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TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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DESCRIPTION OF STRESS-STRAIN CURVES BY THREE PARAMETERS

By Walter Ramberg and William R. Osgood
National Bureau of Standards

FOR REFERENCE

Washington
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DESCRIPTION OF STRESS-STRAIN CURVES BY THREE PARAMETERS

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SUMMARY

A simple formula is suggested for describing the stress-strain curve in terms of three parameters: namely, Young's modulus and two secant yield strengths. Dimensionless charts are derived from this formula for determining the stress-strain curve, the tangent modulus, and the reduced modulus of a material for which these three parameters are given. Comparison with the tensile and compressive data on aluminum-alloy, stainless-steel, and carbon-steel sheet in NACA Technical Note No. 840 indicates that the formula is adequate for most of these materials. The formula does not describe the behavior of alclad sheet, which shows a marked change in slope at low stress. It seems probable that more than three parameters will be necessary to represent such stress-strain curves adequately.

INTRODUCTION

An assembly of the tensile and compressive stress-strain curves for sheet materials characteristic of aircraft construction is being obtained at the National Bureau of Standards as the principal objective of a research project for the National Advisory Committee for Aeronautics. Stress-strain, stress-deviation, secant-modulus, tangent-modulus, and reduced-modulus curves have been presented in reference 1 for various grades of sheet materials of aluminum alloy, carbon steel, and chromium-nickel steel. A second objective of the same research project is a search for yield parameters that give a better description of the stress-strain curve than those in use at present.

The conventional description of the stress-strain curve of metals by the two parameters, Young's modulus and yield strength, is inadequate for the efficient design of members unless the material follows Hooke's law up to a

yield point at which it yields indefinitely under constant stress. This special behavior is approached, for example, by certain steels (fig. 1) and by certain low-strength magnesium alloys, but it is not characteristic of many high-strength alloys for aircraft.

Examination of the stress-strain curves for aluminum-alloy sheet and chromium-nickel-steel sheet given in reference 1 shows, particularly for the compressive stress-strain curves (figs. 2 and 3), a gradual transition from the elastic straight line for low loads toward the horizontal line characterizing plastic behavior. The type of transition varies widely. Hence there is no hope of reducing all stress-strain curves to a single type of curve by uniform stretching, or affine transformation of coordinates. This rules out the possibility, which exists for affinely related stress-strain curves (reference 2), of complete description in terms of only two parameters, Young's modulus and secant yield strength. A minimum of three parameters will be required to describe the changes in shape for different materials.

Several proposals have been made for describing the stress-strain curve in terms of three or more parameters. Donnell (reference 3) suggests as two yield parameters the stresses s_1, s_2, at which the slope of the stress-strain curve is equal to $3/4\ E$ and $1/4\ E$, where E is Young's modulus. The stress-strain curve is then derived from these two parameters on the assumption that the slope varies linearly with the stress. This procedure gives a good description of many tensile stress-strain curves of aluminum alloys, but it does not seem adequate for the highly curved tangent-modulus curves found for the compressive stress-strain properties in reference 1, from which figure 4 is taken. Furthermore there are practical difficulties in determining the stresses corresponding to a tangent modulus of $3/4\ E$ and $1/4\ E$ quickly from the stress-strain curve.

Esser and Ahrend (reference 4) noticed that the stress-strain curves for many materials may be approximated by two straight lines when they are plotted on log-log paper. They proposed to define yield strength as the stress at the intersection of these two lines. Description of the stress-strain curve above the yield strength would be obtained from the slope of the upper straight line. The proposal is doubtless an advance over the description by an offset yield strength. It has the

disadvantage, however, of requiring the plotting of sufficient stress-strain data on log-log paper to determine a straight line through the points. Furthermore it gives no information about the shape of the important transition region near the intersection of the two straight lines.

An analytical expression for the stress-strain curve which is suited for theoretical studies of plastic buckling was proposed by Nadai in 1939 (reference 5). The expression is

$$\left.\begin{aligned} e &= \frac{s}{E} && s < s_p \\ e &= \frac{s}{E} + e_y\left(\frac{s - s_p}{s_y - s_p}\right)^n && s > s_p \end{aligned}\right\} \qquad (1)$$

where

e strain

s stress

e_y strain corresponding to yield strength s_y

s_p proportional limit

n constant

If the logarithm of both sides is taken in equation (1), it can be seen that equation (1) approaches Esser and Ahrend's two straight lines as asymptotes for low and for high stress, respectively. The description of the transition region is obtained by increasing the number of parameters from three to four.

In the study of plastic bending, the second author found an analytical expression containing three parameters that appeared to be well adapted for representing stress-strain curves. Further examination of the expression in the light of the data given in reference 1 confirmed this view.

ANALYTICAL EXPRESSION

Stress-Strain Curve

The proposed analytical expression is

$$e = \frac{s}{E} + K\left(\frac{s}{E}\right)^n \tag{2}$$

where K and n are constants. Nadai's expression (1) becomes the same as equation (2) if

$$\left.\begin{aligned} s_p &= 0 \\ * \quad e_y\left(\frac{E}{s_y}\right)^n &= K \end{aligned}\right\} \tag{3}$$

that is, if the proportional limit is taken as zero, and the requirement is dropped that e_y is the strain corresponding to a yield stress s_y.

The expression (2) may be written in dimensionless form in terms of the following variables (reference 6):

$$\left.\begin{aligned} \epsilon &= \frac{eE}{s_1} \\ \sigma &= \frac{s}{s_1} \end{aligned}\right\} \tag{4}$$

where s_1 is the secant yield strength, equal to the ordinate of the intersection with the stress-strain curve of a line through the origin having a slope equal to $m_1 E$ (fig. 5), m_1 being a chosen constant, $0 < m_1 < 1$. Since m_1 is fixed, the transformation (equation (4)) reduces all affinely related stress-strain curves to a single curve.

The abscissa of the intersection for a stress-strain curve described by equation (2) is

$$e_1 = \frac{s_1}{m_1 E} = \frac{s_1}{E} + K\left(\frac{s_1}{E}\right)^n \tag{5}$$

Inserting equation (4) in equation (2) gives

$$\epsilon = \sigma + K\left(\frac{s_1}{E}\right)^{n-1} \sigma^n \tag{6}$$

From equation (5)

$$K\left(\frac{s_1}{E}\right)^{n-1} = \frac{1 - m_1}{m_1} \tag{7}$$

Inserting equation (7) in equation (6) gives

$$\epsilon = \sigma + \frac{1 - m_1}{m_1} \sigma^n \tag{8}$$

Affinely related stress-strain curves that may be described by equation (8) are characterized by having the same value of n. Figure 6 shows a family of curves for a number of different values of n, and $m_1 = 0.7$.

