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PURDUE UNIVERSITY
SCHOOL OF ELECTRICAL ENGINEERING

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by

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PREFACE

In the last few years the study of nonlinear mechanics has received the attention of numerous investigators, either under the scope of pure mathematics or from the engineering point of view.

Many of the recent developments are based on the early works of H. Poincare [1] and A. Liapunov [2]. As examples can be cited the perturbation method, harmonic balance, the second method of Liapunov, etc.

An approximate technique developed almost simultaneously by C. Goldfarb [3] in the USSR, A. Tustin [4] in England, R. Kochenburger [5] in the USA, W. Oppelt [6] in Germany and J. Dutilh [7] and C. Ecary [8] in France, known as the describing function technique, can be considered as the graphical solution of the first approximation of the method of the harmonic balance.

The describing function technique has reached great popularity, principally because of the relative ease of computation involved and the general usefulness of the method in engineering problems.

However, in the past, the describing function technique has been useful only in analysis. More exactly, it is a powerful tool for the investigation of the possible existence of limit cycles and their approximate amplitudes and frequencies.

Several extensions have been developed from the original describing function technique. Among these can be cited the dual-input describing function, J. C. Douce et al. [9]; the Gaussian-input describing function, R. C. Booton [10]; and the root-mean-square describing function, J. E. Gibson and K. S. Prasanna-Kumar [11].

In a recent work which employs the describing function, C. N. Shen [12] gives one example of stabilization of a nonlinear system by introducing a saturable feedback. However, Shen's work cannot be qualified as a synthesis method since he fixes "a priori" the nonlinearity to be introduced in the feedback loop.

A refinement of the same principle used by Shen has been proposed by R. Haussler [13]. The goal of this new method of synthesis is to find the describing function of the element being synthesized. Therefore, for Haussler's method to be useful, a way must be found to reconstruct the nonlinearity from its describing function. This is called the inverse-describing-function-problem and is essentially a synthesis problem.

This is not the only case in which the inverse-describing-function-problem can be useful. Sometimes, in order to find the input-output characteristic of a physical nonlinear element, a harmonic test can be easier to perform rather than a static one (which also may be insufficient).

The purpose of this report is to present the results of research on a question which may then be concisely stated as: "If the describing function of a nonlinear element is known, what is the nonlinearity?"

The question may be divided into two parts, the first part being the determination of the restrictions on the nonlinearity (or its describing function) necessary to insure that the question has an answer, and the second part the practical determination of that answer when it exists. Accordingly, the material in this report is presented in two parts.

Part I is concerned with determining what types of nonlinearities are (and what types are not) uniquely determined by their conventional (fundamental) describing function.

This is done by first showing the non-uniqueness in general of the describing function, and then constructing a class of null functions with respect to the describing function integral, i.e., a class of nonlinearities not identically zero whose describing functions <u>are</u> identically zero. The defining equations of the describing function are transformed in such a manner as to reduce the inverse describing function problem to the problem of solving a Volterra integral equation, an approach similar to that used by Zadeh [18]. The remainder of Part I presents the solution of the integral equations and studies the effect of including higher order harmonics in the description of the output wave shape. The point of interest here is that inclusion of the second harmonic may cause the describing function to become uniquely invertible in some cases.

Part II presents practical numerical techniques for effecting the inversion of types of describing functions resulting from various engineering assumptions as to the probable form of the nonlinearities from which said describing functions were determined.

The most general method is numerical evaluation of the solution to the Volterra integral equations developed in Part I. A second method, which is perhaps the easiest to apply, requires a least squares curve fit to the given describing function data. Then use is made of the fact that the describing function of a polynomial nonlinearity is itself a polynomial to calculate the coefficients in a polynomial approximation to the nonlinearity. This approach is indicated when one expects that the nonlinearity is a smooth curve, such as a cubic characteristic. The third method presented assumes that the nonlinearity can be approximated by a piecewise linear discontinuous function, and the slopes and y-axis intercepts of each linear segment are computed. This approach is indicated when one expects a nonlinearity with relatively sharp corners.

It may be remarked that the polynomial approximation and the piecewise linear approximation are derived independently of the material in Part I.

All three methods presented in Part II are suited for use with experimental data as well as with analytic expressions for the describing functions involved. Indeed, an analytical expression must be reduced to discrete data for the machine methods to be of use.

To the best of the authors' knowledge, research in the area of describing function inversion has been nonexistent with the exception of Zadeh's paper [18] in 1956. It seems that a larger effort in this area would be desirable in the light of recent extensions of the describing function itself to signal stabilization of nonlinear control systems by Oldenburger and Sridhar [19] and Boyer [20], and the less restrictive study of dual-input describing functions for nonautonomous systems by Gibson and Sridhar [21].

There presently exist techniques for determining a desired describing function for use in avoiding limit cycle oscillations in an already nonlinear system (Haussler [13]), and the methods presented in this report now allow the exact synthesis of the nonlinear element from the describing function data.

Part I

THEORY OF DESCRIBING FUNCTION INVERSION

CHAPTER 1

DEFINITIONS

1.1 Introduction

The purpose of the present chapter is to define a suitable mathematical model for a general type of nonlinearity. The nonlinearities considered will be restricted to the kind known as "gain type nonlinearities." The property of this type of nonlinearity is that the output variable depends only on the actual value of the input, its past history and the sign of its first derivative. However, the output does not depend on the actual value of the first derivative of the output, nor on its higher derivatives.

This mathematical model will be chosen in order to match the nonlinearities that are found in practice. Thus, many of the definitions that in principle can be established arbitrarily, will be chosen with a view toward the physics of the situation.

Once this mathematical model is defined, a formal definition of the describing function will be established.

1.2 Definitions

Let us consider a nonlinear element and define as x and y the input and output variables respectively. Assume that a functional relationship will exist between the input and the output

$$y = f(x) \tag{1.1}$$

Let x be a sinusoidal wave of amplitude E and angular frequence ω.

$$x = E \sin \omega t \tag{1.2}$$

The output y will be a periodic function of time with period $2\pi/\omega$. If the output y satisfies the Dirichlet conditions, it can be expanded in a Fourier series

$$y(t) = \frac{A_0(E)}{2} + A_1(E) \cos \omega t + A_2(E) \cos 2\omega t + \dots$$

$$+ B_1(E) \sin \omega t + B_2(E) \sin 2\omega t + \dots \tag{1.3}$$

Where $A_n(E)$ and $B_n(E)$ are the Fourier coefficients given by the following expressions

$$A_n(E) = \frac{2}{T}\int_0^T f(E \sin \omega t) \cos n\omega t \, dt \tag{1.4}$$

$$B(E) = \frac{2}{T}\int_0^T f(E \sin \omega t) \sin n\omega t \, dt \tag{1.5}$$

If in the equations above we make the change of variable

$$\alpha = \omega t \tag{1.6}$$

and keep in mind that $T = 2\pi/\omega$, equation (1.4) and (1.5) are transformed into

$$A_n(E) = \frac{1}{\pi}\int_0^{2\pi} f(E \sin \alpha) \cos n\alpha d\alpha \qquad (1.7)$$

$$B_n(E) = \frac{1}{\pi}\int_0^{2\pi} f(E \sin \alpha) \sin n\alpha d\alpha \qquad (1.8)$$

The ratio between the nth harmonic to the amplitude of the input will be defined as the nth describing function. Then

$$g_n(E) \triangleq \frac{B_n(E)}{E} = \frac{1}{\pi E}\int_0^{2\pi} f(E \sin \alpha) \sin n\alpha d\alpha \qquad (1.9)$$

$$b_n(E) \triangleq \frac{A_n(E)}{E} = \frac{1}{\pi E}\int_0^{2\pi} f(E \sin \alpha) \cos n\alpha d\alpha \qquad (1.10)$$

For $n = 1$ we have the first (or conventional) describing function or, simply, the describing function. This is generally represented by the complex quantity

$$K_{eq}(E) = g(E) + j\, b(E) \qquad (1.11)$$

Given the definition of the describing function, we can ask if the simple functional relationship (1.1) is sufficient to describe completely the behavior of the nonlinearity; at least with respect to sinusoidal inputs. One simple example will reveal that equation (1.1) is not sufficient to determine, in some cases, the describing function.

Consider the case of a relay with hysteresis and dead band, whose characteristic is shown in figure (1). It is obvious from the figure that the characteristic of the element is double valued for certain ranges of the variable x. Therefore it will be necessary to define a criterion which will permit us to resolve the indeterminacy that appears when x lies within the interval in which f(x) is double valued. The following criterion will prove convenient: Let f(x) be equal to $f_1(x)$ for negative increments of the independent variable x and equal to $f_2(x)$ for positive increments of the same variable, where

$$
\begin{aligned}
f_1(x) &= 0 & -b < x \leq a \\
f_1(x) &= M & x > a \\
f_1(x) &= -M & x < -b
\end{aligned}
\qquad (1.12)
$$

and

$$
\begin{aligned}
f_2(x) &= 0 & -a < x \leq b \\
f_2(x) &= M & x > b \\
f_2(x) &= -M & x < -a
\end{aligned}
\qquad (1.13)
$$

However, this mathematical model is not sufficient to describe the real behavior of a relay with hysteresis and dead band. As a matter of fact the output of such an element not only depends on the actual value of the input, but also on its past history. For the example under consideration,

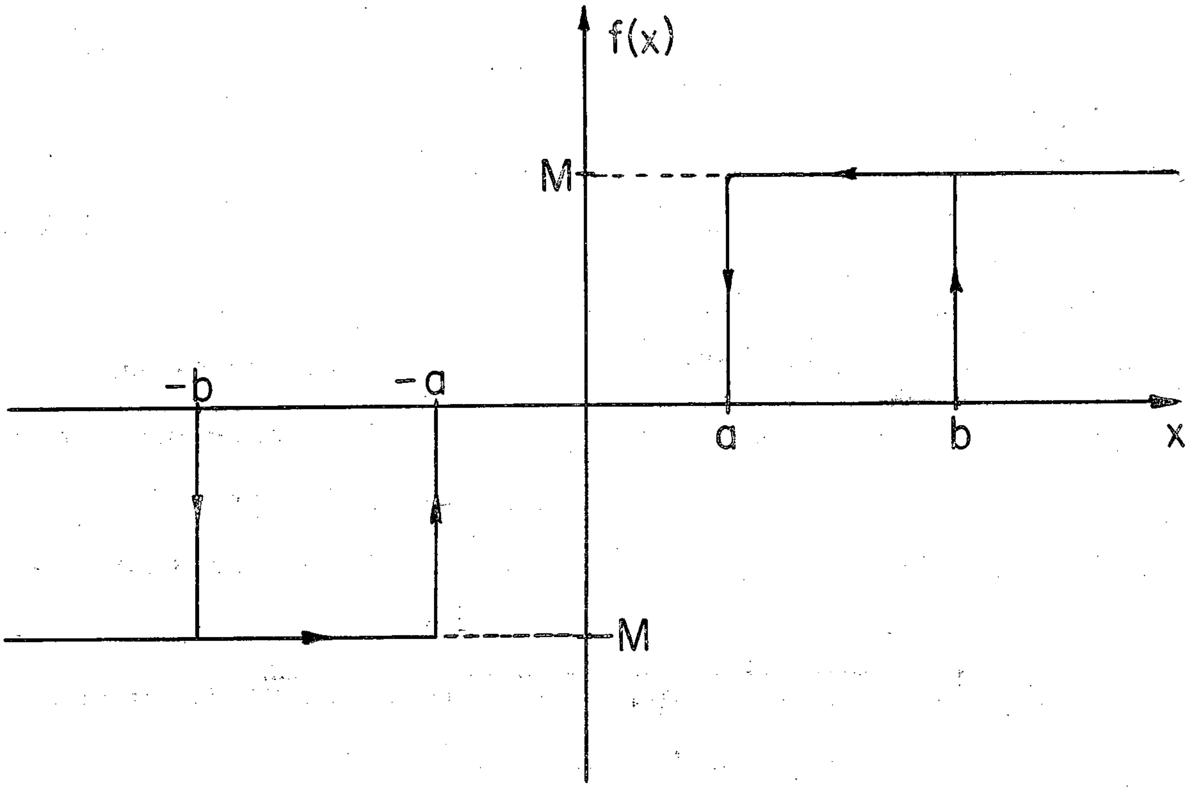

Fig. 1. Characteristic of a Relay with Hysteresis and Dead Band

the output is identically null, if the absolute value of x has never reached a maximum value larger than b. Therefore the describing function is identically zero if $E < b$. This information cannot be given by equation (1.1).

Consider now the more general functional relationship

$$y = f(x)\, h(E) \tag{1.14}$$

where

$$\begin{aligned} h(E) &= 0 \qquad E < b \\ h(E) &= 1 \qquad E > b \end{aligned} \tag{1.15}$$

It is not difficult to show that equation (1.14) is sufficient to describe the behavior of the relay under consideration, at least with respect to sinusoidal inputs. In figure (2) is shown the three dimensional representation of equation (1.14).

The following definition will be established: Given the functional relationship

$$y = F\left[x(t), \max_{\rho \leq t} |x(\rho)|\right] \tag{1.16}$$

for $x = E \sin \alpha$

$$y = F(x, E) \tag{1.17}$$

when (1.16) is double valued with respect to x (independent variable), y is equal to $F_1(x, E)$ for negative increments of x, and equal to $F_2(x, E)$ for positive increments of x. The describing function of $F\left[x(t), \max_{\rho \leq t} |x(\rho)|\right]$ will be de-

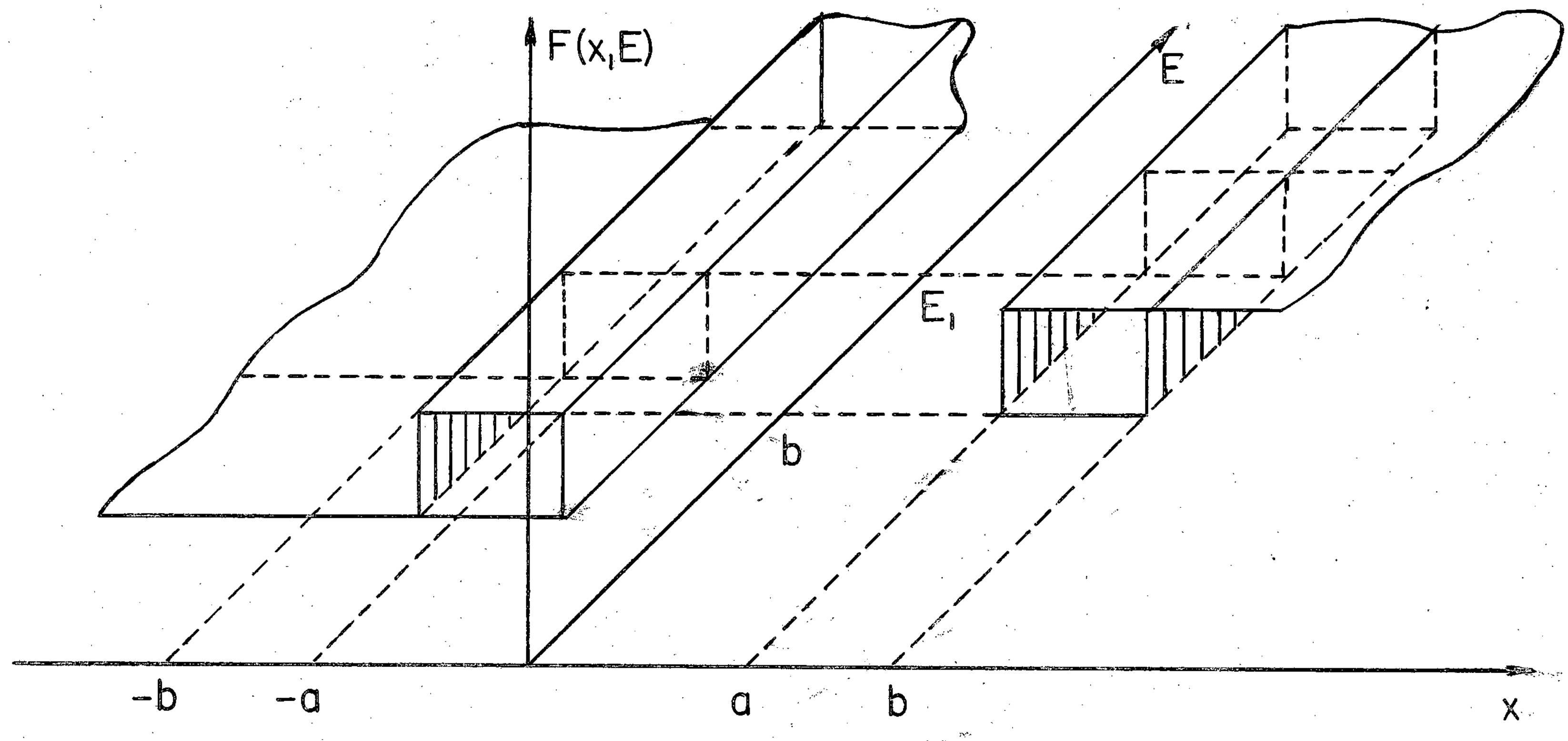

Fig. 2. F(x, E) for a Relay with Hysteresis and Dead Band

fined as the following complex quantity:

$$K_{eq}(E) = g(E) + j\, b(E) \tag{1.18}$$

where

$$g(E) = \frac{1}{\pi E}\int_0^{2\pi} F(E \sin \alpha, E) \sin\alpha d\alpha \tag{1.19}$$

$$b(E) = \frac{1}{\pi E}\int_0^{2\pi} F(E \sin\alpha, E) \cos\alpha d\alpha \tag{1.20}$$

The function $F\left[x(t), \max_{\rho \leq t} |x(\rho)|\right]$ also will be defined as the inverse describing function of $K_{eq}(E)$.

The nonlinearities to be dealt with in this work may be divided into two groups:

I) Nonmemory type nonlinearities, are those for which a mathematical model of the form $y = f(x)$ is sufficient to describe its behavior.

II) Memory type nonlinearities, are those for which a mathematical model of the form $y = F\left[x(t), \max_{\rho \leq t} |x(\rho)|\right]$ is needed to describe completely its behavior.

In both cases $f(x)$ and $F\left[x(t), \max_{\rho \leq t} |x(\rho)|\right]$ can be double valued with respect to x.

1.3 Conclusions

In this chapter a convenient mathematical model to describe the behavior of gain type nonlinearities has been defined. A formal definition of the describing function has also been established. From this definition will be derived, in the next chapters, an analytical approach to the inverse-describing-function-problem. This approach will be, of course, only valid for the type of nonlinearities for which the mathematical model applies. Then, from now on, for "nonlinearity" will be understood, "gain type nonlinearity."

CHAPTER 2

DESCRIBING FUNCTION TECHNIQUE

2.1 Introduction

In Chapter 1 the kinds of functions which can completely describe the behavior of a nonlinearity, in the sense that they are sufficient to determine the Fourier or conventional describing function were discussed. It was concluded that the functional relationship,

$$y = f(x) \qquad (1.1)$$

is not enough for some types of nonlinearities. Among those nonlinearities not included in (1.1) will be those which are said to have memory. Since the behavior of the majority of systems with memory-type nonlinearities depends on the maximum value of the input, the more general relationship

$$y = F\left[x(t), \max_{\rho \leq t} |x(\rho)|\right] \qquad (1.16)$$

was adopted. For sinusoidal inputs Eq. (1.16) reduces to $y = F(x, E)$ where E is the amplitude of the sinusoidal signal at the input of the nonlinearity.

In the present Chapter Eq. (1.19) and (1.20) will be transformed, in order to find a closed expression that will permit us, not only to solve the problem of the inverse describing function, but also gain more insight into the conventional describing function itself.

2.2 Transformation of the Integral Equations (1.19) and (1.20)

Consider the function

$$y = F(x, E) \tag{2.1}$$

which satisfies the conditions stated in Chapter 1. Let $F_1(x, E)$ and $F_2(x, E)$ be the two branches of $F(x, E)$ in the intervals in which it is double valued. According to the definition given in Chapter 1, the describing function will be

$$g(E) = \frac{1}{\pi E}\int_0^{2\pi} F(E \sin \phi, E) \sin \phi \, d\phi$$

$$(2.2)$$

$$b(E) = \frac{1}{\pi E}\int_0^{2\pi} F(E \sin \phi, E) \cos \phi \, d\phi$$

In order to simplify further development the following change of the variable of integration will be performed in Eq. (2.2).

$$\phi = \beta + \pi/2 \tag{2.3}$$

Once the transformation (2.3) is performed, Eqs. (2.2) are reduced to

$$g(E) = \frac{1}{\pi E}\int_{-\pi/2}^{+3/2\pi} F(E \cos \beta, E) \cos \beta \, d\beta$$

(2.4)

$$b(E) = -\frac{1}{\pi E}\int_{-\pi/2}^{+3/2\pi} F(E \cos \beta, E) \sin \beta \, d\beta$$

Because of the periodicity of $F(E \cos \beta, E)$ and $\cos \beta$, Eq. (2.4) can be transformed into

$$g(E) = \frac{1}{\pi E}\int_{0}^{2\pi} F(E \cos \beta, E) \cos \beta \, d\beta$$

(2.5)

$$b(E) = -\frac{1}{\pi E}\int_{0}^{2\pi} F(E \cos \beta, E) \sin \beta \, d\beta$$

In the interval 0 to π the increment of the independent variable is negative and it is positive in the interval π to 2π. Therefore, according to the hypothesis on $F(x, E)$ formulated in Chapter 1, Eq. (2.5) can be rewritten

$$g(E) = \frac{1}{\pi E}\int_{0}^{\pi} F_1(E \cos \beta, E) \cos \beta \, d\beta + \int_{\pi}^{2\pi} F_2(E \cos \beta, E) \cos \beta \, d\beta \qquad (2.6)$$

$$b(E) = -\frac{1}{\pi E}\int_0^{\pi} F_1(E\cos\beta,\ E)\sin\beta\, d\beta$$

$$+\int_{\pi}^{2\pi} F_2(E\cos\beta,\ E)\sin\beta\, d\beta$$

where $F_1(x, E)$ and $F_2(x, E)$ are single-valued functions of(i) x and E.

Let us decompose $F_1(x, E)$ and $F_2(x, E)$ in the following manner:

$$\begin{aligned} F_1(x, E) &= P_1(x, E) + Q_1(x, E) \\ F_2(x, E) &= P_2(x, E) + Q_2(x, E) \end{aligned} \tag{2.7}$$

where $P_1(x, E)$, $P_2(x, E)$, $Q_1(x, E)$ and $Q_2(x, E)$ satisfy the following conditions for all values of x and E

$$\begin{aligned} P_1(x, E) &= P_1(-x, E) \\ P_2(x, E) &= P_2(-x, E) \\ Q_1(x, E) &= -Q_1(-x, E) \\ Q_2(x, E) &= -Q_2(-x, E) \end{aligned} \tag{2.8}$$

i.e., P_1, P_2 are even in x, and Q_1, Q_2 are odd in x. To

(i) $F_1(x,E)$ and $F_1(x,E)$ can be double-valued functions in a set of points of zero measure; i.e., a finite or infinite denumerable number of discontinuities may be allowed.

show that it is always possible to find the set of functions $P_1(x,E)$, $P_2(x,E)$, $Q_1(x,E)$ and $Q_2(x,E)$ that simultaneously verified Eq. (2.7) and (2.8), change the sign of x in Eq. (2.7)

$$F_1(-x,E) = P_1(-x,E) + Q_1(-x,E) = P_1(x,E) - Q_1(x,E)$$
$$F_2(-x,E) = P_2(-x,E) + Q_2(-x,E) = P_2(x,E) - Q_2(x,E) \qquad (2.9)$$

Eq. (2.7) and (2.9) constitute two linear systems of equations in which the unknowns are $P_1(x,E)$, $P_2(x,E)$, $Q_1(x,E)$ and $Q_2(x,E)$. These are,

$$F_1(x,E) = P_1(x,E) + Q_2(x,E)$$
$$F_1(-x,E) = P_1(x,E) - Q_1(x,E) \qquad (2.10)$$

and

$$F_2(x, E) = P_2(x,E) + Q_2(x,E)$$
$$F_2(-x,E) = P_2(x,E) - Q_2(x,E) \qquad (2.11)$$

The solution of system (2.10) and (2.11) will always exist because its determinant is different from zero. Solving Eq. (2.10) and (2.11) results in,

$$P_1(x,E) = \frac{1}{2} \quad F_1(x,E) + F_1(-x,E)$$

$$Q_1(x,E) = \frac{1}{2} \quad F_1(x,E) - F_1(-x,E)$$

$$P_2(x,E) = \frac{1}{2} \quad F_2(x,E) + F_2(-x,E) \tag{2.12}$$

$$Q_2(x,E) = \frac{1}{2} \quad F_2(x,E) - F_2(-x,E)$$

The functions defined by Eq. (2.12) will exist everywhere and will be single-valued functions of x if $F_1(x,E)$ and $F_2(x,E)$ are themselves single valued functions of x. Once the validity of the decomposition of the functions $F_1(x,E)$ and $F_2(x,E)$, given by Eq. (2.7), has been proved, Eq. (2.7) may be substituted into (2.6), in order to transform Eq. (2.6) into the conventional form of the Volterra integral equations. This substitution yields

$$g(E) = \frac{1}{\pi E}\left[\int_0^{\pi} P_1(E\cos\beta,\, E)\cos\beta\, d\beta + \int_{\pi}^{2\pi} P_2(E\cos\beta,\, E)\cos\beta\, d\beta + \int_0^{\pi} Q_1(E\cos\beta,\, E)\cos\beta\, d\beta + \int_{\pi}^{2\pi} Q_2(E\cos\beta,\, E)\cos\beta\, d\beta\right] \tag{2.13}$$

$$b(E) = -\frac{1}{\pi E}\left[\int_0^{\pi} P_1(E\cos\beta, E)\sin\beta\, d\beta\right.$$

$$+\int_{\pi}^{2\pi} P_2(E\cos\beta, E)\sin\beta\, d\beta$$

$$+\int_0^{\pi} Q_1(E\cos\beta, E)\sin\beta\, d\beta$$

$$\left.+\int_{\pi}^{2\pi} Q_2(E\cos\beta, E)\sin\beta\, d\beta\right]$$

If in Eq. (2.13) the change of variable $\beta = \phi + \pi$ is made in all those integrals that are taken over the interval π to 2π, it is found, after some elementary trigonometric transformations, that,

$$g(E) = \frac{1}{\pi E}\left[\int_0^{\pi} P(E\cos\phi, E)\cos\phi\, d\phi + \int_0^{\pi} Q(E\cos\phi, E)\cos\phi\, d\phi\right]$$

$$b(E) = -\frac{1}{\pi E}\left[\int_0^{\pi} P(E\cos\phi, E)\sin\phi\, d\phi + \int_0^{\pi} Q(E\cos\phi, E)\sin\phi\, d\phi\right] \tag{2.14}$$

where

$$P(x, E) = P_1(x, E) - P_2(x,E)$$
$$Q(x, E) = Q_1(x, E) + Q_2(x,E) \tag{2.15}$$

But, because $P_1(x,E)$ and $P_2(x,E)$ are even functions of x, P(x,E) also will be an even function of x. For similar reasons Q(x,E) will be an odd function of x. Keeping in mind the above properties of P(x,E) and Q(x,E), Eq. (2.14) can be considerably simplified.

In Eq. (2.14) divide the interval of integration into two subintervals, the first between 0 and $\pi/2$ and the second between $\pi/2$ and π. In addition make the change of variable $\beta = \pi - \phi$ in all those integrals that are taken over the interval $\pi/2$ to π. After some elementary trigonometric transformations we obtain,

$$g(E) = \frac{2}{\pi E}\int_0^{\pi/2} Q(E\cos\beta, E)\cos\beta\, d\beta \tag{2.16}$$

$$b(E) = -\frac{2}{\pi E}\int_0^{\pi/2} P(E\cos\beta, E)\sin\beta\, d\beta \tag{2.17}$$

To transform Eq. (2.16) and (2.17) into the conventional form of the Volterra integral equations, let

$$E \cos \beta = x \tag{2.18}$$

Finally

$$g(E) = \frac{2}{\pi E^2}\int_0^E \frac{x\, Q(x,E)}{\sqrt{E^2 - x^2}}\, dx \tag{2.19}$$

$$b(E) = -\frac{2}{\pi E^2}\int_0^E P(x,E)\, dx \tag{2.20}$$

Discussion of Eq. (2.19) and (2.20)

The conventional method of computing the describing function of a nonlinear element requires the knowledge of the actual shape of the output signal of the nonlinear element when its input is driven by a sinusoidal wave. Then a Fourier analysis must be performed in order to find the amplitude of the first harmonic. This procedure is sometimes rather tedious, especially in the case in which the characteristic of the nonlinear element is not known by an analytic expression but by experimental data. However, by using Eq. (2.19) and (2.20) it is not necessary to compute the shape of the output, but only the two functions $Q(x,E)$ and $P(x,E)$. These functions, given by Eq. (2.15) and (2.12), can be computed directly from the characteristic of the nonlinear element. This approach appears to possess an advantage over the original expression given by Eq. (1.19) and

(1.20). As a matter of fact, by means of Eq. (2.19) and (2.20) a general method of computation of the describing function can be developed.

This is not the only advantage over the initial form given by Eq. (1.19) and (1.20). By means of Eq. (2.19) and (2.20) it is possible to gain more insight into the mechanism of the describing function. Important properties such as the non-uniqueness of the inverse describing function, conditions of existence of the describing function, etc., can be deduced from them.

From the conceptual point of view, Eq. (2.19) and (2.20) present great interest by themselves. With each single or double-valued (but memoryless) nonlinearity can be associated two single valued functions which give the complete information about the nonlinearity, in the sense that those two functions are sufficient to compute the describing function.

