $\mathcal{S}_{<i}$, i.e.

$$\mathcal{T}_i^{\perp} = \langle \mathcal{P}_{\geq i}, \mathcal{S}_{<i} \rangle.$$

Now we see that $\mathcal{T}_i^{\perp} \subset \mathcal{D}_{i-a}^{\perp}$ and, consequently, $\mathcal{D}_{i-a}$ is the full subcategory of $\mathcal{T}_i$. Moreover, we can describe the right orthogonal to $\mathcal{D}_{i-a}$ in $\mathcal{T}_i$. Actually, there is a decomposition of the form

$$\mathcal{S}_{<i-a} = \langle \mathcal{S}_{<i}, \mathbf{k}(-i), \dots, \mathbf{k}(-i+a+1) \rangle.$$

Therefore, the category $\mathcal{T}_i \cong \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ has a semiorthogonal decomposition of the form

$$\mathcal{T}_i = \langle \mathbf{k}(-i), \dots, \mathbf{k}(-i+a+1), \mathcal{D}_{i-a} \rangle, \tag{13}$$

Since $\mathcal{D}_{i-a} \cong \mathbf{D}^b(\operatorname{qgr} A)$ and $\mathcal{T}_i \cong \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ we obtain the decomposition

$$\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \cong \langle q\mathbf{k}(-i), \dots, q\mathbf{k}(-i+a+1), \Psi_i \mathbf{D}^b(\operatorname{qgr} A) \rangle,$$

where the fully faithful functor Ψ_i can be defined as the composition $\mathbf{D}^b(\operatorname{qgr} A) \xrightarrow{\sim} \mathcal{D}_{i-a} \hookrightarrow \mathbf{D}^b(\mathrm{gr}-A) \xrightarrow{q} \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. If $a=0$, then we get equivalence. □

Remark 2.6. It follows from the construction that the functor Ψ_{i+a} from the bounded derived category $\mathbf{D}^b(\operatorname{qgr} A)$ to $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is the composition of the functor $\mathbf{R}\omega_i : \mathbf{D}^b(\operatorname{qgr} A) \longrightarrow \mathbf{D}^b(\mathrm{gr}-A_{\geq i})$, which is given by formula (5), natural embedding $\mathbf{D}^b(\mathrm{gr}-A_{\geq i}) \hookrightarrow \mathbf{D}^b(\mathrm{gr}-A)$, and the projection $\mathbf{D}^b(\mathrm{gr}-A) \xrightarrow{q} \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$.

Let us consider two limiting cases. The first case is when the algebra A has finite homological dimension. In this case the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is trivial and, hence, the Gorenstein parameter a is non-negative and the derived category $\mathbf{D}^b(\operatorname{qgr} A)$ has a full exceptional collection $\sigma = \big(\pi A(0), \dots, \pi A(a-1)\big)$. More precisely we have the following:

Corollary 2.7. *Let $A = \bigoplus_{i \geq 0} A_i$ be a connected graded noetherian algebra which is Gorenstein with Gorenstein parameter a. Suppose that A has finite homological dimension. Then, $a \geq 0$ and the derived category $\mathbf{D}^b(\operatorname{qgr} A)$ has a full strong exceptional collection $\sigma = (\pi A(0), \dots, \pi A(a-1))$. Moreover, the category $\mathbf{D}^b(\operatorname{qgr} A)$ is equivalent to the derived category $\mathbf{D}^b(\mathrm{mod}-\mathrm{Q}(A))$ of finite (right) modules over the algebra $\mathrm{Q}(A) := \operatorname{End}_{\mathrm{gr}-A}\left(\bigoplus_{i=0}^{a-1} A(i)\right)$ of homomorphisms of σ.*

Proof. Since A has a finite homological dimension the category $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is trivial. By Theorem 2.5 we get that $a \geq 0$ and that $\mathbf{D}^b(\operatorname{qgr} A)$ has a full exceptional collection $\sigma = (\pi A(0), \dots, \pi A(a-1))$. Consider the object $P_\sigma = \bigoplus_{i=0}^{a-1} \pi A(i)$ and the functor

$$\operatorname{Hom}(P_\sigma, -) : \operatorname{qgr} A \longrightarrow \mathrm{mod}-\mathrm{Q}(A),$$

where $\mathrm{Q}(A) = \operatorname{End}_{\operatorname{qgr} A}\left(\bigoplus_{i=0}^{a-1} \pi A(i)\right) = \operatorname{End}_{\mathrm{gr}-A}\left(\bigoplus_{i=0}^{a-1} A(i)\right)$ is the algebra of homomorphisms of the exceptional collection σ. It is easy to see that this functor has a right derived functor

$$\mathbf{R}\operatorname{Hom}(P_\sigma, -) : \mathbf{D}^b(\operatorname{qgr} A) \longrightarrow \mathbf{D}^b(\mathrm{mod}-\mathrm{Q}(A))$$

(e.g. as a composition $\mathbf{R}\omega_0$ and $\operatorname{Hom}\big(\bigoplus_{i=0}^{a-1} A(i), -\big)$). The standard reasoning (see e.g. [6] or [7]) now shows that the functor $\mathbf{R}\operatorname{Hom}(P_\sigma, -)$ is an equivalence. □

Example 2.8. As an application we obtain a well-known result (see [4]) asserting the existence of a full exceptional collection in the bounded derived category of coherent sheaves on the projective space $\mathbb{P}^n$. This result follows immediately if we take $A = \mathbf{k}[x_0, \dots, x_n]$ with its standard grading. More generally, if we take A to be the polynomial algebra $\mathbf{k}[x_0, \dots, x_n]$ graded by $\deg x_i = a_i$, then we get a full exceptional collection $\big(\mathcal{O},\dots,\mathcal{O}(\sum_{i=0}^{n}a_i-1)\big)$ in the bounded derived category of coherent sheaves on the weighted projective space $\mathbb{P}(a_0,\dots,a_n)$ considered as a smooth orbifold (see [3, 2]). It is also true for noncommutative (weighted) projective spaces [2].