Stress-Deviation Curve

The stress-deviation curve is obtained by plotting stress against difference between measured strain and elastic strain corresponding to Hooke's law. For the stress-strain curve given by equation (2) the deviation d is given by

$$d = e - \frac{s}{E} = K\left(\frac{s}{E}\right)^n \tag{9}$$

or

$$\log d = \log K + n \log \frac{s}{E} \tag{10}$$

that is, a log-log plot of deviation against stress would be a straight line. The deviation may be written in dimensionless form as

$$\delta = \frac{Ed}{s_1} \tag{11}$$

From equations (9), (4), and (8)

$$\delta = \epsilon - \sigma = \frac{1 - m_1}{m_1} \sigma^n \tag{12}$$

or

$$\log \delta = \log \frac{1 - m_1}{m_1} + n \log \sigma \tag{13}$$

The family of straight lines corresponding to various values of n and to $m_1 = 0.7$ is shown in figure 7.

Tangent Modulus

The tangent modulus at a given stress is defined as the slope of the tangent to the stress-strain curve at that stress. The reciprocal of the tangent modulus is from equation (2):

$$\frac{1}{E'} = \frac{de}{ds} = \frac{1}{E} + \frac{nK}{s}\left(\frac{s}{E}\right)^n \tag{14}$$

This may be written in dimensionless form by making use of equations (7) and (4):

$$\frac{E}{E'} = 1 + \frac{n(1 - m_1)}{m_1} \sigma^{n-1} \tag{15}$$

Figure 8 shows the tangent-modulus ratio E'/E plotted against stress ratio σ, with $m_1 = 0.7$.

Reduced Modulus for Rectangular Section

The reduction in buckling stress when the stress exceeds the proportional limit is frequently estimated (reference 7, pp. 159 and 274) by replacing Young's modulus E by a reduced modulus E_r. Thus in the case of columns the actual buckling stress s_r would be estimated as

$$s_r = \frac{E_r}{E} s_e \tag{16}$$

where s_e is the buckling stress computed from elastic theory and, for columns of rectangular cross section, from reference 8,

$$E_r = \frac{4EE'}{(\sqrt{E} + \sqrt{E'})^2} \tag{17}$$

Dividing equation (17) by E gives

$$\frac{E_r}{E} = \frac{4E'/E}{(1 + \sqrt{E'/E})^2} \tag{18}$$

Figure 9 shows the reduced-modulus ratio E_r/E plotted against stress ratio σ for different values of n and with $m_1 = 0.7$.

Equation (16) may be solved for E_r/E as follows:

Let $$\sigma_e = \frac{s_e}{s_1}; \qquad \sigma_r = \frac{s_r}{s_1} \tag{19}$$

so that equation (16) becomes

$$\frac{\sigma_r}{\sigma_e} = \frac{E_r}{E} \tag{20}$$

Thus, the desired value of E_r/E is the ordinate of the intersection of the straight line $E_r/E = \sigma/\sigma_e$ with the reduced modulus curve, equation (18), for the material (n) in question. Straight lines with the given slope $1/\sigma_e$ may be drawn conveniently by connecting the origin with the proper point on the circular curve in figure 9.

DERIVATION OF EMPIRICAL CONSTANTS FROM STRESS-STRAIN CURVE

The adequacy of equation (2) was tested by plotting on log-log paper the stress-deviation curves for the sheet materials given in reference 1. The points should lie on a straight line according to equation (10) if equation (2)

is an accurate description of the stress-strain curve. From the slope and intercept of such a straight line the constants K and n can be determined for a best fit.

Straight lines were obtained for all the compressive stress-deviation curves and for all but four of the tensile curves for stresses greater than the stress at which the secant modulus was equal to 90 percent of Young's modulus. The exceptions had stress-strain curves which had a gradual change of slope throughout their entire length. This indicates that any value of m_1 less than 0.90 would give an approximate fit to compressive stress-strain curves and to most tensile stress-strain curves at stresses above that corresponding to $m_1 = 0.90$.

It appeared desirable to choose the value of m_1 such that the secant yield strength s_1 would approximate the widely used yield strength $s_{0.2}$ for 0.2-percent offset. In other words, s_1 should be chosen to satisfy approximately:

$$0.002 = e - \frac{s_1}{E} \tag{21}$$

where (see fig. 5)

$$e = \frac{s_1}{m_1 E} \tag{21a}$$

Inserting equation (21a) in equation (21) and solving for $1/m_1$ gives

$$\frac{1}{m_1} = 1 + \frac{0.002}{s_1/E} = 1 + \frac{0.002}{s_{0.2}/E} \tag{21b}$$

Examination of tables III and IV of reference 1 gave values of $s_{0.2}/E$ for the aluminum alloys and the chromium-nickel steels which ranged from 0.00258 to 0.00675; the 1025 carbon steel in reference 1 was not included because of its relatively low value of this ratio. The average value was

$$\frac{s_{0.2}}{E} = 0.00486 \tag{21c}$$

Substituting this average in equation (21b) and solving for m_1, gives

$$m_1 = 0.709 \tag{21d}$$

It was decided, therefore, to use for m_1 the value

$$m_1 = 0.7 \tag{22}$$

for determining the secant yield strength s_1.

When E is known and s_1 has been determined, it is still necessary to know the shape parameter n in order to establish the shape of the stress-strain curve according to equations (8) and (4).

The shape parameter n is conveniently derived by the use of a second secant yield strength s_2, corresponding to a second secant modulus m_2E, as follows. In analogy to equations (5) and (7),

$$e_2 = \frac{s_2}{m_2 E} = \frac{s_2}{E} + K\left(\frac{s_2}{E}\right)^n \tag{23}$$

$$\frac{1}{m_2} = 1 + K\left(\frac{s_2}{E}\right)^{n-1} \tag{24}$$

Solving both equations (7) and (23) for K gives

$$K = \left(\frac{1}{m_1} - 1\right)\left(\frac{s_1}{E}\right)^{1-n} = \left(\frac{1}{m_2} - 1\right)\left(\frac{s_2}{E}\right)^{1-n} \tag{25}$$

so that

$$\left(\frac{s_1}{s_2}\right)^{1-n} = \frac{\frac{1}{m_2} - 1}{\frac{1}{m_1} - 1} \tag{26}$$

Solving for n gives

$$n = 1 + \frac{\log \frac{m_2}{m_1}\frac{1 - m_1}{1 - m_2}}{\log \frac{s_1}{s_2}} \tag{27}$$

The value of m_2 for the second secant yield strength was chosen as

$$m_2 = 0.85 \tag{28}$$

since this value lies midway between 0.7 and 1.0 and since it is on the safe side of the limiting value $m = 0.90$ up to which equation (2) is an adequate description of most of the stress-strain curves in reference 1. Substituting equations (22) and (28) in equation (27) gives:

$$n = 1 + \frac{\log \frac{17}{7}}{\log \frac{s_1}{s_2}} = 1 + \frac{0.3853}{\log_{10} \frac{s_1}{s_2}} \tag{29}$$

A plot of this relation on log-log paper is given in figure 10.