2.3 Non-Uniqueness of the Inverse Describing Function

Memory Type Nonlinearities

Let us show the non-uniqueness of the solution of the integral equation (2.19) for the case of memory type nonlinearities. It is sufficient to show the existence of a set of functions $Q_0(x,E)$, not identically zero, whose corresponding g(E) are identically zero. Assume $Q_0(x,E)$ to be of the form

$$Q_0(x,E) = h_1(x)\, m_1(E) + h_2(x)\, m_2(E) \qquad (2.21)$$

and attempt to choose $h_1(x)$, $h_2(x)$, $m_1(E)$ and $m_2(E)$ in order to have $g(E) = 0$. Substituting Eq. (2.21) into (2.19)

$$m_1(E)\int_0^E \frac{x\, h_1(x)}{\sqrt{E^2 - x^2}}\,dx + m_2(E)\int_0^E \frac{x\, h_2(x)}{\sqrt{E^2 - x^2}}\,dx = 0 \qquad (2.22)$$

This means that if we choose $h_2(x)$, $m_1(E)$ and $m_2(E)$ arbitrarily (assuming that $\frac{m_2(E)}{m_1(E)}$ has meaning) $h_1(x)$ will be given by the solution of the following integral equation

$$\int_0^E \frac{x\, h_1(x)}{\sqrt{E^2 - x^2}}\,dx = -\frac{m_2(E)}{m_1(E)}\int_0^E \frac{x\, h_2(x)}{\sqrt{E^2 - x^2}}\,dx \qquad (2.23)$$

Solving Eq. (2.23) for $h_1(x)$ (See Appendix II) yields,

$$h_1(x) = -\frac{2}{\pi x}\frac{d}{dx}\int_0^x dz\int_0^z \frac{z\, m_2(z)\, y\, h_2(y)}{m_1(z)\sqrt{x^2-z^2}\sqrt{z^2-y^2}}\,dy \qquad (2.24)$$

Therefore to every function of the type(i)

(i)Because of the symmetry of the original Eq. (2.22) the subindeces 1 and 2 can be interchanged in Eq. (2.25).

$$Q_0(x,E) = h_2(x)\, m_2(E)$$

$$- \frac{2m_1(E)}{\pi x} \frac{d}{dx}\int_0^x y h_2(y) \int_y^x \frac{z\, m_2(z) \quad dz\, dy}{m_1(z)\sqrt{x^2-z^2}\sqrt{z^2-y^2}} \qquad (2.25)^{(i)}$$

will correspond $g(E) \equiv 0$.

To illustrate the procedure let us consider one example. Let us choose arbitrarily

$$\frac{m_2(E)}{m_1(E)} = E^2$$

Eq. (2.25) becomes

$$Q_0(x,E) \equiv h_2(x)\, m_2(E)$$

$$- \frac{2m_1(E)}{\pi x} \frac{d}{dx}\int_0^x y\, h_2(y) \int_y^x \frac{z^3}{\sqrt{x^2-z^2}\sqrt{z^2-y^2}}\, dz\, dy \quad (2.26)$$

but

$$\int_y^x \frac{z^3}{\sqrt{x^2-z^2}\sqrt{z^2-y^2}}\, dz = \frac{\pi}{4}\ (x^2 + y^2) \qquad (2.27)$$

Substituting (2.27) into (2.26)

$$Q_0(x,E) = h_2(x)\, m_2(E) - \frac{m_1(E)}{2x} \frac{d}{dx}\int_0^x y\, h_2(y)\ (x^2+y^2) dy \quad (2.28)$$

which can be reduced to

$$Q_0(x,E) = m_1(E)\left[h_2(x)\ (E^2-x^2) - \int_0^x y\ h_2(y)\ dy\right] \quad (2.29)$$

Suppose that

$$m_1(E) = 1 \quad (2.30)$$

$$h_2(x) = x \quad (2.31)$$

Substituting (2.30) and (2.31) into (2.29)

$$Q_0(x,E) = x\ E^2 - \frac{4}{3}\ x^3 \quad (2.32)$$

In figure 3 is represented the block diagram of this non-linear element.

In an analogous manner the non-uniqueness of the solution of the integral equation (2.20) can be demonstrated.

Nonmemory Type

If the nonlinear element is of the nonmemory type, Eq. (2.19) and (2.20) are reduced to

$$g(E) = \frac{2}{\pi E^2}\int_0^E \frac{x\ Q(x)}{\sqrt{E^2 - x^2}}\ dx \quad (2.33)$$

$$b(E) = -\frac{2}{\pi E^2}\int_0^E P(x)\ dx \quad (2.34)$$

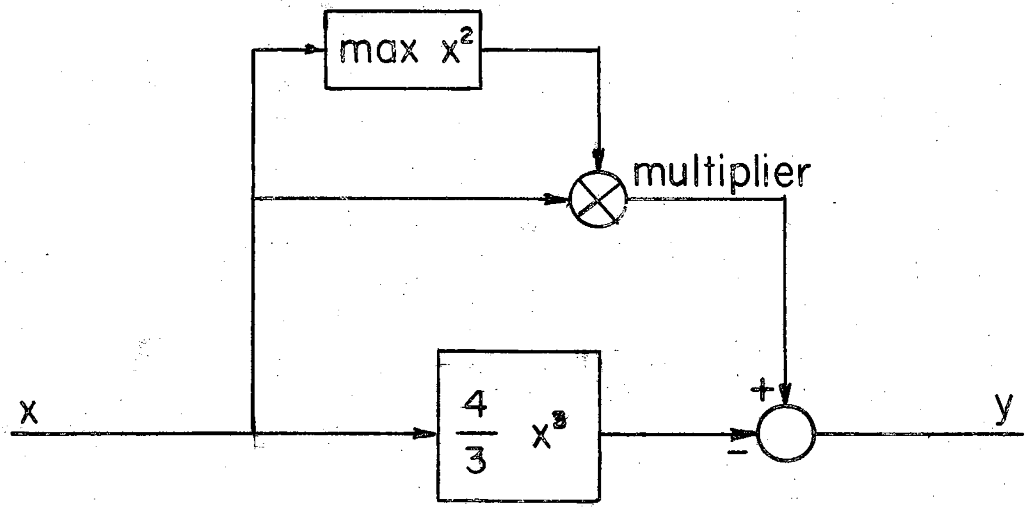

Fig. 3. Example of a Nonlinearity with a Describing Function Identically Null

Both integral equations are Volterra integral equations of the first kind and their solutions will be unique if P(x) and Q(x) are assumed to be continuous. Therefore, given the functions g(E) and b(E) there will exist one and only one pair of functions Q(x) and P(x), that, substituted in Eq. (2.19) and (2.20), will transform these equations in an identity[(i)]. But P(x) and Q(x) are not sufficient to determine the nonlinearity. As a matter of fact, the equation of the nonlinearity will only be determined if $P_1(x)$, $P_2(x)$, $Q_1(x)$ and $Q_2(x)$ are known. From Eq. (2.15) it can be shown that, given P(x) and Q(x), any set of equations $P_1(x)$, $P_2(x)$, $Q_1(x)$ and $Q_2(x)$ that satisfy Eq. (2.15) can generate a different nonlinearity with the same describing function. Therefore the knowledge of the describing function of a nonlinear element is not sufficient to determine the equation of the nonlinear element. Even in the case of single-valued nonlinearities, g(E) is not sufficient to determine the nonlinearity. It will be shown in Chapter 3, that one even harmonic in addition is necessary to determine the nonlinearity uniquely.

2.4 Sufficient and Necessary Conditions for b(E) to be Identically Zero

Another important property can be deduced from Eq.

[(i)] g(E) and b(E) must verify some conditions in order that P(x) and Q(x) exist. Those conditions will be derived in Chapter 3.

(2.19) and (2.20). It is well known, and its demonstration is almost immediate, that a sufficient condition for $b(E) \equiv 0$ is that the nonlinearity must be single-valued. It is not difficult to demonstrate that this condition is not a necessary one. As a matter of fact, for the case of nonmemory type nonlinearities, the necessary and sufficient condition for $b(E) \equiv 0$ is that $P(x) \equiv 0$. (The case of memory-type nonlinearities is not considered here because it is always possible to find a function $P(x,E) \not\equiv 0$ such that, when substituted into Eq. (2.20), $b(E) \equiv 0$.

The sufficiency of the condition is obvious. To prove necessity for $P(x)$ piecewise continuous[(i)], suppose $P(x)$ continuous for x (a,b). Then for x_o (a,b), if $P(x_o) \neq 0$, we also have $P(x) \neq 0$ for x in a sufficiently small neighborhood of x_o:

$$P(x_o + \phi\epsilon) \neq 0, \tag{2.35}$$

$$-1 \leq \phi \leq +1, \tag{2.36}$$

where $\epsilon > 0$ is sufficiently small. Therefore,

$$(x_o + \epsilon)^2\, b(x_o + \epsilon) - (x_o - \epsilon)^2\, b(x_o - \epsilon) =$$

$$-\frac{2}{\pi} \int_{x_o - \epsilon}^{x_o + \epsilon} P(x)dx = \frac{4}{\pi} P(x_o + \phi\epsilon)\,\epsilon \tag{2.37}$$

(i) Restriction of $P(x)$ to be piecewise continuous does not affect the applicability of the method to practical problems.

by the mean-value theorem for integrals, since $P(x)$ will be continuous for $x \in \left[x_0 - \in , x_0 + \in\right]$ for $\in$ sufficiently small.

But from (2.35), the right hand side of (2.37) cannot be zero. Therefore either

$$b(x_0 + \in) \neq 0 \tag{2.38}$$

or

$$b(x_0 - \in) \neq 0 \tag{2.39}$$

or both. Thus it has been shown that for $b(E)$ to be identically zero, $P(x)$ must be zero. Therefore the necessary and sufficient condition that the imaginary part of the describing function of a nonlinear element be identically zero, is that $P(x)$ be also identically zero. Thus,

$$P(x) = P_1(x) - P_2(x) \equiv 0 \tag{2.40}$$

or substituting $P_1(x)$ and $P_2(x)$

$$F_1(x) + F_1(-x) = F_2(x) + F_2(-x) \tag{2.41}$$

Figure 4 shows an example of a double-valued nonlinearity (non-memory type) whose describing function is purely real. In figure 5 is represented the characteristic of a single-valued, nonlinear element whose describing function is the same as in the example shown in figure 4. The difference between them is that in the case of figure 4 the even harmonics are present at the output, while in the case of figure 5 the even harmonics are zero. In figure 6 a non-symmetric nonlinearity and its equivalent, with respect to the describing function, are represented. In figure 7 the

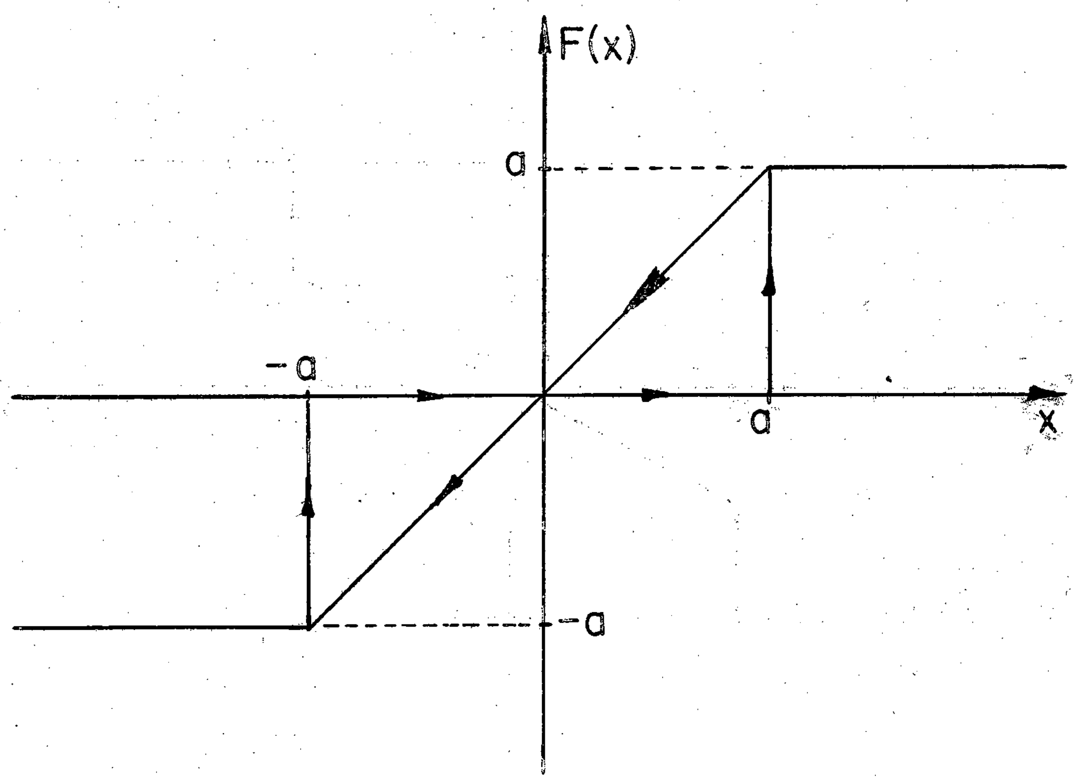

Fig. 4. Example of a Double Valued Nonlinearity with a Real Describing Function

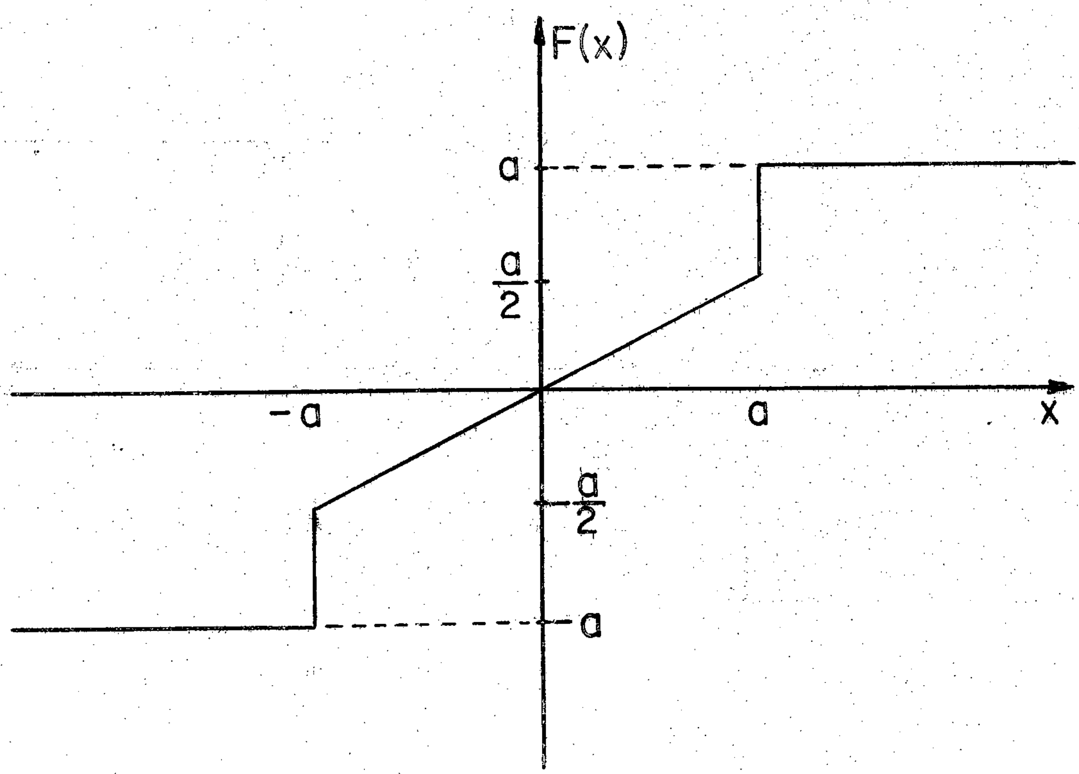

Fig. 5. Single Valued Nonlinearity with the Same Describing Function as the one Shown in Figure 4

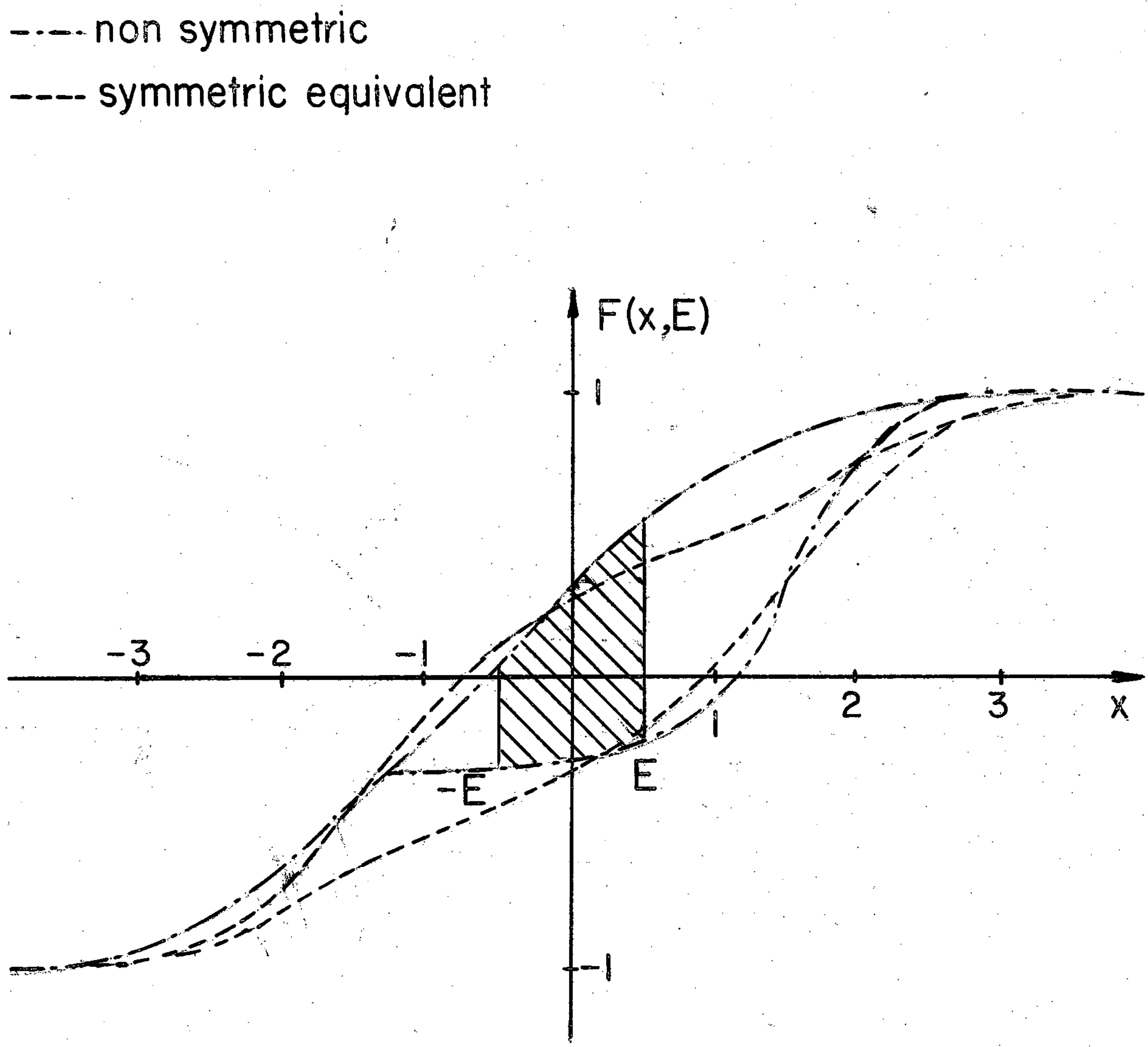

Fig. 6. Double Valued Non-Symmetric Nonlinearity and its Symmetric Equivalent

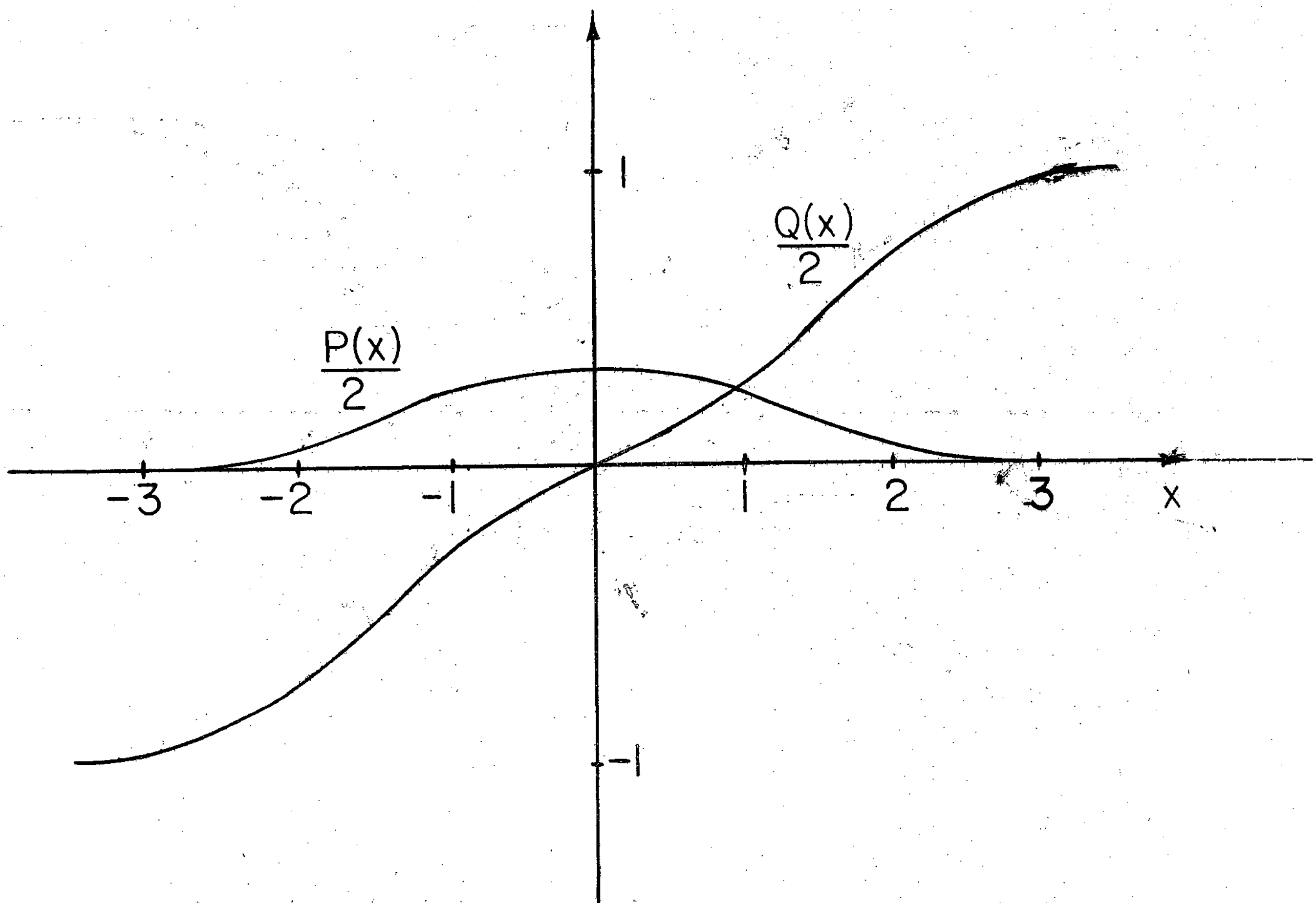

Fig. 7. Functions Q(x) and P(x) for Figure 6

corresponding functions Q(x) and P(x) to the nonlinearity of figure 6 are shown.

2.5 Geometrical Interpretation of the Imaginary Part of the Describing Function

Equation (2.20) permits us to make a geometrical interpretation of the imaginary part of the describing function. From Eq. (2.12) and (2.15)

$$P(x,E) = \frac{1}{2}\left[F_1(x,E) - F_2(x,E) + F_1(-x,E) - F_2(-x,E)\right] \tag{2.42}$$

from which

$$b(E) = -\frac{1}{\pi E^2}\int_0^E \left[F_1(x,E) - F_2(x,E)\right] dx$$

$$-\frac{1}{\pi E^2}\int_0^E \left[F_1(-x,E) - F_2(-x,E)\right] dx$$

$$= -\frac{1}{\pi E^2}\int_{-E}^{+E} \left[F_1(x,E) - F_2(x,E)\right] dx \tag{2.43}$$

But the integral

$$A(E) = \int_{-E}^{+E} \left[F_1(x,E) - F_2(x,E)\right] dx \tag{2.44}$$

represents the area bounded by the curves $F_1(x,E)$ and $F_2(x,E)$ and the straight lines $x = E$ and $x = -E$ (In figure 6 it is the striped area). It will be positive if $F_1(x,E) \geq F_2(x,E)$ and negative if $F_2(x,E) \geq F_1(x,E)$. From Eq. (2.43) and (2.44).

$$b(E) = - \frac{A(E)}{\pi E^2} \tag{2.45}$$

This geometric interpretation facilitates, in some cases, the computation of the imaginary part of the describing function.

To illustrate how this property can be used to compute the imaginary part of the describing function of a non-linear element, let us consider an example. In figure 1 is represented the characteristic of a relay with hysteresis and dead band for $E > b$. For $E < b$ the characteristic of such nonlinear element will be $F(x,E) \equiv 0$. In this case

$$A(E) = 0 \qquad \text{for } E < b \tag{2.46}$$

$$A(E) = 2M(b - a) \qquad \text{for } E > b \tag{2.47}$$

$A(E)$ will be positive because $F_1(x,E) \geq F_2(x,E)$. Therefore, according to Eq. (2.45)

$$b(E) = 0 \qquad \text{for } E < b \tag{2.48}$$

$$b(E) = - \frac{2M}{\pi E^2} (b-a) \qquad \text{for } E > b \tag{2.49}$$

In an analogous manner we could have demonstrated, by inspection, that the imaginary part of the describing function corresponding to the nonlinearity whose characteristic is represented in figure 4, is identically zero.

2.6 Computation of the Describing Function

As we have shown, given a nonlinear element, it is always possible to find the two functions Q(x) and P(x), whose real part of the describing function will depend only on Q(x,E) and whose imaginary part will depend only on P(x,E).

This fact facilitates the computation of the describing function of any nonlinear element. As a matter of fact b(E) can be calculated easily by using the geometric interpretation derived in (2.5), while g(E) can be computed keeping in mind the linearity of the transformation

$$g(E) = \frac{2}{\pi E^2}\int_0^E \frac{x\,Q(x,E)}{\sqrt{E^2 - x^2}}\,dx \qquad (2.19)$$

From the above equation it can be shown that a symmetric, single-valued function

$$f(x,E) = \tfrac{1}{2}\,Q(x,E) \qquad (2.50)$$

and the double-valued nonlinearity that has the same Q(x,E) have describing functions whose real parts are equal. This can be shown keeping in mind the definition of $Q(x,E)$. As a matter of fact

$$Q(x,E) = Q_1(x,E) - Q_2(-x,E)$$

But because $f(x,E)$ is single-valued and symmetric, $Q_1(x,E) = f(x,E)$ and $Q_2(-x,E) = -f(x,E)$, which justifies Eq. (2.50). But since the functional transformation (2.19) is linear the superposition principle applies, and the describing function of the sum is equal to the sum of the describing functions. Therefore the describing function of the original element will be

$$g(E) = \sum_i g_i(E) \tag{2.51}$$

To illustrate the method let us consider an example. Figures 8 and 9 show the characteristics of an amplifier with dead band, saturation and hysteresis. Let us consider that, for $E < b$, the nonlinearity is single-valued (Figure 9). Therefore for $E \leq b$ the describing function will be

$$g(E) = 0 \tag{2.52}$$

$$b(E) = 0 \tag{2.53}$$

For $E > b$ $b(E)$ will be (applying the geometric interpretation),

$$A(E) = 2M\,(b - a) \tag{2.54}$$

Therefore

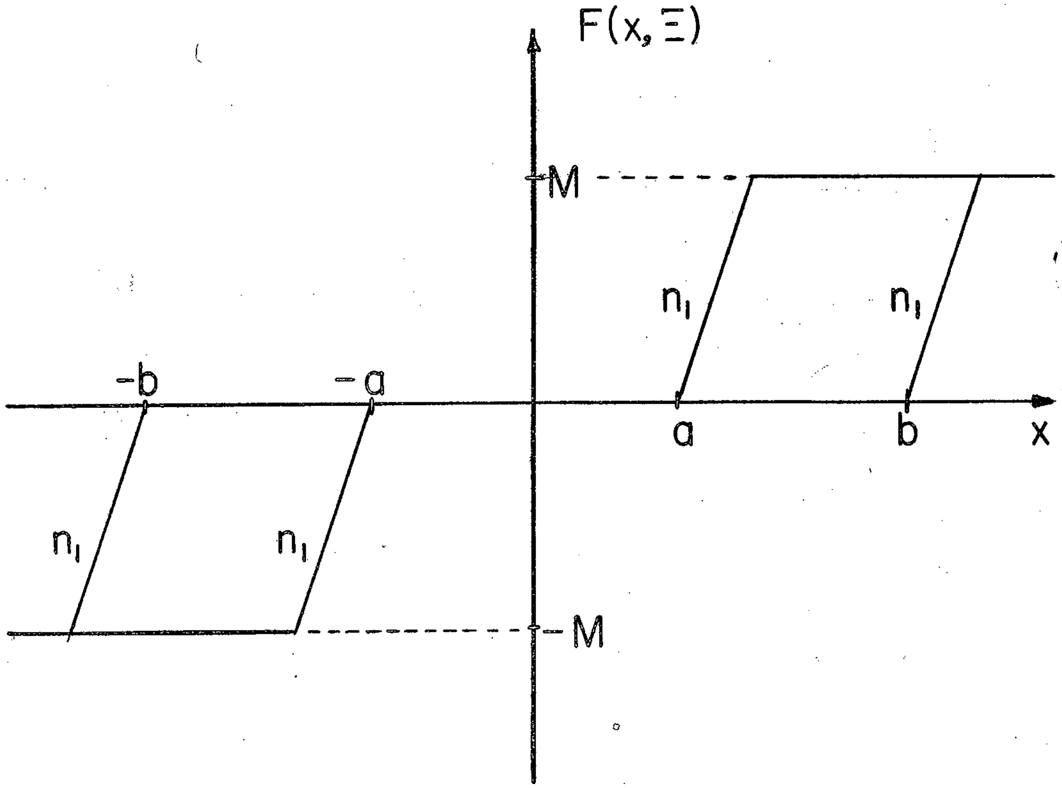

Fig. 8. Characteristic of an Amplifier of Gain n_1 with Saturation Hysteresis and Dead Band for E   b