Another limiting case is when the algebra A is finite dimensional over the base field (i.e. A is a Frobenius algebra). In this case the category $\operatorname{qgr} A$ is trivial and, hence, the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ has a full exceptional collection (compare with [13, 10.10]). More precisely we get the following:

Corollary 2.9. *Let $A=\bigoplus\limits_{i\ge 0}A_i$ be a connected graded noetherian algebra which is Gorenstein with Gorenstein parameter a. Suppose that A is finite dimensional over the field $\mathbf{k}$. Then, $a\le 0$ and the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ has a full exceptional collection $(q\mathbf{k}(0),\dots,q\mathbf{k}(a+1))$, where $q:\mathbf{D}^b(\mathrm{gr}{-}A)\longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is the natural projection. Moreover, the triangulated category $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is equivalent to the derived category $\mathbf{D}^b(\mathrm{mod}{-}\,\mathrm{Q}(A))$ of finite (right) modules over the algebra $\mathrm{Q}(A)=\mathrm{End}_{\mathrm{gr}-A}\left(\bigoplus\limits_{i=a+1}^{0}A(i)\right)$.*

Proof. Since A is finite dimensional the derived category $\mathbf{D}^b(\operatorname{qgr} A)$ is trivial. By Theorem 2.5 we get that $a\le 0$ and $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ has a full exceptional collection $(q\mathbf{k}(0),\dots,q\mathbf{k}(a+1))$. Unfortunately, this collection is not strong. However, we can replace it by the dual exceptional collection which is already strong (see Definition 1.7). By Lemma 2.4 there is a weak semiorthogonal decomposition $\mathbf{D}^b(\mathrm{gr}{-}A_{\ge 0})=\langle\mathcal{P}_{\ge 0},\mathcal{T}_0\rangle$, where $\mathcal{T}_0$ is equivalent to $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. Moreover, by formula (13) we have the following semiorthogonal decomposition for $\mathcal{T}_0$:

$$\mathcal{T}_0=\langle\mathbf{k}(0),\dots,\mathbf{k}(a+1)\rangle.$$

Denote by E_i where $i=0,\dots,-a-1$ the modules $A(i+a+1)/A(i+a+1)_{\ge a}$. These modules belong to $\mathcal{T}_0$ and form a full exceptional collection

$$\mathcal{T}_0=\langle E_0,\dots,E_{-(a+1)}\rangle.$$

Furthermore, this collection is strong and the algebra of homomorphisms of this collection coincides with the algebra $\mathrm{Q}(A)=\mathrm{End}_{\mathrm{gr}-A}\left(\bigoplus\limits_{i=a+1}^{0}A(i)\right)$. As in the previous proposition consider the object $E=\bigoplus\limits_{i=0}^{-(a+1)}E_i$ and the functor

$$\mathbf{R}\operatorname{Hom}(E,-):\mathcal{T}_0=\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)\longrightarrow\mathbf{D}^b(\mathrm{mod}{-}\,\mathrm{Q}(A)).$$

Again the standard reasoning from ([6, 7]) shows that the functor $\mathbf{R}\operatorname{Hom}(E,-)$ is an equivalence of triangulated categories. □

Example 2.10. The simplest example here is $A=\mathbf{k}[x]/x^{n+1}$. In this case the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ has a full exceptional collection and is equivalent to the bounded derived category of finite dimensional representations of the Dynkin quiver of type

$A_n : \underbrace{\bullet - \bullet - \cdots - \bullet}_{n}$, because in this case the algebra $\mathrm{Q}(A)$ is isomorphic to the path algebra of this Dynkin quiver. This example is considered in detail in the paper [26].

Remark 2.11. There are other cases when the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ has a full exceptional collection. It follows from Theorem 2.5 that if $a \le 0$ and the derived category $\mathbf{D}^b(\operatorname{qgr} A)$ has a full exceptional collection, then $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ has a full exceptional collection as well. It happens, for example, in the case when the algebra A is related to a weighted projective line – an orbifold over $\mathbb{P}^1$ (see e.g. [11]).

2.3. **Categories of coherent sheaves for Gorenstein varieties.** Let X be an irreducible projective Gorenstein variety of dimension n and let $\mathcal{L}$ be a very ample line bundle such that the dualizing sheaf ω_X is isomorphic to $\mathcal{L}^{-r}$ for some $r \in \mathbb{Z}$. Denote by A the graded coordinate algebra $\bigoplus_{i\ge 0} H^0(X, \mathcal{L}^i)$. The famous Serre theorem [25] asserts that the abelian category of coherent sheaves $\operatorname{coh}(X)$ is equivalent to the quotient category $\operatorname{qgr} A$.

Assume also that $H^j(X, \mathcal{L}^k) = 0$ for all $k \in \mathbb{Z}$ when $j \ne 0, n$. For example, if X is a complete intersection in $\mathbb{P}^N$ then it satisfies these conditions. In this case, Theorem 2.5 allows us to compare the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ with the bounded derived category of coherent sheaves $\mathbf{D}^b(\operatorname{coh}(X))$. To apply that theorem we need the following lemma.

Lemma 2.12. *Let X be a projective Gorenstein (irreducible) variety of dimension n. Let $\mathcal{L}$ be a very ample line bundle such that $\omega_X \cong \mathcal{L}^{-r}$ for some $r \in \mathbb{Z}$ and $H^j(X, \mathcal{L}^k) = 0$ for all $k \in \mathbb{Z}$ when $j \ne 0, n$. Then the algebra $A = \bigoplus\limits_{i\ge 0} H^0(X, \mathcal{L}^i)$ is Gorenstein with Gorenstein parameter $a = r$.*

Proof. Consider the projection functor $\Pi : \mathrm{Gr}{-}A \to \operatorname{QGr} A$ and its right adjoint $\Omega : \operatorname{QGr} A \to \mathrm{Gr}{-}A$ which is given by the formula (4)

$$\Omega\Pi N \cong \bigoplus_{n=-\infty}^{\infty} \operatorname{Hom}_{\mathrm{QGr}}(\Pi A, \Pi N(n)).$$

The functor Ω has a right derived $\mathbf{R}\Omega$ that is given by the formula

$$\mathbf{R}^j\Omega(\Pi N) \cong \bigoplus_{n=-\infty}^{\infty} \operatorname{Ext}^j_{\mathrm{QGr}}(\Pi A, \Pi N(n))$$