COMPARISON WITH EXPERIMENTAL STRESS-STRAIN AND TANGENT-MODULUS CURVES

The order of approximation with which equation (2) describes the stress-strain curve for materials with values of n from 3.08 to ∞ is brought out in figures 11 to 13, in which the data were taken at random from reference 1. The approximation appears to be adequate for most practical purposes. In these figures E, s_1, and s_2 were obtained from the stress-strain data, and n was computed from equation (27). Equation (8) then gave the relation between ϵ and σ, and the relation between e and s was obtained by multiplying ϵ by s_1/E and σ by s_1. A better fit would have been obtained if n and K had been determined from a plot of the data, as explained in the first paragraph of the preceding section (p. 7); but the procedure used is simpler and probably adequate in most cases.

A much more severe test of the adequacy of equation (2), than the comparison with the stress-strain curve, is a comparison with the tangent modulus - that is, the slope of the stress-strain curve. Such a comparison seems to be advisable since the tangent modulus must be computed for evaluating the reduced modulus in compression. (See equation (17).)

The tangent moduli in compression of reference 1 are plotted on a dimensionless basis in figures 14 to 21 together with computed moduli as given by figure 8. The value of n from equation (29) for each material is given in figures 14 to 21. The computed moduli are shown for integral values of n and for $n = 2.5$. To appreciate the closeness of fit, therefore, it is necessary to interpolate between the curves by using the particular value of n applying to the plotted data. Except for the curves with a very sharp knee ($n > 10$) the experimental values of tangent modulus for stresses below the secant yield strength s_1 differ less than $\pm 0.07\ E$ from the values corresponding to equation (15). In the case of the values with the sharp knee (fig. 21) the maximum difference was considerably greater. These differences do not detract seriously from the usefulness of equation (2), however, since the region in which the agreement is not good comprises a limited stress range. Consequently, in this range the difference between the experimental and the computed values of σ corresponding to a given value of tangent modulus are small.

The comparison was confined to the materials in reference 1, which did not include alclad aluminum alloys. In the alclad aluminum alloys a change in slope at low stress is observed which corresponds to the yielding of the aluminum coating. It seems probable that inclusion of this effect will require the addition of at least one more parameter to the three contained in equation (2).

EXAMPLE FOR APPLICATION OF THREE-PARAMETER METHOD

Computations based on elastic theory give a value of

$$s_e = 87 \times 10^3 \text{ pounds per inch}^2$$

for the critical compressive stress of a given specimen. The material of the specimen has the compressive stress-strain curve shown in figure 22. It is desired to determine the stress:

$$s_r = \frac{E_r}{E} s_e$$

which is an estimate for the critical stress after taking account of the plastic yielding of the material.

From figure 22 are obtained the two secant yield strengths

$$s_1 = 43.0 \times 10^3 \text{ pounds per inch}^2$$

$$s_2 = 38.0 \times 10^3 \text{ pounds per inch}^2$$

so that

$$\frac{s_1}{s_2} = 1.132$$

From figure 10 this corresponds to a shape parameter

$$n = 8.15$$

Entering figure 9 with this value of n and with the ratio

$$\frac{s_1}{s_e} = \frac{43.0}{87.0} = 0.494$$

gives

$$\frac{E_r}{E} = 0.473$$

so that the corrected critical stress is

$$s_r = 0.473 \times 87 \times 10^3 = 41.2 \times 10^3 \text{ pounds per inch}^2$$

National Bureau of Standards,
Washington, D. C., April 8, 1943.

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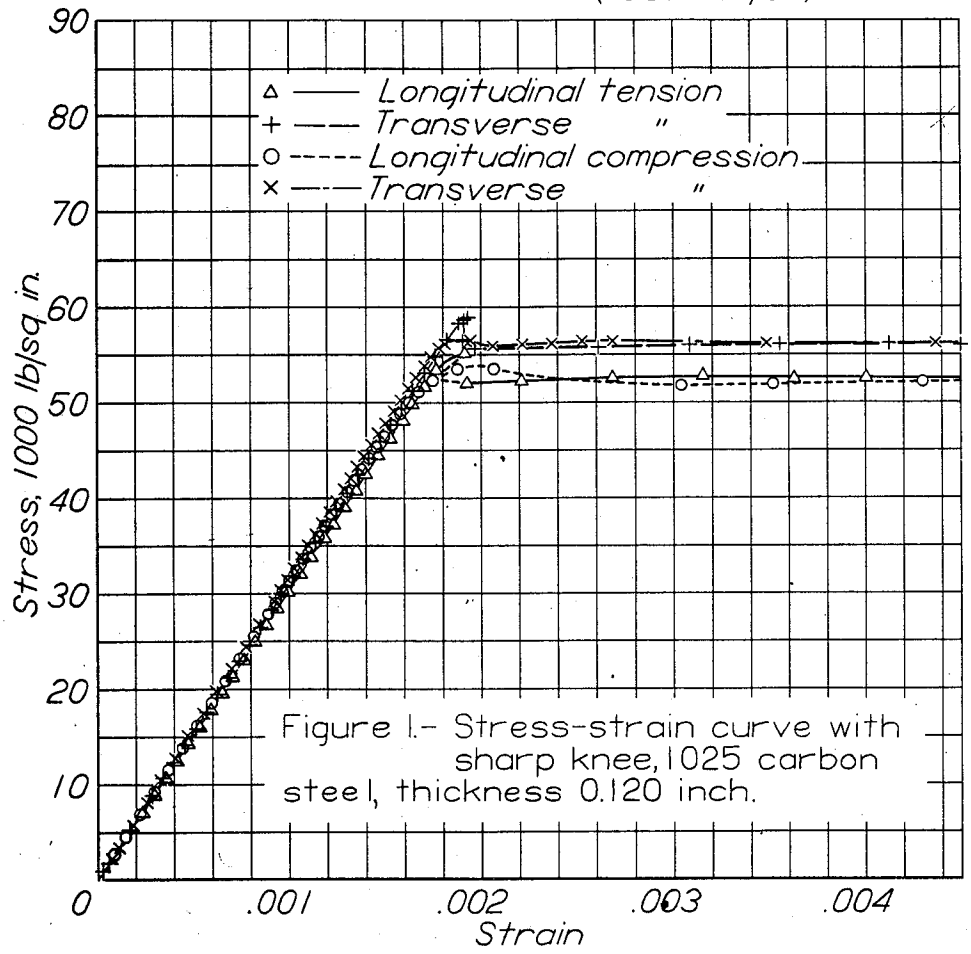


Figure 1.- Stress-strain curve with sharp knee, 1025 carbon steel, thickness 0.120 inch.

