European Journal of Mathematics (2016) 2:196–207
DOI 10.1007/s40879-015-0065-1

RESEARCH ARTICLE

Derived categories of toric varieties III

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Received: 28 December 2014 / Revised: 7 July 2015 / Accepted: 14 July 2015 /
Published online: 28 July 2015

Abstract We prove that the derived McKay correspondence holds for the cases of finite abelian groups and subgroups of $\mathrm{GL}(2, \mathbf{C})$. We also prove that K-equivalent toric birational maps are decomposed into toric flops.

Keywords McKay correspondence · Toric variety · Derived category · Relative exceptional object

Mathematics Subject Classification 14E16 · 14E30 · 14M25

1 Introduction

This is a continuation of papers [16, 18]. We have studied the effect of each elementary birational map in the minimal model program to the derived categories in the case of toric pairs. In this paper we consider problems of more global nature. We add some cases to the list of affirmative answers to the derived McKay correspondence conjecture and its generalization, the "K implies D conjecture" ([14]).

The conjecture states that, if there is an inequality of canonical divisors between birationally equivalent algebraic varieties, then there is a semi-orthogonal decomposition of the derived category with larger canonical divisor into that with smaller canonical divisor and other factors. In particular, if the canonical divisors are equivalent, then so are the derived categories. The reason for the conjecture is the following; the canonical divisor is the key ingredient of the minimal model theory, and we believe

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that the corresponding Serre functor should control the behavior of the derived category.

First we remark that the results in [16,18] are easily extended to the relative situation (Theorem 1.1). Then we remark that the derived McKay correspondence holds for any abelian quotient singularity in the following sense; the derived category defined by a finite abelian group contained in a general linear group is semi-orthogonally decomposed into derived categories of a relative minimal model and subvarieties of the quotient space (Theorem 1.3). As an application, we prove that such derived McKay correspondence also holds for arbitrary quotient singularities in dimension 2 (Theorem 1.4).

Next we prove that any proper birational morphism between toric pairs which is a K-equivalence is decomposed into a sequence of flops (Theorem 1.5). This is a generalization, in the toric case, of a theorem in [17], which states that any birational morphism between minimal models is decomposed into a sequence of flops. We note that the latter case is easier because the log canonical divisor stays at the bottom and cannot be decreased when the models are minimal, while it is more difficult to preserve the level of the log canonical divisor in the former case.

Now we state the theorems in details. We refer the terminology to the next section. The following is the reformulation of the results of [16,18] to the relative case.

Theorem 1.1 *Let (X, B) and (Y, C) be $\mathbf{Q}$-factorial toric pairs whose coefficients belong to the standard set $\{1-1/m : m \in \mathbf{N}\}$, let $\widetilde{X}$ and $\widetilde{Y}$ be smooth Deligne–Mumford stacks associated to the pairs (X, B) and (Y, C), respectively, and let $f\colon X \dashrightarrow Y$ be a toric rational map. Assume one of the following conditions:*

(a) *f is the identity morphism, and $B \geq C$.*
(b) *f is a flip, $C = f_*B$, and $K_X + B \geq K_Y + C$.*
(c) *f is a divisorial contraction, $C = f_*B$, and $K_X + B \geq K_Y + C$.*
(d) *f is a divisorial contraction, $C = f_*B$, and $K_X + B \leq K_Y + C$.*
(e) *f is a Mori fiber space and C is determined by (X, B) and f as explained in* [16].

Then the following hold:

(a), (b), (c) *There are toric closed subvarieties Z_i, $1 \leq i \leq l$, of Y such that $Z_i \neq Y$ for some $l \geq 0$, and fully faithful functors $\Phi\colon D^b(\mathrm{coh}(\widetilde{Y})) \to D^b(\mathrm{coh}(\widetilde{X}))$ and $\Psi_i\colon D^b(\mathrm{coh}(\widetilde{Z}_i)) \to D^b(\mathrm{coh}(\widetilde{X}))$ for the smooth Deligne–Mumford stacks $\widetilde{Z}_i$ associated to Z_i such that there is a semi-orthogonal decomposition of triangulated categories*

$$D^b(\mathrm{coh}(\widetilde{X})) = \langle \Psi_1(D^b(\mathrm{coh}(\widetilde{Z}_1))), \ldots, \Psi_l(D^b(\mathrm{coh}(\widetilde{Z}_l))), \Phi(D^b(\mathrm{coh}(\widetilde{Y}))) \rangle.$$

(d) *There are toric closed subvarieties $Z_i \subset Y$, $1 \leq i \leq l$, as in* (c)*, and fully faithful functors $\Phi\colon D^b(\mathrm{coh}(\widetilde{X})) \to D^b(\mathrm{coh}(\widetilde{Y}))$ and $\Psi_i\colon D^b(\mathrm{coh}(\widetilde{Z}_i)) \to D^b(\mathrm{coh}(\widetilde{Y}))$ such that there is a semi-orthogonal decomposition*

$$D^b(\mathrm{coh}(\widetilde{Y})) = \langle \Psi_1(D^b(\mathrm{coh}(\widetilde{Z}_1))), \ldots, \Psi_l(D^b(\mathrm{coh}(\widetilde{Z}_l))), \Phi(D^b(\mathrm{coh}(\widetilde{X}))) \rangle.$$