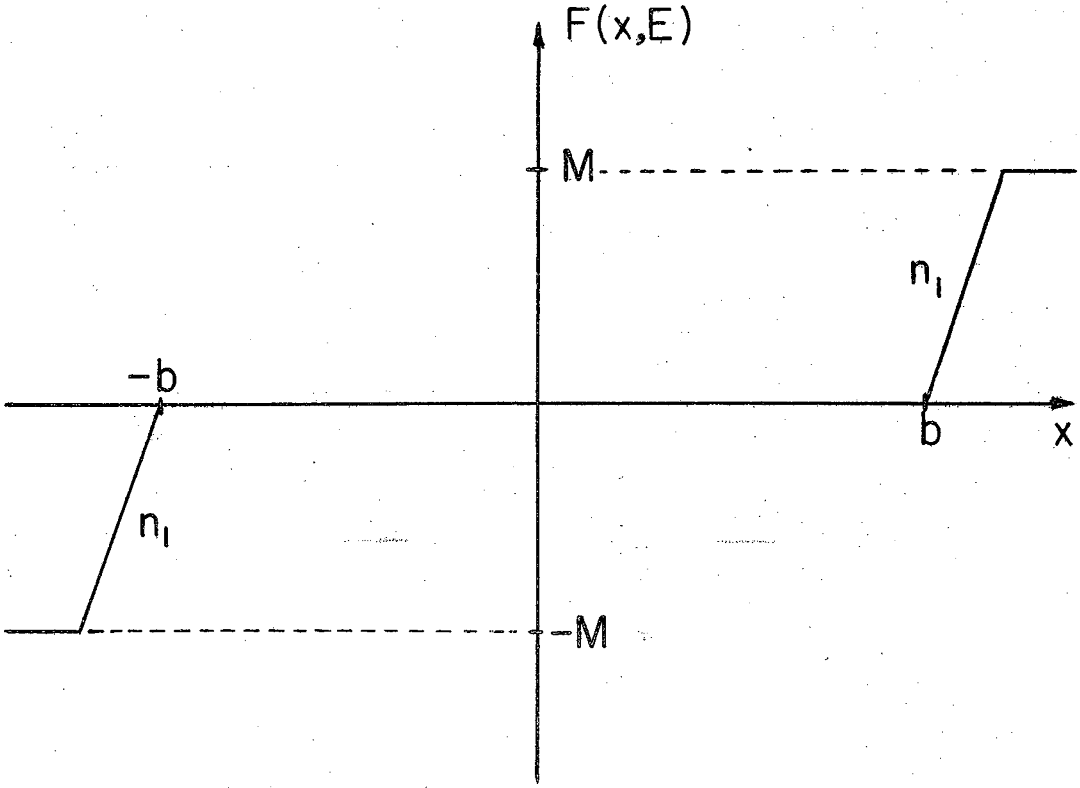

Fig. 9. Characteristic of an Amplifier of Gain n_1 with Saturation Hysteresis and Dead Band for E b

$$b(E) = -\frac{2M}{\pi E^2}(b - a) \tag{2.55}$$

Figure 10 represents the function $F(x,E) = \frac{1}{2}Q(x,E)$. Figure 11 shows the four functions into which $f(x,E)$ can be decomposed. If we call $g_1(E)$, $g_2(E)$, $g_3(E)$, and $g_4(E)$ respectively, the describing function of the original non-linearity (figures 8 and 9) will be

$$g(E) = g_1(E) + g_2(E) + g_3(E) + g_4(E) \tag{2.56}$$

But

$$g_1(E) = \frac{n_1}{2\pi E}\Big[E(\pi - 2\phi_1) + E \sin 2\phi_1 - 4\,a \cos \phi_1\Big] \tag{2.57}$$

$$g_2(E) = \frac{n_1}{2\pi E}\left[E(\pi - 2\,\phi_2) + E \sin 2\,\phi_2 - 4(a + \frac{M}{n_1}) \cos \phi_2\right] \tag{2.58}$$

$$g_3(E) = \frac{n_1}{2\pi E}\left[E(\pi - 2\,\phi_3) + E \sin 2\,\phi_3 - 4\,b \cos \phi_3\right] \tag{2.59}$$

$$g_4(E) = \frac{n_1}{2\pi E}\left[E(\pi - 2\,\phi_4) + E \sin 2\,\phi_4 - 4(b + \frac{M}{n_1}) \cos \phi_4\right] \tag{2.60}$$

where

$$\phi_1 = \text{arc sin}\,\frac{a}{E} \tag{2.61}$$

$$\phi_2 = \text{arc sin}\,\frac{a + \frac{M}{n_1}}{E} \tag{2.62}$$

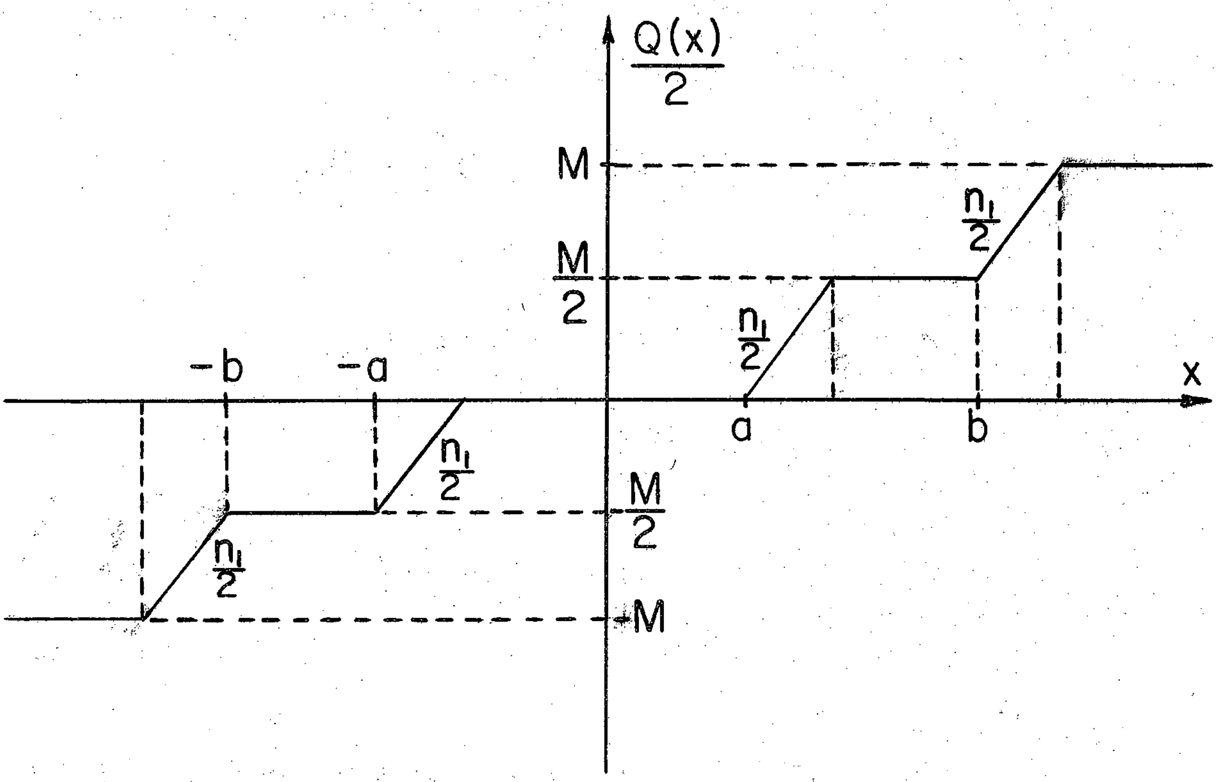

Fig. 10. Function Q(x) Corresponding to the Nonlinearity Shown in Figure 8

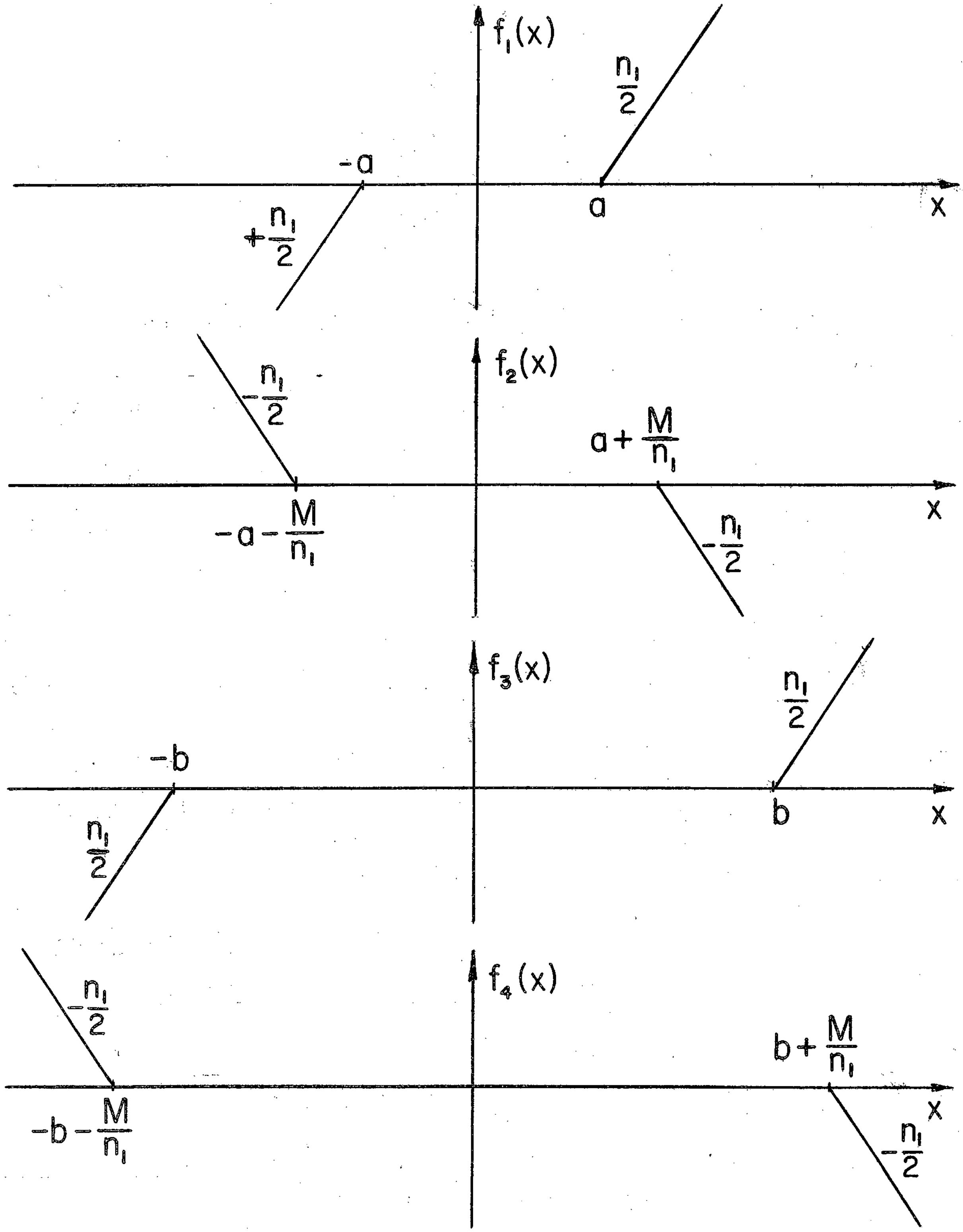

Fig. 11. Functions $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ Corresponding to the Decomposition of $\frac{1}{2}$ Q(x)

$$\phi_3 = \text{arc sin } \frac{b}{E} \tag{2.63}$$

$$\phi_4 = \text{arc sin } \frac{b + \frac{M}{n_1}}{E} \tag{2.64}$$

Therefore

$$g(E) = \frac{n_1}{\pi E}\Big[E(\phi_2-\phi_1+\phi_4-\phi_3) + E/2(\sin 2\,\phi_1 - \sin 2\,\phi_2 + \sin 2\,\phi_3 - \sin 2\,\phi_4) + 2\,a\,(\cos \phi_2 - \cos \phi_1) + 2\,b(\cos \phi_4 - \cos \phi_3) + \frac{2M}{E}(\cos \phi_2 + \cos \phi_4)\Big] \tag{2.65}$$

and

$$b(E) = \frac{-2M}{\pi E^2}\,(b - a) \tag{2.66}$$

This is a well known nonlinearity. The results found by the method illustrated above agree with the results found by conventional techniques (Sridhar, [14]).

2.7 Describing Function of Nonmemory Type Nonlinearities Whose Characteristic can be Represented by Analytical Functions

A particular case presents itself when F(x) is analytic(i).

(i)F(x) will be considered odd, for, as we have shown, g(E) depends only on the odd component of F(x). It will be considered that for x R Eq. (2.67) converges uniformly.

In this case F(x) can be expanded in a Taylor's series

$$F(x) = \sum_{i=0}^{i=\infty} \frac{F^i(0)\ x^i}{i!} \tag{2.67}$$

$$F^i(0) = \left.\frac{d^i F(x)}{dx^i}\right|_{x=0} \tag{2.68}$$

From equations (2.67), (2.50) and (2.33)

$$g(E) = \frac{4}{\pi E^2}\int_0^E \frac{x}{\sqrt{E^2 - x^2}} \sum_{i=0}^{i=\infty} \frac{F^i(0)\ x^i}{i!}\, dx \qquad E \leq R \tag{2.69}$$

But given that the series (2.67) converges uniformly, the equation (2.69) can be written as

$$g(E) = \frac{4}{\pi E^2} \sum_{i=0}^{i=\infty} \frac{F^i(0)}{i!} \int_0^E \frac{x^{i+1}}{\sqrt{E^2 - x^2}}\, dx \tag{2.70}$$

But

$$\int_0^E \frac{x^{i+1}}{\sqrt{E^2 - x^2}}\, dx = \frac{E^{i+1}\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{i+2}{2}\right)}{\Gamma\left(\frac{i+3}{2}\right)} \tag{2.71}$$

Therefore

$$g(E) = \frac{2}{\sqrt{\pi}\; E} \sum_{i=0}^{i=\infty} \frac{\Gamma\frac{i+2}{2}}{i!\,\Gamma\frac{i+3}{2}} F^{i}(0)\, E^{i} \qquad 14\ (2.72)$$

2.8 Existence of the Describing Function

Up to now we have said nothing about the conditions that F(x,E) must satisfy to insure the existence of the describing function. In the next few paragraghs it will be demonstrated that a sufficient condition for the existence of b(E) and g(E) is

$$\int_0^y \left| F_1(x,E) \right|^p dx \leq M(y) \qquad (2.73)$$

$$\int_0^y \left| F_2(x,E) \right|^p dx \leq M(y) \qquad (2.74)$$

where M(y) is any real, single-valued even and finite function of y and p is any positive constant greater than 2. According to equation (2.15)

$$Q_1(x,E) = \frac{F_1(x,E) - F_2(-x,E)}{2} \qquad (2.15)$$

Therefore

$$\left| Q_1(x,E) \right| \leq 1/2 \left[\left| F_1(x,E) \right| + \left| F_1(-x,E) \right| \right] \qquad (2.75)$$

Raising both sides of equation (2.69) to the power p and then integrating between 0 and E we obtain

$$\int_0^E \left|Q_1(x,E)\right|^p dx \leq (\tfrac{1}{2})^p \int_0^E \left[\left|F_1(x,E)\right| + \left|F_1(-x,E)\right|\right]^p dx \tag{2.76}$$

But according to Mincowski's inequality [15]

$$\int_a^b \left[f(x) + h(x)\right]^p dx \leq \left\{\left[\int_a^b \left|f(x)\right|^p dx\right]^{\frac{1}{p}} + \left[\int_a^b \left|h(x)\right|^p dx\right]^{\frac{1}{p}}\right\}^p \tag{2.77}$$

Applying this to equation (2.76) yields

$$\int_0^E \left|Q_1(x,E)\right|^p dx \leq (\tfrac{1}{2})^p \left\{\left[\int_0^E \left|F_1(x,E)\right|^p dx\right]^{\frac{1}{p}} + \left[\int_{-E}^0 \left|F_1(x,E)\right|^p dx\right]^{\frac{1}{p}}\right\}^p \tag{2.78}$$

Therefore from equations (2.73) and (2.74), (2.78) is reduced to

$$\int_0^E \left|Q_1(x,E)\right|^p dx \leq (\tfrac{1}{2})^p \left[M^{\frac{1}{p}}(E) + M^{\frac{1}{p}}(E)\right]^p = M(E) \tag{2.79}$$

In an analogous manner we can show

$$\int_0^E \left|Q_2(x,E)\right|^p dx \leq M(E) \tag{2.80}$$

But, by definition

$$Q(x,E) = Q_1(x,E) + Q_2(x,E) \tag{2.81}$$

Therefore

$$\left| Q\ (x,E) \right| \leq \left| Q_1(x,E) \right| + \left| Q_2(x,E) \right| \tag{2.82}$$

Raising both sides of equation (2.82) to the power p and integrating with respect to x between 0 and E

$$\int_0^E \left| Q(x,E) \right|^p dx \leq \int_0^E \left\{ \left| Q_1(x,E) \right| + \left| Q_2(x,E) \right| \right\}^p dx \tag{2.83}$$

But according to Mincowski's inequality

$$\int_0^E \left\{ \left| Q_1(x,E) \right| + \left| Q_2(x,E) \right| \right\}^p dx \leq \left\{ \left[\int_0^E \left| Q_1(x,E) \right|^p dx \right]^{\frac{1}{p}} + \left[\int_0^E \left| Q_2(x,E) \right|^p dx \right]^{\frac{1}{p}} \right\}^p \leq \left[M^{\frac{1}{p}}(E) + M^{\frac{1}{p}}(E) \right]^p$$

$$= 2^p\, M(E) \tag{2.84}$$

Therefore

$$\int_0^E \left| Q(x,E) \right|^p dx \leq 2^p\, M(E) \tag{2.85}$$

In an analogous manner we can demonstrate

$$\int_0^E \left| Q^*(x,E) \right|^p dx \leq 2^p\, M(E) \tag{2.86}$$

$$\int_0^E \left| P(x,E) \right|^p \; dx \leq 2^p \; M(E) \tag{2.87}$$

$$\int_0^E \left| P^*(x,E) \right|^p \; dx \leq 2^p \; M(E) \tag{2.88}$$

According to equation (2.22)

$$\left| g(E) \right| \leq \frac{2}{\pi E^2} \left| \int_0^E \frac{x \; Q(x,E)}{\sqrt{E^2 - x^2}} dx \right| \tag{2.89}$$

Applying Holder's inequality 15 to (2.83) we obtain

$$\left| g(E) \right| \leq \frac{2}{\pi E^2} \left[\int_0^E \left| Q(x,E) \right|^p \; dx \right]^{\frac{1}{p}} \left[\int_0^E \left| \frac{x}{\sqrt{E^2 - x^2}} \right|^q \; dx \right]^{\frac{1}{q}} \tag{2.90}$$

where

$$p > 1 \qquad\qquad q > 1$$

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{2.91}$$

But

$$\left[\int_0^E \left|\frac{x}{\sqrt{E^2 - x^2}}\right|^q dx\right]^{\frac{1}{q}} = \left|\int_0^E \frac{x^q}{(E^2 - x^2)^{\frac{q}{2}}} dx\right|^{\frac{1}{q}}$$

$$= \left|\frac{E}{2} B\left(\frac{q+1}{3}, \frac{2-q}{2}\right)\right|^{\frac{1}{q}}$$

$$= \left|\frac{E}{2} B\left(\frac{2p-1}{2p-2}, \frac{p-2}{2p-2}\right)\right|^{\frac{1}{q}} \qquad (2.92)$$

where $B(x,y)$ is the beta function [16]. Substituting (2.92) and (2.85) into (2.90)

$$|g(E)| \leq \frac{2}{\pi E^2}\left[2^p \quad M(E)\right]^{\frac{1}{p}} \left[\frac{E}{2} B\left(\frac{2p-1}{2p-2}, \frac{p-2}{2p-2}\right)p\right]^{\frac{p-1}{p}}$$

$$= \frac{2^{\frac{p+1}{p}}}{\pi E^{\frac{p+1}{p}}} M(E)^{\frac{1}{p}} \left[B\left(\frac{2p-1}{2p-2}, \frac{p-2}{2p-2}\right)\right]^{\frac{p-1}{p}} \qquad (2.93)$$

But according to the hypothesis formulated at the beginning, $p > 2$. Therefore the right hand side of (2.93) will be bounded for every value of E. Thus we have shown the existence of $g(E)$. The existence of $b(E)$ can be deduced almost immediately by applying the Holder inequality to the second equation of (2.81). This yields

$$\left| b(E) \right| = \frac{2}{\pi E^2} \left| \int_0^E P(x,E)\, dx \right|$$

$$\leq \frac{2}{\pi E^2} \left[\int_0^E \left| P(x,E) \right|^p dx \right]^{\frac{1}{p}} \left[\int_0^E dx \right]^{\frac{1}{q}}$$

$$\leq \frac{2}{\pi E^2} \left[2^p\, M(E) \right]^{\frac{1}{p}} E^{\frac{1}{q}} = \frac{4}{E^{2-1/q}} M^{\frac{1}{p}}(E) \qquad (2.94)$$

Thus the existence of b(E) has been demonstrated. Equations (2.93) and (2.94) can be simplified in the case of bounded functions to

$$\left| F_1(x,E) \right| \leq R \qquad \left| F_2(x,E) \right| \leq R \qquad (2.95)$$

where R is any positive constant. From (2.95) it can be deduced

$$\int_0^y \left| F_1(x,E) \right|^p dx \leq y\, R^p \qquad \int_0^y \left| F_2(x,E) \right|^p dx \leq y\, R^p \qquad (2.96)$$

Therefore it can be chosen

$$M(y) = y\, R^p \qquad (2.97)$$

Substituting in equation (2.93)

$$\left|g(E)\right| \leq \frac{2^{\frac{p+1}{p}}}{\pi E^{1+\frac{1}{p}}} E^{1/p} R \left[B\left(\frac{2p-1}{2p-2} \cdot \frac{p-2}{2p-2}\right) \right]^{\frac{p-1}{p}}$$

$$= \frac{2^{\frac{p+1}{p}}}{\pi E} R \left[B\left(\frac{2p-2}{2p-2} , \frac{p-2}{2p-2}\right) \right]^{\frac{p-1}{p}} \tag{2.98}$$

Equation (2.98) holds for any value of p larger than 1.

Therefore as $p \longrightarrow \infty$

$$g(E) \leq \frac{2R}{\pi E} B(1,0.5)$$

$$= \frac{4R}{\pi E} \tag{2.99}$$

Substituting equation (2.97) into (2.94), for p ∞

$$\left|b(E)\right| \leq \frac{4R}{\pi E} \tag{2.100}$$

Therefore

$$\left|K_{eq}(E)\right| \leq \frac{4\sqrt{2}R}{\pi E} \tag{2.101}$$

This result can be stated in the following manner: given any bounded nonlinearity, its describing function will be less than or equal to the describing function of a perfect relay with a maximum amplitude equal to $4\sqrt{2}R$, where R is the superior bound of the nonlinearity.

2.9 Conclusions

In the present chapter the original integral transformation given by Eq. (1.19) and (1.20) has been transformed into a more convenient form given by Eq. (2.19) and (2.20). With this, the problem of the inverse describing function has been reduced to the problem of solving a Volterra integral equation. This problem will be solved in the next chapter. However, even without having the closed solution of Eq. (2.19) and (2.20), interesting conclusions have been deduced. Some of them, such as the existence of the describing function, are of interest only from the theoretical point of view. Others, such as the geometric interpretation of the imaginary part of the describing function, and the method of computing the describing function from the decomposition of the original nonlinearity into partial functions, are of more practical interest.

The main point of this chapter consists in the fact that to any single or double-valued nonlinearity, there corresponds two single-valued functions $Q(x,E)$ and $P(x,E)$ that contain all the necessary information to compute the describing function of the nonlinearity.

CHAPTER 3

THE INVERSE DESCRIBING FUNCTION

3.1 Introduction

In Chapter 2 the integrals that generate the describing function have been transformed into the following form:

$$g(E) = \frac{2}{\pi E^2} \int_0^E \frac{x\, Q(x,E)}{\sqrt{E^2 - x^2}}\, dx \qquad (3.1)$$

$$b(E) = -\frac{2}{\pi E^2} \int_0^E P(x,E)\, dx \qquad (3.2)$$

Eq. (3.1) and (3.2) have a more convenient form for our purposes than the original form given in Eq. (1.19) and (1.20).

Also in Chapter 2 the inverse transform that generates Q(x,E) and P(x,E) as a function of g(E) and b(E) was shown to be non-unique, in the case of memory-type nonlinearities. For this reason only the case of nonmemory-type nonlinearities will be considered below. In the present chapter an analytical method to invert the integral equations (3.1) and (3.2) will be developed. From that, some interesting properties of the inverse describing function will be deduced.

3.2 Inversion of the Integral Transformations that Generate the Describing Function

In the case of nonmemory-type nonlinearities, Eq. (3.1) and (3.2) are reduced to

$$g(E) = \frac{2}{\pi E^2} \int_0^E \frac{x\, Q(x)}{\sqrt{E^2 - x^2}}\, dx \tag{3.3}$$

$$b(E) = -\frac{2}{\pi E^2} \int_0^E P(x)\, dx \tag{3.4}$$

These integral equations are of the type of Volterra integral equations of the first kind. In Appendix I the solution of the most general integral equation of this kind,

$$F(z) = \int_0^z \frac{k(x)}{(z^2 - x^2)^{\phi}}\, dx, \qquad \text{for } 0 < \phi < 1 \tag{3.5}$$

is given. The solution of Eq. (3.5) is

$$k(x) = \frac{2 \sin \pi \phi}{\pi} \frac{d}{dx} \int_0^x \frac{z\, F(z)}{(x^2 - z^2)^{1-\phi}}\, dz \tag{3.6}$$

For the case of Eq. (3.3)

$$\phi = \tfrac{1}{2} \;; \qquad F(x) = \frac{\pi x^2 g(x)}{2} \;; \quad k(z) = z\, Q(z) \tag{3.7}$$

Therefore, substituting (3.7) into (3.6) yields,

$$Q(x) = \frac{1}{x}\frac{d}{dx}\int_0^x \frac{z^3\, g(z)}{\sqrt{x^2 - z^2}}\, dz \qquad (3.8)$$

integrating (3.8) by parts, and assuming that g'(z) exists,

$$Q(x) = \frac{1}{x}\frac{d}{dx}\left\{ \left. - z^2 g(z) \quad x^2 - z^2 \right|_0^x + \int_0^x \sqrt{x^2 - z^2}\; d\left[z^2 g(z)\right] \right\}$$

$$= \frac{1}{x}\frac{d}{dx}\int_0^x \sqrt{x^2 - z^2}\; d\left[z^2 g(z)\right] \qquad (3.9)$$

and performing the derivative with respect to x

$$Q(x) = \int_0^x \frac{d\left[z^2 g(z)\right]}{\sqrt{x^2 - z^2}} \qquad (3.10)$$

The solution of the integral equation (3.4) is obvious and yields, assuming that b'(s) exists,

$$P(x) = -\frac{\pi}{2}\frac{d}{dx}\left[x^2 b(x)\right] \qquad (3.11)$$

3.3 Existence of the Inverse Describing Function

Sufficient Condition

Given g(E) and b(E), equations (3.10) and (3.11) generate the pair of functions Q(x) and P(x). But as pointed out in Chapter 2, Q(x) and P(x) are not sufficient, even in the case of nonmemory-type nonlinearities, to determine the nonlinearity. Let us state the problem of the inverse describing function in the following manner. Given a pair of functions g(E) and b(E), is it possible to find a bounded nonlinearity whose real and imaginary parts of the describing function are g(E) and b(E)? What are the conditions that g(E) and b(E) must satisfy to insure boundedness?[i] To investigate this problem let us rewrite equation (3.10) as

$$Q(x) = 2\int_0^x \frac{z\, g(z)}{\sqrt{x^2 - z^2}}\, dz + \int_0^x \frac{z^2\, g'(z)}{\sqrt{x^2 - z^2}}\, dz \tag{3.12}$$

Therefore the sufficient conditions for Q(x) to exist are that both integrals on the right hand side of (3.12) exist. Let us consider one at a time.

$$\left| 2\int_0^x \frac{z\, g(z)}{x^2 - z^2}\, dz \right|$$

$$\leq \quad 2\left[\int_0^x \left| g(z)\right|^p dz\right]^{\frac{1}{p}} \left[\int_0^x \left| \frac{z}{\sqrt{x^2-z^2}}\right|^q dz\right]^{\frac{1}{q}} \tag{3.13}$$

Equation (3.13) was obtained by applying the Holder [15]

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[i] We must impose the condition of boundedness in order to insure physical realizeability.

inequality to the first integral of the right hand side of (3.12). The constants, p and q, satisfy the conditions

$$p > 1 \quad ; \quad q > 1 \quad ; \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{3.14}$$

It is not difficult to show that

$$\int_0^x \left| \frac{z}{\sqrt{x^2-z^2}} \right|^q dz = \int_0^x \left(\frac{z}{\sqrt{x^2-z^2}} \right)^q dz = \frac{x}{2} B\left(\frac{2p-1}{2p-2}, \frac{p-2}{2p-2}\right) \tag{3.15}$$

If Holder's inequality is applied to the second integral in the right hand side of (3.12) we obtain

$$\left| \int_0^x \frac{z^2 g'(z)}{x^2-z^2} dz \right|$$

$$\leq \left[\int_0^x \left| g'(z) \right|^s dz \right]^{\frac{1}{s}} \left[\int_0^x \left| \frac{z^2}{\sqrt{x^2-z^2}} \right|^t dz \right]^{\frac{1}{t}} \tag{3.16}$$

where

$$x > 1 \quad ; \quad t > 1 \text{ and } \frac{1}{s} + \frac{1}{t} = 1 \tag{3.17}$$

But

$$\int_0^x \left| \frac{z^2}{\sqrt{x^2-z^2}} \right|^t dz = \int_0^x z^{2t}(x^2 - z^2)^{-t/2} dz$$

$$= B\left(\frac{3s-1}{2s-2}, \frac{s-2}{2s-2}\right) \frac{x^{\frac{2s-1}{s-1}}}{2} \tag{3.18}$$

Thus from equations (3.12), (3.13), (3.15), (3.16) and (3.18) we obtain

$$\left|Q(x)\right| \leq 2\left(\frac{x}{2}\right)^{\frac{1}{q}}\left[B\left(\frac{2p-1}{2p-2}, \frac{p-2}{2p-2}\right)\right]^{\frac{1}{q}}\left[\int_0^x \left|g(z)\right|^p dz\right]^{\frac{1}{p}}$$
$$+ \left(\frac{x^{t+1}}{2}\right)^{\frac{1}{t}}\left[B\left(\frac{3s-1}{2s-2}, \frac{s-2}{2(s-1)}\right)\right]^{\frac{1}{t}}\left[\int_0^x \left|g'(z)\right|^s dz\right]^{\frac{1}{s}} \tag{3.19}$$

Therefore, a sufficient condition for Q(x) to exist is that the integrals

$$\int_0^x \left|g(z)\right|^p dz \tag{3.20}$$

and

$$\int_0^x \left|g'(z)\right|^p dz \tag{3.21}$$ (i)

exist for all values of x, where p is any positive constant larger than 2. It is interesting to note that the conditions for the existence of the inverse describing function are more restricted than the conditions for the existence of the describing function (derived in Chapter 2).