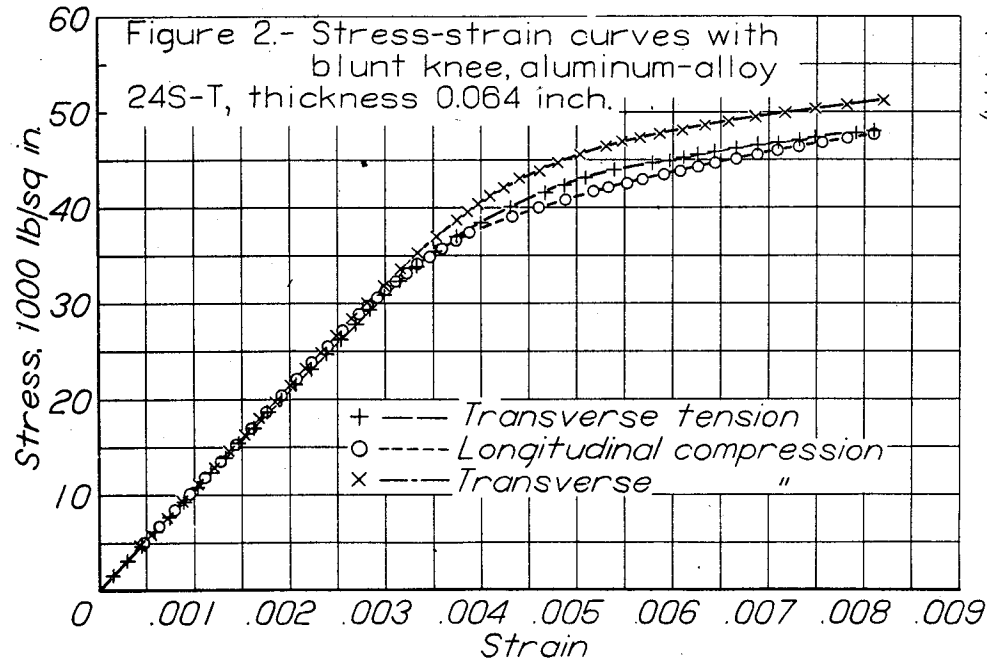


Figure 2.- Stress-strain curves with blunt knee, aluminum-alloy 24S-T, thickness 0.064 inch.

(1 block = 10 divisions on 1/40 Engr. scale)

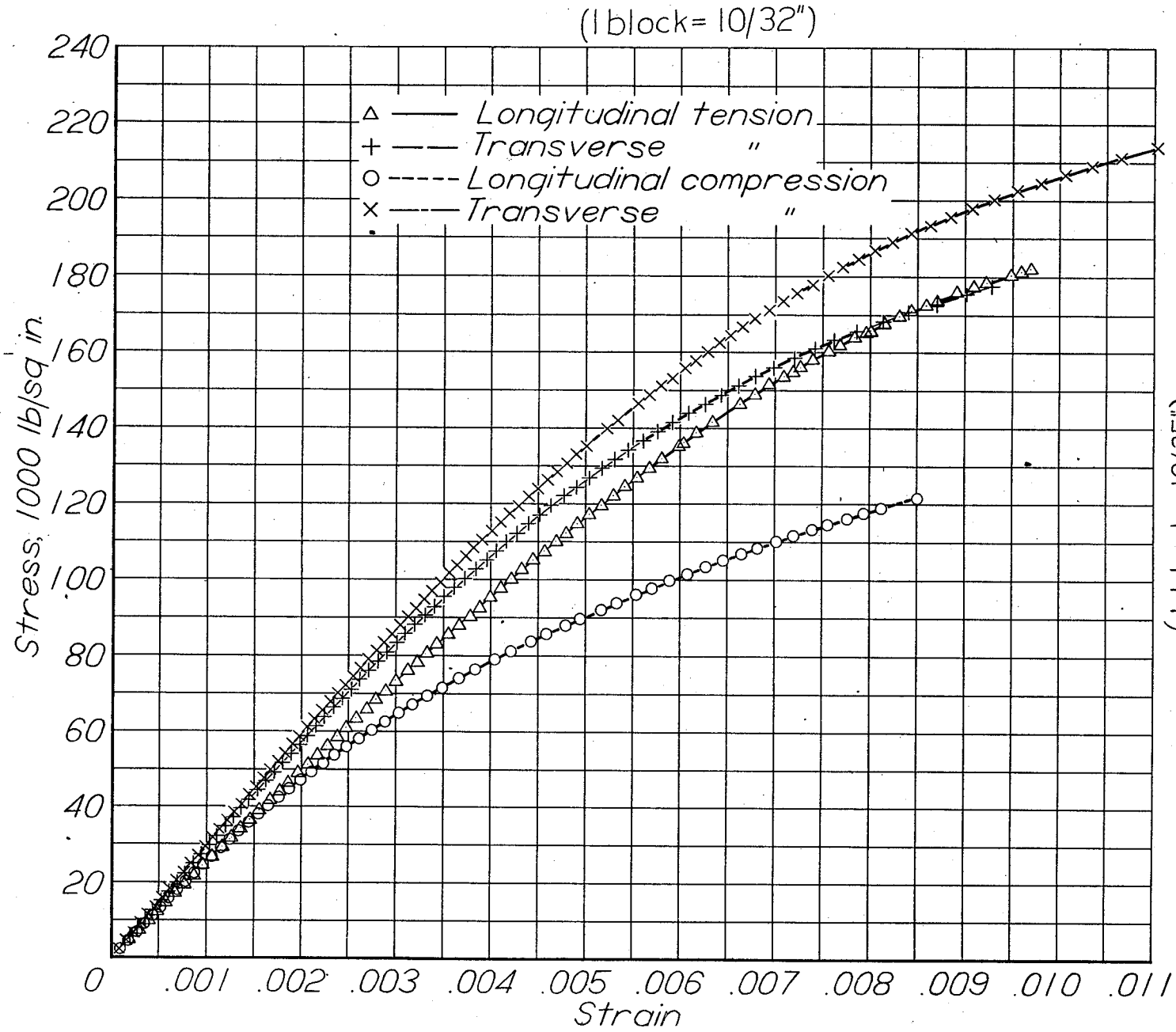


Figure 3.- Stress-strain curves with blunt knee. Chromium-nickel steel, full hard, thickness 0.020 inch.