(e) *There are fully faithful functors* $\Phi_j : D^b(\mathrm{coh}(\widetilde{Y})) \to D^b(\mathrm{coh}(\widetilde{X}))$, $1 \le j \le m$, *for some* $m \ge 2$ *and a semi-orthogonal decomposition*

$$D^b(\mathrm{coh}(\widetilde{X})) = \big\langle \Phi_1\big(D^b(\mathrm{coh}(\widetilde{Y}))\big), \ldots, \Phi_m\big(D^b(\mathrm{coh}(\widetilde{Y}))\big)\big\rangle.$$

Moreover, if $K_X + B = K_Y + C$ *in the cases* (b) *or* (c), *then* Φ *is an equivalence.*

We note that, when f is a divisorial contraction, the direction of the inclusion of the derived categories is unrelated to the direction of the morphism, but is the same as the direction of the inequality of the canonical divisors.

If X and Y are smooth and f is a blowing up of a smooth center in the case (b), or if X and Y are smooth and f is a standard flip in the case (c), then it is a theorem by Bondal–Orlov [4]. The derived equivalence in the case of flops (case (c) with $K_X + B = K_Y + C$) was proved for general (non-toric) case under the additional assumptions: if $\dim X = 3$ and X is smooth by Bridgeland [5], if $\dim X = 3$ and X has only Gorenstein terminal singularities by Chen [8] and Van den Bergh [20], if $\dim X = 3$ and X has only terminal singularities by [14], and if X is a holomorphic symplectic manifold by Kaledin [12]. There are also interesting results concerning the last case by Cautis [7] by using the categorification of linear group actions, and by Donovan–Segal [9], Ballard–Favero–Katzarkov [1] and Halpern-Leistner [10] by using the theory of variations of GIT quotients.

As a corollary we obtain

Corollary 1.2 *Let* (X, B) *be a* $\mathbf{Q}$*-factorial toric pair whose coefficients belong to the standard set* $\{1 - 1/m : m \in \mathbf{N}\}$. *Assume that there is a projective toric morphism* $f : X \to Z$ *to another* $\mathbf{Q}$*-factorial toric variety. Then there exist toric closed subvarieties* Z_i, $1 \le i \le m$, *of* Z *for some* $m \ge 1$ *such that there are fully faithful functors* $\Phi_i : D^b(\mathrm{coh}(\widetilde{Z}_i)) \to D^b(\mathrm{coh}(\widetilde{X}))$ *with a semi-orthogonal decomposition*

$$D^b(\mathrm{coh}(\widetilde{X})) \cong \big\langle \Phi_1\big(D^b(\mathrm{coh}(\widetilde{Z}_1))\big), \ldots, \Phi_m\big(D^b(\mathrm{coh}(\widetilde{Z}_m))\big)\big\rangle,$$

where $\widetilde{X}$ *and the* $\widetilde{Z}_i$ *are smooth Deligne–Mumford stacks associated to* (X, B) *and* Z_i.

In other words, one can say that the derived category $D^b(\mathrm{coh}\,\widetilde{X})$ is generated by a *relative exceptional collection*. The proofs of the above statements are the same as in [15,16,18] except Theorem 1.1 (a). More precisely, the existence of the fully faithful functors Φ in Theorem 1.1 in the cases (a), (b), (c) and (d) is proved in [15]. The semi-orthogonal complements in the cases (b) and (c) are described in [16], and (d) in [18]. Moreover the case (e) is described in [16]. We shall give a proof of Theorem 1.1 (a) in Sect. 4.

The following is the derived McKay correspondence for finite abelian groups.

Theorem 1.3 *Let* $G \subset \mathrm{GL}(n, \mathbf{C})$ *be a finite abelian subgroup acting naturally on an affine space* $\mathbf{A} = \mathbf{C}^n$, *and let* $X = \mathbf{A}/G$ *be the quotient space. Let* $f : Y \to X$ *be a* $\mathbf{Q}$*-factorial terminal relative minimal model of* X. *Then there exist toric closed subvarieties* Z_i, $1 \le i \le m$, *of* X *with* $Z_i \ne X$ *for some* $m \ge 0$, *with possible repetitions, such that there is a semi-orthogonal decomposition*

$$D^b(\mathrm{coh}([\mathbf{A}/G])) \cong \langle D^b(\mathrm{coh}(\widetilde{Z}_1)), \ldots, D^b(\mathrm{coh}(\widetilde{Z}_m)), D^b(\mathrm{coh}(\widetilde{Y}))\rangle,$$

where $\widetilde{Y}$ and $\widetilde{Z}_i$ are smooth Deligne–Mumford stacks associated to Y and Z_i, respectively. Moreover if $G \subset \mathrm{SL}(n, \mathbf{C})$, then $m = 0$.

In the case of dimension 2, we do not need the assumption that the group is abelian.