(i) Since g(z) was assumed continuous Eq. (3.21) implies Eq. (3.20).

Necessary Conditions

It will be shown that, for $Q(x)$ to exist, a necessary condition is that $g(z)$ be a continuous function. Let us assume $g(z)$ to be a bounded function with bounded first derivative everywhere except at the point $z = z_1$ at which it has a finite discontinuity, and call

$$A = \lim_{\epsilon \to 0} g(z_1 - \epsilon) \tag{3.22}$$

$$B = \lim_{\epsilon \to 0} g(z_1 + \epsilon) \qquad \epsilon > 0 \tag{3.23}$$

Define the function

$$h(z) = g(z) + (B - A)\left[\tfrac{1}{2} - u(z - z_1)\right] \tag{3.24}$$

where $u(z)$ is the step function. The function $h(z)$ will be continuous at $z = z_1$ and it will have a bounded derivative everywhere.

From (3.24)

$$g(z) = h(z) + (A - B)\left[\tfrac{1}{2} - u(z - z_1)\right] \tag{3.25}$$

Therefore from (3.8)

$$Q(x) = \frac{1}{x}\frac{d}{dx}\left\{\int_0^x \frac{z^3 h(z)}{\sqrt{x^2 - z^2}}\,dz + (A - B)\int_0^x \frac{z^3\left[\tfrac{1}{2} - u(z - z_1)\right]}{\sqrt{x^2 - z^2}}\,dz\right\} \tag{3.26}$$

The second integral in (3.26) will be

$$\frac{A - B}{2}\int_0^x \frac{z^3}{\sqrt{x^2 - z^2}}\,dz = \frac{A - B}{3}\,x^3 \quad \text{for } x < z_1 \qquad (3.27)$$

and

$$\frac{A - B}{2}\int_0^{z_1} \frac{z^3 dz}{\sqrt{x^2-z^2}} - \frac{A - B}{2}\int_{z_1}^{x} \frac{z^3}{\sqrt{x^2-z^2}}\,dz$$

$$= (A - B)\left[1/3(x^2-z_1{}^2)^{3/2} - x^2\sqrt{x^2-z_1^2} + 1/3\,x^3\right] \text{for } z_1 < x \qquad (3.28)$$

Therefore

$$Q(x) = \frac{1}{x}\frac{d}{dx}\int_0^x \frac{z^3 h(z)}{\sqrt{x^2-z^2}}\,dz + (A - B)\,x \quad \text{for } x < z_1 \qquad (3.29)$$

and

$$Q(x) = \frac{1}{x}\frac{d}{dx}\int_0^x \frac{z^3 h(z)}{\sqrt{x^2-z^2}}\,dz + (A - B)\left(x - \frac{2x^2 - z_1{}^2}{\sqrt{x^2 - z_1{}^2}}\right)$$

$$\text{for } x > z_1 \qquad (3.30)$$

Equations (3.29) and (3.30) show that in spite of the existence of

$$\frac{1}{x}\frac{d}{dx}\int_0^x \frac{z^3\,h(z)}{\sqrt{x^2 - z^2}}\,dz$$

Q(x) is not bounded for $x = z_1$. From this it can be deduced that for the existence of Q(x), g(E) must be continuous.

3.4 Examples

To illustrate the properties deduced above let us consider some examples.

Example 1

Assume

$$g(E) = \frac{3}{4} E^2 \tag{3.31}$$

$$b(E) = 0 \tag{3.32}$$

Applying equation (3.22) we find

$$\begin{aligned} Q(x) &= \frac{1}{x}\frac{d}{dx}\left[\frac{3}{4}\int_0^x \frac{z^5}{\sqrt{x^2 - z^2}}\,dz\right] \\ &= \frac{1}{x}\frac{d}{dx}\left[\frac{3}{4}\left| -\frac{(x^2 - z^2)^{5/2}}{5} + \frac{2(x^2 - z^2)^{3/2}}{3}\right.\right. \\ &\qquad \left.\left. - x^4(x^2 - z^2)^{1/2}\right|_0^x\right] = 2x^3 \end{aligned} \tag{3.33}$$

and F(x) is assumed to be single-valued and symmetric, then $1/2\ Q(x) = Q_1(x) = Q_2(x) = f_1(x) = f_2(x) = f(x)$. Therefore

$$f(x) = x^3 \tag{3.34}$$

Example 2

Consider the following describing function

$$g(E) = 0 \qquad E < a$$

$$g(E) = \frac{4M}{\pi E^2}\sqrt{E^2 - a^2} \qquad E \geq a \tag{3.35}$$

$$b(E) = 0$$

Applying Eq. (3.10),

$$\frac{d}{dz}\left[z^2\ g(z)\right] = \frac{4M}{\pi}\ \frac{z}{\sqrt{z^2 - a^2}}$$

Therefore

$$Q(x) = \frac{4M}{\pi}\int_a^x \frac{z\ dz}{\sqrt{z^2 - a^2}\ \sqrt{x^2 - z^2}}$$

$$= 2M \qquad x \geq a \tag{3.36}$$

and

$$Q(x) = 0 \qquad x < a \tag{3.37}$$

From equation (3.11) it can be deduced that

$$P(x) = 0 \tag{3.38}$$

If the nonlinearity is assumed to be symmetric and single-valued it may be shown (see Eq. (4.5) and Eq. (4.9)) that,

$$f(x) = \frac{Q(x)}{2} = M\, u(|x| - a)\ \mathrm{sgn}\ x \qquad (3.39)$$

where u(x) is the step function.

Example 3

This example will be divided into two parts. In the first, the condition for the existence of the inverse describing function will be illustrated. In the second part a method to synthesize memory-type nonlinearities from discontinuous describing functions is proposed.

Part 1

Consider the following describing function,

$$g(E) = \frac{2M}{\pi E^2}\left[\sqrt{E^2 - a^2} + \sqrt{E^2 - b^2}\right] \qquad E > b \qquad (3.40)$$

$$g(E) = 0 \qquad E < b \qquad (3.41)$$

$$b(E) = -\frac{2M}{\pi E^2}(b - a) \qquad E > b \qquad (3.42)$$

$$b(E) = 0 \qquad E < b \qquad (3.43)$$

This is the describing function of a relay with hysteresis and dead band. It has a discontinuity at E = b and, as it was shown above, Q(x) will be unbounded for this value. To illustrate this point, write Eq. (3.8) for $x > b$.

$$Q(x) = \frac{2M}{\pi x^2}\frac{d}{dx}\left[\int_b^x \frac{z\sqrt{z^2-a^2}}{\sqrt{x^2-z^2}}\,dz + \int_b^x \frac{z\sqrt{z^2-b^2}}{\sqrt{x^2-z^2}}\,dz\right] \qquad (3.44)$$

But

$$\int_b^x \frac{z\sqrt{z^2-a^2}}{\sqrt{x^2-z^2}}\,dz = \frac{\sqrt{b^2-a^2}\,\sqrt{x^2-b^2}}{2} + \frac{\sqrt{x^2-a^2}}{2} \cdot$$

$$\cdot \sin^{-1} \frac{x^2-b^2}{x^2-a^2} \qquad (3.45)$$

and

$$\int_b^x \frac{z\sqrt{z^2-b^2}}{\sqrt{x^2-b^2}}\,dz = \frac{x^2-b^2}{2}\,\frac{\pi}{2} \qquad (3.46)$$

Therefore

$$Q(x) = \frac{2}{\pi}\,\frac{M}{x}\,\frac{d}{dx}\left\{\frac{\sqrt{b^2-a^2}\,\sqrt{x^2-b^2}}{2}\right.$$

$$\left. + \frac{x^2-a^2}{2}\sin^{-1}\sqrt{\frac{x^2-b^2}{x^2-a^2}} + \frac{x^2-b^2}{2}\,\frac{\pi}{2}\right\} \qquad (3.47)$$

And performing the derivative

$$Q(x) = \frac{2M}{\pi}\sqrt{\frac{b^2-a^2}{x^2-b^2}} + \frac{2M}{\pi}\sin^{-1}\sqrt{\frac{x^2-b^2}{x^2-a^2}} + M \qquad (3.48)$$

For this example,

$$Q(x) = 0 \qquad \text{for } x < b \qquad (3.49)$$

Equation (3.48) shows that

$$\lim_{\epsilon \to 0} Q(b+\epsilon) = \infty \qquad (3.50)$$

This result could be foreseen from the fact that $g(E)$ is discontinuous at $E = b$.

Part 2

Attempt now to find a memory-type nonlinearity whose describing function is given by (3.40), (3.41), (3.42) and (3.43)

Define for $E < b$

$$g(E) = h_1(E) \tag{3.51}$$

and

$$b(E) = h_2(E) \tag{3.52}$$

Both functions will be chosen in such a way that g(E) and b(E) are continuous at E = b. With (3.51), (3.52), (3.40), and (3.42) we can find a bounded Q(x) and P(x). The next step will be to define

$$Q(x,E) = P(x,E) = 0 \qquad E < b \tag{3.53}$$

$$Q(x,E) = Q(x) \qquad E > b \tag{3.54}$$

and

$$P(x,E) = P(x) \qquad E > b \tag{3.55}$$

where P(x) and Q(x) are the inverse describing function found with (3.51), (3.52), (3.40) and (3.43), considering the nonlinearity to be of the memory type. For the example under consideration the following choices will be made.

$$h_1(E) = \frac{2M}{\pi b^2}\sqrt{b^2 - a^2} \qquad E < b \tag{3.56}$$

$$h_2(E) = \frac{2M}{\pi b^2}(b - a) \qquad E < b \qquad (3.57)$$

Therefore for $E < b$

$$Q(x) = \frac{2M\sqrt{b^2 - a^2}}{b^2}\frac{1}{x}\frac{d}{dx}\int_0^x \frac{z^3\,dz}{\sqrt{x^2 - z^2}}$$

$$= \frac{4M\sqrt{b^2 - a^2}}{\pi b^2}x \qquad (3.58)$$

and for $E > b$

$$Q(x) = \frac{2M\sqrt{b^2 - a^2}}{\pi b^2}\frac{1}{x}\frac{d}{dx}\int_0^b \frac{z^3\,dz}{\sqrt{x^2 - z^2}}$$

$$+ \frac{2}{\pi}\frac{M}{x}\frac{d}{dx}\int_b^x \frac{z\sqrt{z^2 - a^2}}{\sqrt{x^2 - z^2}}dz$$

$$+ \frac{2}{\pi}\frac{M}{x}\frac{d}{dx}\int_b^x \frac{z\sqrt{z^2 - b^2}}{\sqrt{x^2 - z^2}}dz \qquad (3.59)$$

Performing the integration and derivatives

$$Q(x) = \frac{4M\sqrt{b^2-a^2}}{\pi b^2}\left(x - \sqrt{x^2-b^2}\right) + \frac{2M}{\pi}\sin^{-1}\sqrt{\frac{x^2-b^2}{x^2-a^2}} + M \qquad (3.60)$$

From equations (3.57), (3.42) and (3.11)

$$P(x) = \frac{2M}{b^2}(b - a)\,x \qquad x < b \qquad (3.61)$$

$$P(x) = 0 \qquad x > b \tag{3.62}$$

Therefore for $E < b$

$$Q(x,E) = b \tag{3.63}$$

and

$$P(x,E) = 0 \tag{3.64}$$

For $E > b$

$$Q(x,E) = \frac{2M}{\pi b^2}\sqrt{b^2 - a^2}\,x \qquad x < b \tag{3.65}$$

$$Q(x,E) = \frac{4M\sqrt{b^2-a^2}}{\pi b^2}\left[x - \sqrt{x^2-b^2}\right] + \frac{2M}{\pi}\sin^{-1}\sqrt{\frac{x^2 - b^2}{x^2 - a^2}} + M \qquad x \geq b \tag{3.66}$$

and

$$P(x,E) = \frac{2M}{b^2}(b - a)\,x \qquad x < b \tag{3.67}$$

$$P(x,E) = 0 \qquad x \geq b \tag{3.68}$$

For $M = 1$, $b = 2$ and $a = 1$

$$Q(x) = \frac{\sqrt{3}}{\pi}\,x \qquad \text{for } 0 < x < 2$$

$$Q(x) = \frac{3}{\pi}\left(x - \sqrt{x^2-4}\right) + \frac{2}{\pi}\sin^{-1}\sqrt{\frac{x^2-4}{x^2-1}} + 1 \qquad x > 2$$

$$P(x) = \tfrac{1}{2}\,x \qquad x < 2$$

$$P(x) = 0 \qquad x > 2$$

In figure 12 is presented the characteristic of the nonlinearity for $E > 2$. For $E < 2$

$$F(x,E) = F_1(x,E) = F_2(x,E) = 0$$

This example illustrates the calculation of a bounded inverse describing function, given a discontinuous describing function.

3.5 Higher Describing Functions

As was shown in the preceding paragraphs, even in the case of nonmemory nonlinearities, g(E) and b(E) are not sufficient to uniquely identify the nonlinearity. To overcome this indeterminacy, the nth describing function will be defined as the ratio of the amplitude of the nth harmonic to the amplitude of the input. Denote this by $g_n(E)$ and $b_n(E)$. Then

$$g_n(E) = \frac{1}{\pi E}\int_0^{2\pi} F(E \sin \phi,\ E) \sin n\,\phi \; d\phi \qquad (3.69)$$

$$b_n(E) = \frac{1}{\pi E}\int_0^{2\pi} F(E \sin \phi,\ E) \cos n\,\phi \; d\phi \qquad (3.70)$$

where $F\left[x(t),\ \max_{\rho \le t} |x(\rho)|\right]$ satisfies the hypothesis stated in the first chapter. Following the procedure for the case of the first harmonic, Eqs. (3.69) and (3.70) can be transformed into

Inverse Describing Function of

$$g(E) = \frac{2M}{\pi b^2}\sqrt{b^2 - a^2} \qquad E < b$$

$$b(E) = -\frac{2M}{\pi b^2}(b - a) \qquad E < b$$

$$g(E) = \frac{2M}{\pi E^2}\left[\sqrt{E^2 - a^2} + \sqrt{E^2 - b^2}\right] \quad E \geq b$$

$$b(E) = -\frac{2M}{\pi E^2}(b - a) \qquad E \geq b$$

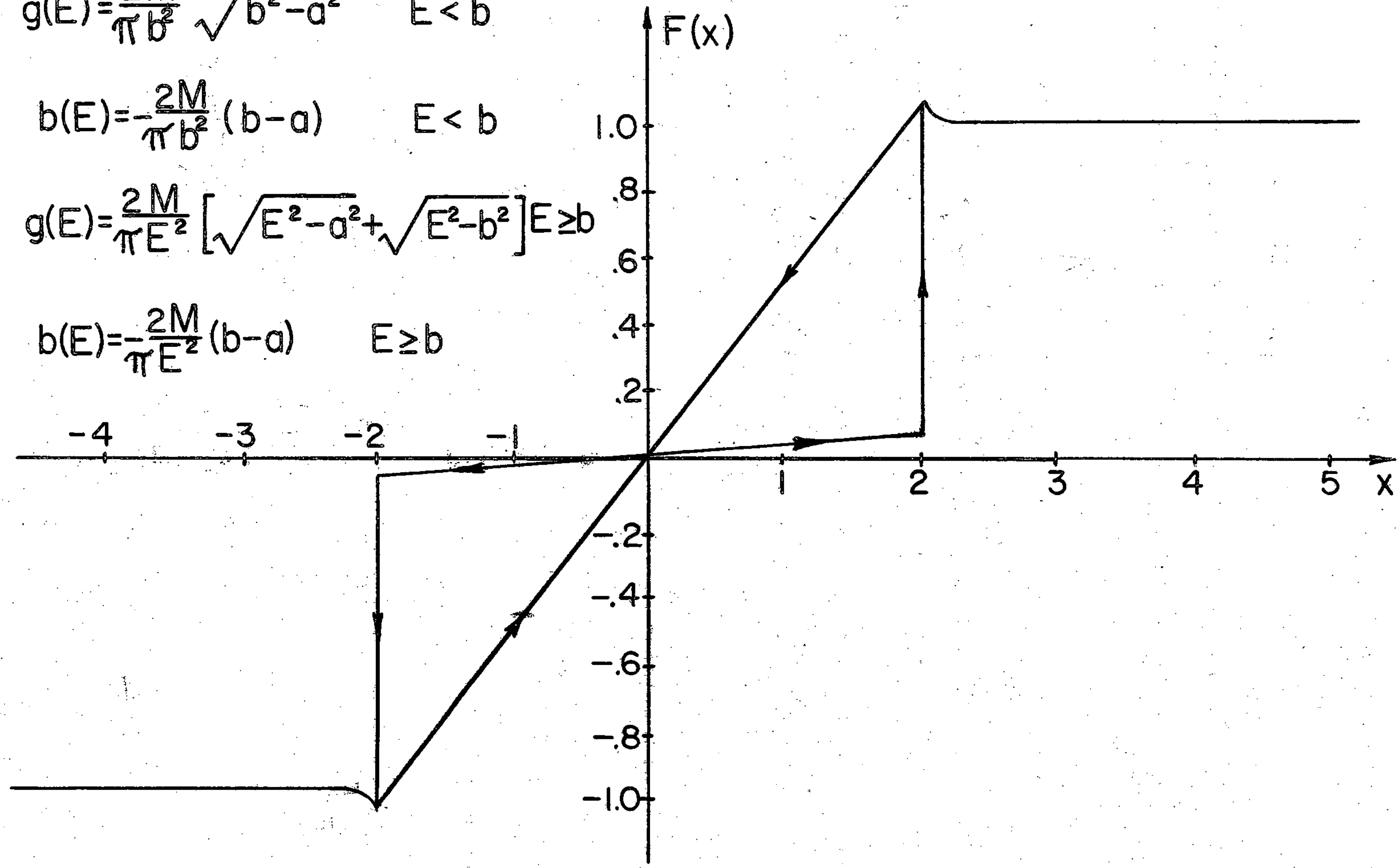

Fig. 12.

$$g_n(E) = \frac{1}{\pi E}\int_0^{\pi/2} F_2(E \sin \phi, E) \sin n \phi \, d\phi$$

$$+ \frac{1}{\pi E}\int_{\pi/2}^{3\pi/2} F_1(E \sin \phi, E) \sin n \phi \, d\phi$$

$$+ \frac{1}{\pi E}\int_{3\pi/2}^{2\pi} F_2(E \sin \phi, E) \sin n \phi \, d\phi \qquad (3.71)$$

and

$$b_n(E) = \frac{1}{\pi E}\int_0^{\pi/2} F_2(E \sin \phi, E) \cos n \phi \, d\phi$$

$$+ \frac{1}{\pi E}\int_{\pi/2}^{3\pi/2} F_1(E \sin \phi, E) \cos n\phi \, d\phi$$

$$+ \frac{1}{\pi E}\int_{3\pi/2}^{2\pi} F_2(E \sin \phi, E) \cos n \phi \, d\phi \qquad (3.72)$$

Introducing in (3.71) and (3.72) the functions $P_1(x, E)$, $P_2(x, E)$, $Q_1(x, E)$ and $Q_2(x, E)$ defined in Chapter 2, equations (3.71) and (3.72) are transformed into

$$g_n(E) = \frac{1}{\pi E}\Bigg[\int_0^{\pi/2} Q_2(E \sin \phi, E) \sin n\phi \, d\phi$$

$$+ \int_0^{\pi/2} P_2(E \sin \phi, E) \sin n\phi \, d\phi$$

$$+ \int_{\pi/2}^{3\pi/2} Q_1(E \sin \phi, E) \sin n\phi \, d\phi$$

$$+ \int_{\pi/2}^{3\pi/2} P_1(E \sin \phi, E) \sin n\phi \, d\phi$$

$$+ \int_{3\pi/2}^{2\pi} Q_2(E \sin \phi, E) \sin n\phi \, d\phi$$

$$+ \int_{3\pi/2}^{2\pi} P_2(E \sin \phi, E) \sin n\phi \, d\phi\Bigg] \qquad (3.73)$$

and

$$b_n(E) = \frac{1}{\pi E}\Bigg[\int_0^{\pi/2} Q_2(E\sin\phi, E)\cos n\phi\, d\phi$$

$$+ \int_0^{\pi/2} P_2(E\sin\phi, E)\cos n\phi\, d\phi$$

$$+ \int_{\pi/2}^{3\pi/2} Q_1(E\sin\phi, E)\cos n\phi\, d\phi$$

$$+ \int_{\pi/2}^{3\pi/2} P_1(E\sin\phi, E)\cos n\phi\, d\phi$$

$$+ \int_{3\pi/2}^{2\pi} Q_2(E\sin\phi, E)\cos n\phi\, d\phi$$

$$+ \int_{3\pi/2}^{2\pi} P_2(E\sin\phi, E)\cos n\phi\, d\phi\Bigg] \quad (3.74)$$

In the above integrals the interval of integration $\pi/2$ to $3\pi/2$ can be divided into two parts, the first between $\pi/2$ and π and the second between π and $3\pi/2$. In addition make the following change of the dummy variable.

$$\beta = \pi - \phi \quad (3.75)$$

in the integrals performed in the interval $\pi/2$ to π.

$$\beta = \phi - \pi \tag{3.76}$$

in the integrals performed in the interval π to $3\pi/2$.

$$\beta = 2\pi - \phi \tag{3.77}$$

in the integrals performed in the interval $3\pi/2$ to 2π.
But according to the definitions, $P_1(x, E)$ and $P_2(x, E)$ are even functions of x and $Q_1(x, E)$ and $Q_2(x, E)$ odd functions of x. Also

$$\sin n(\pi - \phi) = - (-1)^n \sin n \phi \tag{3.78}$$

$$\sin n(\pi + \phi) = (-1)^n \sin n \phi \tag{3.79}$$

$$\cos n(\pi - \phi) = (-1)^n \cos n \phi \tag{3.80}$$

$$\cos n(\pi + \phi) = (-1)^n \cos n \phi \tag{3.81}$$

From (3.73) to (3.81) we find

$$\begin{aligned} g_n(E) = \frac{1}{\pi E}\Bigg[& \int_0^{\pi/2} Q_2(E \sin \phi, E) \sin n \phi \, d\phi \\ & + \int_0^{\pi/2} P_2(E \sin \phi, E) \sin n \phi \, d\phi \\ & + (-1)^n \int_{\pi/2}^{0} Q_1(E \sin \phi, E) \sin n \, n \phi \, d\phi \\ & + (-1)^n \int_{\pi/2}^{0} P_1(E \sin \phi, E) \sin n \phi \, d\phi \\ & - (-1)^n \int_0^{\pi/2} Q_1(E \sin \phi, E) \sin n \phi \, d\phi \\ & + (-1)^n \int_0^{\pi/2} P_1(E \sin \phi, E) \sin n \phi \, d\phi \\ & - \int_{\pi/2}^{0} Q_2(E \sin \phi, E) \sin n \phi \, d\phi \\ & + \int_{\pi/2}^{0} P_2(E \sin \phi, E) \sin n \phi \, d\phi \Bigg] \end{aligned} \tag{3.82}$$

and

$$b_n(E) = \frac{1}{\pi E}\Bigg[\int_0^{\pi/2} Q_2(E \sin \phi, E) \cos n \phi \, d\phi$$

$$+ \int_0^{\pi/2} P_2(E \sin \phi, E) \cos n \phi \, d\phi$$

$$- (-1)^n \int_{\pi/2}^{0} Q_1(E \sin \phi, E) \cos n \phi \, d\phi$$

$$- (-1)^n \int_{\pi/2}^{0} P_1(E \sin \phi, E) \cos n \phi \, d\phi$$

$$- (-1)^n \int_0^{\pi/2} Q_1(E \sin \phi, E) \cos n \phi \, d\phi$$

$$+ (-1)^n \int_0^{\pi/2} P_1(E \sin \phi, E) \cos n \phi \, d\phi$$

$$+ \int_{\pi/2}^{0} Q_2(E \sin \phi, E) \cos n \phi \, d\phi$$

$$- \int_{\pi/2}^{0} P_2(E \sin \phi, E) \cos n \phi \, d\phi \Bigg] \quad (3.83)$$

And rearranging (3.82) and (3.83)

$$g(E) = \frac{2}{\pi E} \int_0^{\pi/2} \left[Q_2(E \sin \phi, E) - (-1)^n Q_1(E \sin \phi, E)\right] \cdot \sin n \phi \, d\phi \quad (3.84)$$

$$b_n(E) = \frac{2}{\pi E} \int_0^{\pi/2} \Big[P_2(E \sin \phi, E) + (-1)^n P_1(E \sin \phi, E) \Big] \cos n \phi \, d\phi \quad (3.85)$$

Define

$$Q(x, E) = Q_1(x, E) + Q_2(x, E) \quad (3.86)$$

$$Q^*(x, E) = Q_1(x, E) - Q_2(x, E) \quad (3.87)$$

$$P(x, E) = P_1(x, E) - P_2(x, E) \quad (3.88)$$

$$P^*(x, E) = P_1(x, E) + P_2(x, E) \quad (3.89)$$

From equations (3.84) to (3.89) the following two sets of equations are obtained.

For n odd:

$$g_n(E) = \frac{2}{\pi E} \int_0^{\pi/2} Q(E \sin \phi, E) \sin n \phi \, d\phi \quad (3.90)$$

$$b_n(E) = -\frac{2}{\pi E} \int_0^{\pi/2} P(E \sin \phi, E) \cos n \phi \, d\phi \quad (3.91)$$

For n even:

$$g_n(E) = -\frac{2}{\pi E} \int_0^{\pi/2} Q^*(E \sin \phi, E) \sin n \phi \, d\phi \quad (3.92)$$

$$b_n(E) = \frac{2}{\pi E} \int_0^{\pi/2} P^*(E \sin \phi, E) \cos n \phi \, d\phi \quad (3.93)$$

But

$$\sin(2n+1)\,\phi = (2n+1)\sum_{i=0}^{i=n} (-1)^i \binom{n+i}{2i} \frac{4^i}{2i+1} \sin^{2i+1}\phi \tag{3.94}$$

$$\cos n\,\phi = 2^{n-1}\cos^n\phi + n\sum_{i=1}^{i\leq n/2} (-1)^i \binom{n-i-1}{i-1} \frac{2^{n-1-2i}}{i} \cos^{n-2i}\phi \tag{3.95}$$

$$\sin 2n\,\phi = \cos\phi \sum_{i=0}^{i=n-1} (-1)^i \binom{n+1}{2i+1} 2^{2i+1} \sin^{2i+1}\phi \tag{3.96}$$

Substituting (3.94), (3.95), and (3.96) into (3.90), (3.91), (3.92) and (3.93),

$$g_{2n+1}(E) = \frac{2(2n+1)}{\pi E} \sum_{i=0}^{i=n} (-1)^i \binom{n+i}{2i} \frac{4^i}{2i+1} \int_0^{\pi/2} Q(E\sin\phi, E)\sin^{2i+1}\phi\, d\phi \tag{3.97}$$

$$b_n(E) = -\frac{2^n}{\pi E}\int_0^{\pi/2} P(E\sin\phi, E)\cos^n\phi\, d\phi - \frac{2n}{\pi E}\sum_{i=1}^{i\leq n/2} (-1)^i \binom{n-i-1}{i-1} \frac{2^{n-1-2i}}{i} \cdot \int_0^{\pi/2} P(E\sin\phi, E)\cos^{n-2i}\phi\, d\phi \tag{3.98}$$

(n odd)

$$g_{2n}(E) = -\frac{2}{\pi E} \sum_{i=0}^{i=n-1} (-1)^i \binom{n+i}{2i+1}$$

$$2^{2i+1} \int_0^{\pi/2} Q^*(E \sin \phi, E) \sin^{2i+1} \phi \cos \phi \, d\phi$$

$$(3.99)$$

$$b_n(E) = \frac{2^n}{\pi E} \int_0^{\pi/2} P^*(E \sin \phi, E) \cos^n \phi \, d\phi$$

$$+ \frac{2n}{\pi E} \sum_{i=1}^{i \leq n/2} (-1)^i \binom{n-i-1}{i-1} \frac{2^{n-1-2i}}{i} \cdot$$

$$\cdot \int_0^{\pi/2} P^*(E \sin \phi, E) \cos^{n-2i} \phi \, d\phi \qquad (3.100)$$

(n even)

If in the above integrals the following change of variable is made,

$$x = E \sin \phi \qquad (3.101)$$

Then

$$g_{2n+1}(E) = \frac{2(2n+1)}{\pi E^2} \sum_{i=0}^{i=n} (-1)^i \binom{n+i}{2i} \frac{4^i}{2i+1}$$

$$\cdot \frac{1}{E^{2i}} \int_0^E \frac{x^{2i+1} \, Q(x, E)}{\sqrt{E^2 - x^2}} \, dx \qquad (3.102)$$

$$g_{2n}(E) = \frac{-2}{\pi E^3} \sum_{i=0}^{i=n-1} (-1)^i \binom{n+i}{2i+1} \frac{2^{i+1}}{E^{2i}} \int_0^E Q^*(x,E)\, x^{2i+1}\, dx \tag{3.103}$$

$$\begin{aligned} b_n(E) = &- \frac{2^n}{\pi E^{n+1}} \int_0^E P(x, E)(E^2 - x^2)^{\frac{n-1}{2}} dx \\ &- \frac{2n}{\pi E^{n+1}} \sum_{i=1}^{i \leq n/2} (-1)^i \binom{n-i-1}{i-1} \\ &\cdot \frac{2^{n-1-2i}}{i} E^{2i} \int_0^E P(x, E)(E^2 - x^2)^{\frac{n-2i-1}{2}} dx \end{aligned} \tag{3.104}$$

(n odd)

$$\begin{aligned} b_n(E) = &\frac{2^n}{\pi E^{n+1}} \int_0^E P^*(x, E)(E^2 - x^2)^{\frac{n-1}{2}} dx \\ &+ \frac{2n}{\pi E^{n+1}} \sum_{i=1}^{i \leq n/2} (-1)^i \binom{n-i-i}{i-1} \frac{2^{n-1-2i}}{i} \\ &\cdot E^{2i} \int_0^E P^*(x, E)(E^2 - x^2)^{\frac{n-2i-1}{2}} dx \end{aligned} \tag{3.105}$$

Equations (3.102) to (3.105) show that (at least for non-memory-type nonlinearities) there must exist a relation between all the functions $g_{2n+1}(E)$, where n 1, 2, 3, ..., and.

the same with $g_{2n}(E)$, $b_{2n+1}(E)$ and $b_{2n}(E)$. Therefore we can expect that, given any one of the functions $g_{2n+1}(E)$, the complete set of functions $g_{2n+1}(E)$ can be found. The same reasoning holds for the other sets of functions. As will be shown below, this fact is true for nonmemory-type nonlinearities.