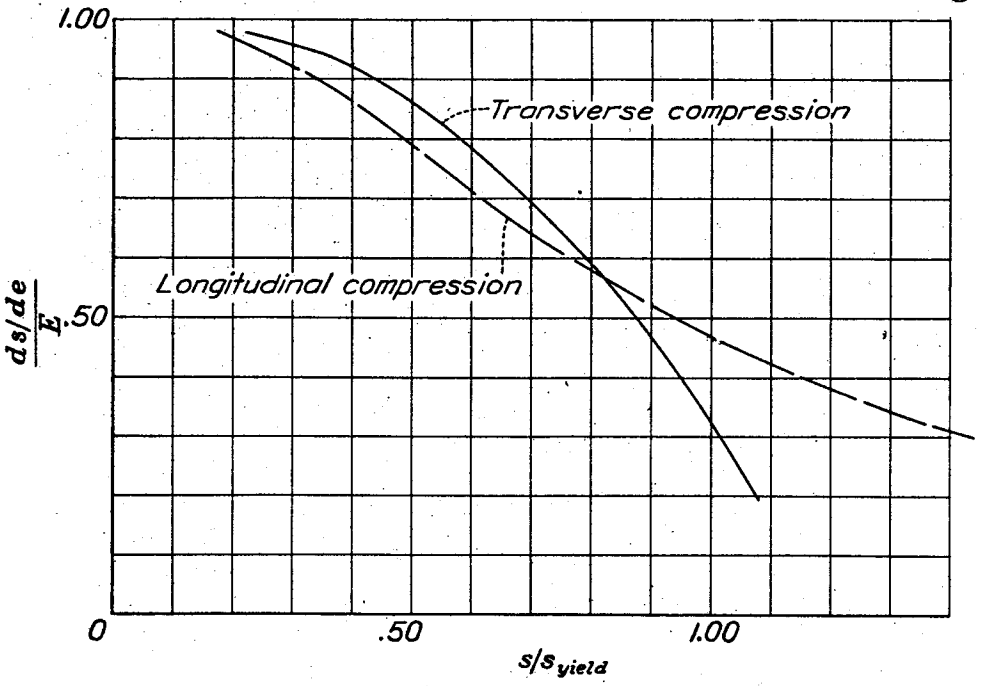


Figure 4.- Non-linear variation of tangent modulus with stress; chromium-nickel steel, full-hard, thickness 0.020 inch.

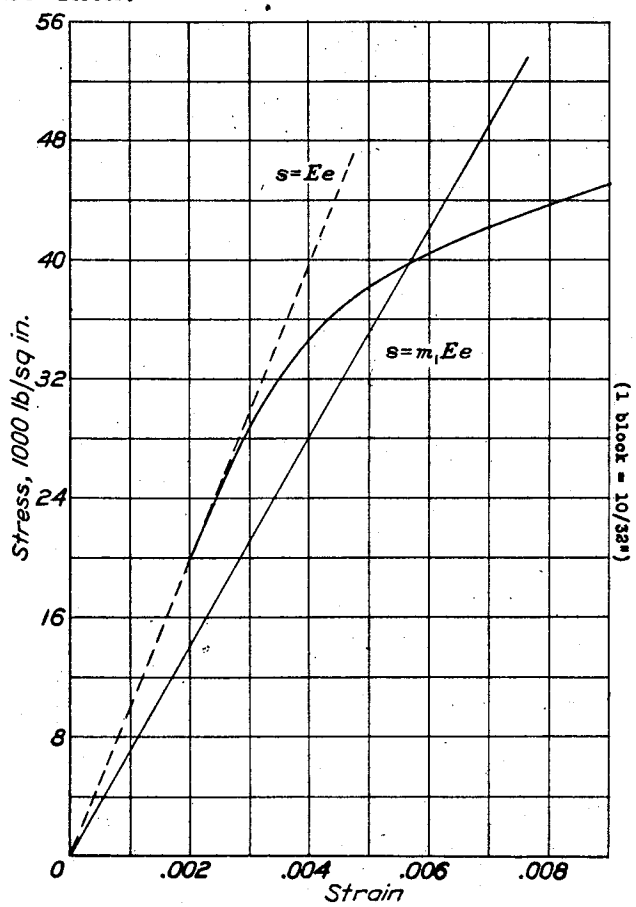


Figure 5.- Determination of secant yield strength.

