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Describing Curved Spaces by Matrices

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It is shown that a covariant derivative on any d-dimensional manifold M can be mapped to a set of d operators acting on the space of functions defined on the principal $Spin(d)$-bundle over M. In other words, any d-dimensional manifold can be described in terms of d operators acting on an infinite-dimensional space. Therefore it is natural to introduce a new interpretation of matrix models in which matrices represent such operators. In this interpretation, the diffeomorphism, local Lorentz symmetry and their higher-spin analogues are included in the unitary symmetry of the matrix model. Furthermore, the Einstein equation is obtained from the equation of motion, if we take the standard form of the action, $S = -\mathrm{tr}\left([A_a, A_b][A^a, A^b]\right)$.

§1. Introduction

Although it is believed that string theory may provide the unification of fundamental interactions, its present formulation based on perturbation theory is not satisfactory. In order to examine whether it really describes our four-dimensional world, a non-perturbative and background independent formulation is needed. Some of the promising candidates are matrix models.[1),2)] They are basically obtained through dimensional reduction of the ten-dimensional $U(N)$ $\mathcal{N}=1$ supersymmetric Yang-Mills theory.

The action of IIB matrix model[2)] is given by

$$S = -\frac{1}{4g^2}\mathrm{tr}\left([A_a, A_b][A^a, A^b]\right) + \frac{1}{2g^2}\mathrm{tr}\left(\bar{\psi}\Gamma^a[A_a, \psi]\right), \tag{1·1}$$

where ψ is a ten-dimensional Majorana-Weyl spinor, and A_a and ψ are $N \times N$ hermitian matrices. The indices a and b are contracted by the flat metric. This action has an $SO(10)$ global Lorentz symmetry and $U(N)$ symmetry. There is evidence that this model describes gravity through a one-loop quantum effect.[2),3)] However, it is unclear how the fundamental principle of general relativity is realized in this model. In order to elucidate this point, we need to clarify the meaning of the flat metric. This should help us to understand the dynamics of this model further.

Let us first point out that this model has several interpretations. First, this action can be regarded as the Green-Schwarz action of IIB superstring in the Schild gauge after the regularization of the functions on the world-sheet by the matrices.[2)]

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In this case, the matrices A_a are simply the coordinates of the target space.*) This interpretation is consistent with the supersymmetry algebra. If we regard a constant shift of matrices as a translation, we have the $\mathcal{N}=2$ supersymmetry.

Another interpretation of this model is based on noncommutative geometry. The matrix model (1·1) has the following noncommutative momenta as a classical solution,

$$A_a = p_a, \quad [p_a, p_b] = iB_{ab}1, \tag{1·2}$$

where B_{ab} is a c-number. Then we introduce noncommutative coordinates x^a as

$$x^a = C^{ab}p_b, \quad C^{ab} = (B^{-1})^{ab}. \tag{1·3}$$

These coordinates satisfy the following commutation relations:

$$[x^a, x^b] = -iC^{ab}1, \quad [x^a, p_b] = i\delta^a{}_b 1. \tag{1·4}$$

As is seen from this, there is no essential difference between coordinates and momenta in noncommutative geometry. If we expand A_a around this background as

$$A_a = p_a + a_a(x), \tag{1·5}$$

$a_a(x)$ becomes a gauge field on the noncommutative space,[5)–7)] and A_a can be regarded as the covariant derivative itself.

In the first interpretation presented above, the matrices represent the spacetime coordinates, while in the second they are regarded as differential operators in a noncommutative space. Therefore it is natural to consider a third interpretation in which matrices represent differential operators in a commutative space. As a first attempt to realize such an interpretation, we regard large N matrices as a linear mapping from W to W, where W is the space of smooth functions from $\mathbb{R}^{10}$ to $\mathbb{C}$. In other words, W is the set of field configurations of a scalar field in ten-dimensional space-time. Therefore matrices are identified with integral kernels, which can formally be expressed as differential operators of infinite order as

$$\begin{aligned} f(x) \to & \int d^{10}y K(x,y) f(y) \\ \simeq & \left(b(x) + b^{\mu}(x)\partial_\mu + b^{\mu\nu}(x)\partial_\mu\partial_\nu + \cdots\right) f(x). \end{aligned} \tag{1·6}$$

In this way, matrices can be naturally regarded as differential operators, and an infinite number of local fields appear as the coefficients of derivatives.

In this interpretation,

$$A_a = i\delta_a{}^{\mu}\partial_\mu \tag{1·7}$$

is a classical solution,**) because the equation of motion is given by $[A_a, [A_a, A_b]] = 0$ when $\psi = 0$. We then expand A_a around this solution,[8)] obtaining

$$A_a = a_a(x) + \frac{i}{2}\{a_a^{\mu}(x), \partial_\mu\} + \frac{i^2}{2}\{a_a^{\mu\nu}(x), \partial_\mu\partial_\nu\} + \cdots, \tag{1·8}$$

*) This action has the same form as the low energy effective action of D-instantons in the flat background,[4)] and the eigenvalues of the matrices represent the positions of the D-instantons. Here, we do not take this point of view.

**) Below, we regard the indices a and μ as the local Lorentz and spacetime indices, respectively. This solution corresponds to the flat background, with $e_a^{\mu} = \delta_a^{\mu}$.

where the coefficients $a_a^{\mu\nu\cdots}(x)$ are real totally symmetric tensor fields, and anti-commutators have been introduced to make each term hermitian. The unitary symmetry of this model is expressed as

$$\delta A_a = i[\Lambda, A_a], \tag{1·9}$$

where the infinitesimal parameter Λ can also be expanded as

$$\Lambda = \lambda(x) + \frac{i}{2}\{\lambda^\mu(x), \partial_\mu\} + \frac{i^2}{2}\{\lambda^{\mu\nu}(x), \partial_\mu\partial_\nu\} + \cdots . \tag{1·10}$$

Let us examine how each local field transforms in the case that $\Lambda = \frac{i}{2}\{\lambda^\mu(x), \partial_\mu\}$, which is expected to give the diffeomorphism. For the first two fields, we obtain

$$\begin{aligned} \delta a_a(x) &= -\lambda^\mu(x)\partial_\mu a_a(x), \\ \delta a_a^\mu(x) &= -\lambda^\nu(x)\partial_\nu a_a^\mu(x) + a_a^\nu(x)\partial_\nu\lambda^\mu(x). \end{aligned} \tag{1·11}$$

From this, we find that $a_a(x)$ and $a_a^\mu(x)$ transform as a scalar and a vector, respectively. The $SO(10)$ index a has no relation to the diffeomorphism. Therefore we attempt to interpret it as a local Lorentz index and regard $a_a^\mu(x)$ as something like the vielbein field. Then it is natural to replace ordinary derivatives with covariant derivatives as

$$A_a \to a_a(x) + \frac{i}{2}\{a_a^b(x), \nabla_b\} + \frac{i^2}{2}\{a_a^{bc}(x), \nabla_b\nabla_c\} + \cdots . \tag{1·12}$$

Here, ∇_a is a covariant derivative, given by

$$\nabla_a = e_a^\mu(x)\left(\partial_\mu + \omega_\mu^{bc}(x)\mathcal{O}_{bc}\right), \tag{1·13}$$

where $e_a^\mu(x)$ and $\omega_\mu^{ab}(x)$ are the vielbein and spin connection, respectively. The operator $\mathcal{O}_{ab}$ is the Lorentz generator, which acts on Lorentz indices. Its explicit form depends on the bundles on which the covariant derivative operates.

There arise two difficulties in carrying out the replacement (1·12). One is the gluing of coordinate patches on curved spaces. In order to define a vector operator like (1·12), we first define it on each patch, and then glue them using transition functions. The latter procedure usually mixes the vector components. Therefore, a vector operator cannot be realized by simply giving a set of ten matrices. Such a set would be identified with a set of ten scalar operators, not a vector operator. The second problem arises when we consider the product of covariant derivatives. Suppose $A_a = i\nabla_a$. Let us consider the $1, 2$ component of A_aA_b, as an example. It is given by

$$\begin{aligned} A_1A_2 = -\nabla_1\nabla_2 &= -\partial_1\nabla_2 - \omega_1^{2c}(x)\nabla_c \\ &= i\partial_1A_2 + i\omega_1^{2c}(x)A_c. \end{aligned} \tag{1·14}$$

The difficulty comes from the second term. Because the sum is taken over the index c, A_1A_2 is not simply a product of A_1 and A_2. This prevents us from directly identifying ∇_a with a matrix and considering naive products. In other words, ∇_a cannot be identified with a matrix component by component. The main goal of this paper is to solve these two problems. We show that it is possible to express covariant derivatives in terms of matrices. In §2.2, we present concrete examples of covariant derivatives on a two-sphere and a two-torus in terms of two matrices.