Theorem 1.4 *Let $G \subset \mathrm{GL}(2, \mathbf{C})$ be a finite subgroup acting naturally on an affine space $\mathbf{A} = \mathbf{C}^2$, let $X = \mathbf{A}/G$, and let $f : Y \to X$ be the minimal resolution. Then there exist closed subvarieties Z_i, $1 \le i \le m$, of X with $Z_i \ne X$ for some $m \ge 0$, with possible repetitions, such that there is a semi-orthogonal decomposition*

$$D^b(\mathrm{coh}([\mathbf{A}/G])) \cong \langle D^b(\mathrm{coh}(Z_1^\nu)), \ldots, D^b(\mathrm{coh}(Z_m^\nu)), D^b(\mathrm{coh}(Y))\rangle,$$

where Z_i^ν are normalizations of Z_i. Moreover if there are no quasi-reflections in G, the elements whose invariant subspaces are of codimension 1, then $\dim Z_i = 0$ for all i. Thus the semi-orthogonal complement of $D^b(\mathrm{coh}(Y))$ in $D^b(\mathrm{coh}([\mathbf{A}/G]))$ is generated by an exceptional collection in this case.

The derived McKay correspondence was already proved for finite subgroups of $\mathrm{SL}(2, \mathbf{C})$, $\mathrm{SL}(3, \mathbf{C})$ (Bridgeland–King–Reid [6]), and $\mathrm{Sp}(2n, \mathbf{C})$ (Bezrukavnikov–Kaledin [2]).

Finally, we prove the following decomposition theorem for K-equivalent toric birational map.

Theorem 1.5 *Let $f : X \dashrightarrow Y$ be a toric birational map between projective $\mathbf{Q}$-factorial toric varieties which is an isomorphism in codimension 1, let B be a toric $\mathbf{R}$-divisor on X whose coefficients belong to the interval $(0, 1)$, and let $f_*B = C$. Assume that $K_X + B = K_Y + C$. Then the birational map $f : (X, B) \dashrightarrow (Y, C)$ is decomposed into a sequence of flops.*

As a corollary of Theorems 1.1 and 1.5, we obtain an affirmative answer to "K implies D conjecture" in the toric case.

Corollary 1.6 *Assume additionally that the coefficients of B belong to a set $\{1-1/m : m \in \mathbf{Z}_{>0}\}$. Let $\widetilde{X}$ and $\widetilde{Y}$ be the smooth Deligne–Mumford stacks associated to the pairs (X, B) and (Y, C), respectively. Then there is an equivalence of triangulated categories $D^b(\mathrm{coh}(\widetilde{X})) \cong D^b(\mathrm{coh}(\widetilde{Y}))$.*

We note that "K implies D conjecture" is known to be true only for 3-dimensional varieties with terminal singularities but without boundary divisors by [5,8,14,20] with the help of Proposition 3.4.

Sasha Kuznetsov kindly informed the author that Ishii and Ueda [11] proved Theorem 1.5 in the case where the action of G is free in codimension 1, i.e., if there are no quasi-reflections, using the method of dimer models.

2 Preliminaries

We fix terminology in this section. A pair (X, B) consisting of a normal algebraic variety and an effective $\mathbf{R}$-divisor is said to be *KLT* (resp. *terminal*) if there is a projective birational morphism $p\colon Z \to X$ from a smooth variety with an **R**-divisor D whose support is a normal crossing divisor such that an equality $p^*(K_X+B) = K_Z + D$ holds and all coefficients of D are smaller than 1 (resp. all coefficients of $D - p_*^{-1}B$ are smaller than 0), where the canonical divisors K_X and K_Z are defined by using the same rational differential forms. It is called **Q**-*factorial* if any prime divisor on X is a **Q**-Cartier divisor.

A rational number is said to be *standard* if it is of the form $1 - 1/m$ for a positive integer m.

A *toric pair* (X, B) consists of a toric variety and an effective **R**-divisor whose irreducible components are invariant under the torus action. It is KLT if and only if the coefficients of B belong to an interval [0, 1). A toric variety X is **Q**-factorial if and only if the corresponding fan is simplicial. In this case it has only abelian quotient singularities. Conversely, the quotient of an affine space by any finite abelian group is a **Q**-factorial toric variety.

A birational map $h\colon X \dashrightarrow Y$ between varieties is said to be *proper* if there exists a third variety Z with proper birational morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ such that $g = h \circ f$.

Let $f\colon (X, B) \dashrightarrow (Y, C)$ be a proper birational map between KLT pairs. We say that there is an *equality of log canonical divisors* $K_X+B = K_Y+C$ (resp. an *inequality* $K_X+B \ge K_Y+C$), if $p^*(K_X+B) = q^*(K_Y+C)$ (resp. $p^*(K_X+B) \ge q^*(K_Y+C)$) as an equality (resp. an inequality) of **R**-divisors for some p, q as above. We note that such equality or inequality can be defined only when the birational map f is fixed, and the definition is independent of the choice of p, q. Such an equality (resp. an inequality) is called a *K-equivalence* (resp. *K-inequality*). The birational map f is also said to be *crepant* if it is a K-equivalence.

A proper birational map $f\colon X \dashrightarrow Y$ is said to be *isomorphic in codimension* 1 (resp. *surjective in codimension* 1) if it induces a bijection (resp. surjection) between the sets of prime divisors.

A *flop* (resp. *flip*) is a proper birational map $f\colon (X, B) \dashrightarrow (Y, C)$ between **Q**-factorial KLT pairs satisfying the following conditions: $f_*B = C$, $K_X + B = K_Y + C$ (resp. $K_X + B > K_Y + C$), and there exist projective birational morphisms $s\colon X \to W$ and $t\colon Y \to W$ to a normal variety which are isomorphisms in codimension 1 and the relative Picard numbers $\rho(X/W)$ and $\rho(Y/W)$ are equal to 1.