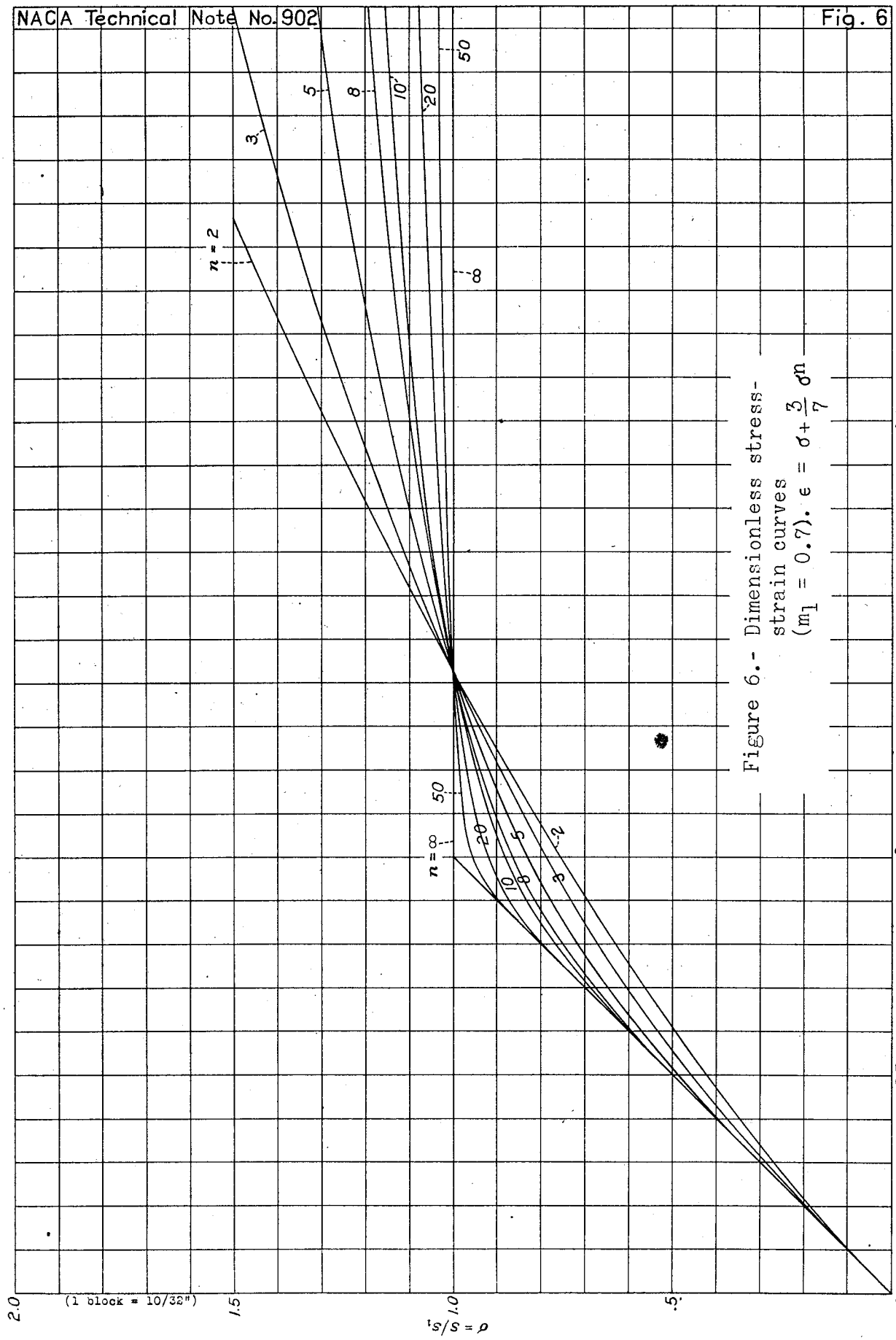


Figure 6.- Dimensionless stress-strain curves ($m_1 = 0.7$). $\epsilon = \sigma + \frac{3}{7}\sigma^n$

Fig. 7

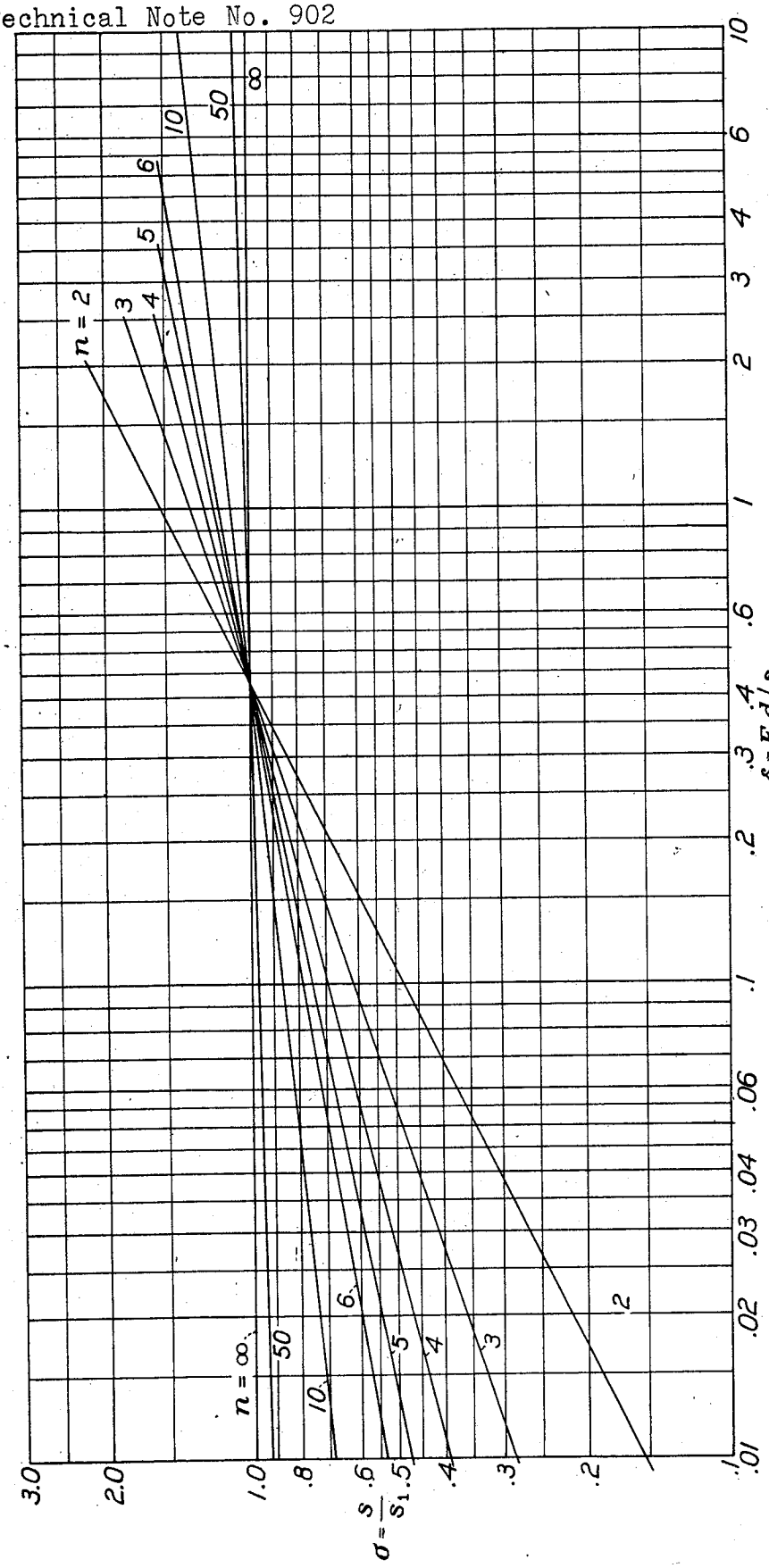


Figure 7.- Dimensionless stress-deviation curves ($m_1 = 0.7$).