In the case of memory-type nonlinearities the problem cannot be solved because of the non-uniqueness of the integral equation relating $g_{2n+1}(E)$ and $Q(x, E)$. The same holds for the other sets of functions $g_{2n}(E)$, $b_{2n+1}(E)$ and $b_{2n}(E)$.

Equations (3.102) through (3.105), also show that if any of $Q(x, E)$, $Q(x, E)$, $P(x, E)$, and/or $P(x, E)$ are zero all the harmonics depending on this function will also be zero. In particular, one can say in the case of nonmemory-type, that if any one of the functions $g_1(E)$, $g_2(E)$, $b_0(E)$ and $b_1(E)$ are identically zero, all the functions belonging to the same set will be identically zero.

3.6 Derivation of the Functional Relation between $g_1(E)$ and $g_{2n+1}(E)$

Let us consider the case of nonmemory type nonlinearities, and find the functional relation between $g_{2n+1}(E)$ and $g_1(E)$. As has been shown previously,

$$Q(x) = \int_0^x \frac{d\left[z^2\, g_1(z)\right]}{\sqrt{x^2 - z^2}} \tag{3.106}$$

Substituting Eq. (3.106) into (3.102)

$$g_{2n+1}(E) = \frac{2(2n+1)}{\pi E^2} \sum_{i=0}^{i=n} C_n^i \; E^{-2i} \, \cdot$$

$$\cdot \int_0^E dx \int_0^x \frac{d\left[z^2\, g_1(z)\right] x^{2i+1}}{\sqrt{x^2 - z^2}\,\sqrt{E^2 - x^2}} \tag{3.107}$$

where

$$C_n^i = (-1)^i \binom{n+i}{2i} \frac{4^i}{2i+1} \tag{3.108}$$

Change the order of integration in equation (3.107)

$$g_{2n+1}(E) = \frac{2(2n+1)}{\pi E^2} \sum_{i=0}^{i=n} C_n^i \; E^{-2i} \int_0^E d\left[z^2\, g_1(z)\right] \cdot$$

$$\cdot \int_z^E \frac{x^{2i+1} \, dx}{\sqrt{E^2 - x^2}\,\sqrt{x^2 - z^2}} \tag{3.109}$$

In Appendix 6 it is shown

$$\int_z^E \frac{x^{2i+1}\,dx}{\sqrt{E^2-x^2}\sqrt{x^2-z^2}} = \frac{\pi}{2}\sum_{p=0}^{p=i} R_i^{\,p}\,(E^2-z^2)^p\,z^{2(i-p)} \qquad (3.110)$$

where

$$R_i^{\,p} = \binom{i}{p}\binom{2p}{p}\,4^{-p} \qquad (3.111)$$

But

$$(E^2-z^2)^p = (-1)^p\sum_{j=0}^{j=p}(-1)^j\binom{p}{j}E^{2j}\,z^{2(p-j)} \qquad (3.112)$$

From Eqs. (3.112) and Eq. (3.110)

$$\int_z^E \frac{x^{2i+1}\,dx}{\sqrt{E^2-x^2}\,\sqrt{x^2-z^2}} = \frac{\pi}{2}\sum_{p=0}^{p=i}\sum_{j=0}^{j=p}$$

$$(-1)^{p+j}\binom{i}{p}\binom{2p}{p}\binom{p}{j}\,4^{-p}\,E^{2j}\,z^{2(i-j)} \qquad (3.113)$$

Changing the order of summation in Eq. (3.113)

$$\int_z^E \frac{x^{2i+1}\,dx}{\sqrt{E^2-x^2}\,\sqrt{x^2-z^2}} = \frac{\pi}{2}\sum_{j=0}^{j=i}\sum_{p=0}^{p=i}$$

$$(-1)^{p+j}\binom{i}{p}\binom{2p}{p}\binom{p}{j}\,4^{-p}E^{2j}\,z^{2(i-j)} \qquad (3.114)$$

If Eq. (3.114) is rearranged, it is possible to remove z^{2k} from the first summation

$$\int_z^E \frac{x^{2i+1}\, dx}{\sqrt{E^2 - x^2}\,\sqrt{x^2 - z^2}} = \frac{\pi}{2} \sum_{k=0}^{k=i} z^{2k}\, E^{2(i-k)} \cdot$$

$$\cdot \sum_{p=i-k}^{p=i} (-1)^{p+i-k} \binom{i}{p}\binom{2p}{p}\binom{p}{i-k} 4^{-p} \qquad (3.115)$$

In Appendix 4 it is shown

$$\sum_{p=i-k}^{p=i} (-1)^p \binom{i}{p}\binom{2p}{p}\binom{p}{i-k} 4^{-p} = \binom{k-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{i-k} \qquad (3.116)$$

From (3.116) and (3.115)

$$\int_z^E \frac{x^{2i+1}\, dx}{\sqrt{E^2-x^2}\,\sqrt{x^2-z^2}} = \sum_{k=0}^{k=i} (-1)^{i-k} \binom{k-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{i-k} \frac{\pi}{2}\, E^{2(i-k)}\, z^{2k} \qquad (3.117)$$

Substituting Eq. (3.117) into (3.109)

$$g_{2n+1}(E) = \frac{2n+1}{E^2} \cdot$$

$$\cdot \sum_{i=0}^{i=n} \sum_{k=0}^{k=i} (-1)^k \binom{n+i}{2i}\binom{k-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{i-k} \frac{4^i E^{-2k}}{2i+1} \int_0^E d\left[z^2\, g_1(z)\right] z^{2k} \qquad (3.118)$$

Performing the integration by parts

$$\int_0^E d\left[z^2\, g_1(z)\right] z^{2k} = g_1(E)\, E^{2k+2-2k} \int_0^E g_1(z)\, z^{2k+1}\, dz \qquad (3.119)$$

From (3.119) and (3.118)

$$g_{2n+1}(E) = (2n+1) \sum_{i=0}^{i=n} \sum_{k=0}^{k=i} (-1)^k \binom{n+i}{2i} \binom{k-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{i-k} \frac{4i}{2i+1} g_1(E)$$

$$- \frac{2(2n+1)}{E} \sum_{i=0}^{i=n} \sum_{k=0}^{k=i} (-1)^k \binom{n+i}{2i} \binom{k-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{i-k} \frac{k\ 4^i}{2i+1} \int_0^E g_1(z) \left(\frac{z}{E}\right)^{2k+1} dz \tag{3.120}$$

In Appendix 3 it is shown

$$(2n+1) \sum_{i=0}^{i=n} \sum_{k=0}^{k=i} (-1)^k \binom{n+1}{2i} \binom{k-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{i-k} \frac{4^i}{2i+1} = (-1)^n \tag{3.121}$$

If the order of summation in Eq. (3.120) is changed and Eq. (3.121) is substituted into Eq. (3.120)

$$g_{2n+1}(E) = (-1)^n g_1(E) - \frac{(2n+1)}{E} \sum_{k=0}^{k=n} (-1)^k\, k \binom{k-\frac{1}{2}}{k} \cdot$$

$$\cdot \sum_{i=k}^{i=n} \binom{n+i}{2i} \binom{-\frac{1}{2}}{i-k} \frac{4^i}{2i+1} \int_0^E g_1(z) \left(\frac{z}{E}\right)^{2k+1} dz \tag{3.122}$$

Now define the polynomial

$$N_n(y) = 2(2n+1) \sum_{k=0}^{k=n} T_{n,k}\, y^{2k+1} \tag{3.123}$$

where

$$T_{n,k} = (-1)^k \, k \binom{k-\frac{1}{2}}{k} \sum_{i=0}^{i=n} \binom{n+i}{2i} \binom{-\frac{1}{2}}{i-k} \frac{4^i}{2i+1} \qquad (3.124)$$

Therefore

$$g_{2n+1}(E) = (-1)^n \, g_1(E) - \frac{1}{E} \int_0^E g_1(z) \, N_n \frac{z}{E} \, dz \qquad (3.125)$$

Eq. (3.125) can be transformed into

$$T_{n,k} = k \binom{2k}{k} \sum_{i=0}^{i=n} (-1)^i \binom{n+i}{2i} \binom{2i-2k}{i-k} \frac{1}{2i+1} \qquad (3.126)$$

Equation (3.125) gives the functional relationship between $g_1(E)$ and all the $g_{2n+1}(E)$. Note that Eq. (3.125) applies for the general case of nonmemory type nonlinearities, single or double valued. It shows that, given the first harmonic of the output of any nonlinearity of the nonmemory type, when the input is driven by a sinusoidal wave, it is always possible to find all the odd harmonics. In Table I are given the numerical values of the coefficients T_n^k .

Example

To illustrate the above results, consider an example. Let

$$g_1(E) = \frac{35}{64} E^6 \qquad (3.127)$$

From Table I is found

$$N_1(y) = -4y^3 \qquad (3.128)$$

$$N_2(y) = -12y^3 + 24y^5 \qquad (3.129)$$

$$N_3(y) = -24y^3 + 120y^5 - 120y^7 \qquad (3.130)$$

$$N_4(y) = -40y^3 + 360y^5 + 840y^7 + 560y^9 \qquad (3.131)$$

Therefore, from Eq. (3.125)

$$g_3(E) = -\frac{35}{64}E^6 + \frac{140}{64E^4}\int_0^E z^9\,dz = -\frac{21}{64}E^6 \qquad (3.132)$$

$$g_5(E) = \frac{35}{64}E^6 + \frac{35 \times 12}{64E}\int_0^E z^6\left[\left(\frac{z}{E}\right)^3 - 2\left(\frac{z}{E}\right)^5\right]dz = \frac{7}{64}E^6 \qquad (3.133)$$

$$g_7(E) = -\frac{35}{64}E^6 + \frac{24 \times 35}{64E}\int_0^E z^6\left[\left(\frac{z}{E}\right)^3 - 5\left(\frac{z}{E}\right)^5 + 5\left(\frac{z}{E}\right)^9\right]dz = -\frac{1}{64}E^6 \qquad (3.134)$$

$$g_9(E) = \frac{35}{64}E^6 + \frac{35 \times 40}{64E}\int_0^E \left[\left(\frac{z}{E}\right)^3 - 9\left(\frac{z}{E}\right)^5 + 21\left(\frac{z}{E}\right)^7 - 14\left(\frac{z}{E}\right)^9\right]dz = 0 \qquad (3.135)$$

For the example under consideration it can be shown (Applying the inversion formula) that the equation of the non-linearity is:

$$y = x^7 \tag{3.136}$$

For

$$x = E \sin \phi$$

$$y = -\frac{E^7}{64} \sin 7\phi + \frac{7}{64} E^7 \sin 5\phi$$

$$- \frac{21}{64} E^7 \sin 3\phi + \frac{35}{64} E^7 \sin \phi \tag{3.137}$$

which verifies the results obtained by the application of Eq. (3.125).

3.7 Derivation of the Functional Relations between $b_1(E)$ and $b_{2n+1}(E)$

Consider the functional relationship between $b_1(E)$ and $b_{2n+1}(E)$ for the case of nonmemory type nonlinearities. From Eq. (3.104)

$$b_{2n+1}(E) = -\frac{2^{2n+1}}{\pi E^{2n+2}} \int_0^E P(x)(E^2 - x^2)^n \, dx$$

$$- \frac{2(2n+1)}{E^{2n+2}} \sum_{i=1}^{i=n} (-1)^i \binom{2n-1}{i-1} \frac{2^{2n-2i}}{i} \cdot$$

$$\cdot \int_0^E E^{2i} P(x) \, (E^2 - x^2)^{n-i} \, dx \tag{3.138}$$

For n = 0, Eq. (3.138) is reduced to

$$b_1(E) = -\frac{2}{\pi E^2} \int_0^E P(x) \, dx \tag{3.139}$$

Therefore

$$P(x) = -\frac{\pi}{2}\ \frac{d}{dx}\left[x^2\ b_1(x)\right] \tag{3.140}$$

From (3.140) and (3.138)

$$b_{2n+1}(E) = \frac{4^n}{E^{2n+2}}\int_0^E (E^2 - x^2)^n\ d\left[x^2\ b\ (x)\right]$$

$$+ \frac{2n+1}{E^{2n+2}} \sum_{i=1}^{i=n} (-1)^i \binom{2n-i}{i-1} \frac{2^{2n-2i}}{i}$$

$$\cdot\ E^{2i}\int_0^E (E^2 - x^2)^{n-i}\ d\left[x^2\ b_1(x)\right] \tag{3.141}$$

Integrating Eq. (3.141) by parts, and rearranging the summation

$$b_{2n+2}(E) = (2n+1)(-1)^n\ b_1(E)$$

$$+ \frac{2n4^n}{E^2}\int_0^E xb_1(x) \sum_{j=0}^{j=n-1} (-1)^j \binom{n-1}{j}\left(\frac{x}{E}\right)^{2j+2} dx$$

$$+ \frac{2(2n+1)}{E^2} \sum_{p=0}^{p=n-2} (-1)^p\int_0^E x\ b_1(x)\left(\frac{x}{E}\right)^{2p+2} \cdot$$

$$\cdot \sum_{i=1}^{i=n-p-1} (-1)^i \binom{2n-i}{i-1}\binom{n-i-1}{p}(n-i)\ \frac{4^{n-i}}{i}\ dx \tag{3.142}$$

In Appendix 5 it is shown that

$$S_n^p = \sum_{i=1}^{i=n-p-1} (-1)^i \binom{2n-i}{i-1}\binom{n-i-1}{p} (n-i) \frac{4^{n-i}}{i}$$

$$= \frac{1}{2n+1}\left[(p+1)\binom{n+p+1}{2p+2} 4^{p+1} - n\binom{n-1}{p} 4^n\right] \qquad (3.143)$$

From (3.143) and (3.142) it may be seen that

$$\frac{2n4^n}{E^2}\int_0^E xb_1(x) \sum_{j=0}^{j=n-2} (-1)^j \binom{n-1}{j} \left(\frac{x}{E}\right)^{2j+2} dx$$

$$- (-1)^n \frac{2n4^n}{E^2}\int_0^E xb_1(x) \left(\frac{x}{E}\right)^{2n} dx$$

$$+ \frac{2(2n+1)}{E^2}\int_0^E xb_1(x) \sum_{p=0}^{p=n-2} (-1)^p S_n^p \left(\frac{x}{E}\right)^{2p+2} dx$$

$$= (2n-1)(-1)^n b_1(E) - \frac{2n4^n}{E^2} (-1)^n \int_0^E xb_1(x)\left(\frac{x}{E}\right)^{2n} dx$$

$$+ \frac{2}{E^2}\int_0^E xb_1(x) \sum_{p=0}^{p=n-2} (-1)^p \left[n\, 4^n \binom{n-1}{p} + (2n+1)S_n^p\right]\left(\frac{x}{E}\right)^{2p+2} dx$$

$$(3.144)$$

But from Eq. (3.143)

$$n\, 4^n \binom{n-1}{p} + (2n+1)\, S_n^p = \binom{n+p+1}{2p+2} (p+1)\, 4^{p+1} \qquad (3.145)$$

Therefore from Eq. (3.144) and (3.145)

$$b_{2n+1}(E) = (-1)^n\ (2n+1)\ b_1(E) + \frac{2}{E^2}\int_0^E xb_1(x)\ Y_n(\frac{x}{E})dx \tag{3.146}$$

where

$$Y_n(w) = \sum_{p=0}^{p=n-1} (-1)^p(p+1)\binom{n+p+1}{2p+2} 4^{p+1}\ w^{2p+2} \tag{3.147}$$

Example

For n = 3, Eq. (3.147) yields

$$Y_3(w) = 24w^2 - 160w^4 + 192w^6 \tag{3.148}$$

Therefore, from Eq. (3.147) the seventh describing function will be given

$$b_7(E) = -\ 7b_1(E) - \frac{2}{E^2}\int_0^E xb_1(x)\left[-24(\frac{x}{E})^2 + 160(\frac{x}{E})^4 - 192(\frac{x}{E})^6\right]dx \tag{3.149}$$

3.8 Conclusions

In the present chapter has been developed a method to synthesize a nonlinearity from its describing function. However, as was shown, the knowledge of the describing function of a nonlinearity is not sufficient to determine uniquely the nonlinearity. Nevertheless, it is always possible to construct a nonlinearity with a prescribed describing function[(i)]. Even in the case in which the describing function is discontinuous, the inverse-describing-function problem can be solved by using a memory-type nonlinearity. The method was illustrated with an example. It was also shown that, for the nonmemory-type nonlinearities, all the odd and even harmonics of the output of the form sin n ωt, depend on the same functions Q(x) and Q*(x) respectively. The same property has been deduced for the odd and the even harmonics of the output with terms of the form cos n ωt. They depend on the same functions P(x) and P*(x) respectively. From this property was deduced a functional relationship between the real part of the describing function, $g_1(E)$, and all the functions $g_{2n+1}(E)$.

An analogous functional relationship was deduced between the imaginary part of the describing function, $b_1(E)$ and all the functions $b_{2n+1}(E)$.

[(i)]The describing function must be such that the conditions for the existence of the inverse describing function are satisfied (See Sec. 3.3).

Table I

Numerical Value of the Coefficients T_n^k

n \ k	0	0	2	3	4
1	0	$-\frac{2}{3}$	0	0	0
2	0	$-\frac{6}{5}$	$\frac{12}{5}$	0	0
3	0	$-\frac{12}{7}$	$\frac{60}{7}$	$-\frac{60}{7}$	0
4	0	$-\frac{20}{9}$	$\frac{180}{9}$	$-\frac{420}{9}$	$\frac{280}{9}$

Part II

COMPUTATIONAL TECHNIQUES

CHAPTER 4

NUMERICAL METHODS

4.1 Introduction

In chapter 3 has been developed an analytical method for the solution of the inverse describing function problem. In chapter 2, the non-uniqueness of the inverse-describing-function problem was also shown. Therefore, to reconstruct a nonlinearity from its describing function, some "a priori" knowledge about the nonlinearity is required. Or, if nothing is known about the nonlinearity, some arbitrary assumptions must be formulated about the nonlinear element.

In this chapter, it will be assumed that the nonlinearity is such that when its input is driven by a sinusoidal wave, the output of the nonlinearity will be periodic with only odd harmonic components. This type of nonlinearity is known as a "symmetric nonlinearity". From Eq. (3.103) and (3.105) the conditions for all the even harmonics to be zero are, in the case of nonmemory type nonlinearities,

$$Q^*(x) = 0 \tag{4.1}$$

and

$$P^*(x) = 0 \tag{4.2}$$

where $Q^*(x)$ and $P^*(x)$ are defined by Eq. (3.87) and (3.89).

Therefore from Eq. (3.87) and (3.89)

$$Q_1(x) = Q_2(x) = q(x) \tag{4.3}$$

and

$$P_1(x) = -P_2(x) = p(x) \tag{4.4}$$

Thus from Eq. (4.3), (4.4), (3.86) and (3.88)

$$Q_{(x)} = Q_1(x) + Q_2(x) = 2q(x) \tag{4.5}$$

$$P_{(x)} = P_1(x) - P_2(x) = 2p(x) \tag{4.6}$$

From Eq. (3.10) and (3.11)

$$q(x) = \frac{1}{2}\int_0^x \frac{\frac{d}{dz}\left[z^2\ g(z)\right]}{\sqrt{x^2 - z^2}}\, dz \tag{4.7}$$

$$p(x) = -\frac{\pi}{4}\frac{d}{dx}\left[x^2 b(x)\right] \tag{4.8}$$

The functions $F_1(x)$ and $F_2(x)$ will be given by

$$F_1(x) = Q_1(x) + P_1(x) = q(x) + p(x) \tag{4.9}$$

and

$$F_2(x) = Q_2(x) + P_2(x) = q(x) - p(x) \tag{4.10}$$

By means of Eq. (4.7), (4.8), (4.9) and (4.10) a symmetric nonmemory-type nonlinearity can be synthesized from its describing function. However, the operations expressed in

Eqs. (4.7) and (4.8), are in general very difficult to perform, if not impossible. Also, in the majority of cases, the describing function is not known by an analytical expression, but by experimental data. Therefore a numerical method is needed.

4.2 Numerical Computation of the Inverse Describing Function

In the next few paragraphs a numerical method for the computation of the inverse describing function will be developed. It will be assumed that the describing function is known only for discrete values of E. Thus

$$g(E_j) = g_j \qquad j = 1,2,\ldots\ n \tag{4.11}$$

and

$$b(E_i) = b_i \qquad i = 1,2,\ldots\ m \tag{4.12}$$

Different subindices will be used for the real and imaginary part of the describing function since, in general, the values of E for which g(E) is known are not the same as the values of E for which b(E) is known. Let us define

$$s(z) = z^2 g(z) \tag{4.13}$$

and

$$r(x) = x^2 b(x) \tag{4.14}$$

From Eqs. (4.13), (4.14), (4.8) and (4.7)

$$q(x) = \frac{1}{2}\int_0^x \frac{\frac{d\,s(z)}{dz}}{\sqrt{x^2-z^2}}\,dz \qquad (4.15)$$

and

$$p(x) = -\frac{\pi}{4}\,\frac{d\,r(x)}{dx} \qquad (4.16)$$

The original problem of finding the inverse describing function has been divided into two parts. The first one consists in performing the first derivative of a function known for discrete values of the independent variable. The second consists in performing the integral expressed by Eq. (4.15).

Numerical Differentiation of Eq. (4.13)
and Eq. (4.14)

The functions s(z) and r(x) are known for discrete values of z and x. Let us define

$$s(z_j) = s_j = z_j^2\, g(z_j) \qquad (4.17)$$

where

$$z_j = Ej$$

and

$$r(x_i) = r_i = x_1^2\, b(x_i) \qquad (4.18)$$

$$x_i = E_i$$

The numerical differentiation of s(z) and r(x) can be achieved by approximating these functions in each one of the intervals, $z_{j-1} < z < z_j$ and $x_{i-1} < x < x_i$, by functions that will be called $s^{j-1}(z)$ and $r^{i-1}(x)$. Therefore

$$s(z) = s^{j-1}(z) \qquad \text{for } z_{j-1} < z < z_j \tag{4.19}$$

and

$$r(x) = r^{i-1}(x) \qquad \text{for } x_{i-1} < x < x_i \tag{4.20}$$

Since the values of z_j and x_i will not be uniformly distributed, a second-degree polynomial makes further development relatively easy to handle. Therefore, if a second degree polynomial is chosen to approximate the functions $s(z)$ and $r(x)$, it is found that

$$s^{j-1}(z) = z^2 \frac{A_{j-1}}{z_{j+1} - z_{j-1}} + \left[\frac{\Delta s_{j-1}}{\Delta z_{j-1}} - \frac{z_j + z_{j-1}}{z_{j+1} - z_{j-1}} A_{j-1} \right] z$$

$$+ \frac{z_j z_{j-1}}{z_{j+1} - z_{j-1}} A_{j-1} - \frac{z_j z_{j-1}}{\Delta z_{j-1}} \Delta \frac{s_{j-1}}{z_{j-1}} \tag{4.21}$$

and

$$r^{i-1}(x) = \frac{B_{i-1}}{x_{i+1} - x_{i-1}} x^2 + \left[\frac{\Delta r_{i-1}}{\Delta x_{i-1}} - \frac{x_i + x_{i-1}}{x_{i+1} - x_{i-1}} B_{i-1} \right] x$$

$$+ x_i x_{i-1} \left[\frac{B_{i-1}}{x_{i+1} - x_{i-1}} - \frac{1}{\Delta x_{i-1}} \Delta \frac{r_{i-1}}{x_{i-1}} \right] \tag{4.22}$$

where

$$A_j = \frac{\Delta s_{j+1}}{\Delta z_{j+1}} - \frac{\Delta s_j}{\Delta z_j} \tag{4.23}$$

and

$$B_i = \frac{\Delta r_{i+1}}{\Delta x_{i+1}} - \frac{\Delta r_i}{\Delta x_i} \tag{4.24}$$

The Δ operator is defined as

$$\Delta Y_k = Y_{k+1} - Y_k \tag{4.25}$$

From Eq. (4.21)

$$m_{j-1} = \left.\frac{ds^{j-1}(z)}{dz}\right|_{z=z_{j-1}} = \frac{\Delta x_{j-1}}{\Delta z_{j-1}} - \frac{\Delta z_{j-1}}{z_{j+1}-z_{j-1}} A_{j-1} \tag{4.26}$$

From Eqs. (4.22), (4.18) and (4.8)

$$p(x) = -\frac{\pi}{4}\left[\frac{B_{i-1}}{x_{i+1}-x_{i-1}}(2x - x_i - x_{i-1}) + \frac{\Delta r_{i-1}}{\Delta x_{i-1}}\right]$$

$$\text{for } x_{i-1} < x < x_i \tag{4.27}$$

Numerical Integration of Eq. (4.15)

The problem of finding q(x) is now reduced to the computation of the integral

$$q(x) = \frac{1}{2}\int_0^x \frac{m(z)}{\sqrt{x^2 - z^2}} \tag{4.28}$$

where m(z) is known for discrete values of the variable z by Eq. (4.26). This can be achieved by approximating m(z) in each of the intervals, $z_{j-1} < z < z_j$, by a suitable function $m^{j-1}(z)$. Then Eq. (4.28) becomes

$$q(x) = \frac{1}{2} \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} \frac{m^{j-1}(z)}{\sqrt{x^2-z^2}} dz + \frac{1}{2} \int_{z_n}^{x} \frac{m^n(z)}{\sqrt{x^2-z^2}} dz \qquad (4.29)$$

for $\quad z_n < x < z_{n+1}$

Since s(z) was approximated by a second-degree polynomial, it seems logical to approximate its derivative m(z) by a first-degree polynomial (linear approximation.)[(i)]

$$m^{j-1}(z) = \frac{m_{j-1}z_j - m_j z_{j-1}}{z_j - z_{j-1}} + \frac{m_j - m_{j-1}}{z_j - z_{j-1}} z$$

$$= \frac{z_j \; z_{j-1}}{z_j - z_{j-1}} \left(\frac{m_{j-1}}{z_{j-1}} - \frac{m_j}{z_j}\right) + \frac{m_j - m_{j-1}}{z_j - z_{j-1}} z$$

$$= - \frac{z_j z_{j-1}}{\Delta z_{j-1}} P_{j-1} + R_{j-1} z \qquad (4.30)$$

where

$$P_j = \Delta \frac{m_j}{z_j} \qquad (4.31)$$

(i) This reasoning holds strictly only in the case that s(z) is really a second order polynomial. However, a linear approximation will be used for the integration to simplify the development.

and

$$R_j = \frac{\Delta m_j}{\Delta z_j} \tag{4.32}$$

Substituting Eq. (4.30) into (4.29)

$$q(x) = \frac{1}{2} \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} \left[- \frac{z_j z_{j-1}}{\Delta z_{j-1}} P_{j-1} + R_{j-1} z \right] \frac{dz}{\sqrt{x^2 - z^2}}$$

$$+ \frac{1}{2} \int_{z_n}^{x} \left[\frac{z_{n+1} z_n}{\Delta z_n} P_n + R_n z \right] \frac{dz}{\sqrt{x^2 - z^2}} \tag{4.33}$$

Once the integral (4.33) is performed,

$$q(x) = \frac{1}{2} \sum_{j=1}^{n} \frac{z_j z_{j-1}}{\Delta z_{j-1}} P_{j-1} \left(\text{arc sin} \frac{z_{j-1}}{x} - \text{arc sin} \frac{z_j}{x}\right)$$

$$+ R_{j-1} \left(\sqrt{x^2 - z_{j-1}^2} - \sqrt{x^2 - z_j^2} \right)$$

$$+ \frac{1}{2} \frac{z_{n+1} z_n}{\Delta z_n} P_n \left[\text{arc sin} \frac{z_n}{x} - \frac{\pi}{2} \right] + R_n \sqrt{x^2 - z_n^2} \tag{4.34}$$

If the summation in Eq. (4.34) is divided into two parts: the first containing all the terms in which the expressions arc sin $\frac{z_{j-1}}{x}$ and $\sqrt{x^2 - z_{j-1}^2}$ appear and the second summation containing all the terms in which the expressions arc sin $\frac{z_j}{x}$ and $\sqrt{x^2 - z_j^2}$ appear, it is found that

$$q(x) = \frac{1}{2}\sum_{j=1}^{j=n}\left\{\frac{z_j z_{j-1}}{\Delta z_{j-1}} P_{j-1} \text{ arc sin } \frac{z_{j-1}}{x} + R_{j-1}\sqrt{x^2 - z_{j-1}^2}\right\} - \frac{1}{2}\sum_{j=1}^{j=n}\left\{\frac{z_j z_{j-1}}{\Delta z_{j-1}} P_{j-1} \cdot \text{ arc sin } \frac{z_j}{x} + R_{j-1}\sqrt{x^2 - z_j^2}\right\}$$
$$+ \frac{1}{2}\left\{\frac{z_{n+1} z_n}{\Delta z_n} P_n \text{ arc sin } \frac{z_n}{x} + R_n\sqrt{x^2 - z_n^2}\right\} - \frac{\pi}{4} P_n \frac{z_{n+1} z_n}{\Delta z_n} \tag{4.35}$$

In the first summation of Eq. (4.35) make the following change of the index of summation

$$i = j - 1$$

Then

$$q(x) = \frac{1}{2}\sum_{i=0}^{n-1}\left\{\frac{z_{i+1} z_i}{\Delta z_i} P_i \text{ arc sin } \frac{z_i}{x} + R_i\sqrt{x^2 - z_i^2}\right\} - \frac{1}{2}\sum_{j=1}^{n}\left\{\frac{z_{j-1} z_j}{\Delta z_{j-1}} P_{j-1} \text{ arc sin } \frac{z_j}{x} + R_{j-1}\sqrt{x^2 - z_j^2}\right\}$$
$$- \frac{\pi}{4} P_n \frac{z_{n+1} z_n}{\Delta z_n} + \frac{1}{2}\left\{\frac{z_{n+1} z_n}{\Delta z_n} P_n \text{ arc sin } \frac{z_n}{x} + R_n\sqrt{x^2 - z_n^2}\right\} \tag{4.36}$$

If the last term of Eq. (4.36) is introduced into the first summation of the same equation

$$q(x) = \frac{1}{2} \sum_{i=0}^{n} \left\{ \frac{z_{i+1} z_i}{\Delta z_i} P_i \text{ arc sin } \frac{z_i}{x} + R_i \sqrt{x^2 - z_i^2} \right\}$$

$$- \frac{1}{2} \sum_{i=1}^{n} \left\{ \frac{z_{j-1} z_j}{\Delta z_{j-1}} P_{j-1} \text{ arc sin } \frac{z_{j-1}}{x} + R_{j-1} \sqrt{x^2 - z_j^2} \right\}$$

$$- \frac{\pi}{4} P_n \frac{z_{n+1} z_n}{\Delta z_n} \qquad (4.37)$$

It is not difficult to transform Eq. (4.37) into

$$q(x) = \frac{1}{2} \sum_{j=1}^{n} \left[\left(\frac{z_{j+1} z_j}{\Delta z_j} P_j - \frac{z_{j-1} z_j}{\Delta z_{j-1}} P_{j-1} \right) \text{ arc sin } \frac{z_j}{x} \right.$$

$$\left. + (R_j - R_{j-1}) \sqrt{x^2 - z_j^2} \right] - \frac{\pi}{4} P_n \frac{z_{n+1} z_n}{\Delta z_n} + \frac{1}{2} R_0 x \qquad (4.38)$$

In order to simplify Eq. (4.38) let us consider the expression

$$\frac{z_{j+1} z_j}{\Delta z_j} P_j - \frac{z_{j-1} z_j}{\Delta z_{j-1}} P_{j-1} \qquad (4.39)$$

Substituting Eq. (4.31) into (4.39)

$$\frac{z_{j+1} z_j}{\Delta z_j} P_j - \frac{z_{j-1} z_j}{\Delta z_{j-1}} P_{j-1} = \frac{m_{j+1} z_j - m_j z_{j+1}}{z_{j+1} - z_j} - \frac{m_j z_{j-1} - m_{j-1} z_j}{z_j - z_{j-1}}$$

$$\frac{m_{j+1} z_j - m_j z_{j+1} - m_{j+1} z_{j-1} + m_j z_{j-1} - m_{j-1} z_j + m_{j-1} z_{j+1}}{(z_{j+1} - z_j)(z_j - z_{j-1})} z_j \qquad (4.40)$$

Adding and subtracting $m_j z_j$ to the numerator of Eq. (4.40) and factoring

$$\frac{z_{j+1} z_j}{\Delta z_j} P_j - \frac{z_{j-1} z_j}{\Delta z_{j-1}} P_{j-1} = z_j \frac{m_{j+1} - m_j}{z_{j+1} - z_j} - \frac{m_j - m_{j-1}}{z_j - z_{j-1}}$$

$$= z_j (R_j - R_{j-1})$$

$$= z_j A_{j-1} \tag{4.41}$$

Since, from Eqs. (4.23) and (4.32)

$$a_j = R_{j+1} - R_j \tag{4.42}$$

Substituting Eq. (4.41) into (4.38)

$$q(x) = \frac{1}{2} \sum_{j=1}^{n} A_{j-1}(z_j \text{ arc sin } \frac{z_j}{x} + \sqrt{x^2 - z_j^2}$$

$$- \frac{\pi}{4} P_n \frac{z_{n+1} z_n}{\Delta z_n} + \frac{1}{2} R_0 x \tag{4.43}$$

for $z_n < x < z_{n+1}$.

