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DESCRIPTION OF ALL SOLUTIONS OF A LINEAR COMPLEMENTARITY PROBLEM*

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Abstract. Description of all solutions of an $n \times n$ linear complementarity problem $x^+ = Mx^- + q$ in terms of 2^n matrices and their Moore-Penrose inverses is given. The result is applied to describe all solutions of the absolute value equation $Ax + B|x| = b$.

Key words. Linear complementarity problem, Moore-Penrose inverse, Verified solution, Absolute value equation.

AMS subject classifications. 90C33.

1. Introduction. In this paper we consider a linear complementarity problem (LCP) in the form

$$x^+ = Mx^- + q, \tag{1.1}$$

where $M \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$; for $x = (x_i)_{i=1}^n$ the vectors x^+ and x^- are defined by $x^+ = (\max(x_i, 0))_{i=1}^n$, $x^- = (\max(-x_i, 0))_{i=1}^n$, so that $x^+ \geq 0$, $x^- \geq 0$, $(x^+)^T x^- = 0$,

$$x = x^+ - x^- \tag{1.2}$$

and

$$|x| = x^+ + x^-, \tag{1.3}$$

where $|x| = (|x_i|)_{i=1}^n$. The linear complementarity problem has been much studied in the last forty years, as evidenced in the monographs by Cottle, Pang and Stone [2], Murty [4] and Schäfer [7]. The traditional approach, as demonstrated e.g., in Lemke's algorithm [3], looks for *some* solution of (1.1). On the contrary, we are interested here in the description of *all* solutions of (1.1). This is done in full generality in Theorem 2.2 of Section 2, where the Moore-Penrose inverses of 2^n matrices F_z are employed for this purpose. In Proposition 2.4 we show that the description essentially simplifies

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if all the matrices F_z are nonsingular, in which case the LCP (1.1) has at most 2^n solutions. Section 3 contains a 10×10 example which has exactly $2^{10} = 1024$ solutions. In Section 4 we show how the main ideas behind the proof of Theorem 2.2 can be used for the description of all solutions of the absolute value equation $Ax + B|x| = b$ (see Theorem 4.1).

We use the following notation. I is the unit matrix and $e = (1, \dots, 1)^T$ is the vector of all ones. $Z_n = \{z \mid |z| = e\}$ is the set of all ±1-vectors in $\mathbb{R}^n$, so that its cardinality is 2^n. For each $z \in Z_n$ we denote

$$T_z = \operatorname{diag}(z_1, \dots, z_n) = \begin{pmatrix} z_1 & 0 & \dots & 0 \\ 0 & z_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z_n \end{pmatrix}.$$

For a matrix F, $F^\dagger$ denotes its Moore-Penrose inverse (see [1], [5]). We shall utilize its property

$$FF^\dagger F = F. \tag{1.4}$$

2. Main result.

The core of our approach consists in reformulating the LCP (1.1) as an absolute value equation.

PROPOSITION 2.1. *A vector $x \in \mathbb{R}^n$ is a solution of the linear complementarity problem (1.1) if and only if it solves the equation*

$$\tfrac{1}{2}(I + M)x + \tfrac{1}{2}(I - M)|x| = q. \tag{2.1}$$

Proof. Let x solve (1.1). From (1.2) and (1.3) we have $x^+ = \frac{1}{2}(|x| + x)$ and $x^- = \frac{1}{2}(|x| - x)$, which, when substituted into (1.1), gives (2.1). Conversely, (2.1) in the light of (1.2) and (1.3) implies (1.1). ∎

Let us denote the solution set of the LCP (1.1) by X, i.e.,

$$X = \{\, x \mid x^+ = Mx^- + q \,\}.$$

Our main result below gives a description of the solution set in the general case. It builds on the ideas of Penrose's description of the solution set of a system of linear equations [6].

THEOREM 2.2. *For any $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ the solution set X of (1.1) is given by*

$$X = \{\, F_z^\dagger q + G_z y \mid T_z G_z y \geq -T_z F_z^\dagger q,\ H_z q = 0,\ y \in \mathbb{R}^n,\ z \in Z_n \,\}, \tag{2.2}$$

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where

$$F_z = \tfrac{1}{2}((I+M)+(I-M)T_z), \tag{2.3}$$
$$G_z = I - F_z^\dagger F_z, \tag{2.4}$$
$$H_z = I - F_z F_z^\dagger \tag{2.5}$$

for each $z \in Z_n$.