A *divisorial contraction* is a projective birational morphism $f\colon (X, B) \to (Y, C)$ between **Q**-factorial KLT pairs satisfying the following conditions: $f_*B = C$, and the exceptional locus of f consists of a single prime divisor. We have always $\rho(X/Y) = 1$, and we have either $K_X+B = K_Y+C$, $K_X+B > K_Y+C$, or $K_X+B < K_Y+C$. The inverse birational map $f^{-1}\colon (Y, C) \dashrightarrow (X, B)$ of a divisorial contraction is called a *divisorial extraction*.

A *Mori fiber space* is a projective morphism $f\colon (X, B) \to Y$ from a **Q**-factorial KLT pair to a normal variety satisfying the following conditions: $\rho(X/Y) = 1$, $-(K_X + B)$ is ample, and $\dim X > \dim Y$.

For a **Q**-factorial KLT pair (X, B), there exists a sequence of crepant divisorial contractions

$$(Y, C) = (X_m, B_m) \to \cdots \to (X_1, B_1) \to (X_0, B_0) = (X, B)$$

such that (Y, C) is terminal and **Q**-factorial [3]. The composition $f : (Y, C) \to (X, B)$ is called a **Q**-*factorial crepant terminalization* of the pair (X, B). We note that even if B has standard coefficients, C may not be so.

Let $f : (Y, C) \to (X, B)$ be a projective birational morphism from a **Q**-factorial terminal pair to a KLT pair. It is called a **Q**-*factorial terminal relative minimal model* if $C = f_*^{-1} B$ and $K_Y + C$ is relatively nef over X. We note that f is not necessarily crepant. The existence of such a model is also proved in [3].

Any **Q**-factorial crepant terminalization and **Q**-factorial terminal relative minimal model of a toric pair are both toric.

A KLT pair (X, B) is said to be of *quotient type*, if there is a quasi-finite surjective morphism $\pi' : X' \to X$ from a smooth variety X', which is not necessarily irreducible, such that $K_{X'} = (\pi')^*(K_X + B)$. In this case B has only standard coefficients, and X is automatically **Q**-factorial. If the pair (X, B) is toric, X is **Q**-factorial, and if B has standard coefficients, then it is of quotient type.

There is a *smooth Deligne–Mumford stack* $\widetilde{X}$ associated to the pair (X, B) of quotient type in such a way that a sheaf $\mathcal{F}$ on $\widetilde{X}$ is nothing but a sheaf $\mathcal{F}'$ on X' equipped with an isomorphism $h_{21} : p_1^*\mathcal{F}' \cong p_2^*\mathcal{F}'$, where $p_i : (X' \times_X X')^\nu \to X'$, $i = 1, 2$, are the projections from the normalization of the fiber product, satisfying the cocycle condition $h_{32}h_{21} = h_{31}$ for $p_i : (X' \times_X X' \times_X X')^\nu \to X'$, $i = 1, 2, 3$. If $B = 0$, we simply say that $\widetilde{X}$ is an associated Deligne–Mumford stack to X.

By the construction, there is a finite birational morphism $\pi : \widetilde{X} \to X$ such that $\pi^*(K_X + B) = K_{\widetilde{X}}$, cf. [15].

We note that the construction of $\widetilde{X}$ does not depend on the choice of the covering $\pi' : X' \to X$. Indeed if $\pi_1' : X_1' \to X$ is another covering, then the normalization of the fiber product $X_2' = (X' \times_X X_1')^\nu$ is étale over X' and X_1', so that X_2' defines the same set of sheaves as X' and X_1' by the étale descent.

For an algebraic variety or a Deligne–Mumford stack X, let $D^b(\mathrm{coh}(X))$ denote the bounded derived category of coherent sheaves on X. There are variants of this notation; if a finite group G acts on X, then $D^b(\mathrm{coh}^G(X))$ denotes the bounded derived category of G-equivariant coherent sheaves on X. Thus $D^b(\mathrm{coh}^G(X)) \cong D^b(\mathrm{coh}([X/G]))$.

A triangulated category $\mathcal{T}$ is said to have a *semi-orthogonal decomposition* $\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_m \rangle$ by triangulated full subcategories $\mathcal{T}_i$ if the following conditions are satisfied: (a) $\mathrm{Hom}(a, b) = 0$ for $a \in \mathcal{T}_i$ and $b \in \mathcal{T}_j$ if $i > j$, and (b) for any object $a \in \mathcal{T}$ there is a sequence of objects $a_i \in \mathcal{T}$ for $1 \le i \le m + 1$ such that $a = a_1$, $a_{m+1} = 0$, and that there are distinguished triangles $a_{i+1} \to a_i \to b_i$ with $b_i \in \mathcal{T}_i$ for $1 \le i \le m$. In particular, if $\mathcal{T}_i$ are equivalent to $D^b(\mathrm{Spec}\, k)$ for all i, then $\mathcal{T}$ is said to have a *full exceptional collection*.

For a pair of quotient type, we can define an abelian category whose homological dimension is finite. Therefore we would like to ask the following

Question 2.1 Let $f : (X, B) \dashrightarrow (Y, C)$ be a flop or a flip. If (X, B) is of quotient type, then is (Y, C) too?