With the method described above, using the RPC 4000 digital computer,(i) the inverse describing function corresponding

---

(i)It was found in the initial computation that the discontinuities in the first derivative of the describing function resulted in large errors in the inverse describing function. Therefore, the programming of the differentiation was performed in sections, with end points at the discontinuities of the first derivative of the describing function. (E. diTada[17]) to the describing function shown in figures 13 and 14 was calculated. In Table II the experimental and exact values are listed. In figure 15 the experimental values from Table II are plotted.

4.3 Numerical Computation of the Describing Function

In chapter 2, it was shown that the real and imaginary parts of the describing function of a nonlinearity can be expressed by the following integral transformations:

$$g(E) = \frac{2}{\pi E^2} \int_0^E \frac{x\, Q(x)}{\sqrt{E^2 - x^2}} dx \qquad (4.44)$$

$$b(E) = -\frac{2}{\pi E^2} \int_0^E P(x)\, dx \qquad (4.45)$$

For the same reasons discussed in paragraph 4.1 for the case of the inverse describing function, it is interesting to find a numerical method to compute the above integrals. In a manner similar to that used for the cases of the inverse describing function, the functions Q(x) and P(x) may be approximated by polynomials. In order to simplify the procedure a linear approximation will be used. Therefore, for the interval $x_{j-1} < x < x_j$

$$Q^{j-1}(x) = \frac{1}{x_{j-1} - x_j} \left[Q_j x_{j-1} - Q_{j-1} x_j + (Q_{j-1} - Q_j)\, x \right] \qquad (4.46)$$

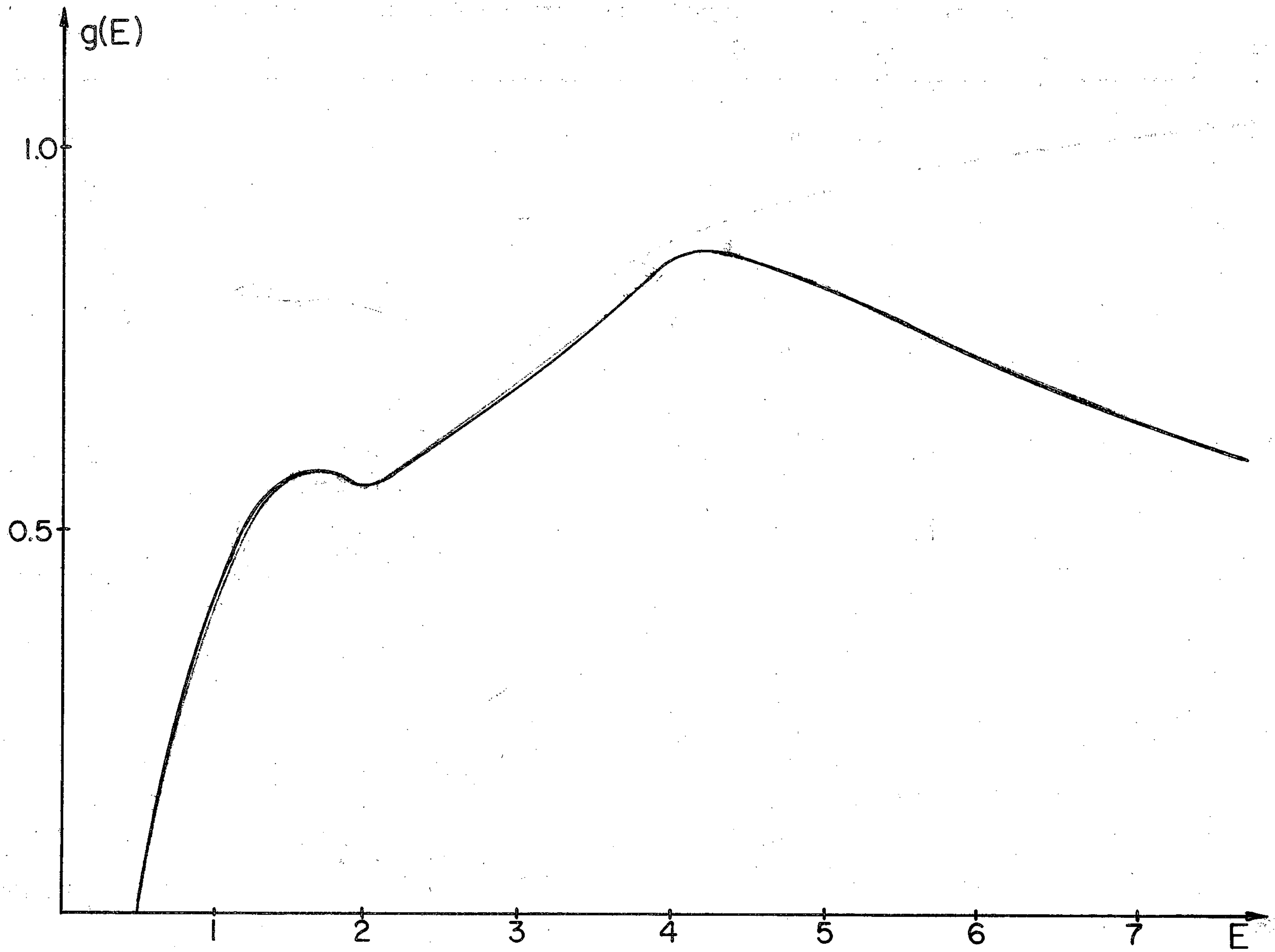

Fig. 13. Describing Function Corresponding to the Nonlinearity Shown in Figure 15

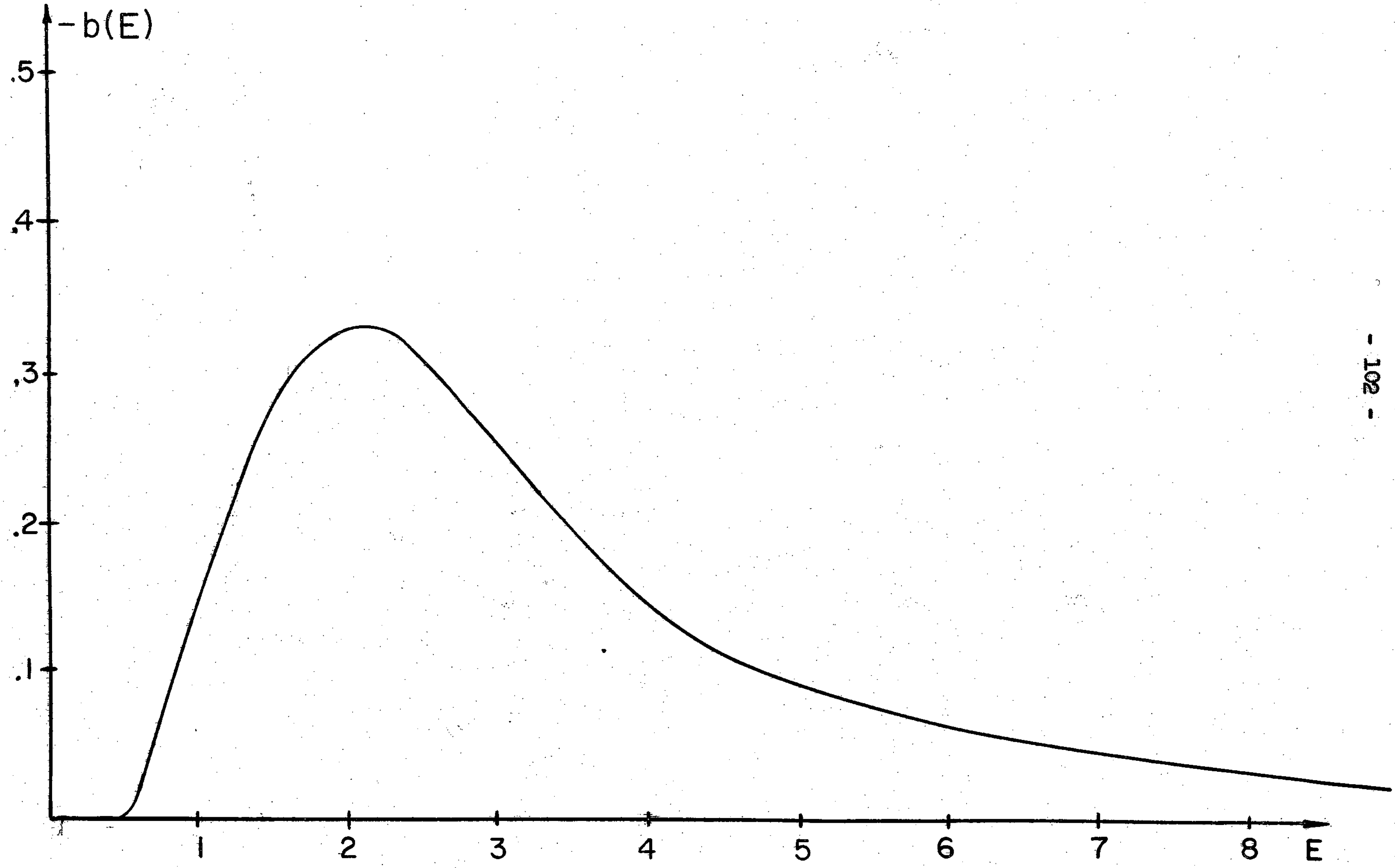

Fig. 14. Describing Function Corresponding to the Nonlinearity Shown in Figure 15

TABLE II

INVERSE DESCRIBING FUNCTION OF THE DESCRIBING FUNCTION SHOWN IN FIGURES 13 AND 14 EXPERIMENTAL AND THEORETICAL RESULTS

x	$F_1(x)$		$F_2(x)$	
	Experimental	Theoretical	Experimental	Theoretical
-5.00	-4.0017	-4.00	-3.9972	-4.00
-4.80	-3.9920	-4.00	-4.0045	-4.00
-4.60	-3.9971	-4.00	-3.9993	-4.00
-4.40	-4.0047	-4.00	-3.9921	-4.00
-4.20	-3.9951	-4.00	-3.9954	-4.00
-4.00	-3.9912	-4.00	-4.-127	-4.00
-3.80	-3.6030	-3.60	-3.6052	-3.60
-3.60	-3.2020	-3.20	-3.2079	-3.20
-3.40	-2.8027	-2.80	-2.9505	-2.95
-3.20	-2.4020	-2.40	-2.8516	-2.85
-3.00	-2.0034	-2.00	-2.7500	-2.75
-2.80	-1.6020	-1.60	-2.6537	-2.65
-2.60	-1.2025	-1.20	-2.5528	-2.55
-2.40	-0.80520	-0.80	-2.4521	-2.45
-2.20	-0.40372	-0.40	-2.3542	-2.35
-2.00	-0.00499	0.00	-2.2475	-2.25
-1.80	-0.00130	0.00	-2.1518	-2.15
-1.60	-0.00100	0.00	-2.0517	-2.05
-1.40	-0.02379	0.00	-1.8228	-1.80
-1.20	-0.00046	0.00	-1.4007	-1.40
-1.00	-0.00082	0.00	-1.0008	-1.00
-0.800	-0.00140	0.00	-0.60107	-0.60
-0.600	-0.00202	0.00	-0.20202	-0.20
-0.400	0.0000	0.00	0.0000	0.00
-0.200	0.0000	0.00	0.0000	0.00
0.00	0.0000	0.00	0.0000	0.00
0.20	0.0000	0.00	0.0000	0.00
0.40	0.0000	0.00	0.0000	0.00
0.60	0.20202	0.20	0.00202	0.00
0.80	0.60107	0.60	0.00141	0.00

TABLE II (Continued)

x	$F_1(x)$		$F_2(x)$	
	Experimental	Theoretical	Experimental	Theoretical
1.00	1.0008	1.00	0.00082	0.00
1.20	1.4007	1.40	0.00046	0.00
1.40	1.8228	0.80	0.02379	0.00
1.60	2.0517	2.05	0.00100	0.00
1.80	2.1518	2.15	0.00130	0.00
2.00	2.2475	2.25	0.00499	0.00
2.20	2.3542	2.35	0.40372	0.40
2.40	2.4521	2.45	0.80520	0.80
2.60	2.5528	2.55	1.2025	1.20
2.80	2.6537	2.65	1.6020	1.60
3.00	2.7500	2.75	2.0034	2.00
3.20	2.8516	2.85	2.4020	2.40
3.40	2.9505	2.95	2.8027	2.80
3.60	3.2079	3.20	3.2020	3.20
3.80	3.6052	3.60	3.6030	3.60
4.00	4.0127	4.00	3.9912	4.00
4.20	3.9954	4.00	3.9951	4.00
4.40	3.9921	4.00	4.0047	4.00
4.60	3.9993	4.00	3.9971	4.00
4.80	4.0045	4.00	3.9920	4.00
5.00	3.9972	4.00	4.0017	4.00

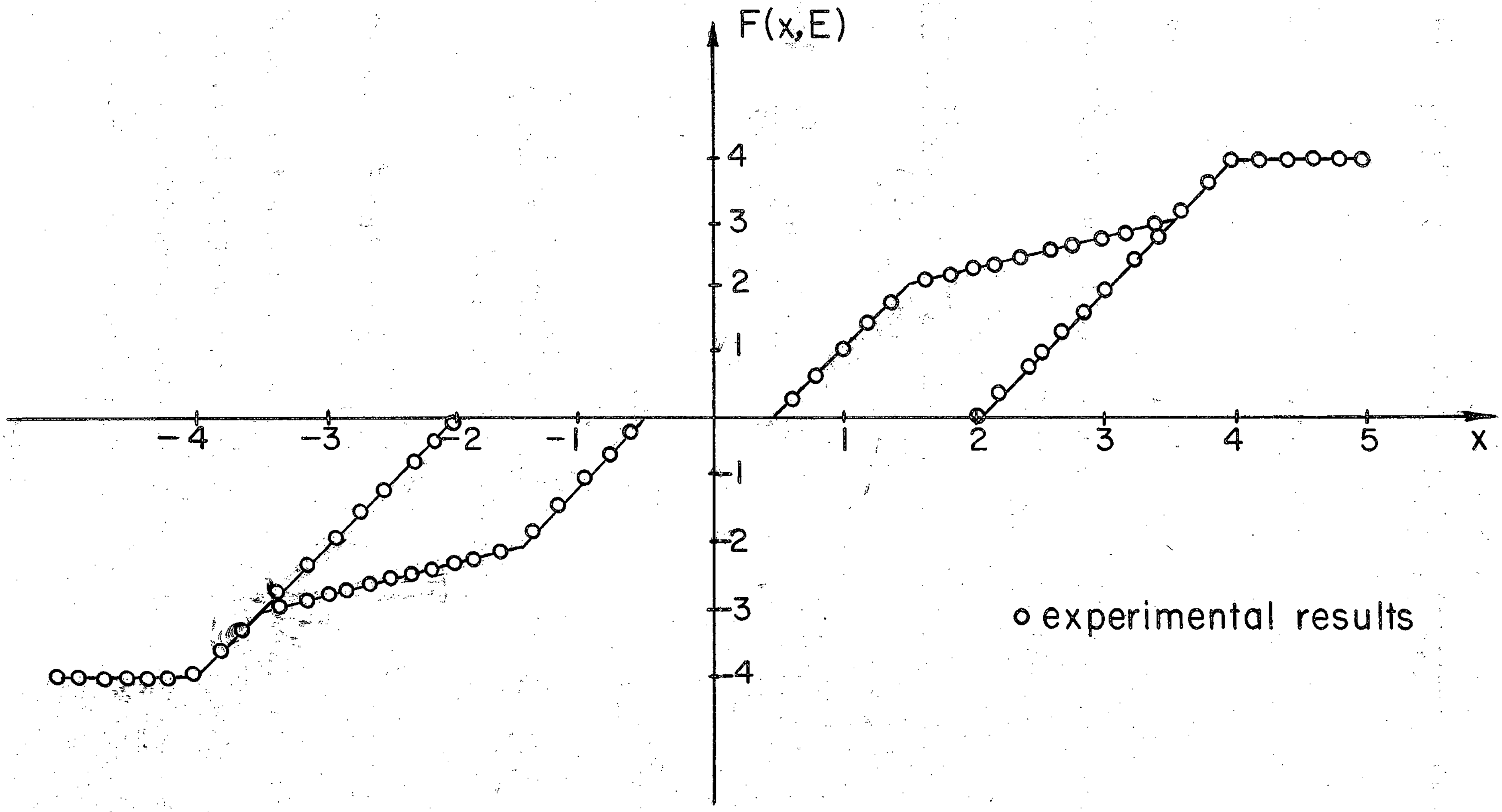

Fig. 15. Experimental Inverse Describing Function Corresponding to the Describing Function Shown in Figures 14 and 13

$$P^{j-1}(x) = \frac{1}{x_{j-1}-x_j}\left[P_j x_{j-1} - P_{j-1}x_j + (P_{j-1} - P_j)x\right] \qquad (4.47)$$

where

$$Q_j = Q(x_j) \qquad (4.48)$$

and

$$P_j = P(x_j) \qquad (4.49)$$

Substituting (4.36), (4.37) into (4.34) and (4.35) and after some transformations

$$g(E) = \sum_{j=1}^{j=n} \frac{Q_{j-1}(x_j-x_{j+1})+Q_j(x_{j+1}-x_{j-1})+Q_{j+1}(x_{j-1}-x_j)}{\pi(x_j-x_{j+1})(x_{j-1}-x_j)}$$

$$\left[\frac{x_j\sqrt{E^2-x_j^2}}{E^2} + \text{arc sin}\,\frac{x_j}{E}\right] + \frac{Q_n - Q_{n+1}}{2(x_n-x_{n+1})} + \frac{2Q_0}{\pi E} \qquad (4.50)$$

$$b(E) = \frac{1}{E^2}\sum_{j=1}^{n}$$

$$\left\{x_j^2 \; \frac{P_{j-1}(x_j-x_{j+1})+P_j(x_{j+1}-x_{j-1})+P_{j+1}(x_{j-1}-x_j)}{\pi(x_{j-1}-x_j)\,(x_j-x_{j+1})}\right\}$$

$$- \frac{P_{n+1}x_n-P_n x_{n+1}}{x_n-x_{n+1}}\,\frac{2}{\pi E} - \frac{P_n-P_{n+1}}{x_n-x_{n+1}}\,\frac{1}{\pi} \text{ for } x_n < E < x_{n+1} \qquad (4.51)$$

With the method described above, using the 4000 digital computer, the describing function corresponding to the relay with hysteresis and dead band whose characteristic is shown in figure 16 was computed. The results are listed in

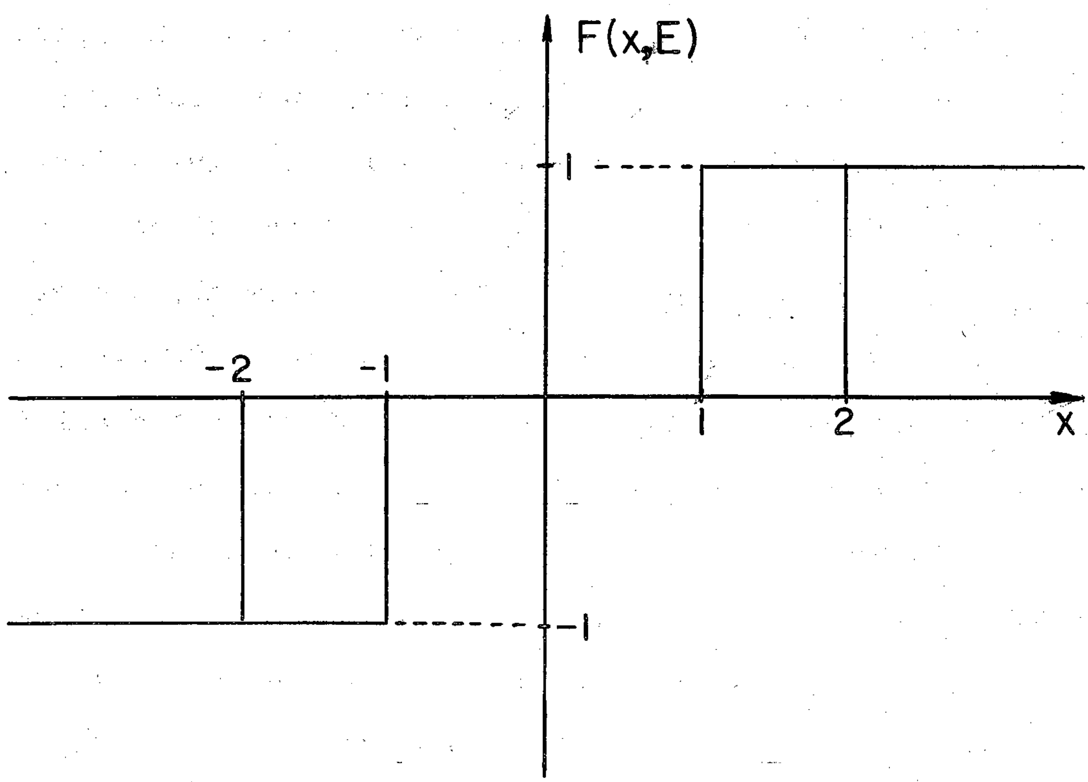

Fig. 16. Characteristic of a Relay with Hysteresis and Dead Band

Table III. In figures 17 and 18 the experimental values listed in Table III are plotted. In reference 17 the program of computation is discussed.

4.4 Conclusions

In the present chapter, numerical methods to compute the inverse describing function and the describing function have been developed. Both methods are based on the approximation with a polynomial of a function, known only for discrete values of the independent variable. The selection of a polynomial to fit the original function is arbitrary. Other kinds of functions, besides the polynomial, can be used in the method presented in this chapter. However, to simplify the development, polynomial fitting is convenient. In this chapter the polynomial was restricted to a second degree only to simplify the development. If a better accuracy is needed a higher degree polynomial can be chosen. The philosophy of the method will remain the same. The development will be more involved however.