(see, e.g. [1, Prop. 7.2]). The assumptions on X and $\mathcal{L}$ imply that $\mathbf{R}^j\Omega(\Pi A) \cong 0$ for all $j \ne 0, n$. Moreover, since X is Gorenstein and $\omega_X \cong \mathcal{L}^{-r}$, Serre duality for X yields that

$$\mathbf{R}^0\Omega(\Pi A) \cong \bigoplus_{i=-\infty}^{\infty} H^0(X, \mathcal{L}^i) \cong A \qquad \text{and} \qquad \mathbf{R}^n\Omega(\Pi A) \cong \bigoplus_{i=-\infty}^{\infty} H^n(X, \mathcal{L}^i) \cong A^*(r),$$

where $A^* = \operatorname{Hom}_{\mathbf{k}}(A, \mathbf{k})$. As X is irreducible, the algebra A is connected. Since Π and $\mathbf{R}\Omega$ are adjoint functors we have

$$\mathbf{R}\operatorname{Hom}_{\mathrm{Gr}}(\mathbf{k}(s), \mathbf{R}\Omega(\Pi A)) \cong \mathbf{R}\operatorname{Hom}_{\mathrm{QGr}}(\Pi\mathbf{k}(s), \Pi A) = 0$$

for all s. Furthermore, we know that $\mathbf{R}\operatorname{Hom}_A(\mathbf{k}, A^*) \cong \mathbf{R}\operatorname{Hom}_A(A, \mathbf{k}) \cong \mathbf{k}$. This implies that $\mathbf{R}\operatorname{Hom}_A(\mathbf{k}, A) \cong \mathbf{k}(r)[-n-1]$. This isomorphism implies that the affine cone $\mathbf{Spec}A$ is Gorenstein at the vertex and the assumption on X now implies that $\mathbf{Spec}A$ is Gorenstein scheme ([14, V, §9,10]). Since $\mathbf{Spec}A$ has a finite Krull dimension, the algebra A is a dualizing complex for itself, i.e. it has a finite injective dimension. Thus, the algebra A is Gorenstein with parameter r. □

Theorem 2.13. *Let X be an irreducible projective Gorenstein variety of dimension n. Let $\mathcal{L}$ be a very ample line bundle such that $\omega_X \cong \mathcal{L}^{-r}$ for some $r \in \mathbb{Z}$. Suppose $H^j(X, \mathcal{L}^k) = 0$ for all $k \in \mathbb{Z}$ when $j \neq 0, n$. Set $A := \bigoplus\limits_{i \geq 0} H^0(X, \mathcal{L}^i)$. Then, the derived category of coherent sheaves $\mathbf{D}^b(\operatorname{coh}(X))$ and the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ are related as follows:*

(i) *if $r > 0$, i.e. if X is a Fano variety, then there is a semiorthogonal decomposition*

$$\mathbf{D}^b(\operatorname{coh}(X)) = \langle \mathcal{L}^{-r+1}, \dots, \mathcal{O}_X, \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \rangle,$$

(ii) *if $r < 0$, i.e. if X is a variety of general type, then there is a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) = \langle q\mathbf{k}(r+1), \dots, q\mathbf{k}, \mathbf{D}^b(\operatorname{coh}(X)) \rangle,$$

where $q : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is the natural projection,

(iii) *if $r = 0$, i.e. if X is a Calabi-Yau variety, then there is an equivalence*

$$\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{coh}(X)).$$

Proof. Since $\mathcal{L}$ is very ample Serre's theorem implies that the bounded derived category $\mathbf{D}^b(\operatorname{coh}(X))$ is equivalent to the category $\mathbf{D}^b(\operatorname{qgr} A)$, where $A = \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i)$. Since $H^j(X, \mathcal{L}^k) = 0$ for $j \neq 0, n$ and all $k \in \mathbb{Z}$, Lemma 2.12 implies that A is Gorenstein. Now, the theorem immediately follows from Theorem 2.5. □

Corollary 2.14. *Let X be an irreducible projective Gorenstein Fano variety of dimension n with at most rational singularities. Let $\mathcal{L}$ be a very ample line bundle such that $\omega_X^{-1} \cong \mathcal{L}^r$ for some $r \in \mathbb{N}$. Set $A = \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i)$. Then the category $\mathbf{D}^b(\operatorname{coh}(X))$ admits a semiorthogonal decomposition of the form*

$$\mathbf{D}^b(\operatorname{coh}(X)) = \langle \mathcal{L}^{-r+1}, \dots, \mathcal{O}_X, \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \rangle.$$

Proof. The Kawamata-Viehweg vanishing theorem (see, e.g. [18, Th. 1.2.5]) yields $H^j(X, \mathcal{L}^k) = 0$ for $j \neq 0, n$ and all k. Hence, we can apply Theorem 2.13(i). □

Corollary 2.15. *Let X be a Calabi-Yau variety. That is, X is an irreducible projective variety with at most rational singularities, with trivial canonical sheaf $\omega_X \cong \mathcal{O}_X$, and such that $H^j(X, \mathcal{O}_X) = 0$ for $j \neq 0, n$. Let $\mathcal{L}$ be a some very ample line bundle on X. Set $A = \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i)$. Then there is an equivalence*

$$\mathbf{D}^b(\operatorname{coh}(X)) \cong \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A).$$

Proof. The variety X has rational singularities hence it is Cohen-Macaulay. Moreover, X is Gorenstein, because $\omega_X \cong \mathcal{O}_X$. The Kawamata-Viehweg vanishing theorem ([18, Th. 1.2.5]) yields $H^j(X, \mathcal{L}^k) = 0$ for $j \neq 0, n$ and all $k \neq 0$. Since by assumption $H^j(X, \mathcal{O}_X) = 0$ for $j \neq 0, n$, we can apply Theorem 2.13 (iii). $\square$

Proposition 2.16. *Let* $X \subset \mathbb{P}^N$ *be a complete intersection of* m *hypersurfaces* $D_1, \dots, D_m$ *of degrees* $d_1, \dots, d_m$ *respectively. Then* X *and* $\mathcal{L} = \mathcal{O}_X(1)$ *satisfy the conditions of Theorem 2.13 with Gorenstein parameter* $r = N + 1 - \sum_{i=1}^m d_i$.

Proof. Since the variety X is a complete intersection it is Gorenstein. The canonical class ω_X is isomorphic to $\mathcal{O}(\sum d_i - N - 1)$. It can be easily proved by induction on m that $H^j(X, \mathcal{O}_X(k)) = 0$ for all k and $j \neq 0, n,$ where $n = N - m$ is the dimension of X. Indeed, The base of the induction is clear. For the induction step, assume that for $Y = D_1 \cap \dots \cap D_{m-1}$ these conditions hold. Then, consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y(k - d_m) \longrightarrow \mathcal{O}_Y(k) \longrightarrow \mathcal{O}_X(k) \longrightarrow 0.$$

Since the cohomologies $H^j(Y, \mathcal{O}_Y(k)) = 0$ for all k and $j \neq 0, n + 1$ we obtain that $H^j(X, \mathcal{O}_X(k)) = 0$ for all k and $j \neq 0, n$. $\square$

Theorem 2.13 can be extended to the case of quotient stacks. To do this we will need an appropriate generalization of Serre's theorem [25]. The usual Serre theorem says that if a commutative connected graded algebra $A = \bigoplus_{i \geq 0} A_i$ is generated by its first component, then the category $\operatorname{qgr} A$ is equivalent to the category of coherent sheaves $\operatorname{coh}(X)$ on the projective variety $X = \mathbf{Proj}\, A$. (Such equivalence holds for the categories of quasicoherent sheaves $\operatorname{Qcoh}(X)$ and $\operatorname{QGr} A$ too.)