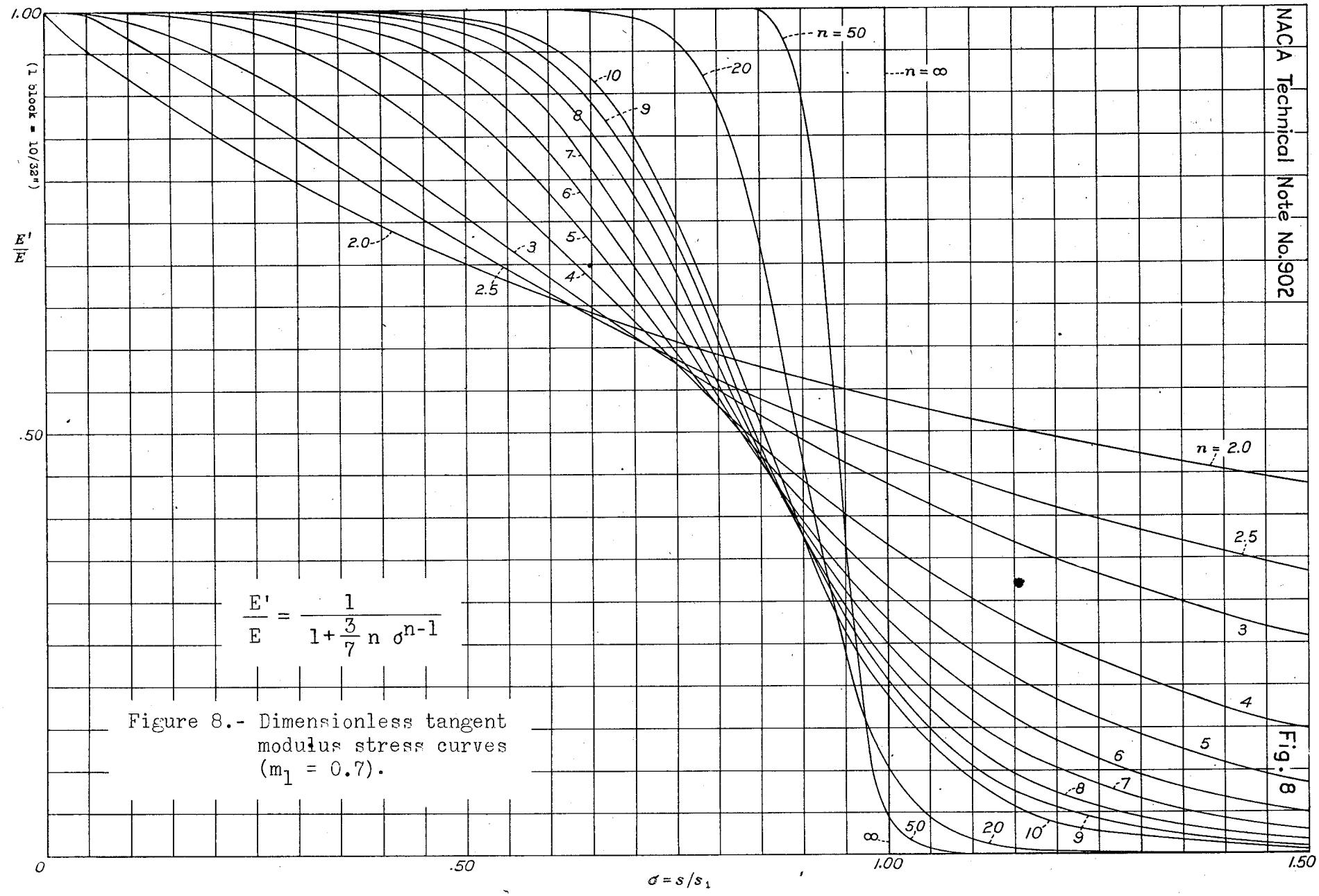
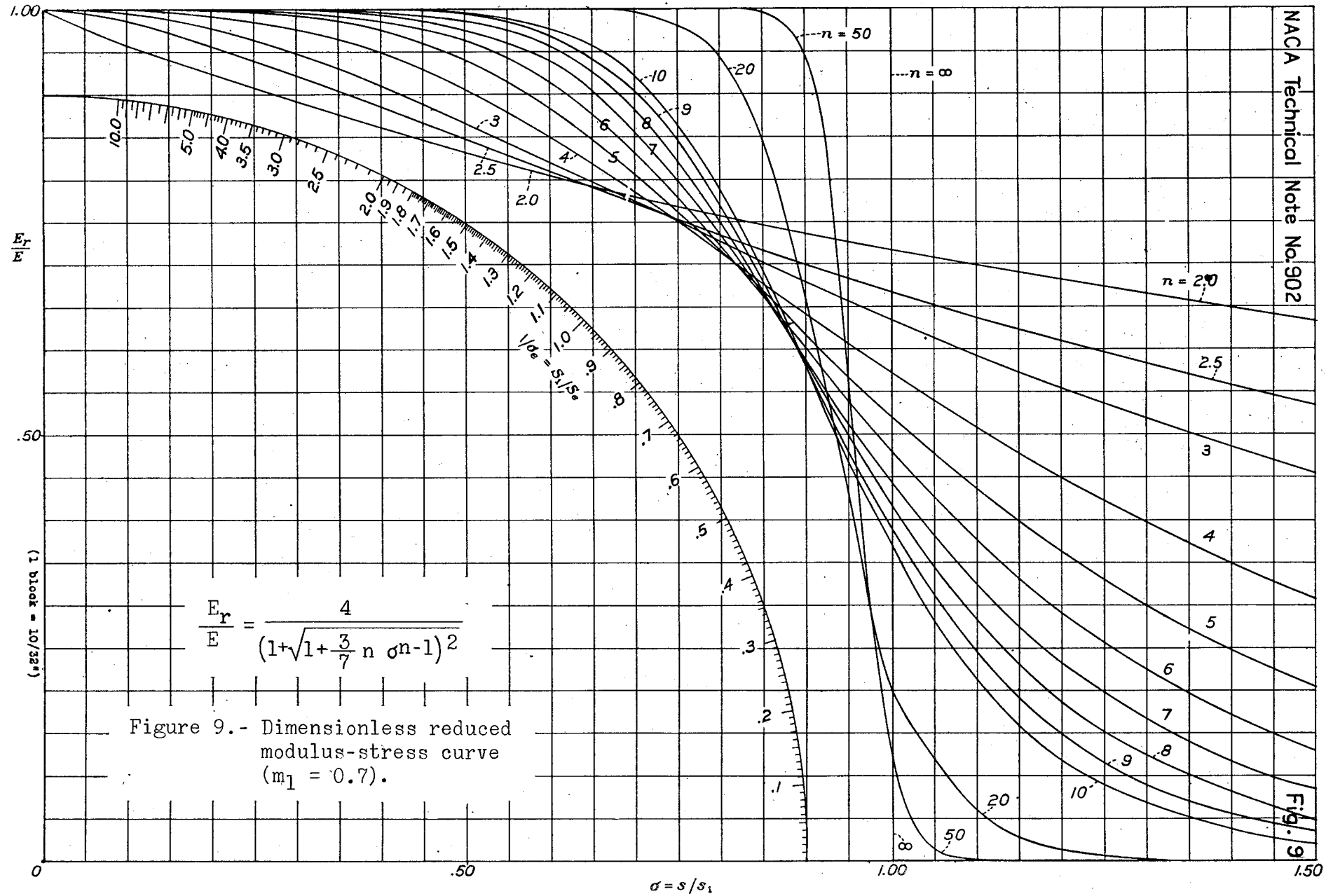


Figure 8.- Dimensionless tangent modulus stress curves ($m_1 = 0.7$).



$$\frac{E_r}{E} = \frac{4}{\left(1+\sqrt{1+\frac{3}{7}\, n\, \sigma^{n-1}}\right)^2}$$

Figure 9.- Dimensionless reduced modulus-stress curve ($m_1 = 0.7$).