The organization of this paper is as follows. In the next section, we explain how covariant derivatives on any curved space can be described by matrices. The regular representation of the Lorentz group plays an important role in this description. We regard matrices as mappings from the function space on the principal $Spin(10)$ bundle to itself. Then we present the above-mentioned examples. In §3, we apply this idea to matrix models. It is shown that the Einstein equation can be obtained from the equation of motion of the matrix model. If we introduce a mass term, we obtain a cosmological constant. We then consider the expansion of matrices with respect to covariant derivatives, and examine the local fields that appear as its coefficients. We find that both the diffeomorphism and the local Lorentz symmetry are naturally included in the unitary symmetry of the matrix model. Section 4 is devoted to discussion. In Appendix A, we give the proof of (2·3). In Appendix B, we discuss higher-spin fields.

§2. Differential operators on curved spaces

In order to express covariant derivatives in terms of matrices, the vector space on which they act should be sufficiently large. A covariant derivative maps a tensor field of rank n to one of rank $(n+1)$. Thus, if a space is invariant under the actions of covariant derivatives, it should contain at least tensor fields of any rank. In this section, we show that we can indeed find such a good space, in which covariant derivatives are expressed as endomorphisms on it. Here we use the term *endomorphism on* V in reference to a linear map from a vector space V to itself, and the expression $End(V)$ to represent the set of such maps. We further show that the use of this space resolves both of the difficulties mentioned in the previous section.

In §2.1, we consider the regular representation of a group and examine some important properties of a vector bundle whose fiber is its representation space. We then apply it to the Lorentz group and give a prescription for embedding covariant derivatives into matrices in §2.2.

2.1. *Preliminaries: regular representation*

Let us begin by considering the space of smooth functions from a group G to $\mathbb{C}$:

$$V_{\rm reg} = \{f : G \to \mathbb{C}\}. \tag{2·1}$$

We assume G to be compact and later choose G as $Spin(10)$ or $Spin^c(10)$. The action of an element h of G is given by

$$(\hat{h}f)(g) = f(h^{-1}g). \tag{2·2}$$

The space $V_{\rm reg}$ is called *the regular representation*. This representation is reducible and is decomposed into irreducible representations as

$$V_{\rm reg} = \bigoplus_r (\underbrace{V_r \oplus \cdots \oplus V_r}_{d_r}), \tag{2·3}$$

where V_r is the space of the irreducible representation r of G, and d_r is its dimension. This decomposition is proven in Appendix A.

The following interesting isomorphism holds for any representation r:

$$V_r \otimes V_{\rm reg} \cong \underbrace{V_{\rm reg} \oplus \cdots \oplus V_{\rm reg}}_{d_r}. \tag{2·4}$$

This is the key equation for expressing covariant derivatives as endomorphisms. The concrete form of (2·4) is as follows. Let $\Phi^i(g)$ be an element of $V_r \otimes V_{\rm reg}$, where the index i transforms as the representation r. That is, the action of G is given by

$$(\hat{h}\Phi^i)(g) = R^i{}_j(h)\Phi^j(h^{-1}g), \tag{2·5}$$

where $R^i{}_j(h)$ is the representation matrix corresponding to r. Then the isomorphism (2·4) is given by*)

$$\Phi^{(i)}(g) = R^{(i)}{}_j(g^{-1})\Phi^j(g), \tag{2·6}$$

where $i = 1, \cdots, d_r$. We can verify that each component of $\Phi^{(i)}(g)$ belongs to $V_{\rm reg}$ as

$$\begin{aligned}(\hat{h}\Phi^{(i)})(g) &= R^{(i)}{}_j(g^{-1})(\hat{h}\Phi^j)(g)\\ &= R^{(i)}{}_j(g^{-1})R^j{}_k(h)\Phi^k(h^{-1}g)\\ &= \Phi^{(i)}(h^{-1}g).\end{aligned} \tag{2·7}$$

The index (i) is not transformed by the action of $\hat{h}$ but, rather, it merely labels the copies of $V_{\rm reg}$ on the right-hand side of (2·4).

Here we emphasize that this isomorphism is constructed in such a way that the action of G commutes with the conversion of indices between (i) and i. If $\Phi(g)$ has some number of indices, each index can be converted in a similar manner. For example, if $\Phi(g)$ has two indices, we have

$$\Phi^{(i)j}(g) = R^{(i)}{}_k(g^{-1})\Phi^{kj}(g), \tag{2·8}$$

where $\Phi^{(i)j}(g)$ is an element of $V_{r'} \otimes V_{\rm reg}$ for each $i = 1, \cdots, d_r$, and $\Phi^{kj}(g)$ is an element of $V_r \otimes V_{r'} \otimes V_{\rm reg}$.

So far we have considered the regular representation $V_{\rm reg}$ of G. We next consider a fiber bundle over a manifold $M = \cup_i U_i$, where the fiber is $V_{\rm reg}$ and the structure group is G. We denote this fiber bundle by $E_{\rm reg}$. A global section of $E_{\rm reg}$ is defined

*) Note that $R^i{}_j$ and $R^{(i)}{}_j$ are the same quantity. However, we distinguish them, because the indices i and (i) obey different transformation laws. Specifically, i is transformed by the action of G, while (i) is not.

as a set of smooth maps from each coordinate patch U_i to $V_{\rm reg}$. Because an element of $V_{\rm reg}$ is a function from G to $\mathbb{C}$, such map is simply a function from $U_i \times G$ to $\mathbb{C}$. In order to be glued globally, they must satisfy the following condition on each overlapping region $x \in U_i \cap U_j$:*)

$$\tilde{f}^{[j]}(x,g) = \tilde{f}^{[i]}(x, t_{ij}(x)g), \tag{2·9}$$

where $t_{ij}(x)$ is a transition function. We denote the set of sections by $\Gamma(E_{\rm reg})$.**)

We next consider the principal bundle $E_{\rm prin}$ which is associated with $E_{\rm reg}$, and introduce the space of smooth functions on it:

$$C^\infty(E_{\rm prin}) = \{f : E_{\rm prin} \to \mathbb{C}\}. \tag{2·11}$$

Here we show that $\Gamma(E_{\rm reg})$ is isomorphic to $C^\infty(E_{\rm prin})$. To define an element f of $C^\infty(E_{\rm prin})$, we first introduce a function $f^{[i]}$ from $U_i \times G$ to $\mathbb{C}$ for each patch U_i. They should be related on each overlapping region of U_i and U_j as

$$f^{[j]}(x,g) = f^{[i]}(x, t_{ij}(x)g). \tag{2·12}$$

This follows from the fact that $E_{\rm prin}$ is constructed from the set of $U_i \times G$ by identifying $(x_{[i]}, g_{[i]})$ and $(x_{[j]}, g_{[j]})$ on each overlapping region, when they satisfy the relations

$$x_{[i]} = x_{[j]}, \quad g_{[i]} = t_{ij}(x) g_{[j]}. \tag{2·13}$$

Because f and $\tilde{f}$ satisfy the same gluing condition, they must be the same object. Thus we have shown the isomorphism

$$\Gamma(E_{\rm reg}) \cong C^\infty(E_{\rm prin}). \tag{2·14}$$

In the next subsection, we regard covariant derivatives as operators acting on such space.

An isomorphism similar to (2·4) exists for fiber bundles. It is given by

$$\Gamma(E_r \otimes E_{\rm reg}) \cong \underbrace{\Gamma(E_{\rm reg}) \oplus \cdots \oplus \Gamma(E_{\rm reg})}_{d_r}, \tag{2·15}$$

where E_r is a fiber bundle whose fiber is V_r and is associated with $E_{\rm reg}$. Let $f_k^{[i]}(x,g)$ and $f_{(k)}^{[i]}(x,g)$ be elements of the left-hand side and right-hand side, respectively. The isomorphism (2·15) is expressed by the relation

$$f_{(k)}^{[i]}(x,g) = R_{(k)}{}^{l}(g^{-1}) f_l^{[i]}(x,g). \tag{2·16}$$

In this way, we can convert the two kinds of indices (i.e. indices with and without parentheses) using $R(g)$.

Using this isomorphism, we can naturally lift an element of $End(\Gamma(E_{\rm reg}))$ to one of $End(\Gamma(E_r \otimes E_{\rm reg}))$. This is illustrated in Fig. 1. The action of $B \in End(\Gamma(E_{\rm reg}))$

*) We use the symbol $[i]$ to indicate quantities associated with U_i.