TABLE III

DESCRIBING FUNCTION OF THE RELAY
WHOSE CHARACTERISTIC IS SHOWN IN FIGURE 16

EXPERIMENTAL AND THEORETICAL RESULTS

E	g(E)		b(E)	
	Experimental	Theoretical	Experimental	Theoretical
0.00	0.0000	0.0000	0.0000	0.0000
0.20	0.0000	0.0000	0.0000	0.0000
0.40	0.0000	0.0000	0.0000	0.0000
0.60	0.0000	0.0000	0.0000	0.0000
0.80	0.0000	0.0000	0.0000	0.0000
1.00	0.0000	0.0000	0.0000	0.0000
1.20	0.0000	0.0000	0.0000	0.0000
1.40	0.0000	0.0000	0.0000	0.0000
1.60	0.0000	0.0000	0.0000	0.0000
1.80	0.0000	0.0000	0.0000	0.0000
2.00	0.2862	0.2760	-0.1587	-0.1590
2.20	0.3783	0.3695	-0.1315	-0.1315
2.40	0.3877	0.389	-0.1105	-0.1105
2.60	0.3824	0.3825	-0.09418	-0.0942
2.80	0.3715	0.372	-0.08120	-0.0812
3.00	0.3582	0.358	-0.07073	-0.0707
3.20	0.3443	0.348	-0.06217	-0.0620
3.40	0.3303	0.330	-0.05507	-0.0550
3.60	0.3169	0.317	-0.04912	-0.0491
3.80	0.3040	0.303	-0.04408	-0.0441
4.00	0.2919	0.292	-0.03979	-0.0398
4.20	0.2805	0.280	-0.03609	-0.0361
4.40	0.2697	0.270	-0.03288	-0.0329
4.60	0.2597	0.260	-0.03008	-0.0301
4.80	0.2503	0.250	-0.02763	-0.0276
5.00	0.2414	0.241	-0.02546	-0.0255

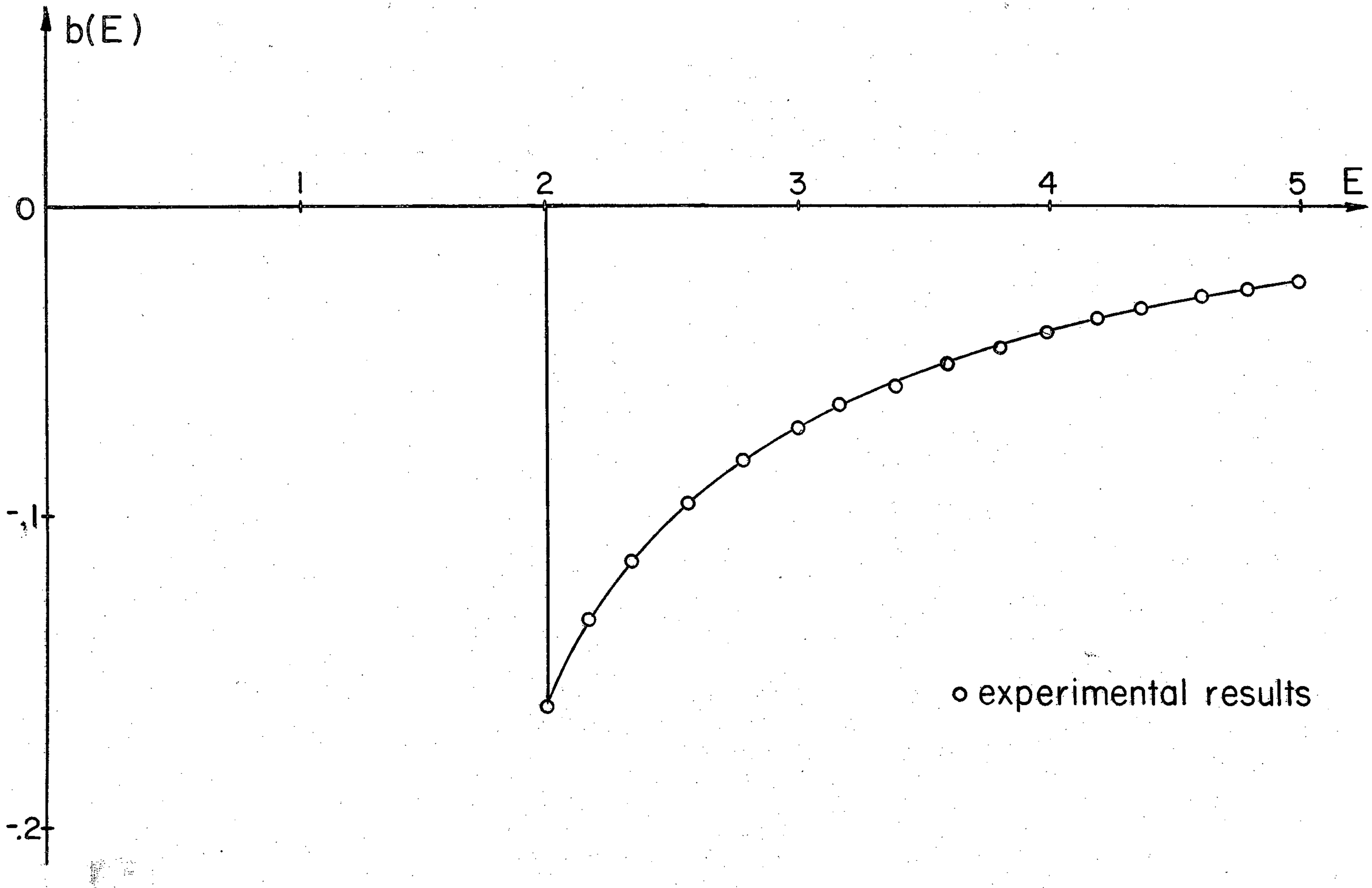

Fig. 17. Experimental Describing Function Corresponding to the Nonlinearity Whose Characteristic is Shown in Figure 16

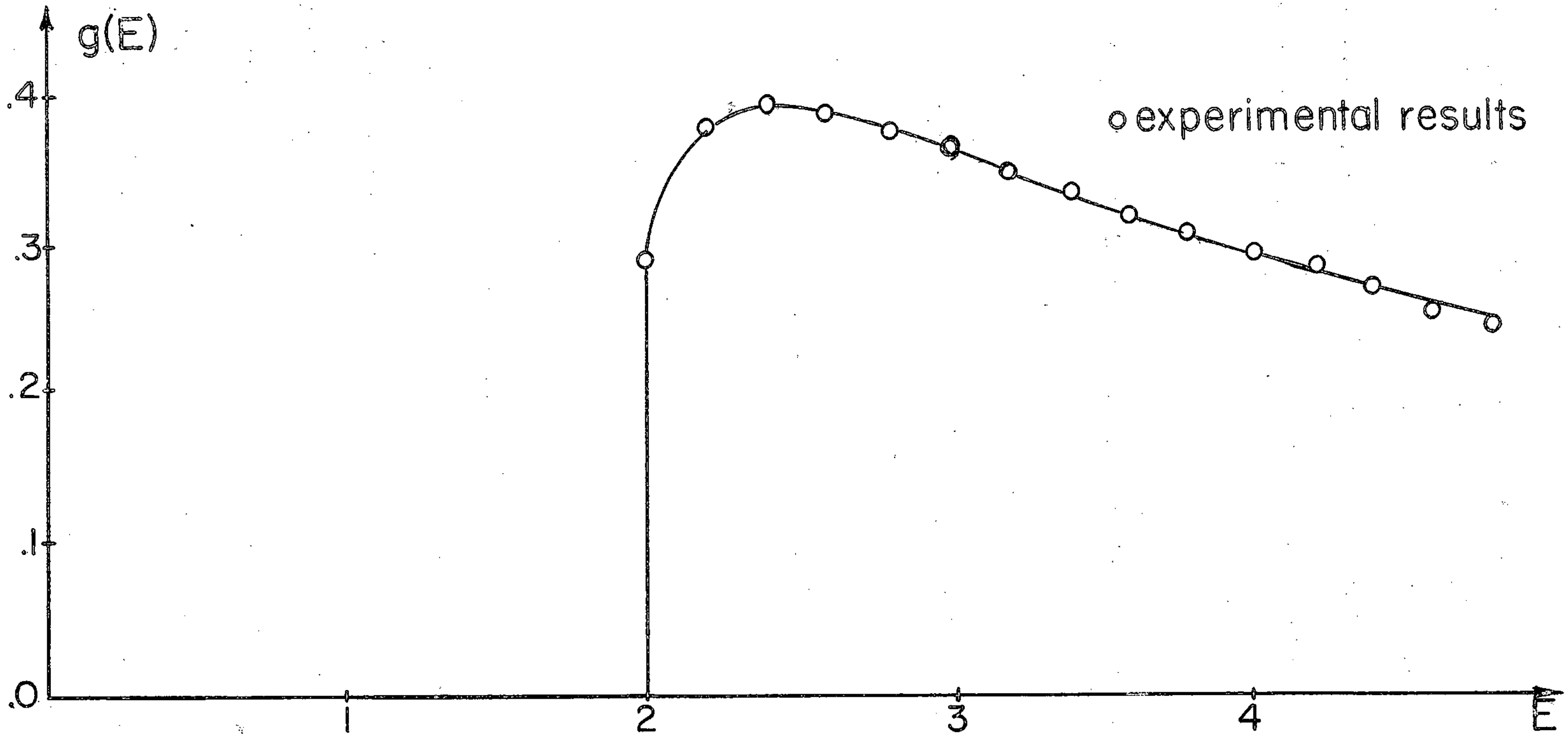

Fig. 18. Experimental Describing Function Corresponding to the Nonlinearity Whose Characteristic is Shown in Figure 16

CHAPTER 5

NUMERICAL INVERSION: POLYNOMIAL TYPE NONLINEARITIES

5.1 Introduction

Given a nonlinear element characterized by the functional relationship between its output and its input

$$y = f(x) \tag{5.1}$$

the describing function is defined by the integral transformation

$$K_{eq} = \frac{1}{\pi E}\int_0^{2\pi} f(E \sin \alpha) \sin \alpha \, d\alpha + \frac{j}{\pi E}\int_0^{2\pi} f(E \sin \alpha) \cos \alpha \, d\alpha \tag{5.2}$$

where

$$x = E \sin \alpha \tag{5.3}$$

or

$$K_{eq} = g(E) + j\, b(E) \tag{5.4}$$

In general, for symmetric, single-valued nonlinearities, there will be no phase shift in the output fundamental, and thus b(E) is zero. The problem to be investigated is the development of a procedure for carrying out the inverse transformation, i.e., given K_{eq} as defined by (5.2), what is the functional relationship of the nonlinearity.

The problem will be simplified initially by restricting f(x) to the class of single-valued odd functions.

5.2 Polynomial Describing Functions and Nonlinearities

The input-output characteristic of a general polynomial type nonlinearity is described by the following equation.

$$f(x) = a_1 x + a_2 x^2 + \ldots + a_n x^n$$

$$= \sum_{k=1}^{n} a_k x^k \tag{5.5}$$

Substituting (5.5) in (5.2), the describing function components are

$$g(E) = \frac{1}{\pi E}\int_0^{2\pi} \sum_{k=1}^{n} a_k (E \sin \alpha)^k \sin \alpha \, d\alpha$$

$$= \frac{1}{\pi} \sum_{k=1}^{n} a_k E^{k-1} \int_0^{2\pi} \sin^{k+1} \alpha \, d\alpha$$

$$= \frac{2}{\pi} \sum_{k=1}^{n} a_k E^{k-1} \frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+3}{2}\right)}$$

and

$$b(E) = \frac{1}{\pi E}\int_0^{2\pi} \sum_{k=1}^{n} a_k (E \sin \alpha)^k \cos \alpha \, d\alpha$$

$$= \frac{1}{\pi} \sum_{k=1}^{n} a_k E^{k-1} \int_0^{2\pi} \sin^k \alpha \cos \alpha \, d\alpha = 0$$

so that

$$K_{eq} = g(E) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{n} a_k E^{k-1} \frac{\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+3}{2})} \qquad (5.6)$$

as in Sridhar [14].

The describing function is also a polynomial and (5.6) may be written

$$E K_{eq} = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{n} a_k \frac{\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+3}{2})} E^k$$

$$= \sum_{k=1}^{n} a_k b_k E^k$$

$$= \sum_{k=1}^{n} C_k E^k \qquad (5.7)$$

where

$$b_k = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+3}{2})} \qquad k = 1,2,3,\ldots, n$$

$$C_k = a_k b_k \qquad k = 1,2,3,\ldots, n$$

Equation (5.7) is of the same form as (5.5). Therefore, the characteristic polynomial describing f(x) is given by

$$f(x) = \sum_{k=1}^{n} \frac{C_k}{b_k} x^k \qquad (5.8)$$

A table of b_k is shown below, where

$$\Gamma(k + 1) = k\,\Gamma(k)$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

and in general

$$\Gamma\left(\frac{k + 1}{2}\right) = \frac{1\cdot3\cdot5\cdot\cdots\cdot(k - 1)}{2^{n/2}}\sqrt{\pi}$$

$$b_1 = 1$$

$$b_2 = \frac{8}{3\pi}$$

$$b_3 = \frac{3}{4}$$

$$b_4 = \frac{32}{15\pi}$$

$$b_5 = \frac{5}{8}$$

$$b_6 = \frac{64}{35\pi}$$

$$\vdots$$

$$b_k = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{k + 2}{2}\right)}{\Gamma\left(\frac{k + 3}{2}\right)}$$

5.2 Examples

Example 1. Consider the describing function

$$K_{eq} = g(E) = .75E^2 = 1.2582E^3 \qquad (5.9)$$

Find the nonlinearity whose describing function is given in (5.9). From (5.7) and (5.9)

$$E\ K_{eq} = .75E^3 + 1.2582E^4 \qquad (5.10)$$

$$k = 3,\ 4$$

From (5.8), (5.10) and tables of b_k

$$f(x) = x^3 + 2x^4 \qquad (5.11)$$

If an analytical expression of the describing function is not given and tabular data for the describing function is known, the problem then reduces to finding a polynomial of best fit. Once a polynomial of best fit is obtained the previous analysis is effected.

Example 2. Consider the plot of the describing function

E	0	1	2	3	4	5	6	7	8
K_{eq}	0	.75	3	6.75	12	18.75	27	36.75	48

Find the nonlinearity.

An assumed 4th order polynomial was programmed on the digital computer using the method of least squares. The resulting coefficients of the assumed polynomial is

$$K_{eq} = -\ .07635 + .12192E + .69166E^2 + .01048E^3 - .000617\ E^4 \qquad (5.12)$$

From (5.8), (5.12) and tables of b_k, $k = 1,2,3,4,5$, the nonlinearity is

$$f(x) = -.07635x + .14363x^2 + .9222x^3 + .01543x^4 - .000987x^5 \qquad (5.13)$$

Note that the predominant coefficient in (5.13) is that of x^3. A plot of (5.13) is shown in figure 19, coinciding with that of $f(x) = x^3$.

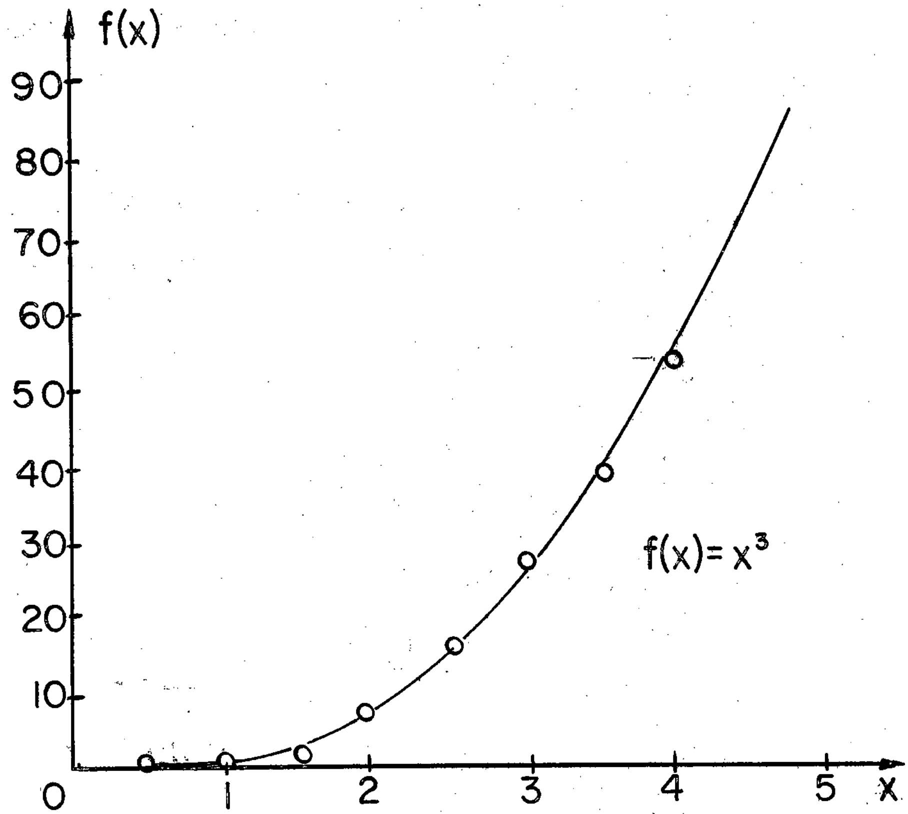

Fig. 19. Example of Inversion of a Polynomial Type Describing Function

CHAPTER 6

NUMERICAL INVERSION: PIECEWISE LINEAR DISCONTINUOUS NONLINEARITIES

6.1 Introduction

This chapter will describe the analytical formulation and computational technique used to attack the inverse describing function problem for a general class of nonlinearities - specifically those representable by a relation $y = f(x)$ which, if drawn in the xy plane, may be approximated by a finite number of line segments in the plane. This implies that the nonlinearity will not be frequency sensitive, but it may be asymmetrical and discontinuous.

The extension to asymmetrical nonlinearities should be apparent from this analysis, and since the asymmetrical case is principally of academic interest it will not be presented here.

6.2 Piecewise Linear Single-Valued[1] Nonlinearities

In this section an analytical formulation of the problem of describing function inversion for single-valued symmetrical nonlinearities will be presented. The problem

---

[1]Single-valued except possibly at a set of points of zero measure; specifically, finite discontinuities are allowed.

is defined as "Given the describing function of a non-linearity, find an analytic representation for the non-linearity". The analysis here is for the simplest case in order to provide intuitive feel for the problem.

The fundamental assumption is that the nonlinearity may be approximated by a finite number of piecewise linear segments. The slopes and y-axis intercepts of these segments will be chosen so that the piecewise linear approximation will have the same describing function as the original nonlinearity for a specified number of input signal amplitudes.

Consider a piecewise linear, single-valued, symmetrical nonlinearity as shown in figure 20, which is an N/2 segment approximation to a nonlinearity whose describing function is known. This describing function will be matched for the input amplitudes $E_k = x_k$, $k = 1, 2, \ldots, N$.

For the k/2th segment

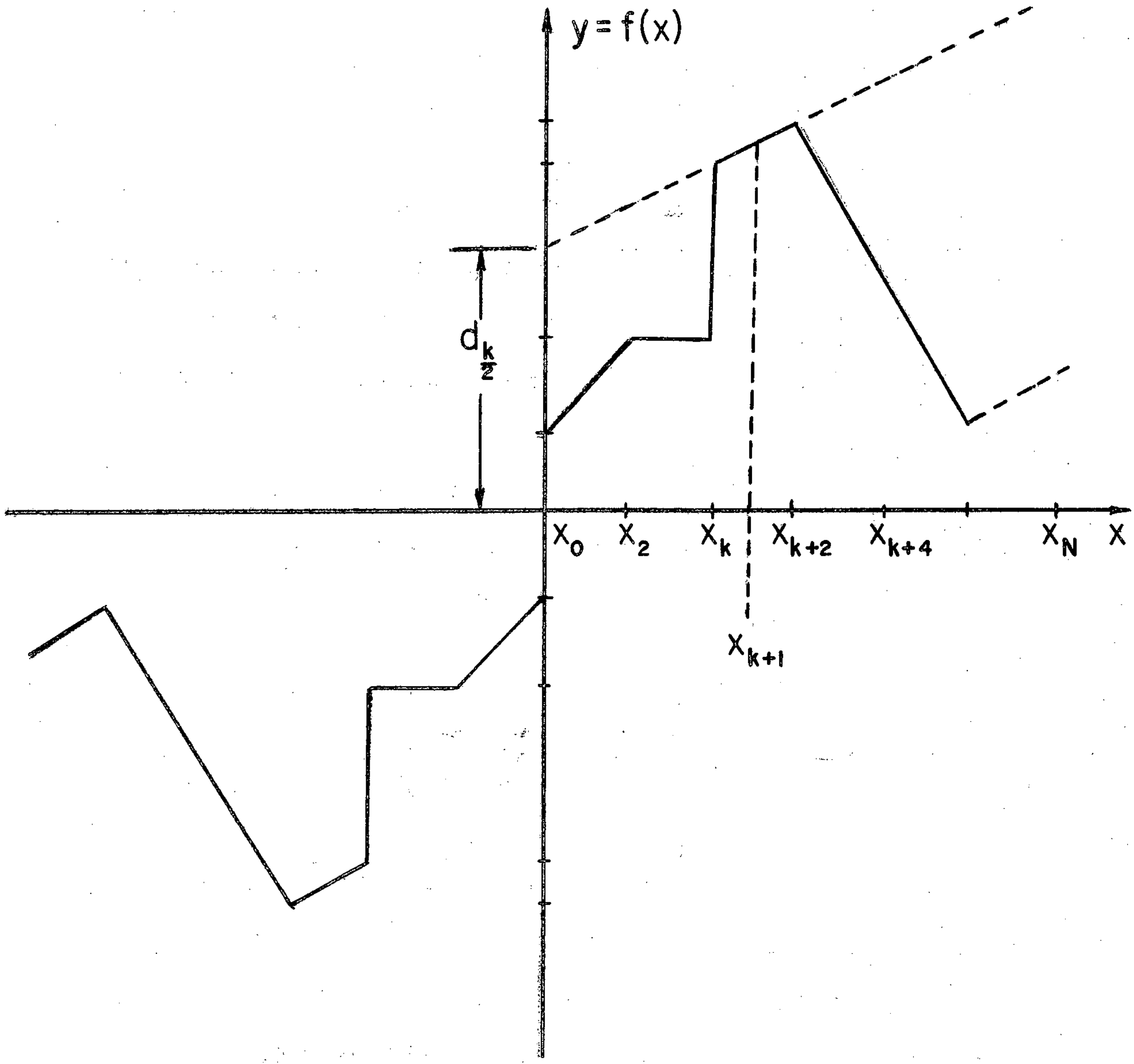

Fig. 20. Single-Valued[1] Symmetrical Nonlinearity

$$y_{k/2} = m_{k/2}\, x + d_{k/2}\,, \quad x_{k-2} \leq x \leq x_k \quad (k = 2,4,6,\ldots,N) \tag{6.1}$$

where N is the number of subdivisions on x.

Let

$$x = E \sin \omega t = E \sin \alpha \tag{6.2}$$

For the restricted N. L. (nonlinearity) considered here, the D. F. (describing function) is real and has no dc term, hence consideration of the first coefficient is sufficient [22], [23]:

$$g(E) \overset{\Delta}{=} \text{describing function} \overset{\Delta}{=} \frac{1}{\pi E}\int_0^{2\pi} f(x) \sin \alpha \, d\alpha \tag{6.3}$$

Since y = f(x) is single-valued and symmetrical,

$$g(E) = \frac{4}{\pi E}\int_0^{\pi/2} f(x) \sin \alpha \, d\alpha \tag{6.4}$$

This integral will be evaluated by the insertion of the relation given in (6.1), and will be shown to lead to a set of N simultaneous linear algebraic equations in the slopes and y-axis intercepts, whose solution will determine the nonlinearity from experimental or analytical describing function data.

Inserting (6.1) leads to

$$g(E) = \frac{4}{\pi E}\left\{\int_{0}^{\alpha_1} (m_1\,x + d_1)\,\sin\alpha\,d\alpha + \int_{\alpha_1}^{\alpha_2} (m_1\,x + d_1)\sin\alpha\,d\alpha\right.$$

$$+ \ldots + \int_{\alpha_{k-2}}^{\alpha_k} (m_{k/2}\,x + d_{k/2})\,\sin\alpha\,d\alpha + \ldots$$

$$\left. + \int_{\alpha_n}^{\pi/2} \left(\frac{m_{n+2}}{2}\,x + \frac{d_{n+2}}{2}\right)\,\sin\alpha\,d\alpha\right\} \qquad (6.5)$$

where

$$\begin{aligned} \alpha_1 &= \text{Arcsin}\,\frac{x_1}{E} \\ \alpha_2 &= \text{Arcsin}\,\frac{x_2}{E} \\ &\vdots \\ \alpha_k &= \text{Arcsin}\,\frac{x_k}{E} \\ &\vdots \\ \alpha_n &= \text{Arcsin}\,\frac{x_n}{E} \end{aligned} \qquad (6.6)$$

The subscript n is such that, for a given value of the variable A, the nonlinearity is driven into operation on the $\frac{n+2}{2}$ th segment.

Now substitute $x = E \sin\alpha$ into (6.5)

$$g(E) = \frac{4}{\pi E}\left\{\int_{o}^{\alpha_1} (m_1\, E \sin\alpha + d_1) \sin\alpha\, d\alpha + \ldots\right.$$

$$\ldots + \int_{\alpha_{k-2}}^{\alpha_k} (m_{k/2}\, E \sin\alpha + d_{k/2}) \sin\alpha\, d\alpha + \ldots$$

$$\left.\ldots + \int_{\alpha_n}^{\pi/2} (\frac{m_{n+2}}{2}\, E \sin\alpha + \frac{d_{n+2}}{2}) \sin\alpha\, d\alpha\right\} \qquad (6.7)$$

Now

$$\int \sin^2\alpha\, d\alpha = \tfrac{1}{2}\left\{\alpha - \tfrac{1}{2}\sin 2\alpha\right\} \qquad (6.8)$$

$$\int \sin\alpha\ \ d\alpha = -\cos\alpha \qquad (6.9)$$

and the integration can be performed:

$$g(E) = \frac{4}{\pi E}\left\{\frac{m_1 E}{2}\Big[\alpha - \tfrac{1}{2}\sin 2\alpha\Big]_o^{\alpha_1} - d_1\Big[\cos\alpha\Big]_o^{\alpha_1} + \ldots\right.$$

$$\ldots + \frac{m_{k/2}E}{2}\Big[\alpha - \tfrac{1}{2}\sin 2\alpha\Big]_o^{\alpha_k} - d_{k/2}\Big[\cos\alpha\Big]_{\alpha_{k-2}}^{\alpha_k} + \ldots$$

$$\left.\ldots + \frac{\frac{m_{n+2}}{2}\, E}{2}\Big[\alpha - \tfrac{1}{2}\sin 2\alpha\Big]_{\alpha_n}^{\pi/2} - \frac{d_{n+2}}{2}\Big[\cos\alpha\Big]_{\alpha_n}^{\pi/2}\right\}$$

$$(6.10)$$

Now notice that from equations (6.6), the angles

$\alpha_1, \alpha_2, .., \alpha_k, \ldots, \alpha_n$ can be evaluated <u>for a given value of E</u>.

We now define

$$\alpha_{ij} = \text{Arcsin}\,\frac{x_j}{E_i} \qquad (6.11)$$

and evaluate (6.10) successively for i = 1,2,..., N, thereby forming an Nth order set of simultaneous linear algebraic equations which must be satisfied by the $m_{k/2}$ and $d_{k/2}$: (the subscripts on the limits of the integrals in (6.10) have been relegated to the second ordered subscript in order to conform more closely with standard matrix notation).

$$g(E_i) = \frac{4}{\pi E_i}\left\{ \frac{m_1 E_i}{2}\left[\alpha - \tfrac{1}{2}\sin 2\alpha\right]_{o}^{\alpha_{i1}} - d_1\left[\cos\alpha\right]_{o}^{\alpha_{i1}} + \ldots \right.$$

$$\ldots + \frac{m_{k/2}E_i}{2}\left[\alpha - \tfrac{1}{2}\sin 2\alpha\right]_{\alpha_{i(k-2)}}^{\alpha_{ik}} - d_{k/2}\left[\cos\alpha\right]_{\alpha_{i(k-2)}}^{\alpha_{ik}} + \ldots$$

$$\left. \ldots + \frac{m_{\frac{n+2}{2}}E_i}{2}\left[\alpha - \tfrac{1}{2}\sin 2\alpha\right]_{\alpha_{in}}^{\pi/2} - d_{\frac{n+2}{2}}\left[\cos\alpha\right]_{\alpha_{in}}^{\pi/2}\right\} \qquad (6.12)$$

with i = 1,2,..., N.

By evaluating the α_{ij} from (6.11), the coefficients of the unknowns ($m_{k/2}$ and $d_{k/2}$) may be determined, and (6.12) will appear as

$$
\begin{aligned}
g(E_1) &= a_{11}\, m_1 + b_{11}\, d_1 + a_{12}\, m_2 + b_{12}\, d_2 + \dots \\
g(E_2) &= a_{21}\, m_1 + b_{21}\, d_1 + a_{22}\, m_2 + b_{22}\, d_2 + \dots \\
&\vdots \\
g(E_N) &= a_{N1}\, m_1 + b_{N1}\, d_1 + a_{N2}\, m_2 + b_{N2}\, d_2 + \dots
\end{aligned} \tag{6.13}
$$

where many of the a's and b's are zero because the signal will not get into the higher ordered intervals for small inputs.

Choose the E_i such that $E_i = x_i$, $i = 1,2, \dots, N$; and choose the x_i such that

$$\Delta_i x = x_{i+1} - x_i \equiv \Delta x \tag{6.14}$$

i.e., let the x increment be constant.

The set (6.13) then appears as follows:

$$
\begin{aligned}
g(E_1) &= a_{11}\, m_1 + b_{11}\, d_1 \\
g(E_2) &= a_{21}\, m_1 + b_{21}\, d_1 \\
g(E_3) &= a_{31}\, m_1 + b_{31}\, d_1 + a_{32}\, m_2 + b_{32}\, d_2 \\
g(E_4) &= a_{41}\, m_1 + b_{41}\, d_1 + a_{42}\, m_2 + b_{42}\, d_2 \\
&\vdots
\end{aligned} \tag{6.15}
$$

The set (6.15) may be written in matrix form as

$$\underset{\sim}{M}\, \underset{\sim}{\phi} = \underset{\sim}{g} \tag{6.16}$$

with $\underset{\sim}{g}$ being the describing function data, where

$$\underset{\sim}{g} \overset{\Delta}{=} \left\{ g(E_1),\ g(E_2),\ \dots,\ g(E_N) \right\} \tag{6.17}$$

and

$$\underset{\sim}{\phi} \stackrel{\Delta}{=} \left\{ m_1, d_1, m_2, d_2, \ldots, m_{N/2}, d_{N/2} \right\} \tag{6.18}$$

are N dimensional column vectors, and

$$\underset{\sim}{M} \stackrel{\Delta}{=} \begin{bmatrix} a_{11} & b_{11} & a_{12} & b_{12} & \cdots & a_{1(N/2)} & b_{1(N/2)} \\ a_{21} & b_{21} & a_{22} & b_{22} & \cdots & a_{2(N/2)} & b_{2(N/2)} \\ \vdots & & & & & & \\ a_{N1} & b_{N1} & a_{N2} & b_{N2} & \cdots & a_{N(N/2)} & b_{N(N/2)} \end{bmatrix} \tag{6.19}$$

is an N x N square matrix.

Notice that M will be "quasi-diagonal", the equations (6.15) may be solved pairwise. Therefore

$$\underset{\sim}{M} = \begin{bmatrix} a_{11} & b_{11} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & b_{21} & 0 & 0 & 0 & \cdots & 0 \\ a_{31} & b_{31} & a_{32} & b_{32} & 0 & \cdots & 0 \\ a_{41} & b_{41} & a_{42} & b_{42} & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ a_{N1} & b_{N1} & a_{N2} & b_{N2} & \cdots & a_{N(N/2)} & b_{N(N/2)} \end{bmatrix} \tag{6.20}$$

For the more complex nonlinearities the matrix $\underset{\sim}{M}$ will not have this characteristic appearance.

The use of N/2 segments to approximate the nonlinearity in the right half-plane leads to N equations in N unknowns, requiring N^2 coefficients in the $\underset{\sim}{M}$ matrix and knowledge of the D. F. at the N amplitudes.

$$E_i = x_i \quad , \quad i = 1,2, \ldots, N \tag{6.21}$$

A digital computer program to generate $\underset{\sim}{M}$ and perform the inversion required by (6.16) has been written for N = 50 using the Burroughs Datatron. An important point to make is that the matrix $\underset{\sim}{M}$ does not depend on $\underset{\sim}{g}$, so that $\underset{\sim}{M}$ can be generated once, inverted to get $\underset{\sim}{M}^{-1}$, and $\underset{\sim}{M}^{-1}$ can be stored on tape. One then has

$$\underset{\sim}{\phi} = \underset{\sim}{M}^{-1} \underset{\sim}{q} \tag{6.22}$$

and the production routine need only perform matrix multiplication.