Consider now a commutative connected graded $\mathbf{k}$-algebra $A = \bigoplus_{i \geq 0} A_i$ which is not necessary generated by its first component. The grading on A induces an action of the group $\mathbf{k}^*$ on the affine scheme $\mathbf{Spec} A$. Let $\mathbf{0}$ be the closed point of $\mathbf{Spec} A$ that corresponds to the ideal $A_+ = A_{\geq 1} \subset A$. This point is invariant under the action.

Denote by $\mathbb{P}\mathrm{roj}\, A$ the quotient stack $\left[(\mathbf{Spec} A \backslash \mathbf{0})/\mathbf{k}^*\right]$. (Note that there is a natural map $\mathbb{P}\mathrm{roj}\, A \to \mathbf{Proj}\, A,$ which is an isomorphism if the algebra A is generated by A_1 .)

Proposition 2.17. (see also [2]) *Let* $A = \underset{i \geq 0}{\oplus} A_i$ *be a connected graded finitely generated algebra. Then the category of (quasi)coherent sheaves on the quotient stack* $\mathbb{P}\mathrm{roj}\,(A)$ *is equivalent to the category* $\operatorname{qgr} A$ *(respectively* $\operatorname{QGr} A$ *).*

Proof. Let $\mathbf{0}$ be the closed point on the affine scheme $\mathbf{Spec} A$ which corresponds to the maximal ideal $A_+ \subset A$. Denote by U the complement $\mathbf{Spec} A \backslash \mathbf{0}$. We know that the category of (quasi)coherent sheaves on the stack $\mathbb{P}\mathrm{roj}\, A$ is equivalent to the category of $\mathbf{k}^*$-equivariant (quasi)coherent sheaves on U. The category of (quasi)coherent sheaves on U is equivalent to the quotient of the category of (quasi)coherent sheaves on $\mathbf{Spec} A$ by the subcategory of (quasi)coherent sheaves with support on $\mathbf{0}$ (see [9]). This is also true for the categories of $\mathbf{k}^*$-equivariant sheaves. But the category of (quasi)coherent $\mathbf{k}^*$-equivariant sheaves on $\mathbf{Spec}A$ is just the category $\mathrm{gr}{-}A$ (resp. $\mathrm{Gr}{-}A$) of graded modules over A, and the subcategory of (quasi)coherent sheaves with support on $\mathbf{0}$ coincides with the subcategory $\mathrm{tors}{-}A$ (resp. $\mathrm{Tors}{-}A$). Thus, we obtain that $\mathrm{coh}(\mathbb{P}\mathrm{roj}\, A)$ is equivalent to the quotient category $\mathrm{qgr}\, A = \mathrm{gr}{-}A/\,\mathrm{tors}{-}A$ (and $\mathrm{Qcoh}(\mathbb{P}\mathrm{roj}\, A)$ is equivalent to $\mathrm{QGr}\, A = \mathrm{Gr}{-}A/\,\mathrm{Tors}{-}A$). □

Corollary 2.18. *Assume that the noetherian Gorenstein connected graded algebra A from Theorem 2.5 is finitely generated and commutative. Then instead the bounded derived category $\mathbf{D}^b(\mathrm{qgr}\, A)$ in Theorem 2.5 we can substitute the category $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}\mathrm{roj}\, A))$, where $\mathbb{P}\mathrm{roj}\, A$ the quotient stack $\left[(\mathbf{Spec}A\backslash\mathbf{0})/\mathbf{k}^*\right]$.*

3. Categories of graded D-branes of type B in Landau-Ginzburg models.

3.1. **Categories of graded pairs.** Let $B = \bigoplus_{i\ge 0} B_i$ be a finitely generated connected graded algebra over a field $\mathbf{k}$. Let $W \in B_n$ be a central element of degree n which is not a zero-divisor, i.e. $Wb = bW$ for any $b \in B$ and $bW = 0$ only for $b = 0$. Denote by J the two-sided ideal $J := WB = BW$ and denote by A the quotient graded algebra B/J.

With any such element $W \in B_n$ we can associate two categories: an exact category $\mathrm{GrPair}(W)$ and a triangulated category $\mathrm{DGrB}(W)$.[1] Objects of these categories are ordered pairs

$$\overline{P} := \left(P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \right)$$

where $P_0, P_1 \in \mathrm{gr}{-}B$ are finitely generated free graded right B-modules, p_1 is a map of degree 0 and p_0 is a map of degree n (i.e a map from P_0 to $P_1(n)$) such that the compositions p_0p_1 and $p_1(n)p_0$ are the left multiplications by the element W . A morphism $f : \overline{P} \to \overline{Q}$ in the category $\mathrm{GrPair}(W)$ is a pair of morphisms $f_1 : P_1 \to Q_1$ and $f_0 : P_0 \to Q_0$ of degree 0 such that $f_1(n)p_0 = q_0 f_0$ and $q_1 f_1 = f_0 p_1$. The morphism $f = (f_1, f_0)$ is null-homotopic if there are two morphisms $s : P_0 \to Q_1$ and $t : P_1 \to Q_0(-n)$ such that $f_1 = q_0(n)t + sp_1$ and $f_0 = t(n)p_0 + q_1 s$. Morphisms in the category $\mathrm{DGrB}(W)$ are the classes of morphisms in $\mathrm{GrPair}(W)$ modulo null-homotopic morphisms.

In other words, objects of both categories are quasi-periodic infinite sequences

$$\underline{K}^{\bullet} := \{\cdots \longrightarrow K^i \xrightarrow{k^i} K^{i+1} \xrightarrow{k^{i+1}} K^{i+2} \longrightarrow \cdots\},$$

of morphisms in $\mathrm{gr}{-}B$ of *free* graded right B-modules so that the composition $k^{i+1}k^i$ of any two consecutive morphisms is equal to multiplication by W . The quasi-periodicity property here means that $\underline{K}^{\bullet}[2] = \underline{K}^{\bullet}(n)$. In particular

$$K^{2i-1} \cong P_1(i \cdot n),\ K^{2i} \cong P_0(i \cdot n),\ k^{2i-1} = p_1(i \cdot n),\ k^{2i} = p_0(i \cdot n).$$

A morphism $f : \underline{K}^{\bullet} \longrightarrow \underline{L}^{\bullet}$ in the category $\mathrm{GrPair}(W)$ is a family of morphisms $f^i : K^i \longrightarrow L^i$ in $\mathrm{gr}{-}B$ which is quasi-periodic, i.e $f^{i+2} = f^i(n)$, and which commutes with k^i and l^i, i.e. $f^{i+1}k^i = l^i f^i$.