We note that a divisorial contraction of a smooth threefold may produce a singularity of non-quotient type even if it is crepant. We note also that even if X is smooth and f is a flop, Y is not necessarily smooth, but we have still the derived equivalence in some examples when we consider the associated Deligne–Mumford stacks, cf. [13]. The following lemma is standard.

Lemma 2.2 *Let $f\colon (X,B) \dashrightarrow (Y,C)$ be a proper birational map between* **Q**-*factorial terminal pairs, and assume that $K_X + B \geq K_Y + C$. Then f is surjective in codimension* 1*, and $f_*B \geq C$. Moreover if $K_X + B = K_Y + C$, then f is an isomorphism in codimension* 1.

Proof Suppose that there is a prime divisor C_1 on Y which is not the strict transform of a prime divisor on X. Let $c_1 \geq 0$ be the coefficient of C_1 in C. Let $p\colon Z \to X$ and $q\colon Z \to Y$ be projective birational morphisms from a smooth variety, and let D_1 be the strict transform of C_1. Since we have an inequality $p^*(K_X+B) \geq q^*(K_Y+C)$, the coefficient of D_1 in $p^*(K_X+B) - K_Z$ should be non-negative, a contradiction to the fact that (X,B) is terminal. Therefore f is surjective in codimension 1. If we apply the first assertion to f and f^{-1}, then we obtain the second assertion. □

3 Decomposition theorem for toric pairs

We prove Theorem 1.5 in this section. First we prove a condition for the decomposition theorem that is also valid for the non-toric case. Let $f\colon (X,B) \dashrightarrow (Y,C)$ be a birational map between projective **Q**-factorial KLT pairs which is an isomorphism in codimension 1 and such that $K_X + B = K_Y + C$. Let $p\colon Z \to X$ and $q\colon Z \to Y$ be projective birational morphisms such that $f \circ p = q$.

Lemma 3.1 *Assume that there is a set of curves $\{l_\lambda\}$ on X whose classes generate the cone of curves $\overline{\mathrm{NE}}(X)$ as a closed convex cone and such that the following condition is satisfied: for any fixed λ, if there is an ample divisor L_Y on Y such that $(L_X, l_\lambda) < 0$ for the strict transform $L_X = f_*^{-1}L_Y$, then $((K_X+B), l_\lambda) = 0$. Then f is decomposed into a sequence of flops.*

It is important to note that $\{l_\lambda\}$ is not necessarily the set of all curves.

Proof We shall find suitable extremal rays in order to find a path from (X,B) to (Y,C).

We take an ample and effective divisor L_Y on Y such that $L_Y - (K_Y+C)$ is also ample, and let $L_X = f_*^{-1}L_Y$ be the strict transform.

Assume first L_X is nef. We claim that $L_X - (K_X+B)$ is also nef. Otherwise, there exists a curve l_λ such that $((L_X - (K_X+B)), l_\lambda) < 0$. By the assumption, it follows that $((K_X+B), l_\lambda) = 0$. Therefore $(L_X, l_\lambda) < 0$, a contradiction.

By the base point free theorem [19], L_X is semiample. Since f is an isomorphism in codimension 1, the associated morphism coincides with f. Since both X and Y are **Q**-factorial, we conclude that f is an isomorphism.

Next assume that L_X is not nef. We take a small positive number ϵ such that the pair $(X, B+\epsilon L_X)$ is KLT. We would like to run a minimal model program for this pair in order to move from the pair (X,B) to the pair (Y,C). Here we have to note that there may be extremal rays of $(X, B+\epsilon L_X)$ which do not come from the negativity of L_X.

Since L_X is not nef, there exists a curve l_λ such that $(L_X, l_\lambda) < 0$. By the assumption, $((K_X+B), l_\lambda) = 0$ for such curve. Therefore $K_X + B + \epsilon L_X$ is not nef.

The part of the closed cone of curves $\overline{\mathrm{NE}}(X)$ where L_X is negative is contained in the part where $K_X + B + \epsilon L_X$ is negative. Thus there exists an extremal ray R of $\overline{\mathrm{NE}}(X)$ on which L_X is negative. Since $K_X + B$ is numerically trivial on R, the flip of R for the pair $(X, B+\epsilon L_X)$ is a flop for (X, B). Since $B + \epsilon L_X$ is big, this process terminates by [3], and f is decomposed into the flops. $\square$

It is important note that $\{l_\lambda\}_{\lambda\in\Lambda}$ is not necessarily the set of all curves.

Now we consider the toric case. We take $p\colon Z \to X$, $q\colon Z \to Y$ and L_Y to be also toric. Since the coefficients of B belong to (0, 1), the pairs (X, B) and (Y, C) are KLT.

Let Δ_X and Δ_Y be the simplicial fans corresponding to X and Y, respectively, in the same vector space $N_{\mathbf{R}}$. By assumption, the sets of primitive vectors on the edges of Δ_X and Δ_Y coincide. Since $\overline{\mathrm{NE}}(X)$ is generated by toric curves, Theorem 1.5 is a consequence of the following result.