6.3 Examples

In figure 21 the behavior of the inversion technique is indicated for the case of simple ideal saturation. The nonlinearity has a gain of unity, and it saturates at $\pm$ 15. The dotted line is the original nonlinearity, with the solid straight line segments being the results of the piecewise linear inversion. The nonlinearity was deliberately chosen to have a break in the middle of a seg-

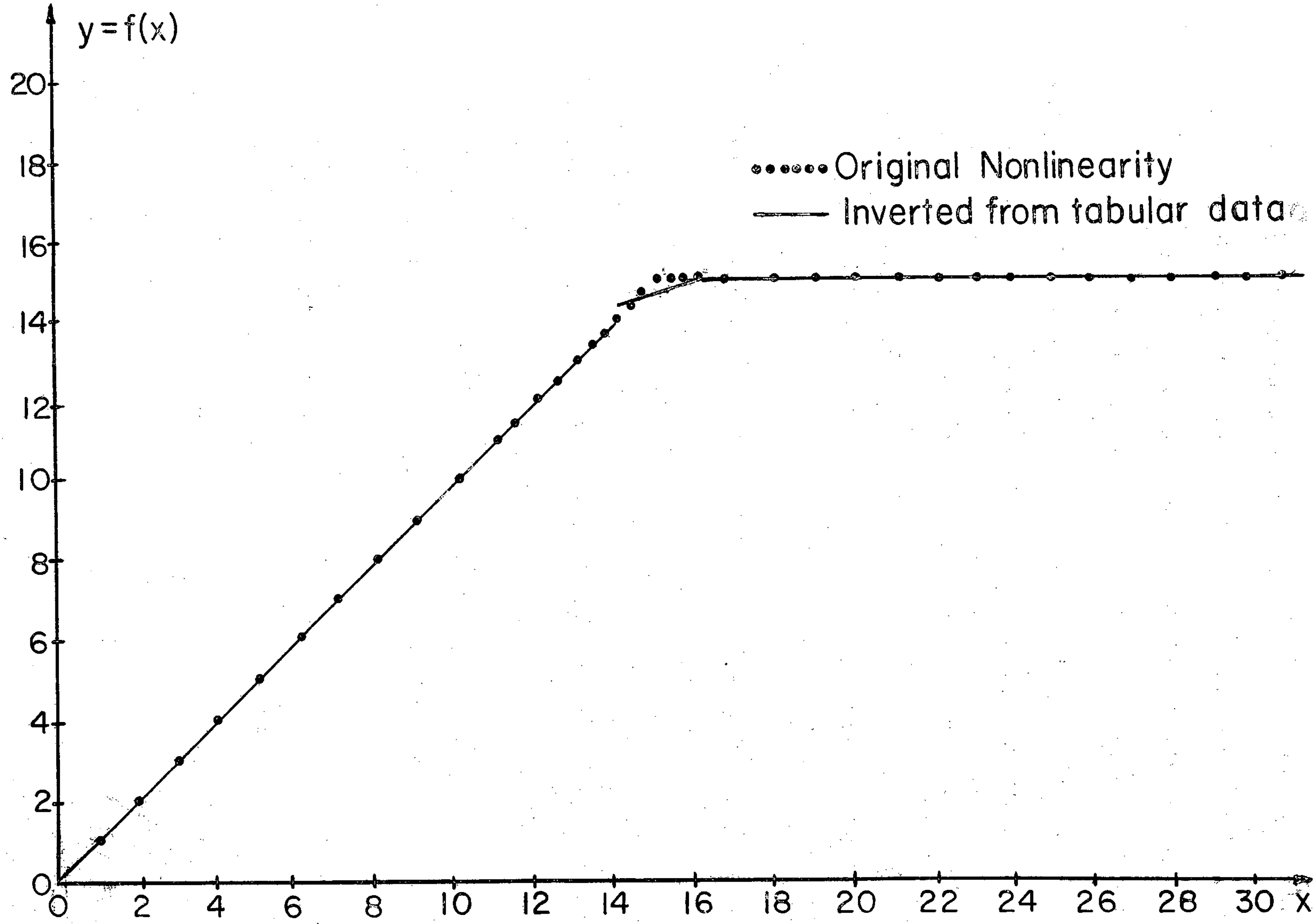

Fig. 21. Inverse Describing Function
Saturation Nonlinearity, Gain = 1, Saturation Level = 15

ment so that the inversion could not possibly match the nonlinearity exactly, in order to see whether or not the resulting approximation would be usable. If the corner had occurred at x = 16, the resulting fit would have been exact. The conclusion is that the piecewise linear approximation is certainly good enough for engineering use, since the only visible error is in the range $14 \leq x \leq 16$. It should be noted that the actual "dynamic range" of the inversion used here extends to + 50, but only a part of this range is shown in figure 21.

In figure 22 a similar presentation is made of the results for a saturating cubic nonlinearity, where again the saturation level was chosen so as to have a sharp corner in the middle of a segment. The fit of the piecewise linear approximation to the cubic part of the nonlinearity is clearly visible, and the only region of error is $10 \leq x \leq 12$. Again the range of inversion extends to + 50.

In figure 23, the nonlinearity is a more difficult relay with dead band, the discontinuity being chosen in the middle of a segment. Here the error near the discontinuity is fairly large, but the inversion solution quickly converges to the desired nonlinearity. The inversion is exact if the discontinuity value is even. The accuracy here could be

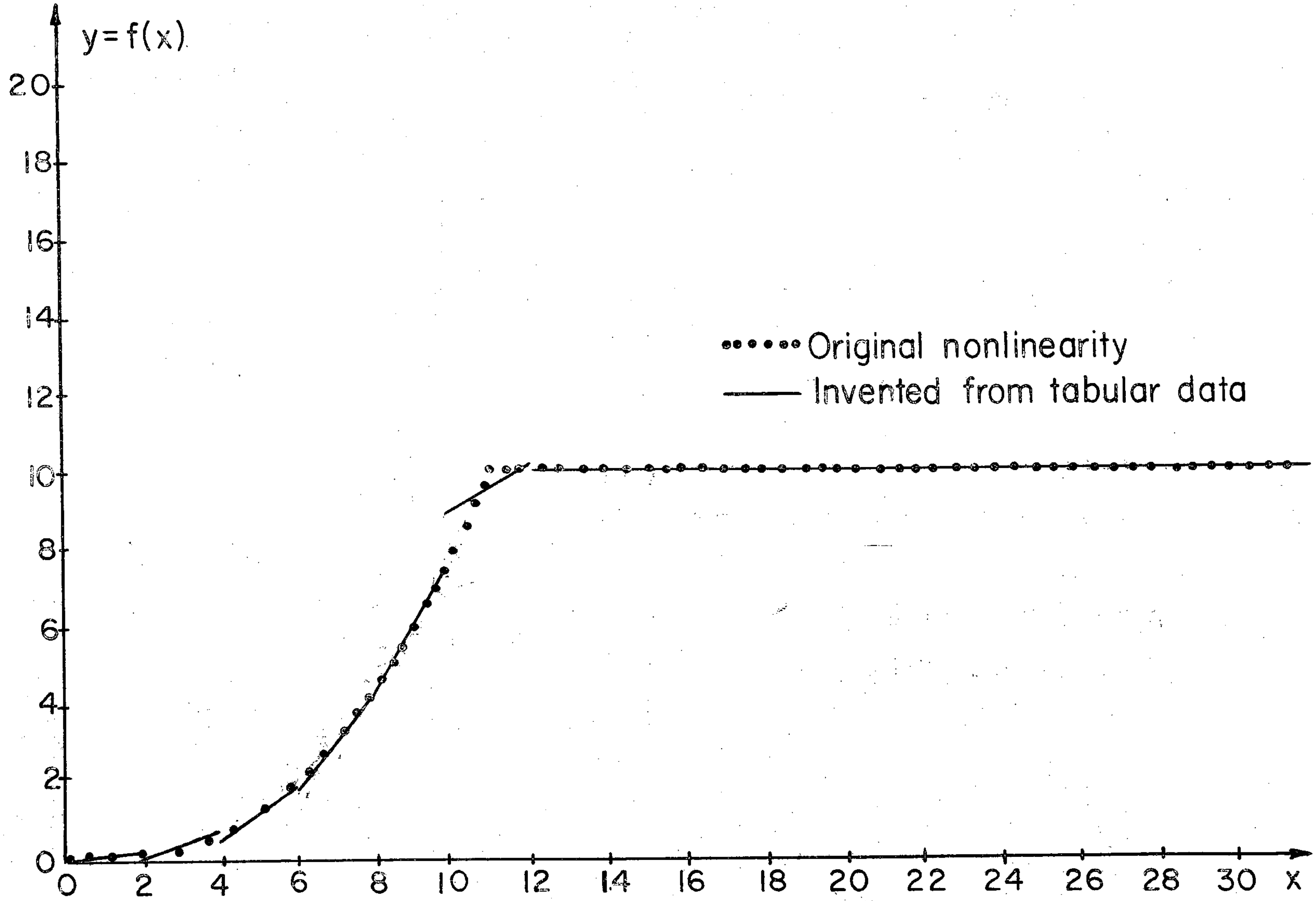

Fig. 22. Inverse Describing Function
Saturating Cubic Nonlinearity, Saturation Level = 10

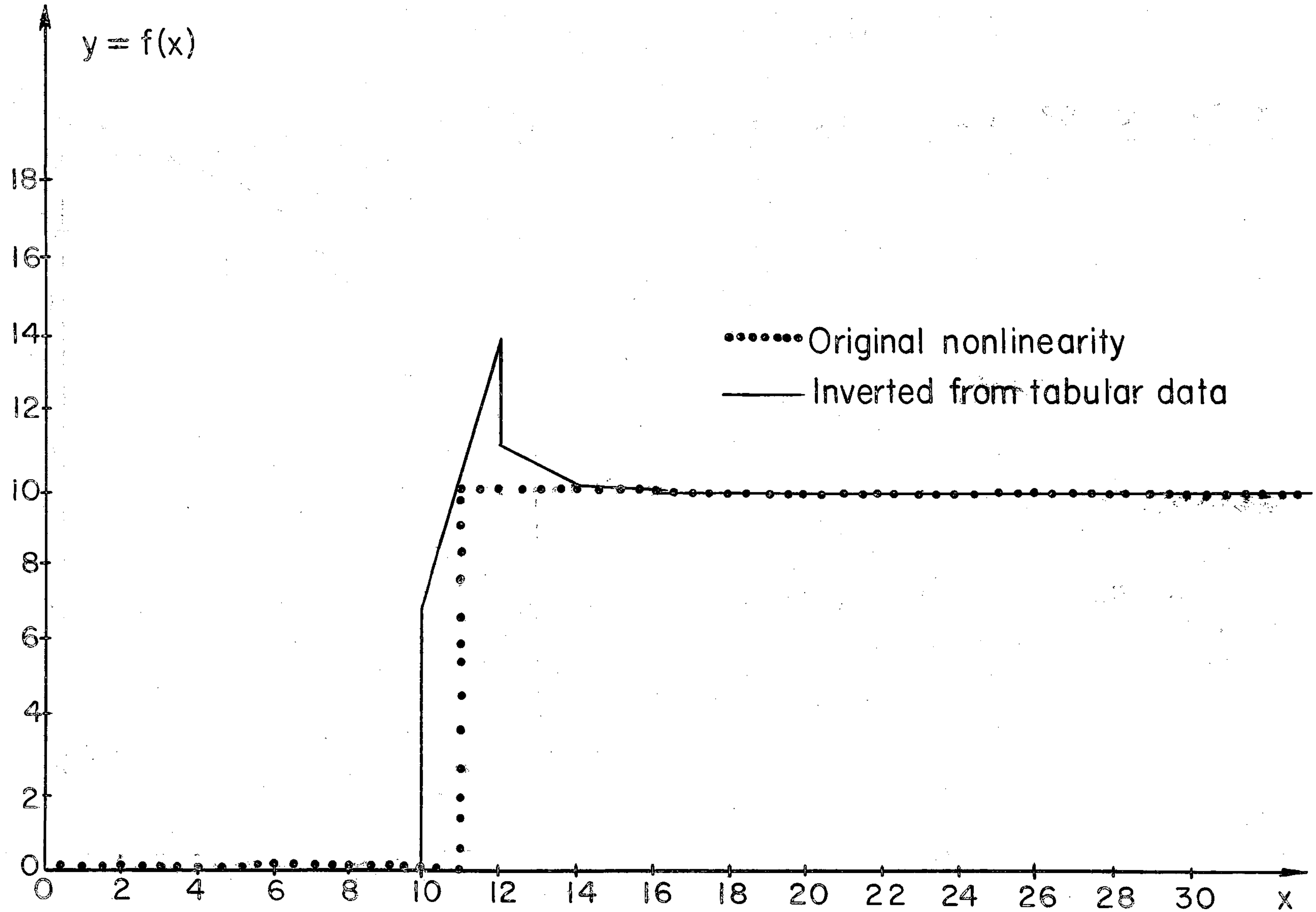

Fig. 23. Inverse Describing Function Relay with Dead Band
Half-dead-band width = 11 |y| = 10 for x 11

markedly improved by grouping more segments near points where the slope of the describing function changes rapidly, i.e., near $x = 11$. The range of inversion extends to + 50.

CHAPTER 7

CONCLUSIONS

7.1 Conclusions

An analytical approach to the inverse-describing-function problem has been developed in this report. To accomplish this task a suitable mathematical model was defined to describe the behavior of a nonlinear element. The definition of this mathematical model was chosen in order to be compatible with physical nonlinear elements. This enhances the practical usefulness of the work. With this mathematical model the mechanism of the describing function has been investigated. The principal result is that the real and the imaginary parts of the describing function of a nonlinear element are related by an integral transformation to a pair of functions that are called $Q(x,E)$ and $P(x, E)$.

These functions are completely determined by the characteristic of the nonlinear element. Sufficient conditions for the existence of the describing function have been deduced.

For the case of memory type nonlinearities it was found that the solution of the integral transformation that relates $g(E)$ as a functional of $Q(x,E)$ is not unique. As a result of this work a method of constructing nonlinearities with a describing function identically null was developed.

For the case of nonmemory type nonlinearities the functions $Q(x,E)$ and $P(x,E)$ do not depend on E. Therefore the functional relationship between the describing function and the functions $Q(x)$ and $P(x)$ is reduced to a Volterra integral equation of the first kind. In general it is not possible to find the solution of a Volterra integral equation in closed form. Nevertheless, for the case of the describing function, an analytical solution was found for $Q(x)$ and $P(x)$ as a function of $g(E)$ and $b(E)$ respectively.

Even if it is true that to any nonlinearity there corresponds one and only one pair of functions $Q(x)$ and $P(x)$, the inverse transformation does not have the same property. To any pair of functions $Q(x)$ and $P(x)$ there corresponds an infinity of nonlinearities. From this was concluded the non-uniqueness of the solution to the inverse-describing-function problem.

Sufficient conditions for the existence of the inverse describing function have been deduced. It is interesting to note that the conditions for the existence of the inverse describing function are more restrictive than the conditions for the existence of the describing function. It was also found that a necessary condition for the existence of a bounded inverse describing function is the continuity of the describing function in the case of nonmemory type nonlinearities.

Nevertheless it was illustrated with one example how a bounded memory-type nonlinearity can be synthesized from a discontinuous describing function.

The original definition of the describing function has been extended to the higher harmonics of the output. These new functions have been called <u>higher describing functions</u>.

It has been found that, when the input of a nonlinear element is a sinusoidal wave, all the odd and even harmonics of the type $B_n \sin n\,\omega t$ $(n = 1,2, \ldots)$ depend on the same functions Q(x) and Q*(x) respectively. The same property has been found for all the odd and even harmonics of the type $A_n \cos n\,\omega t$ $(n = 1,2,3,\ldots)$. They depend on the same functions P(x) and P*(x). The functions Q(x), Q*(x), P(x) and P*(x) are completely determined by the characteristic of the nonlinear element. Therefore, since $g_1(E)$ determines uniquely Q(x), a functional relationship must exist between $g_1(E)$ and all the functions $g_{2n+1}(E)$, $(n=1,2,3,\ldots)$. Similarly, a functional relationship must exist between $b_1(E)$ and all the functions $b_{2n+1}(E)$. Both relationships have been deduced.

The goal of the method developed is to find the analytical expression of the inverse describing function. However, if the describing function has not a simple mathematical expression, the integrals that result from this method are difficult, if not impossible, to perform. Also, in many practical cases the analytical expression of the describing function of the nonlinearity being synthesized is not known. It is known as experimental data or as a result of a graphical or a numerical computation. Thus the necessity of a numerical method of computation for the inverse describing function is apparent. This problem has been solved by approximating the describing function of the nonlinearity being synthesized by a polynomial.

Also the integral that generates the real and imaginary parts of the describing function appears particularly appropriate to a numerical computation. A numerical method to compute the describing function for a general type of nonlinearity has been developed and two other independent numerical techniques have been presented.

Several extensions may be made to the method proposed in this report. First of all the method may be extended to non-conventional describing functions, as the root mean square describing function, the Gaussian input describing function R. C. Booton, [10], etc.(i)

(i)Some unpublished research has been done by E. G. diTada in this area.

In the case of a symmetric nonmemory type nonlinearity the describing function gives the complete information about the nonlinearity. Thus it seems that an exact method of analysis and synthesis may be developed on the basis of the describing function.

Also it seems that a series development of a function in terms of its describing function, or functionals of it, may exist.[i]

The method may also be extended to a more general class of nonlinearities. This type may be the one in which the functional relationship between the input and the output is not restricted to a simple function, but to a differential or difference equation.

If these extensions can be made, a great insight in the analysis, synthesis and identification of nonlinear systems may be gained.

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[i] Some unpublished research has been done by E. G. diTada in this area.

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APPENDICES

APPENDIX 1

Let us consider the following integral equation

$$F(x) = \int_0^x \frac{\phi(z)}{(x^2 - z^2)^\alpha} dz \quad 0 < \alpha < 1 \tag{1}$$

Equation (1) is a general case of (3.3) and (3.4) and a special case of a Volterra integral equation of the first kind. Multiply both sides of (1) by $H(x,\xi)$ and integrate with respect to x between 0 and ξ

$$\int_0^\xi F(x)\, H(x,\xi) dx = \int_0^\xi dx \int_0^x \frac{(z)\, H(x,\xi)}{(x^2 - z^2)^\alpha} dz \tag{2}$$

where $H(x,\xi)$ will be determined later.

If the order of integration in (26) (Dirichlet's formula) is changed, we have

$$\int_0^\xi F(x)\, H(x,\xi) dx = \int_0^\xi \phi(z)\, dz \int_z^\xi \frac{H(x,\xi)}{(x^2 - z^2)^\alpha} dx \tag{3}$$

If $H(x,\xi)$ is chosen properly, the integral

$$I(z,\xi) = \int_z^\xi \frac{H(x,\ )}{(x^2 - z^2)^\alpha} dx \tag{4}$$

can be reduced to a constant.

Make the following change of variable in equation (4)

$$x^2 = y$$

Thus

$$I(\xi, \sqrt{\eta}) = \frac{1}{2}\int_{\eta}^{\xi^2} \frac{P(y,\xi)}{(y-\eta)^{\alpha}}\, dy \tag{6}$$

where

$$z^2 = \eta \tag{7}$$

$$P(y, \xi) = \frac{H(\sqrt{y}, \xi)}{\sqrt{y}} \tag{8}$$

Let us make another change of variable in equation (6)

$$t = \frac{y - \eta}{\xi^2 - \eta} \tag{9}$$

then

$$dt = \frac{dy}{\xi^2 - \eta} \tag{10}$$

and

$$1 - t = \frac{\xi^2 - y}{\xi^2 - \eta} \tag{11}$$

From equations (6), (9), (10) and (11), we find

$$\begin{aligned} I(\xi, \sqrt{\eta}) &= \frac{1}{2}\int_0^1 \frac{P(y, \xi)}{t^{\alpha}(\xi^2 - \eta)^{\alpha-1}}\, dt \\ &= \frac{1}{2}\int_0^1 \frac{P(y, \xi)(1-t)^{\alpha-1}}{t^{\alpha}(\xi^2 - y)^{\alpha-1}}\, dt \end{aligned} \tag{12}$$

Therefore if we choose

$$P(y, \xi) = (\xi^2 - y)^{\alpha-1} \tag{13}$$

the integral (12) is reduced to

$$I(\xi, \sqrt{\eta}) = \frac{1}{2}\int_0^1 t^{-\alpha}(1 - t)^{\alpha-1}\, dt$$

$$= \frac{1}{2} B(1 - \alpha, \alpha)$$

$$= \frac{1}{2} \Gamma(1 - \alpha) \Gamma(\alpha)$$

$$= \frac{1}{2} \frac{\pi}{\sin \alpha\pi} \tag{14}$$

From (5), (7) and (8)

$$H(x, \xi) = \frac{x}{(\xi^2 - x^2)^{1-\alpha}} \tag{15}$$

Using (14) and (15) the equation (3) is reduced

$$\int_0^{\xi} \frac{F(x)\, x}{(\xi^2 - x^2)^{1-\alpha}}\, dx = \frac{\pi}{2 \sin \alpha \pi} \int_0^{\xi} \phi(z)\, dz \tag{16}$$

Therefore

$$\phi(\xi) = \frac{2 \sin \alpha \pi}{\pi} \frac{d}{d\xi} \int_0^{\xi} \frac{x\, F(x)}{(\xi^2 - x^2)^{1-\alpha}}\, dx \tag{17}$$

Equation 17 gives the solution of the integral equation (1).

APPENDIX 2

In Appendix I the solution of the integral equation

$$F(x) = \int_0^x \frac{f(z)}{(x^2 - z^2)^\alpha} dz \qquad 0 < \alpha < 1 \tag{1}$$

was shown to be

$$f(\xi) = \frac{2 \sin \alpha \pi}{\pi} \frac{d}{d\xi} \int_0^\xi \frac{x F(x)}{(\xi^2 - x^2)^{1-\alpha}} dx \tag{2}$$

Therefore for

$$\alpha = \tfrac{1}{2}$$

$$f(z) = z\, h_1(z)$$

$$F(x) = - \frac{m_2(x)}{m_1(x)} \int_0^x \frac{\eta\, h_2(\eta)}{\sqrt{x^2 - \eta^2}} d\eta \tag{3}$$

Equation (2) becomes

$$h_1(x) = - \frac{2}{\pi x} \frac{d}{dx} \int_0^x dz \int_0^z \frac{z\, m_2(z) \eta\, h_2(\eta)\, d\eta}{m_1(z) \sqrt{z^2 - \eta^2} \sqrt{x^2 - z^2}} \tag{4}$$

APPENDIX 3

Let us consider the following summation

$$S_n = (2n+1) \sum_{i=0}^{i=n} \sum_{k=0}^{k=i} (-1)^k \binom{n+i}{2i} \binom{k-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{i-k} \frac{4^i}{2i+1} \qquad (1)$$

The above summation can be rewritten as

$$S_n = (2n+1) \sum_{i=0}^{i=n} \binom{n+i}{2i} \frac{4^i}{2i+1} \sum_{k=0}^{k=i} (-1)^k \binom{k-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{i-k} \qquad (2)$$

But

$$\binom{s-t}{k} = \sum_{j=0}^{j=k} (-1)^j \binom{s}{k-j} \binom{t+j-1}{j} \qquad (3)$$

Therefore

$$\sum_{k=0}^{k=i} (-1)^k \binom{k-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{i-k} = \binom{-1}{i} = (-1)^i \binom{i}{i} = (-1)^i \qquad (4)$$

From (4) and (2)

$$S_n = (2n+1) \sum_{i=0}^{i=n} (-1)^i \binom{n+i}{2i} \frac{4^i}{2i+1} \qquad (5)$$

But

$$\sin(2n+1)\phi = (2n+1)\sum_{i=0}^{i=n} (-1)^i \binom{n+i}{2i} \frac{4^i}{2i+1} \sin^{2i+1}\phi \tag{6}$$

For $\phi = \pi/2$

$$\sum_{i=0}^{i=n} (-1) \binom{n+i}{2i} \frac{4^i}{2i+1} = \frac{(-1)^n}{2n+1} \tag{7}$$

From (7) and (5) we find

$$S_n = (-1)^n \tag{8}$$

APPENDIX 4

Let us consider the summation

$$M_i^k = \sum_{p=i-k}^{i} (-1)^p \binom{i}{p}\binom{2p}{p}\binom{p}{i-k}\frac{1}{4^p} \qquad (1)$$

Make the following transformation:

$$\binom{2p}{p}\frac{1}{4^p} = \binom{p-\frac{1}{2}}{p} \qquad (2)$$

From eqs. (1) and (2)

$$M_i^k = \sum_{p=i-k}^{i} (-1)^p \binom{i}{p}\binom{p}{i-k}\binom{p-\frac{1}{2}}{p} \qquad (3)$$

But

$$\binom{p}{i-k}\binom{i}{p} = \frac{p!}{(i-k)!\,(p-i+k)!} \cdot \frac{i!}{p(i-p)!} = \binom{i}{k}\binom{k}{i-p} \qquad (4)$$

Substituting Eq. (4) into (3)

$$M_i^k = \binom{i}{k} \sum_{p=i-k}^{i} (-1)^p \binom{k}{i-p}\binom{p-\frac{1}{2}}{p} \qquad (5)$$

But

$$\binom{k}{i-p} = 0 \qquad (6)$$

for $i - p > k$ or $p < i - k$.

Therefore the lower limit in (5) can be extended to zero.

$$M_i^k = \binom{i}{k} \sum_{p=0}^{p=i} (-1) \binom{k}{i-p} \binom{p-\frac{1}{2}}{p} = \binom{i}{k} \binom{k-\frac{1}{2}}{i} = \binom{k-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{i-k} \qquad (7)$$

APPENDIX 5

Let us consider the following summation

$$S_n^j = \sum_{i=1}^{i=n-j-1} (-1)^i \binom{2n-i}{i-1}\binom{n-i-1}{j} (n-i) \frac{4^{n-i}}{i} \tag{1}$$

But

$$(n-i)\binom{n-i-1}{j} = (j+1)\binom{n-i}{j+1} \tag{2}$$

From eq. (1) and (2) we obtain

$$S_n^j = (j+1) \sum_{i=1}^{i=n-j-1} (-1)^i \binom{2n-i}{i-1}\binom{n-i}{j+1} \frac{4^{n-i}}{i} \tag{3}$$

But

$$\binom{2n-1}{i-1} = \binom{2n-i}{i} \frac{i}{2n-2i+1} \tag{4}$$

From (4) and (3)

$$S_n^j = (j+1) \sum_{i=1}^{n-j-1} (-1)^i \binom{2n-1}{i}\binom{n-i}{j+1} \frac{4^{n-i}}{2(n-i)+1} \tag{5}$$

Let us make in (5) the following change of the index i

$$p = n - i \tag{6}$$

From (6) and (5)

$$S_n^j = (-1)^n (j+1) \sum_{p=j+1}^{n-1} (-1)^p \binom{n+p}{2p} \binom{p}{j+1} \frac{4^p}{2p+1} \tag{7}$$

Let us define

$$T_n^j = \sum_{p=0}^{n} (-1)^p \binom{n+p}{2p} \binom{p}{j} \frac{4^p}{2p+1} \tag{8}$$

But

$$\sin(2n+1)\,\phi = (2n+1) \sin\phi \sum_{i=0}^{i=n} (-1)^i \binom{n+i}{2i} \frac{4^i}{2i+1} (\sin^2\phi)^i$$

$$= (2n+1) \sin\phi \sum_{i=0}^{n} (-1)^i \binom{n+i}{2i} \frac{4^i}{2i+1} (1-\cos^2\phi)^i$$

(9)

Expanding $(1 - \cos^2\phi)^i$, and rearranging the summation in increasing powers of $\cos\phi$ we find

$$\sin(2n+1)\phi = (2n+1)\sin\phi \sum_{j=0}^{j=n} (-1)^j \cos^{2j}\phi \sum_{i=0}^{i=n} (-1)^i \binom{n+i}{2i}\binom{i}{j} \frac{4^i}{2i+1}$$

$$= (2n+1)\sin\phi \sum_{j=0}^{n} (-1)^j\, T_n^j \cos^{2j}\phi \qquad (10)$$

Therefore from (10)

$$\frac{\sin(2n+1)\phi}{\sin\phi} = (2n+1)\sum_{j=0}^{n} (-1)^j\, T_n^j \cos^{2j}\phi \qquad (11)$$

But

$$\frac{\sin(2n+1)\phi}{\sin\phi}$$

can be expanded in the following fashion

$$\frac{\sin(2n+1)\phi}{\sin\phi} = (-1)^n \sum_{j=0}^{j=n} (-1)^j\, 4^j \binom{n+j}{2j} \cos^{2j} \qquad (12)$$

From Eq. (11) and (12)

$$(2n+1)\, T_n^j = (-1)^n\, 4^j \binom{n+j}{2j} \qquad (13)$$

Now from Eqs. (8) and (7)

$$S_n^j = (-1)^n (j + 1) T_n^{j+1} - (j + 1) \binom{n}{j+1} \frac{4^n}{2n + 1} \quad (14)$$

Substituting Eq. (13) into (14)

$$S_n^j = \frac{1}{2n + 1} \left[4^{j+1} (j + 1) \binom{n+j+1}{2j+2} - n4^n \binom{n-1}{j} \right]$$

APPENDIX 6

Let us consider the integral

$$\int_z^E \frac{x^{2n+1}\,dx}{\sqrt{E^2 - x^2}\,\sqrt{x^2 - z^2}} \tag{1}$$

Make the change of the dummy variable

$$t = \frac{x^2 - z^2}{E^2 - z^2} \tag{2}$$

From (2)

$$dt = \frac{2x\,dx}{E^2 - z^2} \tag{3}$$

$$1 - t = \frac{E^2 - x^2}{E^2 - z^2} \tag{4}$$

From (2) and (4) we find

$$(E^2 - x^2)(x^2 - z^2) = (1 - t)(E^2 - z^2)^2\, t \tag{5}$$

Therefore

$$\sqrt{E^2 - x^2}\,\sqrt{x^2 - z^2} = (E^2 - z^2)\sqrt{t(1-t)} \tag{6}$$

and

$$\frac{x\,dx}{\sqrt{E^2 - x^2}\,\sqrt{x^2 - z^2}} = \frac{dt}{2\sqrt{t(1 - t)}} \tag{7}$$

From (2)

$$x^2 = t(E^2 - z^2) + z^2 \tag{8}$$

Therefore

$$x^{2n} = \left[t(E^2 - z^2) + z^2\right]^n = \sum_{i=0}^{i=n} \binom{n}{i} t^i (E^2 - z^2)^i z^{2(n-i)} \tag{9}$$

Substituting (9) into (1)

$$\int_z^E \frac{x^{2n+1}\,dx}{\sqrt{E^2-x^2}\sqrt{x^2-z^2}} = \tfrac{1}{2} z^{2n} \sum_{i=0}^{i=n} \binom{n}{i} \left[\left(\frac{E}{z}\right)^2 - 1\right]^i \int_0^1 t^{i-\frac{1}{2}} (1-t)^{-\frac{1}{2}}\,dt$$

$$= \tfrac{1}{2} z^{2n} \sum_{i=0}^{i=n} \binom{n}{i} \left[\left(\frac{E}{z}\right)^2 - 1\right]^i \frac{\Gamma(i + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(i + 1)} \tag{10}$$

But

$$\frac{\Gamma(i + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(i + 1)} = \frac{(2i)!}{(i!)^2 4^i}\,\pi = \pi \binom{2i}{i} \frac{1}{4^i} \tag{11}$$

From (11) and (10)

$$\int_z^E \frac{x^{2n+1}\,dx}{\sqrt{E^2 - x^2}\sqrt{x^2 - z^2}} = \frac{\pi}{2} \sum_{i=0}^{i=n} R_i^n \,(E^2 - z^2)^i \, z^{2(n-i)} \qquad (12)$$

where

$$R_i^n = \binom{n}{i}\binom{2i}{i}\frac{1}{4^i} \qquad (13)$$