**) From (2·3), the set of sections is equivalent to the space of the field configurations

$$\bigoplus_r \underbrace{(\mathcal{V}_r \oplus \cdots \oplus \mathcal{V}_r)}_{d_r}, \tag{2·10}$$

$$
\begin{tikzcd}
\Gamma(E_r \otimes E_{\rm reg}) \arrow[r, "B", "\spadesuit"'] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \Gamma(E_r \otimes E_{\rm reg}) \arrow[d] \\
\Gamma(E_{\rm reg}) \oplus \cdots \oplus \Gamma(E_{\rm reg}) \arrow[r, "B\oplus\cdots\oplus B", "\spadesuit\spadesuit"'] & \Gamma(E_{\rm reg}) \oplus \cdots \oplus \Gamma(E_{\rm reg})
\end{tikzcd}
$$

Fig. 1. The action of B commutes with the conversions of indices. The latter are expressed by the vertical arrows and given by the isomorphism (2·15).

on $\Gamma(E_{\rm reg}) \oplus \cdots \oplus \Gamma(E_{\rm reg})$ is well-defined, because B acts on each component of the direct sum. This action is denoted by ♠♠ in Fig. 1. The action denoted by ♠ is defined in such a way that the commutative diagram in Fig. 1 holds; that is, the action ♠ is given in terms of ♠♠ by

$$B^{[i]} f_l^{[i]}(x,g) = R_l{}^{(k)}(g) B^{[i]} f_{(k)}^{[i]}(x,g). \tag{2·17}$$

Here, B belongs to $End(\Gamma(E_r \otimes E_{\rm reg}))$ on the left-hand side, while it belongs to $End(\Gamma(E_{\rm reg}))$ on the right-hand side. In this way, we can lift $End(\Gamma(E_{\rm reg}))$ to $End(\Gamma(E_r \otimes E_{\rm reg}))$. Generalizations, such as a lift from $End(\Gamma(E_r \otimes E_{\rm reg}))$ to $End(\Gamma(E_r \otimes E_{r'} \otimes E_{\rm reg})$, are straightforward. As we see in the next subsection, covariant derivatives are a typical example of this lifting mechanism.

2.2. *Covariant derivatives as matrices*

As discussed at the beginning of this section, we need to prepare a good space in which covariant derivatives are expressed as endomorphisms. We now show that $\Gamma(E_{\rm reg})$ is such space. From now on, we take G to be $Spin(10)$ or $Spin^c(10)$.*) We consider a ten-dimensional manifold M, and take a spin structure or a spin-c structure.

A covariant derivative is given by

$$\nabla_a^{[i]} = e_a^{\mu[i]}(x)\left(\partial_\mu + \omega_\mu^{bc[i]}(x)\mathcal{O}_{bc} - i a_\mu{}^{[i]}(x)\right), \tag{2·19}$$

where $\mathcal{O}_{bc}$ is the generator of G and $a_\mu(x)$ is a $U(1)$ gauge field belonging to the $U(1)$ part of $Spin^c(10)$.**) When we consider its action on $\Gamma(E_{\rm reg})$, it can be regarded as the map

$$\nabla_a : \Gamma(E_{\rm reg}) \to \Gamma(T \otimes E_{\rm reg}), \tag{2·20}$$

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where $\mathcal{V}_r$ is the space of fields on M, which transforms as the irreducible representation r of G.

*) The definition of $Spin^c(10)$ is given by

$$Spin^c(10) = (Spin(10) \times U(1))/Z_2. \tag{2·18}$$

Every manifold does not necessarily admit a spin structure. However, we can introduce a spin-c structure into arbitrary manifolds. If a manifold we consider admits a spin structure, we choose $Spin(10)$. If not, we choose $Spin^c(10)$.

**) When we consider $Spin(10)$, we can set $a_\mu = 0$.

where T is the tangent bundle. In general, the action of a covariant derivative changes the space. Here we use the isomorphism given in (2·15),

$$\Gamma(T \otimes E_{\mathrm{reg}}) \cong \underbrace{\Gamma(E_{\mathrm{reg}}) \oplus \cdots \oplus \Gamma(E_{\mathrm{reg}})}_{10}. \qquad (2·21)$$

By composing the two maps (2·20) and (2·21), we obtain the following map:

$$\nabla_{(a)} : \Gamma(E_{\mathrm{reg}}) \to \underbrace{\Gamma(E_{\mathrm{reg}}) \oplus \cdots \oplus \Gamma(E_{\mathrm{reg}})}_{10}. \qquad (2·22)$$

Here, the index (a) is a label indicating one of the ten copies of $\Gamma(E_{\mathrm{reg}})$. Therefore, each component of $\nabla_{(a)}$ can be regarded as a mapping from $\Gamma(E_{\mathrm{reg}})$ to $\Gamma(E_{\mathrm{reg}})$:

$$\nabla_{(a)} \in End(\Gamma(E_{\mathrm{reg}})) \qquad (2·23)$$

for $a = 1, \cdots, 10$. Explicitly, $\nabla_{(a)}$ and ∇_a are related on each patch as

$$\nabla^{[i]}_{(a)} = R_{(a)}{}^{b}(g^{-1}_{[i]})\nabla^{[i]}_{b}. \qquad (2·24)$$

Let us confirm the gluing condition for them. In the region in which two patches overlap, i.e. $x \in U_i \cap U_j$, $\nabla^{[i]}_a$ and $\nabla^{[j]}_a$ are related as

$$\nabla^{[i]}_{a} = R_{a}{}^{b}(t_{ij}(x))\nabla^{[j]}_{b}. \qquad (2·25)$$

Therefore, we obtain

$$\begin{aligned}\nabla^{[i]}_{(a)} &= R_{(a)}{}^{b}(g^{-1}_{[i]})\nabla^{[i]}_{b} \\ &= R_{(a)}{}^{b}(g^{-1}_{[i]})R_{b}{}^{c}(t_{ij}(x))\nabla^{[j]}_{c} \\ &= R_{(a)}{}^{c}((t^{-1}_{ij}g_{[i]})^{-1})\nabla^{[j]}_{c} \\ &= \nabla^{[j]}_{(a)}. \end{aligned} \qquad (2·26)$$

This confirms that each component of $\nabla_{(a)}$ is globally defined and gives an endomorphism. In other words, $\nabla_{(a)}$ can be expressed as ten infinite-dimensional matrices.

We now examine explicitly how each component of $\nabla_{(a)}$ acts on $\Gamma(E_{\mathrm{reg}})$. It acts on an element f of $\Gamma(E_{\mathrm{reg}})$ as follows:

$$\nabla^{[i]}_{(a)} f^{[i]}(x,g) = R_{(a)}{}^{b}(g^{-1})e^{\mu[i]}_{b}(x)\left(\partial_\mu + \omega^{bc[i]}_{\mu}(x)\mathcal{O}_{bc} - ia_{\mu}{}^{[i]}(x)\right) f^{[i]}(x,g). \qquad (2·27)$$

The action of $\mathcal{O}_{bc}$ on $\Gamma(E_{\mathrm{reg}})$ is given by

$$i\epsilon^{ab}\left(\mathcal{O}_{ab}f^{[i]}\right)(x,g) = f^{[i]}\left(x, \left(1 + i\epsilon^{ab}M_{ab}\right)^{-1} g\right) - f^{[i]}(x,g), \qquad (2·28)$$

where M_{ab} is the matrix of the fundamental representation. From the relation $\Gamma(E_{\mathrm{reg}}) \cong C^\infty(E_{\mathrm{prin}})$, we see that this covariant derivative is a first-order differential operator on $C^\infty(E_{\mathrm{prin}})$.

To this point, we have considered a covariant derivative as a set of endomorphisms on $\Gamma(E_{\rm reg})$. By an argument similar to that given in the last part of the previous subsection, we can naturally lift an endomorphism on $\Gamma(E_{\rm reg})$ to one on $\Gamma(E_r \otimes E_{\rm reg})$ for any representation r. Let us consider a specific example in which E_r is the tangent bundle T. Let $f_a^{[i]}(x,g)$ and $f_{(a)}^{[i]}(x,g)$ be elements of the left-hand side and the right-hand side of (2·21), respectively. Because we know how $\nabla_{(a)} \in End(\Gamma(E_{\rm reg}))$ acts on $f_{(b)}^{[i]}(x,g)$, we can define an endomorphism on $\Gamma(T \otimes E_{\rm reg})$ through the relation

$$\nabla_{(a)}^{[i]} f_b^{[i]}(x,g) = R_b{}^{(c)}(g) \nabla_{(a)}^{[i]} f_{(c)}^{[i]}(x,g). \tag{2·29}$$

This equation indicates that

$$\epsilon^{de} \mathcal{O}_{de} f_b^{[i]}(x,g) = R_b{}^{(c)}(g) \epsilon^{de} \mathcal{O}_{de} f_{(c)}^{[i]}(x,g). \tag{2·30}$$

However, this is merely an infinitesimal version of (2·7). In this sense, the lifting is naturally realized for covariant derivatives.