[1] One can also construct a differential graded category the homotopy category of which is equivalent to DGrB .

Morphisms in the category $\mathrm{DGrB}(W)$ are morphisms in $\mathrm{GrPair}(W)$ modulo null-homotopic morphisms, and we consider only quasi-periodic homotopies, i.e. such families $s^i : K^i \longrightarrow L^{i-1}$ that $s^{i+2} = s^i(n)$.

Definition 3.1. *The category* $\mathrm{DGrB}(W)$ *constructed above will be called* the category of graded D-branes of type B *for the pair* $(B = \bigoplus_{i\geq 0} B_i, W)$.

Remark 3.2. If B is commutative, then we can consider the affine scheme $\mathbf{Spec}B$. The grading on B corresponds to an action of the algebraic group $\mathbf{k}^*$ on $\mathbf{Spec}B$. The element W can be viewed as a regular function on $\mathbf{Spec}B$ which is semi-invariant with respect to this action. This way, we get a singular Landau-Ginzburg model $(\mathbf{Spec}B, W)$ with an action of torus $\mathbf{k}^*$. Thus, Definition 3.1 is a definition of the category of *graded* D-branes of type B for this model (see also [16, 28]).

It is clear that the category $\mathrm{GrPair}(W)$ is an exact category (see [24] for the definition) with monomorphisms and epimorphisms being the componentwise monomorphisms and epimorphisms. The category $\mathrm{DGrB}(W)$ can be endowed with a natural structure of a triangulated category. To exhibit this structure we have to define a translation functor $[1]$ and a class of exact triangles.

The translation functor as usually is defined as a functor that takes an object $\underline{K}^\bullet$ to the object $\underline{K}^\bullet[1]$, where $K[1]^i = K^{i+1}$ and $d[1]^i = -d^{i+1}$, and takes a morphism f to the morphism $f[1]$ which coincides with f componentwise.

For any morphism $f : \underline{K}^\bullet \to \underline{L}^\bullet$ from the category $\mathrm{GrPair}(W)$ we define a mapping cone $\underline{C}^\bullet(f)$ as an object

$$\underline{C}^\bullet(f) = \{\cdots \longrightarrow L^i \oplus K^{i+1} \xrightarrow{c^i} L^{i+1} \oplus K^{i+2} \xrightarrow{c^{i+1}} L^{i+2} \oplus K^{i+3} \longrightarrow \cdots\}$$

such that

$$c^i = \begin{pmatrix} l^i & f^{i+1} \\ 0 & -k^{i+1} \end{pmatrix}.$$

There are maps $g : \underline{L}^\bullet \to \underline{C}^\bullet(f)$, $g = (\mathrm{id}, 0)$ and $h : \underline{C}^\bullet(f) \to \underline{K}^\bullet[1]$, $h = (0, -\mathrm{id})$.

Now we define a standard triangle in the category $\mathrm{DGrB}(W)$ as a triangle of the form

$$\underline{K}^\bullet \xrightarrow{f} \underline{L}^\bullet \xrightarrow{g} \underline{C}^\bullet(f) \xrightarrow{h} \underline{K}^\bullet[1].$$

for some $f \in \mathrm{GrPair}(W)$.

Definition 3.3. *A triangle* $\underline{K}^\bullet \to \underline{L}^\bullet \to \underline{M}^\bullet \to \underline{K}^\bullet[1]$ *in* $\mathrm{DGrB}(W)$ *will be called an exact (distinguished) triangle if it is isomorphic to a standard triangle.*

Proposition 3.4. *The category* $\mathrm{DGrB}(W)$ *endowed with the translation functor* $[1]$ *and the above class of exact triangles becomes a triangulated category.*

We omit the proof of this proposition which is more or less the same as the proof of the analogous result for a usual homotopic category (see, e.g. [12]).

3.2. **Categories of graded pairs and categories of singularities.** With any object $\underline{K}^{\bullet}$ as above, one associates a short exact sequence

$$0 \longrightarrow K^{-1} \xrightarrow{k^{-1}} K^0 \longrightarrow \mathrm{Coker}\, k^{-1} \longrightarrow 0. \tag{14}$$

We can attach to an object $\underline{K}^{\bullet}$ the right B-module $\mathrm{Coker}\, k^{-1}$. It can be easily checked that the multiplication by W annihilates it. Hence, the module $\mathrm{Coker}\, k^{-1}$ is naturally a right A-module, where $A = B/J$ with $J = WB = BW$. Any morphism $f : \underline{K}^{\bullet} \to \underline{L}^{\bullet}$ in $\mathrm{GrPair}(W)$ induces a morphism between cokernels. This construction defines a functor $\mathrm{Cok} : \mathrm{GrPair}(W) \longrightarrow \mathrm{gr}{-}A$. Using the functor Cok we can construct an exact functor between triangulated categories $\mathrm{DGrB}(W)$ and $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$.

Proposition 3.5. *There is a functor F which completes the following commutative diagram*

$$\begin{tikzcd}
\mathrm{GrPair}(W) \arrow[r, "\mathrm{Cok}"] \arrow[d] & \mathrm{gr}{-}A \arrow[d] \\
\mathrm{DGrB}(W) \arrow[r, "F"] & \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A).
\end{tikzcd}$$

Moreover, the functor F is an exact functor between triangulated categories.

Proof. We have the functor $\mathrm{GrPair}(W) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ which is the composition of Cok and the natural functor from $\mathrm{gr}{-}A$ to $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. To prove the existence of a functor F we need to show that any morphism $f : \underline{K}^{\bullet} \to \underline{L}^{\bullet}$ which is null-homotopic goes to the 0-morphism in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. Fix a homotopy $s = (s^i)$ with $s^i : K^i \to L^{i-1}$. Consider the following decomposition of f :

$$\begin{tikzcd}
K^{-1} \arrow[r, "k^{-1}"] \arrow[d, "(s^{-1}, f^{-1})"'] & K^0 \arrow[r] \arrow[d, "(s^0, f^0)"] & \mathrm{Coker}\, k^{-1} \arrow[d] \\
L^{-2} \oplus L^{-1} \arrow[r, "u^{-1}"] \arrow[d, "pr"'] & L^{-1} \oplus L^0 \arrow[r] \arrow[d, "pr"] & L^0 \otimes_B A \arrow[d] \\
L^{-1} \arrow[r, "l^{-1}"] & L^0 \arrow[r] & \mathrm{Coker}\, l^{-1}
\end{tikzcd} \qquad \text{where} \qquad u^{-1} = \begin{pmatrix} -l^{-2} & \mathrm{id} \\ 0 & l^{-1} \end{pmatrix},$$