Lemma 3.2 (i) *Let L_Y be an ample divisor on Y and $L_X = f_*^{-1}L_Y$. Let w be a wall in Δ_X and let l_w be the corresponding curve on X. Assume that $(L_X, l_w) \le 0$. Then w is not a wall in Δ_Y.*

(ii) *Let w be a wall in Δ_X which does not belong to Δ_Y. Then $((K_X+B), l_w) = 0$.*

Proof (i) Since f is an isomorphism in codimension 1, the sets of primitive vectors $\{v_i\}$ on the edges of Δ_X and Δ_Y coincide. Let g_X and g_Y be functions on $N_{\mathbf{R}}$ corresponding to the divisors L_X and L_Y, respectively. The functions g_X and g_Y have the same values at v_i, and are linear inside each simplex belonging to Δ_X and Δ_Y, respectively. Since L_Y is ample, g_Y is convex along the walls of Δ_Y.

Assuming that w is also a wall in Δ_Y, we shall derive a contradiction. Under this assumption, the values of g_X and g_Y are equal on w.

Let v_1, v_2 be two vertexes adjacent to w in Δ_X. We take a general point P on w such that the line segments I_i connecting P to v_i for $i = 1, 2$ pass only through the interiors of simplexes and walls of Δ_Y. Then the function g_Y is convex along I_i.

Since g_X is linear on I_i and coincides with g_Y at the end points, we have an inequality $g_X \ge g_Y$ on $I_1 \cup I_2$. On the other hand, $g_X = g_Y$ on w. Since g_Y is convex along w, we conclude that g_X is also convex along w, hence $(L_X, l_w) > 0$, a contradiction.

(ii) Let D_i be the prime divisors on X corresponding to v_i. We write $B = \sum d_i D_i$, and let h be the function on the vector space $N_{\mathbf{R}}$ which takes values $1 - d_i$ at v_i and linear inside each simplex of Δ_X. Since $K_X + B = K_Y + C$, h is also linear inside each simplex of Δ_Y. There is a point inside w which is an interior point of a simplex in Δ_Y. Then h is linear across w. Therefore $((K_X+B), l_w) = 0$. $\square$

Remark 3.3 The relative version is similarly proved. We need to assume that X and Y are projective and toric over a toric variety S and f is a toric birational map over S. We only consider curves relative over S, i.e., those mapped to single points on S.

If $\dim X = 3$, then the condition of Lemma 3.1 is easily verified even in the non-toric case.

Proposition 3.4 *Assume that* $\dim X = 3$. *Then any* K*-equivalent birational map between projective* $\mathbf{Q}$*-factorial KLT pairs* $f : (X, B) \dashrightarrow (Y, C)$ *which is an isomorphism in codimension* 1 *is decomposed into a sequence of flops.*

Proof Assume that $(L_X, l) \le 0$ for a curve l. Since $\dim X = 3$, there are only finitely many such curves l. Then l is the image of a fiber of q by p. Therefore we have $((K_X + B), l) = 0$. □

4 Derived McKay correspondence

We prove Theorems 1.3 and 1.4 as well as Theorem 1.1 (a) in this section.

Proof of Theorem 1.3 Let B be a $\mathbf{Q}$-divisor on X such that $p^*(K_X + B) = K_{\mathbf{A}}$. Since the action of G is diagonalizable, the pair (X, B) is toric. We have $D^b(\mathrm{coh}([\mathbf{A}/G]) \cong D^b(\mathrm{coh}(\widetilde{X}))$ for the smooth Deligne–Mumford stack $\widetilde{X}$ associated to the pair (X, B).

There exists a toric relative minimal model of X. Any other relative minimal model is obtained by a sequence of flops from the toric relative minimal model by [17]. Since the extremal rays are toric, the sequence is automatically toric, hence any relative minimal model is again toric. By Theorem 1.1, it is sufficient to prove the theorem for a particular relative minimal model.

Since X is $\mathbf{Q}$-factorial and (X, B) is KLT, we can construct a sequence toric birational morphisms

$$(X_t, B_t) \to (X_{t-1}, B_{t-1}) \to \cdots \to (X_0, B_0) = (X, B)$$

which satisfy the following conditions:

- Each step $f_i : (X_i, B_i) \to (X_{i-1}, B_{i-1})$ is a divisorial contraction such that $B_i = f_{i*}^{-1} B_{i-1}$ and $K_{X_i} + B_i \le K_{X_{i-1}} + B_{i-1}$ for $1 \le i \le t$.
- The pair (X_t, B_t) is terminal.

We note that the coefficients of B_i are standard because the exceptional divisors have coefficients 0.

Next we run a toric minimal model program for X_t over X to obtain a sequence of birational maps

$$X_t = Y_0 \dashrightarrow Y_1 \dashrightarrow \cdots \dashrightarrow Y_s = Y$$

consisting of divisorial contractions and flips such that K_Y is nef over X. We have $K_{Y_{j-1}} > K_{Y_j}$ for $1 \le j \le s$.

We apply Theorem 1.1 and Corollary 1.2 to each step of the above sequences to obtain our result.