Proof. Let x solve (1.1), then, by Proposition 2.1, it also solves (2.1). Set $z_i = 1$ if $x_i \geq 0$ and $z_i = -1$ otherwise ($i = 1, \ldots, n$), then $z \in Z_n$ and $T_z x = (z_i x_i)_{i=1}^n = (|x_i|)_{i=1}^n = |x| \geq 0$. Substituting $|x| = T_z x$ into (2.1), we get that x satisfies

$$\tfrac{1}{2}((I+M)+(I-M)T_z)x = q, \tag{2.6}$$

i.e.,

$$F_z x = q, \tag{2.7}$$

where F_z is given by (2.3). Then

$$H_z q = (I - F_z F_z^\dagger)q = (I - F_z F_z^\dagger)F_z x = (F_z - F_z F_z^\dagger F_z)x = 0 \tag{2.8}$$

because of (1.4). Now set $y = x - F_z^\dagger q$, then we have

$$F_z y = F_z x - F_z F_z^\dagger q = q - F_z F_z^\dagger q = H_z q = 0$$

by (2.8), therefore x can be written as

$$x = F_z^\dagger q + y = F_z^\dagger q + (I - F_z^\dagger F_z)y = F_z^\dagger q + G_z y,$$

and $T_z x \geq 0$ implies that y satisfies

$$T_z G_z y \geq -T_z F_z^\dagger q.$$

In this way we have proved that

$$X \subseteq \{\, F_z^\dagger q + G_z y \mid T_z G_z y \geq -T_z F_z^\dagger q,\ H_z q = 0,\ y \in \mathbb{R}^n,\ z \in Z_n \,\}$$

holds. To prove the converse inclusion, let x be of the form $x = F_z^\dagger q + G_z y$ for some $y \in \mathbb{R}^n$ and $z \in Z_n$ satisfying $T_z G_z y \geq -T_z F_z^\dagger q$ and $H_z q = 0$. Then

$$F_z x = F_z F_z^\dagger q + F_z G_z y = q - H_z q + (F_z - F_z F_z^\dagger F_z)y = q$$

and

$$T_z x = T_z F_z^\dagger q + T_z G_z y \geq 0,$$

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hence x solves (2.6) and since $T_z x = |x|$, it satisfies (2.1) and thus also (1.1). This concludes the proof of (2.2). □

Let us note that the columns of a matrix F_z can be expressed by

$$(F_z)_{\cdot j} = \begin{cases} I_{\cdot j} & \text{if } z_j = 1, \\ M_{\cdot j} & \text{if } z_j = -1 \end{cases} \qquad (j = 1, \dots, n).$$

Taking into account the singularity/nonsingularity of F_z, we can bring the description of X to a more complex, but also a more specific form.

Proposition 2.3. *For any $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ the solution set X of (1.1) is given by*

$$\begin{aligned} X = \{\, & F_z^\dagger q + G_z y \mid F_z \text{ singular}, T_z G_z y \ge -T_z F_z^\dagger q,\ H_z q = 0,\ y \in \mathbb{R}^n,\ z \in Z_n \,\} \cup \\ & \{\, F_z^{-1} q \mid F_z \text{ nonsingular}, T_z F_z^{-1} q \ge 0,\ z \in Z_n \,\}, \end{aligned} \tag{2.9}$$

where F_z, G_z, H_z are as in Theorem 2.2.

Proof. If F_z is nonsingular for some $z \in Z_n$, then we have $F_z^\dagger = F_z^{-1}$, $G_z = 0$ and $H_z = 0$, hence (2.2) becomes (2.9). □

In particular, if each matrix F_z, $z \in Z_n$, is nonsingular, we have the following simplified description of X.

Proposition 2.4. *Let $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. If each F_z, $z \in Z_n$, is nonsingular, then*

$$X = \{\, F_z^{-1} q \mid T_z F_z^{-1} q \ge 0,\ z \in Z_n \,\}. \tag{2.10}$$

Hence, if each F_z, $z \in Z_n$, is nonsingular, then the linear complementarity problem (1.1) has a finite number of solutions (at most 2^n). It is easy to show that this upper bound can really be attained. Consider the LCP

$$x^+ = -x^- + e, \tag{2.11}$$

which, in view of (1.3), is equivalent to

$$|x| = e.$$

This shows that the solution set X of (2.11) consists of all the ± 1-vectors, i.e., $X = Z_n$. A less obvious example is given in the next section.

Finally we give a sufficient condition for (1.1) to have infinitely many solutions.