This yields a decomposition of $F(f)$ through a locally free object $L^0 \otimes_B A$. Hence, $F(f) = 0$ in the category $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. By Lemma 3.7, which is proved below, the tensor product $\underline{K}^{\bullet} \otimes_B A$ is an acyclic complex. Hence, there is an exact sequence $0 \to \mathrm{Coker}\, k^{-1} \to K^1 \otimes_B A \to \mathrm{Coker}\, k^0 \to 0.$ Since $K^1 \otimes_B A$ is free, we have $\mathrm{Coker}\, k^0 \cong \mathrm{Coker}\, k^{-1}[1]$ in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. But, $\mathrm{Coker}\, k^0 = F(\underline{K}^{\bullet}[1])$. Hence, the functor F commutes with translation functors. It is easy to see that F takes a standard triangle in $\mathrm{DGrB}(W)$ to an exact triangle in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. Thus, F is exact. □

Lemma 3.6. *The functor* Cok *is full.*

Proof. Any map $g : \mathrm{Coker}\ k^{-1} \to \mathrm{Coker}\ l^{-1}$ between A-modules can be considered as the map of B-modules and can be extended to a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{-1} & \xrightarrow{k^{-1}} & K^0 & \longrightarrow & \mathrm{Coker}\ k^{-1} & \longrightarrow & 0 \\ & & {\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & L^{-1} & \xrightarrow{l^{-1}} & L^0 & \longrightarrow & \mathrm{Coker}\ l^{-1} & \longrightarrow & 0, \end{array}$$

because K^0 is free. This gives us a sequence of morphisms $f = (f^i), i \in \mathbb{Z}$, where $f^{2i} = f^0(in)$ and $f^{2i-1} = f^{-1}(in)$. To prove the lemma it is sufficient to show that the family f is a map from $\underline{K}^\bullet$ to $\underline{L}^\bullet$, i.e $f^1k^0 = l^0f^0$. Consider the sequence of equalities

$$l^1(f^1k^0 - l^0f^0) = f^2k^1k^0 - Wf^0 = f^2W - Wf^0 = f^0(2)W - Wf^0 = 0.$$

Since l^1 is an embedding, we obtain that $f^1k^0 = l^0f^0$. □

Lemma 3.7. *For any sequence* $\underline{K}^\bullet \in \mathrm{GrPair}(W)$ *the tensor product* $\underline{K}^\bullet \otimes_B A$ *is an acyclic complex of* A*-modules and the* A*-module* $\mathrm{Coker}\ k^{-1}$ *satisfies the condition*

$$\mathrm{Ext}^i_A(\mathrm{Coker}\ k^{-1}, A) = 0 \qquad \textit{for all} \quad i > 0.$$

Proof. It is clear that $\underline{K}^\bullet \otimes_B A$ is a complex. Applying the Snake Lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{i-2} & \xrightarrow{k^{i-2}} & K^{i-1} & \longrightarrow & \mathrm{Coker}\ k^{i-2} & \longrightarrow & 0 \\ & & {\scriptstyle W}\downarrow & & \downarrow{\scriptstyle W} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & K^{i} & \xrightarrow{k^{i}} & K^{i+1} & \longrightarrow & \mathrm{Coker}\ k^{i} & \longrightarrow & 0, \end{array}$$

we obtain an exact sequence

$$0 \to \mathrm{Coker}\ k^{i-2} \longrightarrow K^i \otimes_B A \xrightarrow{k_i|_W} K^{i+1} \otimes_B A \longrightarrow \mathrm{Coker}\ k^i \to 0.$$

This implies that $\underline{K}^\bullet \otimes_B A$ is an acyclic complex.

Further, consider the dual sequence of left B-modules $\underline{K}^{\bullet\vee}$, where $\underline{K}^{\bullet\vee} \cong \mathrm{Hom}_B(\underline{K}^\bullet, B)$. By the same reasons as above $A \otimes_B \underline{K}^{\bullet\vee}$ is an acyclic complex. On the other hand, the cohomologies of the complex $\{(K^0)^\vee \longrightarrow (K^{-1})^\vee \longrightarrow (K^{-2})^\vee \longrightarrow \cdots\}$ are isomorphic to $\mathrm{Ext}^i_A(\mathrm{Coker}\ k^{-1}, A)$. And so, by the acyclicity of $A \otimes_B \underline{K}^{\bullet\vee}$, they are equal to 0 for all $i > 0$. □

Lemma 3.8. *If* $F\underline{K}^\bullet \cong 0$, *then* $\underline{K}^\bullet \cong 0$ *in* $\mathrm{DGrB}(W)$.

Proof. If $F\underline{K}^\bullet \cong 0$, then the A-module $\mathrm{Coker}\ k^{-1}$ is perfect as a complex of A-modules. Let us show that $\mathrm{Coker}\ k^{-1}$ is projective in this case. Indeed, there is a natural number m such that $\mathrm{Ext}^i_A(\mathrm{Coker}\ k^{-1}, N) = 0$ for any A-module N and any $i \geq m$. Considering the exact sequence

$$0 \longrightarrow \mathrm{Coker}\ k^{-2m-1} \longrightarrow K^{-2m} \otimes_B A \longrightarrow \cdots \longrightarrow K^{-1} \otimes_B A \longrightarrow K^0 \otimes_B A \longrightarrow \mathrm{Coker}\ k^{-1} \longrightarrow 0$$

and taking into account that all A-modules $K^i \otimes_B A$ are free, we find that for all modules N $\mathrm{Ext}^i_A(\mathrm{Coker}\ k^{-2m-1}, N) = 0$ when $i > 0$. Hence, $\mathrm{Coker}\ k^{-2m-1}$ is a projective A-module. This implies that $\mathrm{Coker}\ k^{-1}$ is also projective, because it is isomorphic to $\mathrm{Coker}\ k^{-2m-1}(-mn)$.