If $G \subset \mathrm{SL}(n, \mathbf{C})$, then it follows that $B = 0$ and X is Gorenstein and canonical. In this case, the morphisms f_i are crepant with $B_i = 0$ for all i, and K_{X_t} is already nef over X because it is trivial. Therefore $m = 0$. □

Proof of Theorem 1.4 Let H be the normal subgroup of G defined by $H = G \cap \mathrm{SL}(2, \mathbf{C})$. The residue group is a cyclic group $\mathbf{Z}_r = G/H$ for $r = [G : H]$. By [6], a component of the Hilbert scheme of H-invariant subschemes of $\mathbf{A}$

gives a crepant resolution $f: M \to \mathbf{A}/H$, and there is a derived equivalence $\Phi: D^b(\mathrm{coh}(M)) \to D^b(\mathrm{coh}^H(\mathbf{A})) = D^b(\mathrm{coh}([\mathbf{A}/H]))$. More precisely, M is the moduli space of closed subschemes $P \subset \mathbf{A}$ which are H-invariant and such that length $\mathcal{O}_P = \# H$. For a generic point $[P] \in M$, P is a reduced subscheme, and corresponds to a generic orbit of H in $\mathbf{A}$. There exists a universal subscheme $W \subset M \times \mathbf{A}$ with projections $p: W \to M$ and $q: W \to \mathbf{A}$ such that the functor $\Phi = q_* p^*$ gives the equivalence.

For $[P] \in M$ and $g \in G$, gP is also H-invariant, since H is a normal subgroup. Hence $[gP] \in M$, and G acts on M such that its subgroup H acts trivially. The equivariant derived category $D^b(\mathrm{coh}^{\mathbf{Z}_r}(M))$ is identified with the category where the objects are G-equivariant complexes on which H acts trivially, and the morphisms are G-equivariant morphisms.

The subgroup G acts on the product $M \times \mathbf{A}$ diagonally and preserves the subscheme W. We claim that the functor

$$\Phi' = q_* p^* : \ D^b(\mathrm{coh}^{\mathbf{Z}_r}(M)) \to D^b(\mathrm{coh}^G(\mathbf{A}))$$

is an equivalence. Indeed if a, b are $\mathbf{Z}_r$-equivariant objects, then the isomorphism

$$\mathrm{Hom}_M(a, b) \cong \mathrm{Hom}_{\mathbf{A}}(\Phi(a), \Phi(b))^H$$

implies an isomorphism

$$\mathrm{Hom}_M(a, b)^{\mathbf{Z}_r} \cong \left(\mathrm{Hom}_{\mathbf{A}}(\Phi(a), \Phi(b))^H\right)^{\mathbf{Z}_r} = \mathrm{Hom}_{\mathbf{A}}(\Phi(a), \Phi(b))^G.$$

Let $g: M \to M/\mathbf{Z}_r$ be the map to the quotient space. We define a boundary divisor $B_{M/\mathbf{Z}_r}$ on $M/\mathbf{Z}_r$ by $g^*(K_{M/\mathbf{Z}_r} + B_{M/\mathbf{Z}_r}) = K_M$. Since M is smooth, $M/\mathbf{Z}_r$ has only cyclic quotient singularities. Let $h: Y' \to M/\mathbf{Z}_r$ be the minimal resolution, and let $k: Y' \to Y$ be the contraction morphism to the minimal resolution of $\mathbf{A}/G$. The former is a toroidal morphism, and decomposed into toroidal divisorial contractions. The latter is a composition of divisorial contractions of (-1)-curves, which are toroidal too.

The canonical divisors satisfy the following inequalities:

$$K_Y \le K_{Y'} \le K_{M/\mathbf{Z}_r} \le K_{M/\mathbf{Z}_r} + B_{M/\mathbf{Z}_r} = K_{\mathbf{A}/G}.$$

By Theorem 1.1, we obtain fully faithful embeddings

$$D^b(\mathrm{coh}(\widetilde{Y})) \to D^b(\mathrm{coh}(\widetilde{Y}')) \to D^b(\mathrm{coh}([M/\mathbf{Z}_r])) = D^b(\mathrm{coh}([\mathbf{A}/G]))$$

whose semi-orthogonal complements are generated by the derived categories of subvarieties of $X = \mathbf{A}/G$.

The positive dimensional subvarieties among Z_i appear in the process to decrease the coefficients of the irreducible components of $B_{M/\mathbf{Z}_r}$ which are mapped to positive dimensional subvarieties of X. All other Z_i coincide with the singular point of X, and each corresponding semi-orthogonal component is generated by an exceptional object. If there are no quasi-reflections, then the action of G is free outside the origin of $\mathbf{A}$, and only the latter case occurs. □

Proof of Theorem 1.1 (a) It is sufficient to consider the following situation: $X = Y$ is an affine toric variety corresponding to a simplicial cone $\sigma \subset N_{\mathbf{R}}$ generated by primitive vectors $v_1, \dots, v_n \in N$. Let B_i be prime divisors on X corresponding to v_i, and let $B = \sum(1 - 1/r_i)B_i$ and $C = \sum(1 - 1/s_i)B_i$, where $r_1 > s_1$ and $r_i = s_i$ for $i = 2, \dots, n$. Let $t_1 = \mathrm{LCM}(r_1, s_1)$ and $t_i = r_i$ for $i = 2, \dots, n$.