We can now show the hermiticity of $i\nabla_{(a)}$ for each component. Let u and v be elements of $\Gamma(E_{\rm reg})$. We first define an inner product on $\Gamma(E_{\rm reg})$ as

$$(u,v) = \int e d^d x dg u^{[i]}(x,g)^* v^{[i]}(x,g), \tag{2·31}$$

where $e = \det e_a^\mu$ and dg is the Haar measure. This inner product is independent of the choice of the patch. Then, for any $u_{(a)} \in \Gamma(E_{\rm reg}) \oplus \cdots \oplus \Gamma(E_{\rm reg})$, we can show

$$\begin{aligned}
(u_{(a)}, i\nabla_{(a)} v) &= \int e d^d x dg u_{(a)}^{[i]}(x,g)^* i\nabla_{(a)}^{[i]} v^{[i]}(x,g) \\
&= \int e d^d x dg u_a^{[i]}(x,g)^* i\nabla_a^{[i]} v^{[i]}(x,g) \\
&= \int e d^d x dg (i\nabla_a^{[i]} u_a^{[i]}(x,g))^* v^{[i]}(x,g) \\
&= \int e d^d x dg (i\nabla_{(a)}^{[i]} u_{(a)}^{[i]}(x,g))^* v^{[i]}(x,g) \\
&= (i\nabla_{(a)} u_{(a)}, v).
\end{aligned} \tag{2·32}$$

To derive this, we have used two important relations. The first is

$$\begin{aligned}
u^{(a)[i]}(x,g) \nabla_{(a)}^{[i]} &= \delta^{(a)(b)} R_{(a)}{}^c(g^{-1}) R_{(b)}{}^d(g^{-1}) u_c^{[i]}(x,g) \nabla_d^{[i]} \\
&= u^{c[i]}(x,g) \nabla_c^{[i]},
\end{aligned} \tag{2·33}$$

which follows from $R^T R = 1$, and the second is

$$\begin{aligned}
\nabla^{(a)[i]} u_{(a)}^{[i]}(x,g) &= \delta^{(a)(b)} \nabla_{(a)}^{[i]} u_{(b)}^{[i]}(x,g) \\
&= \delta^{(a)(b)} R_{(b)}{}^d(g^{-1}) \nabla_{(a)}^{[i]} u_d^{[i]}(x,g) \\
&= \delta^{(a)(b)} R_{(b)}{}^d(g^{-1}) R_{(a)}{}^c(g^{-1}) \nabla_c^{[i]} u_d^{[i]}(x,g)
\end{aligned}$$

$$= \nabla^{c[i]} u_c^{[i]}(x, g), \tag{2·34}$$

where we have used the equation (2·29) in going from the first line to the second line. Note that $\nabla_{(a)}$ belongs to $End(\Gamma(E_{\mathrm{reg}})) \oplus \cdots \oplus End(\Gamma(E_{\mathrm{reg}}))$ in the first line, while it belongs to $End(\Gamma(T \otimes E_{\mathrm{reg}}))$ in the second line. Finally, by setting such as $u_{(1)} = u$ and $u_{(2)} = \cdots = u_{(10)} = 0$ in (2·32), we complete the proof of the hermiticity of $i\nabla_{(a)}$.

In the following, we present the explicit forms of covariant derivatives on S^2 and T^2, from which we can understand how curved spaces are globally described by matrices.

Here we take G to be $Spin(2)$. Its representation of spin s is given by

$$R^{\langle s \rangle}(\theta) = e^{2is\theta}. \quad (\theta \in [0, 2\pi)) \tag{2·35}$$

As shown in (2·28), the Lorentz generator $\mathcal{O}_{+-}$ is expressed in terms of the derivative with respect to θ as

$$\mathcal{O}_{+-} = \frac{1}{4i}\frac{\partial}{\partial\theta}. \tag{2·36}$$

Here, $+$ and $-$ indicate the linear combinations of the Lorentz indices $1 + i2$ and $1 - i2$, respectively.

First we consider S^2 with a homogeneous and isotropic metric. In the stereographic coordinates projected from the north pole, we have

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{(1 + z\bar{z})^2}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0. \tag{2·37}$$

The simplest choice of the zweibein consistent with this metric is

$$e_{+\bar{z}} = e_{-z} = \frac{1}{1 + z\bar{z}}, \quad e_{+z} = e_{-\bar{z}} = 0, \tag{2·38}$$

and the spin connection is given by

$$\omega_z^{+-} = -\frac{\bar{z}}{1 + z\bar{z}}, \quad \omega_{\bar{z}}^{+-} = \frac{z}{1 + z\bar{z}}. \tag{2·39}$$

Following the general procedure (2·24), we obtain

$$\begin{aligned} \nabla^{[z]}_{(+)} &= e^{-2i\theta}\left((1 + z\bar{z})\partial_z + \frac{i}{2}\bar{z}\partial_\theta\right), \\ \nabla^{[z]}_{(-)} &= e^{2i\theta}\left((1 + z\bar{z})\partial_{\bar{z}} - \frac{i}{2}z\partial_\theta\right). \end{aligned} \tag{2·40}$$

We can explicitly check that these satisfy the gluing condition given in (2·26). We introduce stereographic coordinate $(w, \bar{w})$ projected from the south pole. In the region in which the two patches overlap, they are related by $z = 1/w$. The coordinate θ' of the fiber on this patch is related to θ by the transition function

$$\theta' = \theta + \arg(z) + \frac{\pi}{2}. \tag{2·41}$$

Rewriting (2·40) in terms of (w, θ'), we obtain

$$\nabla^{[w]}_{(+)} = e^{-2i\theta'}\left((1+w\bar{w})\partial_w + \frac{i}{2}\bar{w}\partial_{\theta'}\right),$$
$$\nabla^{[w]}_{(-)} = e^{2i\theta'}\left((1+w\bar{w})\partial_{\bar{w}} - \frac{i}{2}w\partial_{\theta'}\right), \tag{2·42}$$

which have the same forms as (2·40) as expected. This confirms that the index (a) does not transform under a local Lorentz transformation, and each component of $\nabla_{(a)}$ is a scalar operator. Thus, a two-sphere can be expressed in terms of two endomorphisms.

Because $\nabla_{(+)}$ and $\nabla_{(-)}$ are endomorphisms, their products are well-defined, and we can consider their commutators. A simple calculation gives

$$[\nabla_{(+)}, \nabla_{(-)}] = -i\frac{\partial}{\partial\theta} = 4\mathcal{O}_{+-} \tag{2·43}$$

and

$$[\mathcal{O}_{+-}, \nabla_{(\pm)}] = \mp\frac{1}{2}\nabla_{(\pm)}. \tag{2·44}$$

Thus $\nabla_{(\pm)}$ and $\mathcal{O}_{+-}$ form the $\mathfrak{su}(2)$ algebra. Therefore we have

$$[\nabla_{(b)}, [\nabla^{(b)}, \nabla_{(a)}]] = -\nabla_{(a)}, \tag{2·45}$$

for $a = 1, 2$, and it turns out that the homogeneous isotropic two-sphere is a classical solution of a two-dimensional matrix model with a negative mass term (see also §3.1).

Here we emphasize the difference from a fuzzy two-sphere. It is described by an embedding into the three-dimensional flat space, and we identify the $SO(3)$ symmetry of this space with the isometry of that sphere. We thus need three matrices, i.e. derivatives are given by angular momentum operators. It is not described in terms of two matrices, although it is a two-dimensional manifold.

We next consider a second example, T^2. To cover all regions of T^2, we need four coordinate patches. We denote them by $(z_i, \bar{z}_i)$ $(i = 1, \cdots, 4)$. The simplest metric in this case is

$$g_{z_i\bar{z}_i} = g_{\bar{z}_i z_i} = 1, \quad g_{z_i z_i} = g_{\bar{z}_i\bar{z}_i} = 0, \tag{2·46}$$

and the corresponding covariant derivatives are given by

$$\nabla^{[z_i]}_{(+)} = e^{-2i\theta_i}\partial_{z_i},$$
$$\nabla^{[z_i]}_{(-)} = e^{2i\theta_i}\partial_{\bar{z}_i}. \tag{2·47}$$

In a region in which coordinate patches overlap, they are related by $z_i = z_j + c_{ij}$, where c_{ij} is a constant. The fiber is trivially glued as

$$\theta_i = \theta_j. \tag{2·48}$$

We have seen that two-dimensional manifolds with different topologies can be uniformly described by two infinite-dimensional matrices. In a similar way, all d-dimensional manifolds can be described by d infinite-dimensional matrices. (See Fig. 2.)

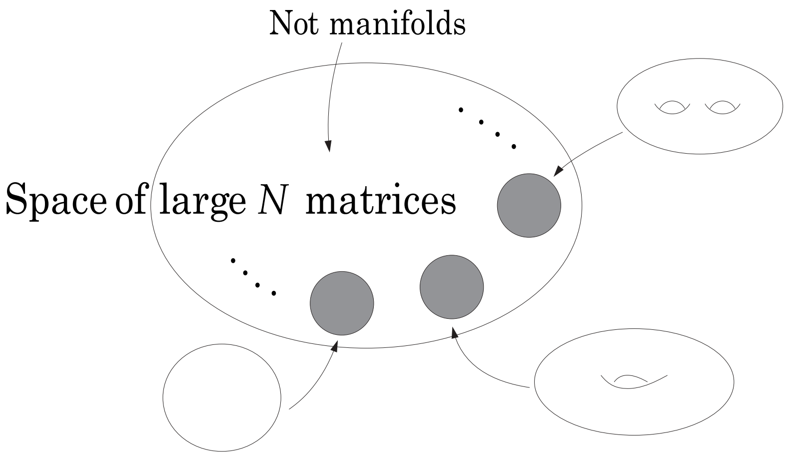

Fig. 2. All d-dimensional manifolds can be described by d large N matrices.