Since $\operatorname{Coker} k^{-1}$ is projective there is a map $f : \operatorname{Coker} k^{-1} \to K^0 \otimes_B A$ which splits the epimorphism $\mathrm{pr} : K^0 \otimes_B A \to \operatorname{Coker} k^{-1}$. It can be lifted to a map from the complex $\{K^{-1} \xrightarrow{k^{-1}} K^0\}$ to the complex $\{K^{-2} \xrightarrow{W} K^0\}$. Denote the lift by (s^{-1}, u). Consider the following diagram

$$\begin{array}{ccccc}
K^{-1} & \xrightarrow{k^{-1}} & K^0 & \longrightarrow & \operatorname{Coker} k^{-1} \\
{\scriptstyle s^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle f} \\
K^{-2} & \xrightarrow{W} & K^0 & \longrightarrow & K^0 \otimes_B A \\
{\scriptstyle k^{-2}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle pr} \\
K^{-1} & \xrightarrow{k^{-1}} & K^0 & \longrightarrow & \operatorname{Coker} k^{-1}.
\end{array}$$

Since the composition $\mathrm{pr}\, f$ is identical, the map $(k^{-2}s^{-1}, u)$ from the pair $\{K^{-1} \xrightarrow{k^{-1}} K^0\}$ to itself is homotopic to the identity map. Hence, there is a map $s^0 : K^0 \to K^{-1}$ such that

$$\mathrm{id}_{K^{-1}} - k^{-2}s^{-1} = s^0 k^{-1} \qquad \text{and} \qquad k^{-1}s^0 = \mathrm{id}_{K^0} - u.$$

Moreover, we have the following equalities

$$0 = (uk^{-1} - Ws^{-1}) = (uk^{-1} - s^{-1}(n)W) = (u - s^{-1}(n)k^0)k^{-1}.$$

This gives us that $u = s^{-1}(n)k^0$, because there are no maps from $\operatorname{Coker} k^{-1}$ to K^0 . Finally, we get the sequence of morphisms $s^i : K^i \longrightarrow K^{i-1}$, where $s^{2i-1} = s^{-1}(in), s^{2i} = s^0(in)$, such that $k^{i-1}s^i + k^i s^{i+1} = \mathrm{id}$. Thus the identity morphism of the object $\underline{K}^{\bullet}$ is null-homotopic. Hence, the object $\underline{K}^{\bullet}$ is isomorphic to the zero object in the category $\mathrm{DGrB}_0(W)$. $\square$

Theorem 3.9. *The exact functor* $F : \mathrm{DGrB}(W) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *is fully faithful.*

Proof. By Lemma 3.7 we have $\operatorname{Ext}^i_A(\operatorname{Coker} k^{-1}, A) = 0$ for $i > 0$. Now, Proposition 1.11 gives an isomorphism

$$\operatorname{Hom}_{\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(\operatorname{Coker} k^{-1}, \operatorname{Coker} l^{-1}) \cong \operatorname{Hom}_{\mathrm{gr}-A}(\operatorname{Coker} k^{-1}, \operatorname{Coker} l^{-1})/\mathcal{R},$$

where $\mathcal{R}$ is the subspace of morphisms factoring through projective modules. Since the functor Cok is full we get that the functor F is also full.

Next we show that F is faithful. The reasoning is standard. Let $f : \underline{K}^{\bullet} \to \underline{L}^{\bullet}$ be a morphism for which $F(f) = 0$. Include f in an exact triangle $\underline{K}^{\bullet} \xrightarrow{f} \underline{L}^{\bullet} \xrightarrow{g} \underline{M}^{\bullet}$. Then the identity map of $F\underline{L}^{\bullet}$ factors through the map $F\underline{L}^{\bullet} \xrightarrow{Fg} F\underline{M}^{\bullet}$. Since F is full, there is a map $h : \underline{L}^{\bullet} \to \underline{L}^{\bullet}$ factoring through $g : \underline{L}^{\bullet} \to \underline{M}^{\bullet}$ such that $Fh = \mathrm{id}$. Hence, the cone $\underline{C}^{\bullet}(h)$ of map h goes to zero under the functor F . By Lemma 3.8 the object $\underline{C}^{\bullet}(h)$ is the zero object as well, so h is an isomorphism. Thus $g : \underline{L}^{\bullet} \to \underline{M}^{\bullet}$ is a split monomorphism and $f = 0$. $\square$

Theorem 3.10. *Suppose that the algebra* B *has a finite homological dimension. Then the functor* $F : \mathrm{DGrB}(W) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *is an equivalence.*

Proof. We know that F is fully faithful. To prove the theorem we need to show that each object $T \in \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is isomorphic to $F\underline{K}^{\bullet}$ for some $\underline{K}^{\bullet} \in \mathrm{DGrB}(W)$.

The algebra B has a finite homological dimension and, as consequence, it has a finite injective dimension. This implies that $A = B/J$ has a finite injective dimension too. By Lemma 1.10 any object $T \in \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is isomorphic to the image of an A-module M such that $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. This means that the object $D(M) = \mathbf{R}\operatorname{Hom}_A(M, A)$ is a left A-module. We can consider a projective resolution $\underline{Q}^\bullet \to D(M)$. The dual of $\underline{Q}^\bullet$ is a right projective A-resolution

$$0 \longrightarrow M \longrightarrow \{P^0 \longrightarrow P^1 \longrightarrow \cdots\}.$$

Consider M as B-module and chose an epimorphism $K^0 \twoheadrightarrow M$ from free B-module K^0. Denote by $k^{-1} : K^{-1} \to K^0$ the kernel of this map.

The short exact sequence $0 \to B \xrightarrow{W} B \to A \to 0$ implies that for a projective A-module P and any B-module N we have equalities $\mathrm{Ext}^i_B(P, N) = 0$ when $i > 1$. This also yields that $\mathrm{Ext}^i_B(M, N) = 0$ for $i > 1$ and any B-module N, because M has a right projective A-resolution and the algebra B has finite homological dimension. Therefore, $\mathrm{Ext}^i_B(K^{-1}, N) = 0$ for $i > 0$ and any B-module N, i.e. B-module K^{-1} is projective. Since A is connected and finitely generated, any graded projective module is free. Hence, K^{-1} is free.

Since the multiplication on W gives the zero map on M, there is a map $k^0 : K^0 \to K^{-1}(n)$ such that $k^0 k^{-1} = W$ and $k^{-1}(n) k^0 = W$. This way, we get a sequence $\underline{K}^\bullet$ with

$$K^{2i} \cong K^0(i \cdot n),\ K^{2i-1} = K^{-1}(i \cdot n),\ k^{2i} = k^0(i \cdot n),\ k^{2i-1} = k^{-1}(i \cdot n).$$

and this sequence is an object of $\mathrm{DGrB}(W)$ for which $F\underline{K}^\bullet \cong T$. □

3.3. **Graded D-branes type B and coherent sheaves.** By a Landau-Ginzburg model we mean the following data: a smooth variety X equipped with a symplectic Kähler form ω, closed real 2-form $\mathcal{B}$, which is called B-field, and a regular nonconstant function W on X . The function W is called the superpotential of the Landau-Ginzburg model. Since for the definition of D-branes of type B a symplectic form and B-field are not needed we do not fix them.