The stacks $\widetilde{X}, \widetilde{Y}$ and $\widetilde{W}$ are defined by the coverings which correspond to the sublattices N_X, N_Y and N_W of N generated by $r_i v_i$, $s_i v_i$ and $t_i v_i$, respectively. Let $\widetilde{B}_i^X$, $\widetilde{B}_i^Y$ and $\widetilde{B}_i^W$ be prime divisors on $\widetilde{X}, \widetilde{Y}$ and $\widetilde{W}$ over B_i, respectively. They correspond to primitive vectors $r_i v_i$, $s_i v_i$ and $t_i v_i$, respectively. Let $p\colon \widetilde{W} \to \widetilde{X}$ and $q\colon \widetilde{W} \to \widetilde{Y}$ be the natural morphisms. Then the fully faithful functor $\Phi\colon D^b(\mathrm{coh}(\widetilde{Y})) \to D^b(\mathrm{coh}(\widetilde{X}))$ is constructed as $\Phi = p_*q^*$ in [15, Theorem 4.2 (1)]. We have

$$\Phi\left(\mathcal{O}_{\widetilde{Y}}\left(\sum_{i=1}^{n} k_i \widetilde{B}_i^Y\right)\right) = \mathcal{O}_{\widetilde{X}}\left(\left\lfloor \frac{k_1 r_1}{s_1} \right\rfloor \widetilde{B}_1^X + \sum_{i=2}^{n} k_i \widetilde{B}_i^X\right)$$

for $k_i = 0, \dots, s_i - 1$.

We define a toric boundary D on B_1 from the pair (X, B) as follows (cf. [16, Section 5]). Let $\overline{N} = N/\mathbf{Z}v_1$, and write $v_i + \mathbf{Z}v_1 = d_i \bar{v}_i \in \overline{N}$ for primitive vectors $\bar{v}_i$, $i = 2, \dots, n$. Then define $D = \sum_{i=2}^{n}(1 - 1/d_i r_i)D_i$ for $D_i = B_1 \cap B_i$. Let $\widetilde{B}_1$ be the associated smooth Deligne–Mumford stack above (B_1, D). Let $\widetilde{B}_1'$ be the normalization of the fiber product $\widetilde{B}_1 \times_X \widetilde{X}$ with natural morphisms $p_1\colon \widetilde{B}_1' \to \widetilde{B}_1$ and $p_2\colon \widetilde{B}_1' \to \widetilde{X}$.

We consider the semi-orthogonal complement of $\Phi(D^b(\mathrm{coh}(\widetilde{Y})))$. We have exact sequences

$$\begin{aligned} 0 \to \mathcal{O}_{\widetilde{X}}\left((k_1 - 1)\widetilde{B}_1^X + \sum_{i=2}^{n} k_i \widetilde{B}_i^X\right) &\to \mathcal{O}_{\widetilde{X}}\left(\sum_{i=1}^{n} k_i \widetilde{B}_i^X\right) \\ &\to p_{2*}p_1^*\mathcal{O}_{\widetilde{B}_1^X} \otimes \mathcal{O}_{\widetilde{X}}\left(\sum_{i=1}^{n} k_i \widetilde{B}_i^X\right) \to 0. \end{aligned}$$

Therefore $D^b(\mathrm{coh}(\widetilde{X}))$ is generated by $\Phi(D^b(\mathrm{coh}(\widetilde{Y})))$ and the sheaves $p_{2*}p_1^*\mathcal{O}_{\widetilde{B}_1^X} \otimes \mathcal{O}_{\widetilde{X}}\left(\sum_{i=1}^{n} l_i \widetilde{B}_i^X\right)$ for $0 \le l_i \le r_i - 1$ such that $l_1 \ne \lfloor k_1 r_1/s_1 \rfloor$ for any k_1.

We have the following vanishing:

- If $l_1 \ne \lfloor k_1' r_1/s_1 \rfloor$ for any k_1', then

$$R\mathrm{Hom}_{\widetilde{X}}\left(\mathcal{O}_{\widetilde{X}}\left(\left\lfloor \frac{k_1 r_1}{s_1} \right\rfloor \widetilde{B}_1^X + \sum_{i=2}^{n} k_i \widetilde{B}_i^X\right), p_{2*}p_1^*\mathcal{O}_{\widetilde{B}_1^X} \otimes \mathcal{O}_{\widetilde{X}}\left(\sum_{i=1}^{n} l_i \widetilde{B}_i^X\right)\right) = 0.$$

- If $l_1, l'_1 \neq \lfloor k'_1 r_1/s_1 \rfloor$ for any k'_1 and if $l < l'$, then

$$R\mathrm{Hom}_{\widetilde{X}}\left(p_{2*}p_1^*\mathcal{O}_{\widetilde{B}_1^X}\otimes\mathcal{O}_{\widetilde{X}}\left(\sum_{i=1}^n l_i \widetilde{B}_i^X\right),\ p_{2*}p_1^*\mathcal{O}_{\widetilde{B}_1^X}\otimes\mathcal{O}_{\widetilde{X}}\left(\sum_{i=1}^n l_i \widetilde{B}_i^X\right)\right) = 0.$$

Therefore the semi-orthogonal complement is generated by subcategories

$$p_{2*}p_1^* D^b(\mathrm{coh}(\widetilde{B}_1)) \otimes \mathcal{O}_{\widetilde{X}}(l_1 \widetilde{B}_1^X),$$

$1 \le l_1 \le r_1 - 1$, such that $l_1 \neq \lfloor k_1 r_1/s_1 \rfloor$ for any k_1 as in [16], and there is no morphism between objects belonging to different l_1. Therefore we conclude the proof. $\square$

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