§3. New interpretation of matrix model

In this section, we apply the idea developed in the previous section to matrix models. We have seen that covariant derivatives can be expressed as endomorphisms if their indices are converted from a to (a). For this reason, it is natural to regard the matrix model (1·1) to be written in terms of indices with parentheses:

$$S = -\frac{1}{4g^2}\mathrm{tr}\left([A_{(a)}, A_{(b)}][A_{(c)}, A_{(d)}]\delta^{(a)(c)}\delta^{(b)(d)}\right) + \frac{1}{2g^2}\mathrm{tr}\left(\bar{\psi}^{(\alpha)}(\Gamma^{(a)})_{(\alpha)}{}^{(\beta)}[A_{(a)}, \psi_{(\beta)}]\right). \tag{3·1}$$

Each component of the dynamical variables can be identified with an endomorphism as

$$A_{(a)} : \Gamma(E_{\mathrm{reg}}) \to \Gamma(E_{\mathrm{reg}}), \tag{3·2}$$

where $(a) = 1, \cdots 10$, and

$$\psi_{(\alpha)} : \Gamma(E_{\mathrm{reg}}) \to \Gamma(E_{\mathrm{reg}}), \tag{3·3}$$

where $(\alpha) = 1, \cdots 16$. According to the equation (2·16), we can convert the indices as

$$A_a = R_a{}^{(b)}(g)A_{(b)}, \tag{3·4}$$

where $R_a{}^{(b)}(g)$ is the vector representation, and we have

$$\psi_\alpha = R_\alpha{}^{(\beta)}(g)\psi_{(\beta)}, \tag{3·5}$$

where $R_\alpha{}^{(\beta)}(g)$ is the spinor representation. In particular, A_a provides a mapping from $\Gamma(E_{\mathrm{reg}})$ to $\Gamma(T \otimes E_{\mathrm{reg}})$, where T is the tangent bundle, and ψ_α provides a mapping from $\Gamma(E_{\rm reg})$ to $\Gamma(S\otimes E_{\rm reg})$, where S is a spin bundle. Note that $A_{(a)}$ and $\psi_{(\alpha)}$ can be expressed as matrices, while A_a and ψ_α cannot.

We next attempt to rewrite the above action in terms of A_a and ψ_α. We first point out the following fact. In a quantity that is constructed from the product of matrices $A_{(a)}$, the indices with parentheses can be converted to those without parentheses by multiplying $R(g)$ on the left:

$$A_{(a)}A_{(b)}A_{(c)}\cdots = {R_{(a)}}^{a'}(g^{-1}){R_{(b)}}^{b'}(g^{-1}){R_{(c)}}^{c'}(g^{-1})A_{a'}A_{b'}A_{c'}\cdots . \qquad (3{\cdot}6)$$

Here, the right-hand side is defined by use of the lifting discussed in the last part of §2.1. For an endomorphism $A_{(a)} \in End(\Gamma(E_{\rm reg}))$, we can naturally lift it to $A_{(a)} \in End(\Gamma(E_r\otimes\cdots\otimes E_{\rm reg}))$ in such a way that the action of $A_{(a)}$ commutes with the conversion of indices. For example, $A_{(a)} \in End(\Gamma(T\otimes E_{\rm reg}))$ is defined in such a way that the following equation is satisfied:

$$A_{(a)}f_{(b)}(x,g) = {R_{(b)}}^{c}(g^{-1})A_{(a)}f_c(x,g). \qquad (3{\cdot}7)$$

Here, $A_{(a)}$ belongs to $End(\Gamma(E_{\rm reg}))$ on the left-hand side, while it belongs to $End(\Gamma(T\otimes E_{\rm reg}))$ on the right-hand side. From this, we can show

$$\begin{aligned} A_{(a)}A_{(b)}f(x,g) &= A_{(a)}{R_{(b)}}^{b'}(g^{-1})A_{b'}f(x,g) \\ &= {R_{(b)}}^{b'}(g^{-1})A_{(a)}A_{b'}f(x,g) \\ &= {R_{(a)}}^{a'}(g^{-1}){R_{(b)}}^{b'}(g^{-1})A_{a'}A_{b'}f(x,g). \end{aligned} \qquad (3{\cdot}8)$$

We stress that the product $A_{a'}A_{b'}$ is not the conventional matrix product. Rather, $A_{a'}$ acts on the index b' as does an ordinary covariant derivative. In a similar manner, we can prove the equation (3·6), which can be straightforwardly extended to include the fermionic matrices $\psi_{(\alpha)}$.

When indices are contracted on the left-hand side of (3·6), some of the representation matrices R on the right-hand side cancel out, due to the relation $R^TR=1$. For instance, we obtain

$$\delta^{(a)(b)}A_{(a)}A_{(b)} = \delta^{ab}A_aA_b \qquad (3{\cdot}9)$$

and

$$\begin{aligned} \bar\psi^{(\alpha)}{(\Gamma^{(a)})_{(\alpha)}}^{(\beta)}A_{(a)}\psi_{(\beta)} &= {R^{(\alpha)}}_{\gamma}{(\Gamma^{(a)})_{(\alpha)}}^{(\beta)}{R_{(a)}}^{b}{R_{(\beta)}}^{\delta}\bar\psi^{\gamma}A_b\psi_\delta \\ &= \bar\psi^{\gamma}{(\Gamma^{b})_{\gamma}}^{\delta}A_b\psi_\delta. \end{aligned} \qquad (3{\cdot}10)$$

Thus, the above matrix model action can be brought to the following form:

$$S = -\frac{1}{4g^2}\mathrm{tr}\left([A_a,A_b][A^c,A^d]\delta^{ac}\delta^{bd}\right) + \frac{1}{2g^2}\mathrm{tr}\left(\bar\psi^{\alpha}{(\Gamma^a)_{\alpha}}^{\beta}[A_a,\psi_\beta]\right). \qquad (3{\cdot}11)$$

We stress that A_a and ψ_α are no longer matrices. Their product is taken like that of ordinary covariant derivatives.

3.1. *Classical solutions*

In this subsection, we study classical solutions. The equation of motion is given by

$$[A^{(a)}, [A_{(a)}, A_{(b)}]] = 0 \tag{3·12}$$

in the case $\psi = 0$. Taking account of (3·6) and (3·9), we find that this is equivalent to

$$[A^a, [A_a, A_b]] = 0. \tag{3·13}$$

We search for classical solutions having the following form:

$$A_a = i\nabla_a(x) = ie_a^\mu(x)(\partial_\mu + \omega_\mu^{bc}(x)\mathcal{O}_{bc} - ia_\mu(x)). \tag{3·14}$$

Here we assume that there is no torsion, for simplicity. Then, substituting this into the left-hand side of (3·13), we obtain*)

$$\begin{aligned}[][\nabla^a, [\nabla_a, \nabla_b]] &= [\nabla^a, R_{ab}{}^{cd}\mathcal{O}_{cd} - if_{ab}] \\ &= [\nabla^a, R_{ab}{}^{cd}]\mathcal{O}_{cd} + R_{ab}{}^{cd}[\nabla^a, \mathcal{O}_{cd}] - i[\nabla^a, f_{ab}] \\ &= (\nabla^a R_{ab}{}^{cd})\mathcal{O}_{cd} - R_b{}^c\nabla_c - i(\nabla^a f_{ab}).\end{aligned} \tag{3·15}$$

Therefore, (3·13) is satisfied if the following hold:

$$\nabla^a R_{ab}{}^{cd} = 0, \quad R_{bc} = 0, \quad \nabla^a f_{ab} = 0. \tag{3·16}$$

The first equation follows from the second by the Bianchi identity $\nabla_{[a}R^{bc}{}_{de]} = 0$. Thus we have derived the Ricci-flat condition and the Maxwell equation from the equation of motion of the matrix model.

The above result seems unnatural in the sense that the energy-momentum tensor of a_μ does not appear in the second equation. It might be possible to resolve with the following argument. If we expand the exact equations with respect to the string scale α', they would take forms like the following:

$$\begin{aligned}&R_{ab} - \frac{1}{2}\delta_{ab}R \sim \alpha'\left(f_a{}^c f_{cb} + \frac{1}{4}\delta_{ab}f^2\right), \\ &\nabla^a f_{ab} \sim O(\alpha').\end{aligned} \tag{3·17}$$

We thus see that the equations we have obtained can be regarded as the lowest-order forms of the exact equations. If the loop expansion of the matrix model is in some manner related to the α' expansion, the classical solution naturally has the form of that which we have obtained. And the contribution from the energy momentum tensor comes from quantum corrections.**) This result might suggest a closer relation between our new interpretation of the matrix model and the world-sheet picture of string theory, because in the latter, the loop expansion is nothing but the α' expansion.

*) Note that $[\mathcal{O}_{cd}, \nabla_a] = \frac{1}{2}(\eta_{ca}\nabla_d - \eta_{da}\nabla_c)$ and $R_{ab}{}^{ac} = R_b{}^c$.

**) This is consistent with the original interpretation of the matrix model,[1),2)] where the coupling between the gravity and the energy momentum tensor arises as a one-loop quantum effect.