With any point $\lambda \in \mathbb{A}^1$ we can associate a triangulated category $DB_\lambda(W)$. We give a construction of this categories under the condition that $X = \mathbf{Spec}(B)$ is affine (see [17, 22]). The category of coherent sheaves on an affine scheme $X = \mathbf{Spec}(B)$ is the same as the category of finitely generated B-modules. The objects of the category $DB_\lambda(W)$ are ordered pairs $\overline{P} := \Big(P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \Big)$, where P_0, P_1 are finitely generated projective B-modules and the compositions $p_0 p_1$ and $p_1 p_0$ are the multiplications by the element $(W - \lambda) \in B$. The morphisms in the category $DB(W)$ are the classes of morphisms between pairs modulo null-homotopic morphisms, where a morphism $f : \overline{P} \to \overline{Q}$ between pairs is a pair of morphisms $f_1 : P_1 \to Q_1$ and $f_0 : P_0 \to Q_0$ such that $f_1 p_0 = q_0 f_0$ and $q_1 f_1 = f_0 p_1$. The morphism f is null-homotopic if there are two morphisms $s : P_0 \to Q_1$ and $t : P_1 \to Q_0$ such that $f_1 = q_0 t + s p_1$ and $f_0 = t p_0 + q_1 s$.

We define a category of D-branes of type B (B-branes) on $X = \mathbf{Spec}(B)$ with the superpotential W as the product $DB(W) = \prod_{\lambda \in \mathbb{A}^1} DB_\lambda(W)$.

It was proved in the paper [22, Cor. 3.10] that the category $DB_{\lambda}(W)$ for smooth affine X is equivalent to the triangulated category of singularities $\mathbf{D}_{\mathrm{Sg}}(X_{\lambda})$, where X_{λ} is the fiber over $\lambda \in \mathbb{A}^1$. Therefore, the category of B-branes $DB(W)$ is equivalent to the product $\prod_{\lambda\in\mathbb{A}^1}\mathbf{D}_{\mathrm{Sg}}(X_{\lambda})$. For non-affine X the category $\prod_{\lambda\in\mathbb{A}^1}\mathbf{D}_{\mathrm{Sg}}(X_{\lambda})$ can be considered as a definition of the category of D-branes of type B. Note that, in the affine case, X_{λ} is $\mathbf{Spec}(A_{\lambda})$, where $A_{\lambda} = B/(W-\lambda)B$ and, hence, the triangulated categories of singularities $\mathbf{D}_{\mathrm{Sg}}(X_{\lambda})$ is the same that the category $\mathbf{D}_{\mathrm{Sg}}(A_{\lambda})$.

Assume now that there is an action of the group $\mathbf{k}^*$ on the Landau-Ginzburg model (X, W) such that the superpotential W is semi-invariant of the weight d. Thus, $X = \mathbf{Spec}(B)$ and $B = \bigoplus_i B_i$ is a graded algebra. The superpotential W is an element of B_d. Let us assume that B is positively graded and connected. In this case, we can consider the triangulated category of graded B-branes $\mathrm{DGrB}(W)$, which was constructed in subsection 3.1 (see Definition 3.1).

Denote by A the quotient graded algebra B/WB. We see that the affine variety $\mathbf{Spec}(A)$ is the fiber X_0 of W over the point 0. Denote by Y the quotient stack $[(\mathbf{Spec}(A)\setminus \mathbf{0})/\mathbf{k}^*]$, where $\mathbf{0}$ is the point on $\mathbf{Spec}(A)$ corresponding to the ideal A_+. Theorems 2.5, 3.10 and Proposition 2.17 allow us to establish a relation between triangulated category of graded B-branes $\mathrm{DGrB}(W)$ and the bounded derived category of coherent sheaves on the stack Y.

First, Theorem 3.10 gives us the equivalence F between the triangulated category of graded B-branes $\mathrm{DGrB}(W)$ and the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. Second, Theorem 2.5 describes the relationship between the category $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and the bounded derived category $\mathbf{D}^b(\mathrm{qgr}\, A)$. Third, the category $\mathbf{D}^b(\mathrm{qgr}\, A)$ is equivalent to the derived category $\mathbf{D}^b(\mathrm{coh}(Y))$ by Proposition 2.17. In the particular case, when X is the affine space $\mathbb{A}^N$ with the standard action of the group $\mathbf{k}^*$, we get the following result.

Theorem 3.11. *Let X be the affine space $\mathbb{A}^N$ and let W be a homogeneous polynomial of degree d. Let $Y \subset \mathbb{P}^{N-1}$ be the hypersurface of degree d which is given by the equation $W = 0$. Then, there is the following relation between the triangulated category of graded B-branes* $\mathrm{DGrB}(W)$ *and the derived category of coherent sheaves* $\mathbf{D}^b(\mathrm{coh}(Y))$:

(i) *if $d < N$, i.e. if Y is a Fano variety, there is a semiorthogonal decomposition*

$$\mathbf{D}^b(\mathrm{coh}(Y)) = \langle \mathcal{O}_Y(d-N+1), \dots, \mathcal{O}_Y, \mathrm{DGrB}(W)\rangle,$$

(ii) *if $d > N$, i.e. if X is a variety of general type, there is a semiorthogonal decomposition*

$$\mathrm{DGrB}(W) = \langle F^{-1}q(\mathbf{k}(r+1)), \dots, F^{-1}q(\mathbf{k}), \mathbf{D}^b(\mathrm{coh}(Y))\rangle,$$

where $q : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is the natural projection, and $F : \mathrm{DGrB} \xrightarrow{\sim} \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is the equivalence constructed in Proposition 3.5.

(iii) *if $d = N$, i.e. if Y is a Calabi-Yau variety, there is an equivalence*

$$\mathrm{DGrB}(W) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{coh}(Y)).$$

Remark 3.12. We can also consider a weighted action of the torus $\mathbf{k}^*$ on the affine space $\mathbb{A}^N$ with positive weights $(a_1, \dots, a_N)$, $a_i > 0$ for all i. If the superpotential W is quasi-homogeneous then we have the category of graded B-branes $\mathrm{DGrB}(W)$. The polynomial W defines an orbifold (quotient stack) $Y \subset \mathbb{P}^{N-1}(a_1, \dots, a_N)$. The orbifold Y is the quotient of $\mathbf{Spec}(A)\backslash\mathbf{0}$ by the action of $\mathbf{k}^*$, where $A = \mathbf{k}[x_1, \dots, x_N]/W$. Proposition 2.17 gives the equivalence between $\mathbf{D}^b(\mathrm{coh}(Y))$ and $\mathbf{D}^b(\mathrm{qgr}\, A)$. And Theorem 2.5 shows that we get an analogue of Theorem 3.11 for the weighted case as well.

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