Proposition 2.5. *Let $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. If $T_z F_z^\dagger q > 0$, $G_z \ne 0$ and $H_z q = 0$ for some $z \in Z_n$, then (1.1) has infinitely many solutions.*

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Proof. Under the assumptions, the inequality $T_zG_zy \geq -T_zF_z^{\dagger}q$ has not only the solution $y = 0$, but also a whole neighborhood of it, hence by Theorem 2.2 there are infinitely many solutions to (1.1). □

3. Example. At the author's web page [10] there is a freely available verification software package VERSOFT written in MATLAB, currently consisting of more than 50 verification programs. One of them, called VERLCPALL, is dedicated to the present problem; it can be directly assessed at [9]. Its syntax is

```
[X,all]=verlcpall(M,q)
```

where M, q are the data of (1.1) and X is a matrix whose columns are interval vectors each of whom is guaranteed to contain a solution of (1.1). The parameter *all* satisfies $all = 1$ if it is verified that all solutions have been found, and $all = -1$ otherwise.

Consider the following example with random (but somewhat structured) data

```
>> n=10; M=-eye(n,n)+0.03*(2*rand(n,n)-1); M=round(100*M)
M =
  -100    -1    -2     2    -1     3     2    -1     2     3
     2   -99     2     1     2     1    -2    -1     1     1
     0     0  -101     0    -1     2     3    -1    -2     3
    -2     2    -1  -102     0    -3    -1     2    -2    -2
    -1     3     2     1   -99    -1    -1     0     2    -2
    -1    -1     0    -3    -2  -102     3     1     2    -1
     3     0     2     0     0     0  -100    -1     1     2
     1    -2    -3    -1     2     0     1  -101    -2     2
    -1     1    -1    -2     0     3     1     1  -102    -2
    -3    -2     0     0     2    -3    -2    -2     1   -97

>> q=rand(n,1); q=round(100*q)
q =
    32
    21
    74
    41
    48
    32
    10
    87
    86
    96
```

Running the program gives the following result:

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```
>> tic, [X,all]=verlcpall(M,q); sols=size(X,2), all, toc
sols =
        1024
all =
     1
Elapsed time is 9.563235 seconds.
```

Here, *sols* is the number of columns of X, i.e., the number of solutions found. We see that $sols = 1024 = 2^{10}$, and "$all = 1$" indicates that all the solutions have been found, all of them verified. The rather long computation time is due to the verification procedure involved. We have suppressed the output of X since it is a 10×1024 interval matrix. But we can look e.g., at the last solution:

```
>> X(:,1024)
intval ans =
[  -0.32628969214138,  -0.32628969214137]
[  -0.24832082740008,  -0.24832082740006]
[  -0.71247684314471,  -0.71247684314469]
[  -0.38263062365052,  -0.38263062365051]
[  -0.52000025193934,  -0.52000025193933]
[  -0.31502584347121,  -0.31502584347120]
[  -0.12420810850764,  -0.12420810850763]
[  -0.82951280893732,  -0.82951280893731]
[  -0.84650089698698,  -0.84650089698697]
[  93.55847130433784,  93.55847130433789]
```

This interval vector is guaranteed to contain a solution of (1.1). Observe the high accuracy of the result.

4. The equation $Ax+B|x| = b$. The method employed in the proof of Theorem 2.2 can be extended to describe all solutions of the absolute value equation

$$Ax + B|x| = b \tag{4.1}$$

($A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$), which is more general but less frequently used than (1.1). Denote

$$X_a = \{\, x \mid Ax + B|x| = b \,\}.$$

Then we have this description.

THEOREM 4.1. *For any $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ the solution set X_a of (4.1) is given by*

$$X_a = \{\, F_z^{\dagger} b + G_z y \mid T_z G_z y \geq -T_z F_z^{\dagger} b,\, H_z b = 0,\, y \in \mathbb{R}^n,\, z \in Z_n \,\},$$

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where

$$F_z = A + BT_z,$$
$$G_z = I - F_z^\dagger F_z,$$
$$H_z = I - F_z F_z^\dagger$$

for each $z \in Z_n$.

Proof. If x solves (4.1), then it satisfies $(A+BT_z)x = b$, where z is the sign vector of x, hence

$$F_z x = b.$$

The rest of the proof runs exactly as in Theorem 2.2 (with q replaced by b) because from (2.7) on, it does not depend on the actual form of F_z. □

We do not formulate here the analogues of Propositions 2.3, 2.4 and 2.5 as they are obvious. We note in passing that VERSOFT [10] also contains a program VERABSVALEQNALL for finding and verifying all solutions of (4.1); it works similarly to VERLCPALL and can be directly assessed at [8].

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