We next consider a matrix model with a mass term,

$$S = -\frac{1}{4g^2}\mathrm{tr}\left([A_{(a)}, A_{(b)}][A^{(a)}, A^{(b)}]\right) - \frac{m^2}{g^2}\mathrm{tr}\left(A_{(a)}A^{(a)}\right). \tag{3·18}$$

The equation of motion is now given by

$$[A_{(b)}, [A_{(a)}, A^{(b)}]] + 2m^2 A_{(a)} = 0, \tag{3·19}$$

and a calculation similar to (3·15) gives

$$\nabla^a R_{ab}{}^{cd} = 0, \quad R_{bc} = -2m^2\delta_{bc} \quad \nabla^a f_{ab} = 0. \tag{3·20}$$

The first condition again follows from the Bianchi identity and the second equation. The second equation gives the Einstein equation with the cosmological constant

$$\Lambda = -(d-2)m^2. \tag{3·21}$$

This implies that maximally symmetric spaces are classical solutions. A positive (negative) mass term corresponds to a negative (positive) cosmological constant.

3.2. *Expansion with respect to covariant derivatives*

As we have seen, each component of the matrices $A_{(a)}$ and $\psi_{(\alpha)}$ can be regarded as an element of $End(C^\infty(E_{\mathrm{prin}}))$. Therefore it can be expanded by derivatives acting on E_{prin}, that is, the covariant derivatives ∇_a and the Lorentz generators $\mathcal{O}_{ab}$.*) In general, the coefficients in such an expansion are functions on $C^\infty(E_{\mathrm{prin}})$, which are expanded as

$$f^{[i]}(x,g) = \sum_{r:\mathrm{rep}} c_{ij}^{[i]\langle r\rangle}(x) R_{ij}^{\langle r\rangle}(g), \tag{3·22}$$

where the summation is taken over all irreducible representations. For details, see Appendix A. As there seem to be too many fields, it would be better if we could restrict this space to a smaller one. We show here that it is indeed possible to remove the g-dependence in (3·22) by imposing a physically natural constraint.

First, we introduce the right action of G on $C^\infty(E_{\mathrm{prin}})$, and denote it by $\hat{r}(h)$:

$$\hat{r}(h) : f^{[i]}(x,g) \mapsto f^{[i]}(x,gh). \tag{3·23}$$

Note that this right action commutes with the left one. For $A_{(a)}$, we require that it transforms as a vector under the right action of G:

$$\hat{r}(h^{-1})A_{(a)}\hat{r}(h) = R_{(a)}{}^{(b)}(h)A_{(b)}. \tag{3·24}$$

Because the left-hand side can be rewritten as

$$\begin{aligned}\hat{r}(h^{-1})A_{(a)}\hat{r}(h) &= \hat{r}(h^{-1})R_{(a)}{}^{b}(g^{-1})A_b\hat{r}(h)\\ &= R_{(a)}{}^{b}(hg^{-1})\hat{r}(h^{-1})A_b\hat{r}(h),\end{aligned} \tag{3·25}$$

*) If G is $Spin^c(10)$, we have a $U(1)$ generator in addition to them. Here we ignore it for simplicity.

the requirement (3·24) is equivalent to

$$\hat{r}(h^{-1})A_a\hat{r}(h) = A_a. \tag{3·26}$$

If A_a is a function given by $A_a = a_a^{[i]}(x,g)$, the above equation imposes the condition $a_a^{[i]}(x,g) = a_a^{[i]}(x,gh^{-1})$, which means that $a_a^{[i]}$ is independent of g. The covariant derivative ∇_a also satisfies the constraint (3·26), because $\mathcal{O}_{ab}$ commutes with the right action. Therefore, the expansion of A_a is given by*)

$$A_a = \hat{a}_a^{[i]}(x) + \frac{i}{2}\{\hat{a}_a^{b[i]}(x), \nabla_b^{[i]}\} + \frac{i^2}{2}\{\hat{a}_a^{bc[i]}(x), \nabla_b^{[i]}\nabla_c^{[i]}\} + \cdots, \tag{3·27}$$

where

$$\begin{aligned}\hat{a}_a^{[i]}(x) &= a_a^{[i]}(x) + \frac{i}{2}\left\{a_{(1)a}{}^{bc[i]}(x), \mathcal{O}_{bc}\right\} + \frac{i^2}{2}\left\{a_{(2)a}{}^{bc,de[i]}(x), \mathcal{O}_{bc}\mathcal{O}_{de}\right\} + \cdots,\\ \hat{a}_a^{b\cdots[i]}(x) &= a_a^{b\cdots[i]}(x) + \frac{i}{2}\left\{a_{(1)a}{}^{b\cdots,cd[i]}(x), \mathcal{O}_{cd}\right\}\\ &\quad + \frac{i^2}{2}\left\{a_{(2)a}{}^{b\cdots,cd,de[i]}(x), \mathcal{O}_{cd}\mathcal{O}_{de}\right\} + \cdots.\end{aligned} \tag{3·28}$$

Here, the hat symbol ˆ indicates a power series in $\mathcal{O}_{ab}$. Here, the anti-commutators have been introduced to make each term manifestly hermitian.

We require that $\psi_{(\alpha)}$ transforms as a spinor under the right action of G:

$$\hat{r}(h^{-1})\psi_{(\alpha)}\hat{r}(h) = R_{(\alpha)}{}^{(\beta)}(h)\psi_{(\beta)}. \tag{3·29}$$

This is again equivalent to the requirement that ψ_α transforms as a scalar,

$$\hat{r}(h^{-1})\psi_\alpha\hat{r}(h) = \psi_\alpha, \tag{3·30}$$

and therefore ψ_α can be expanded similarly to (3·27).

Finally, we require that the infinitesimal parameter of the unitary transformation Λ transforms as a scalar under the right action of G:

$$\hat{r}(h^{-1})\Lambda\hat{r}(h) = \Lambda. \tag{3·31}$$

Therefore Λ can also be expanded similarly to (3·27).

The transformation law under the right action of G is summarized as follows: Λ, $A_{(a)}$ and $\psi_{(\alpha)}$ transform as a scalar, vector and spinor, respectively, while A_a and ψ_α transform as scalars.

Next, we confirm the background independence of the expansion (3·27). If we ignore the hermiticity for simplicity, the expansion can be expressed as**)

$$\hat{a}_a(x) + \hat{a}_a^b(x)\nabla_b + \hat{a}_a^{bc}(x)\nabla_b\nabla_c + \cdots. \tag{3·32}$$

*) We can assume that these coefficients are symmetric under permutations of ∇_a and $\mathcal{O}_{bc}$. For example, we have $a_{(2)a}{}^{bc,de} = a_{(2)a}{}^{de,bc}$ and $a_a^{bc} = a_a^{cb}$. Their anti-symmetric parts can be absorbed into $a_{(1)a}{}^{bc}$, because we have $a_{(2)a}{}^{[bc,de]}\mathcal{O}_{bc}\mathcal{O}_{de} = -a_{(2)a}{}^{[ce,cb]}\mathcal{O}_{eb}$ and $a_a^{[bc]}\nabla_b\nabla_c = \frac{1}{2}a_a^{[bc]}R_{bc}{}^{de}\mathcal{O}_{de}$.

**) For the remainder of this paper, we suppress the symbol $[i]$.

Let us see what happens if we express this using a different background $e'^{\mu}_{a}(x)$ and $\omega'^{ab}_{\mu}(x)$. Then, a simple calculation shows

$$
\begin{aligned}
\hat{a}^{b}_{a}(x)\nabla_b &= \hat{a}^{b}_{a}(x)e^{\mu}_{b}(x)\left(\partial_\mu + \omega^{cd}_{\mu}(x)\mathcal{O}_{cd}\right) \\
&= \hat{a}^{\mu}_{a}(x)\left(\partial_\mu + \omega^{cd}_{\mu}(x)\mathcal{O}_{cd}\right) \\
&= \hat{a}^{\mu}_{a}(x)\left(\partial_\mu + \omega'^{cd}_{\mu}(x)\mathcal{O}_{cd}\right) + \hat{a}^{\mu}_{a}(x)\delta\omega^{cd}_{\mu}(x)\mathcal{O}_{cd} \\
&= \hat{a}^{\mu}_{a}(x)\nabla'_{\mu} + \hat{a}^{\mu}_{a}(x)\delta\omega^{cd}_{\mu}(x)\mathcal{O}_{cd} \\
&= \hat{a}'^{b}_{a}(x)\nabla'_{b} + \hat{a}'^{b}_{a}(x)e'^{\mu}_{b}(x)\delta\omega^{cd}_{\mu}(x)\mathcal{O}_{cd},
\end{aligned}
\tag{3·33}
$$

where $\delta\omega^{cd}_{\mu}(x) = \omega^{cd}_{\mu}(x) - \omega'^{cd}_{\mu}(x)$. Although the second term in the last line of (3·33) seems to be an extra one, taking account of the fact that $\hat{a}_a(x)$ is given by a power series in $\mathcal{O}_{ab}$ as

$$\hat{a}_a(x) = a_a(x) + a^{bc}_{a}(x)\mathcal{O}_{bc} + a^{bc,de}_{a}(x)\mathcal{O}_{bc}\mathcal{O}_{de} + \cdots , \tag{3·34}$$

we find that it can be absorbed into $a^{bc}_{a}(x)$. Through similar calculations, the expansion (3·32) can be brought into the same form, after we change the background:

$$\hat{a}'_{a}(x) + \hat{a}'^{b}_{a}(x)\nabla'_{c} + \hat{a}'^{bc}_{a}(x)\nabla'_{b}\nabla'_{c} + \cdots . \tag{3·35}$$

This demonstrates the background independence of (3·32).

Finally, we comment on supersymmetry. The matrix model (3·1) is invariant under the following supersymmetry transformation,

$$
\begin{aligned}
\delta^{(1)}A_{(a)} &= i\bar{\epsilon}^{(\alpha)}(\Gamma_{(a)})_{(\alpha)}{}^{(\beta)}\psi_{(\beta)}, \\
\delta^{(1)}\psi_{(\alpha)} &= \frac{i}{2}[A_{(a)}, A_{(b)}](\Gamma^{(a)(b)})_{(\alpha)}{}^{(\beta)}\epsilon_{(\beta)},
\end{aligned}
\tag{3·36}
$$

where $\epsilon^{(\alpha)}$ is a constant parameter. We have restricted the matrix space by requiring that $A_{(a)}$ and $\psi_{(\alpha)}$ transform as a vector and a spinor, respectively, under the right action of G. Accordingly, we have to require that $\epsilon_{(\alpha)}$ transforms as a spinor. There is, however, no such constant parameter. Therefore, the restriction under the right action of G destroys the supersymmetry. We discuss this point further in §4.

3.3. *Diffeomorphism and local Lorentz symmetry*

In this subsection, we analyze the unitary symmetry of the matrix model. In particular, we show that the differmorphism and the local Lorentz symmetry are included in it. The matrix model (3·1) has the unitary symmetry

$$\delta A_{(a)} = i[\Lambda, A_{(a)}], \quad \delta\psi_{(\alpha)} = i[\Lambda, \psi_{(\alpha)}], \tag{3·37}$$

or, equivalently,

$$\delta A_a = i[\Lambda, A_a], \quad \delta\psi_\alpha = i[\Lambda, \psi_\alpha]. \tag{3·38}$$

We first consider the $U(1)$ gauge transformation, which is generated by $\Lambda = \lambda(x)$. Because a_a transforms according to

$$\delta a_a(x) = \partial_a\lambda(x), \tag{3·39}$$

we can identify a_a with the corresponding gauge field.

The regularization is generated by $\Lambda = \frac{i}{2}\{\lambda^\mu(x), \partial_\mu\}$. For example, the fields that appear in the first terms in the expansion (3·28) transform according to

$$\delta a_a(x) = -\lambda^\mu(x)\partial_\mu a_a(x), \tag{3·40}$$

$$\delta a_a^\mu(x) = -\lambda^\nu(x)\partial_\nu a_a^\mu(x) + a_a^\nu(x)\partial_\nu\lambda^\mu(x), \tag{3·41}$$

$$\delta a_a^{\mu\nu}(x) = -\lambda^\rho(x)\partial_\rho a_a^{\mu\nu}(x) + 2a_a^{\rho(\mu}(x)\partial_\rho\lambda^{\nu)}(x), \tag{3·42}$$

where the parentheses ( ) represent the symmetrization operation.[*)] In general, the field $a_a^{\mu_1\cdots\mu_{s-1}}(x)$ transforms as a rank-$(s-1)$ symmetric tensor field.

Finally, we discuss the local Lorentz symmetry, which is obtained by taking $\Lambda = {\lambda_{(1)}}^{ab}(x)\mathcal{O}_{ab}$. Here, the subscript (1) is added to indicate the first power of $\mathcal{O}_{ab}$. The background vielbein $e_a^\mu(x)$ and the spin connection $\omega_\mu^{ab}(x)$ indeed transform in accordance with

$$\delta e_a^\mu(x) = -{\lambda_{(1)a}}^b(x)e_b^\mu(x), \tag{3·43}$$

$$\delta {\omega_\mu}^{ab}(x) = \partial_\mu {\lambda_{(1)}}^{ab}(x) + 2{\lambda_{(1)}}^{[a}{}_c(x)\omega_\mu^{b]c}(x), \tag{3·44}$$

which are obtained from $\delta\nabla_a = -[{\lambda_{(1)}}^{bc}(x)\mathcal{O}_{bc}, \nabla_a]$. The transformations for the gauge field $a_a(x)$, the vielbein field $a_a^b(x)$ and the fluctuation of the spin connection ${a_{(1)}}_a^{bc}(x)$ are also obtained as

$$\delta a_a(x) = -{\lambda_{(1)a}}^b(x)a_b(x), \tag{3·45}$$

$$\delta {a_a}^b(x) = -{\lambda_{(1)a}}^c(x){a_c}^b(x) - {\lambda_{(1)}}^b{}_c(x){a_a}^c(x), \tag{3·46}$$

$$\delta {a_{(1)}}_a^{bc}(x) = -{\lambda_{(1)a}}^d(x){a_{(1)}}_d^{bc}(x) + 2\lambda^{[b}_{(1)d}(x){a_{(1)}}_a^{c]d}(x). \tag{3·47}$$

For the fermionic field $\chi(x)$, we obtain

$$\delta\chi(x) = -\frac{1}{4}{\lambda_{(1)}}^{bc}(x)\Gamma_{bc}\chi(x). \tag{3·48}$$

In this way, the local Lorentz transformation is correctly reproduced.

Other gauge symmetries are related to higher-spin fields. They are discussed in Appendix B.

§4. Discussion

In this section, we discuss some issues which remain to be clarified.

We first comment on the massless fields that appear in string theory. We have considered only the gravity and $U(1)$ gauge field. First, we have not yet understood how anti-symmetric tensor fields arise in this model. For the dilaton, one possibility is to introduce it as an overall factor:

$$A_a = e^\phi\left(ia_a^b\nabla_b + \cdots\right). \tag{4·1}$$

[*)] Note that $a^{(\mu_1\cdots\mu_n)} \equiv (a^{\mu_1\cdots\mu_n} + \text{all permutations of the indices})/n!$.

Substituting this ansatz into the equation of motion (3·13), we find that only constant solutions solve it. As in the case of the equation (3·16), we probably need to take account of loop corrections in order to obtain non-trivial solutions. Second, we assumed that there is no torsion in §3.1. Solving the equation of motion (3·13) with non-zero torsion, we find that it propagates as a rank-three anti-symmetric tensor field. This may provide a clue to understanding how anti-symmetric tensor fields appear in our model. Third, although we have considered only the $U(1)$ gauge group, it does not seem difficult to introduce other gauge symmetries by extending G.

Next, let us discuss what happens if we try to express the action in terms of the local fields that appear in the expansion (3·27). Because matrices are regarded as differential operators defined on a manifold E_{prin}, a naive definition of the trace gives a divergent result. We need to introduce the regularization in such a way that the cutoff corresponds to the matrix size N. In this case, we can introduce the heat kernel, using the Laplacian on the principal bundle, and we could probably use it as the regularization. It would be interesting if the action can be obtained in a good approximation.

As mentioned in the last paragraph of §3.2, the global supersymmetry is destroyed if we restrict the matrix space. There would be two possible ways to implement the supersymmetry in our formulation. One is to use supergroups as G, and the other is to use supermanifolds as the base space M. Here we discuss the latter possibility. In this case, the bosonic variable would be expanded as

$$A_a \sim i a_a^{\mu}(x)\nabla_\mu + \frac{i}{2}\bar{\chi}_a^{\alpha}(x)\nabla_\alpha, \tag{4·2}$$

where ∇_μ and ∇_α are the covariant derivatives on the superspace. If we consider the unitary transformation generated by

$$\Lambda = i\bar{\epsilon}^{\alpha}(x)\nabla_\alpha + 2(\bar{\epsilon}(x)\gamma^\mu\theta)\nabla_\mu, \tag{4·3}$$

the fields introduced in (4·2) transform according to

$$\delta a_a^{\mu}(x) = 2i\bar{\epsilon}(x)\gamma^\mu\chi_a(x), \quad \delta\chi_a^{\alpha}(x) = 2\nabla_a\epsilon^{\alpha}(x). \tag{4·4}$$

Thus we can regard $a_a^{\mu}(x)$ and $\chi_a^{\alpha}(x)$ as the vielbein and the gravitino. In this manner, it is possible to include the local supersymmetry into the unitary symmetry of the matrix model. Because the bosonic variable A_a contains the gravitino, there is no reason to consider the fermionic variable ψ_α. Therefore there is no guiding principle to determine the form of the action. If we take a Yang-Mills type action, it gives a second-order equation for the gravitino, although it gives the Einstein equation as we have seen in §3.1. To pursue this possibility further, we should search for a better action.

In Appendix B, we discuss the fact that matrix models contain the higher-spin gauge fields with corresponding gauge symmetries. There remains a problem concerning the fields that appear as the coefficients of higher powers of $\mathcal{O}_{ab}$. At first order in $\mathcal{O}_{ab}$, we find that they are higher-spin analogues of the spin connection (see

Appendix B.2). On the other hand, we have not obtained a complete understanding for higher orders. One of the advantages of our formulation is the manifestation of the gauge invariance even with the interaction. For this reason, it may be interesting to analyze higher-spin gauge theories based on matrix models.

In our model, higher-spin fields do not have mass terms at the tree level. To obtain string theory, we should clarify how they acquire masses without explicitly breaking higher-spin gauge symmetries. If we are able to elucidate the origin of the masses and resolve the problem of supersymmetry, we will get a deeper understanding of string theory and matrix models.

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Appendix A
—— *Proof of (2·3)* ——

In this appendix, we give the proof of (2·3). We assume the group G to be compact. We use the orthonormality of the representation matrices expressed as

$$\frac{1}{Vol(G)}\int dg R^{\langle r\rangle}_{ij}(g)^* R^{\langle r'\rangle}_{kl}(g) = \frac{1}{d_r}\delta^{\langle r\rangle\langle r'\rangle}\delta_{ik}\delta_{jl}, \tag{A·1}$$

where dg is the Harr measure and $R^{\langle r\rangle}_{ij}(g)$ is the representation matrix for the irreducible representation r. Furthermore, the representation matrices $R^{\langle r\rangle}_{ij}(g)$ form a complete set of smooth functions from G to $\mathbb{C}$. Therefore, any smooth function $f(g)$ can be expanded as

$$f(g) = \sum_r c^{\langle r\rangle}_{ij}\sqrt{d_r}R^{\langle r\rangle}_{ij}(g), \tag{A·2}$$

where the sum is taken over all irreducible representations. We now see how $c^{\langle r\rangle}_{ij}$ transform under the action of G. The right-hand side transforms as

$$\begin{aligned}\hat{h} : c^{\langle r\rangle}_{ij}R^{\langle r\rangle}_{ij}(g) &\to c^{\langle r\rangle}_{ij}R^{\langle r\rangle}_{ij}(h^{-1}g)\\ &= c^{\langle r\rangle}_{ij}R^{\langle r\rangle}_{ik}(h^{-1})R^{\langle r\rangle}_{kj}(g),\end{aligned} \tag{A·3}$$

which shows that $c^{\langle r\rangle}_{ij}$ transforms as

$$c^{\langle r\rangle}_{ij} \to {}^T R^{\langle r\rangle}_{ik}(h^{-1})c^{\langle r\rangle}_{kj}. \tag{A·4}$$

The index i of c_{ij} transforms as the dual representation of r, while the index j is invariant. It follows that $c^{\langle r\rangle}_{ij}$ represents d_r copies of the dual representation of r. Thus we have found that the regular representation is decomposed as (2·3).

Appendix B
—— *Speculation on Higher-Spin Gauge Fields* ——

In this appendix, we discuss the higher-spin gauge fields that appear in the expansion of A_a with respect to covariant derivatives.

B.1. *Interpretation of* $a_a^{(\mu_1\cdots\mu_{s-1})}$ *and higher-spin gauge symmetries*

As we have seen from the transformation law under the differmorphism, the fields $a_a^{\mu_1\cdots\mu_{s-1}}$ transform as rank-$(s-1)$ totally symmetric tensor fields (see (3·42)). The field $a_a^\mu(x)$ is identified with the vielbein, which can be regarded as a gauge field associated with the symmetry generated by $\Lambda \sim i\lambda^\mu(x)\partial_\mu$. Therefore, it is natural to regard $a_a^{\mu_1\cdots\mu_{s-1}}(x)$ as a gauge field associated with

$$\Lambda = \frac{i^{s-1}}{2}\{\lambda^{\mu_1\cdots\mu_{s-1}}(x), \partial_{\mu_1}\cdots\partial_{\mu_{s-1}}\}. \tag{B·1}$$

We first examine the inhomogeneous term in such a transformation. It comes from the commutator between this Λ and the term $\frac{i}{2}\{a_a^\mu(x), \partial_\mu\}$ in A_a. Indeed, we can show that the field $a_a^{\mu_1\cdots\mu_{s-1}}(x)$ transforms according to

$$\delta a_a^{\mu_1\cdots\mu_{s-1}}(x) = a_a^\nu\partial_\nu\lambda^{\mu_1\cdots\mu_{s-1}}(x) + \cdots, \tag{B·2}$$

which is a natural generalization of (3·39) and (3·41).

We next discuss the homogeneous terms. The higher order derivatives in Λ given in (B·1) cause notrivial complexity compared to the case of the lower-spin gauge symmetries. Fields with different spins mix in a complicated manner, and also higher-order derivative terms appear. The latter implies the existence of higher-order derivative interactions.

B.2. *Interpretation of* ${a_{(1)}}_a^{b\cdots,cd}$ *and higher-spin analogue of local Lorentz symmetry*

It is natural to generalize the relation between the vielbein and the metric,

$$g_{\mu\nu}(x) = a_\mu^a(x)a_\nu^b(x)\delta_{ab}, \tag{B·3}$$

to higher-spin fields as

$$\phi^{\mu_1\cdots\mu_s}(x) = a_a^{(\mu_1}(x)a_b^{\mu_2\cdots\mu_s)}(x)\delta^{ab}. \tag{B·4}$$

Here $\phi^{\mu_1\cdots\mu_s}(x)$ is a rank-s totally symmetric tensor field[9),10)] which transforms according to

$$\delta\phi^{\mu_1\cdots\mu_s}(x) = \partial^{(\mu_1}\lambda^{\mu_2\cdots\mu_s)}(x) + \cdots \tag{B·5}$$

under the transformation generated by (B·1). There are some studies of higher-spin gauge theories based on the relation (B·4). See 11) for reviews.

We now recall the role of the local Lorentz symmetry. It removes the anti-symmetric part of the vielbein $a_\mu^a(x)$, and the remaining symmetric part corresponds to the metric tensor. We find that a mechanism similar to the higher-spin gauge fields $a_a^{(\mu_1\cdots\mu_{s-1})}$ exists in our model.

We consider the case of spin three ($s=3$) as an example. Because the ordinary local Lorentz symmetry is generated by $\Lambda \sim i\lambda_{(1)}^{ab}\mathcal{O}_{ab}$, it is natural to consider

$$\Lambda = \frac{1}{4} i\{i\{\lambda_{(1)}{}^{a,bc}, \mathcal{O}_{bc}\}, \partial_a\}. \tag{B·6}$$

Then $a_a^{\mu\nu}(x)$ transforms as

$$\delta a_a^{\mu\nu}(x) \sim -\lambda_{(1)}{}^{(\mu,}{}_{ab}(x) a_b^{\nu)}(x), \tag{B·7}$$

where $\lambda_{(1)}^{\mu,ab}(x) = \lambda_{(1)}^{c,ab}(x) e_c^{\mu}(x)$. This is a generalization of (3·43). In this case, however, we have a mixing of fields with different spins ($a_a^{\mu\nu}(x)$ and $a_b^{\nu}(x)$) due to the existence of the derivative in (B·6). Under this transformation, the spin-three field $\phi^{\mu\nu\rho}(x)$ is invariant:

$$\delta\phi^{\mu\nu\rho}(x) \to a_a^{(\mu}(x)\delta a_b^{\nu\rho)}(x)\delta^{ab} \sim -\lambda_{(1)}{}^{(\nu,}{}_{ab}(x) a_a^{\mu}(x) a_b^{\rho)}(x) = 0, \tag{B·8}$$

where we have used $\lambda_{(1)}{}^{\mu,ab}(x) = -\lambda_{(1)}{}^{\mu,ba}(x)$. This symmetry removes unnecessary components from the field $a_a^{\mu\nu}(x)$ and leaves the field $\phi^{\mu\nu\rho}(x)$ as the dynamical degrees of freedom.

Because the spin connection appears in A_a as the term $i\{a_{(1)}{}_a^{bc}, \mathcal{O}_{bc}\}$, we expect that a term like $\{\{a_{(1)}{}_a^{b,cd}, \mathcal{O}_{cd}\}, \partial_b\}$ plays an analogous role as the spin connection. Actually, we can check that $a_{(1)}{}_a^{b,cd}$ transforms inhomogeneously under the transformation (B·6) as

$$\delta a_{(1)}{}_a^{b,cd} = \partial_a \lambda_{(1)}{}^{b,cd} + \cdots . \tag{B·9}$$

A similar mechanism works also for $s \geq 4$.

In this manner, higher-spin sectors are included in our formulation. However, the positivity of the theory is still unclear. We have not yet discussed fields appearing with the second order in $\mathcal{O}_{bc}$, such as $a_{(2)}{}_a^{bc,de}(x)$. In such cases, analyses become difficult, because the higher-spin gauge symmetries mix fields with different spins in a complicated manner.

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