Fig. 9

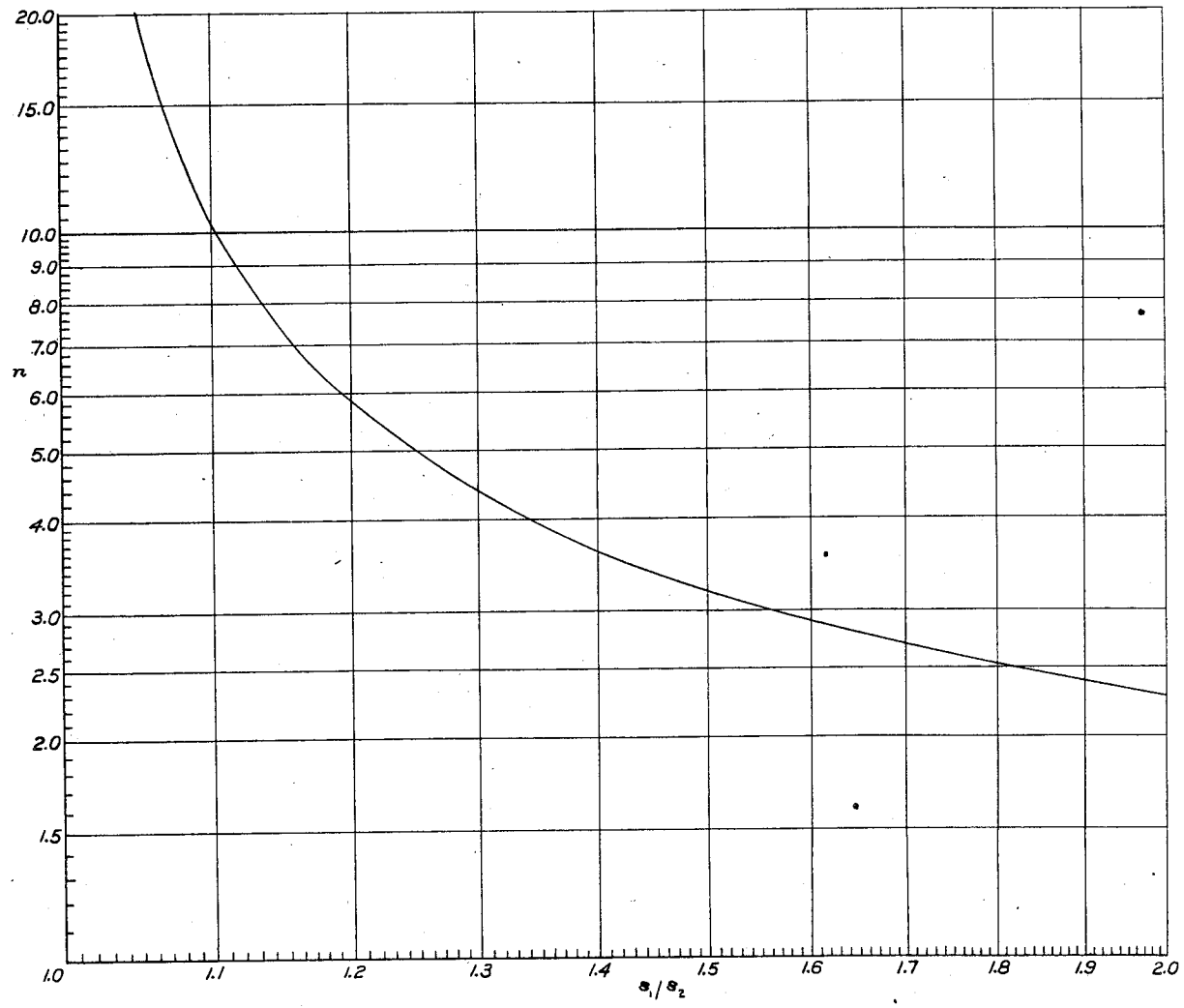


Figure 10.- Relation between n and s_1/s_2.

$$n = 1 + \frac{\log \frac{17}{7}}{\log \frac{s_1}{s_2}}$$

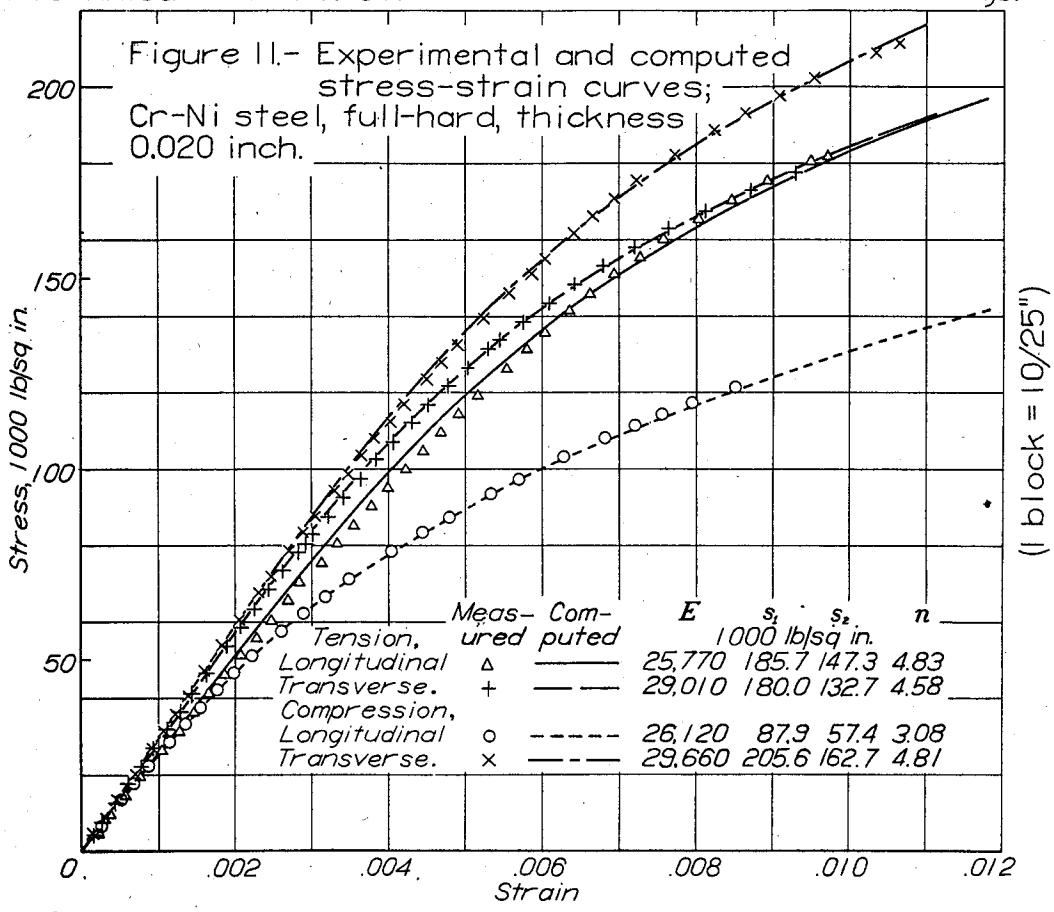


Figure 11.- Experimental and computed stress-strain curves; Cr-Ni steel, full-hard, thickness 0.020 inch.

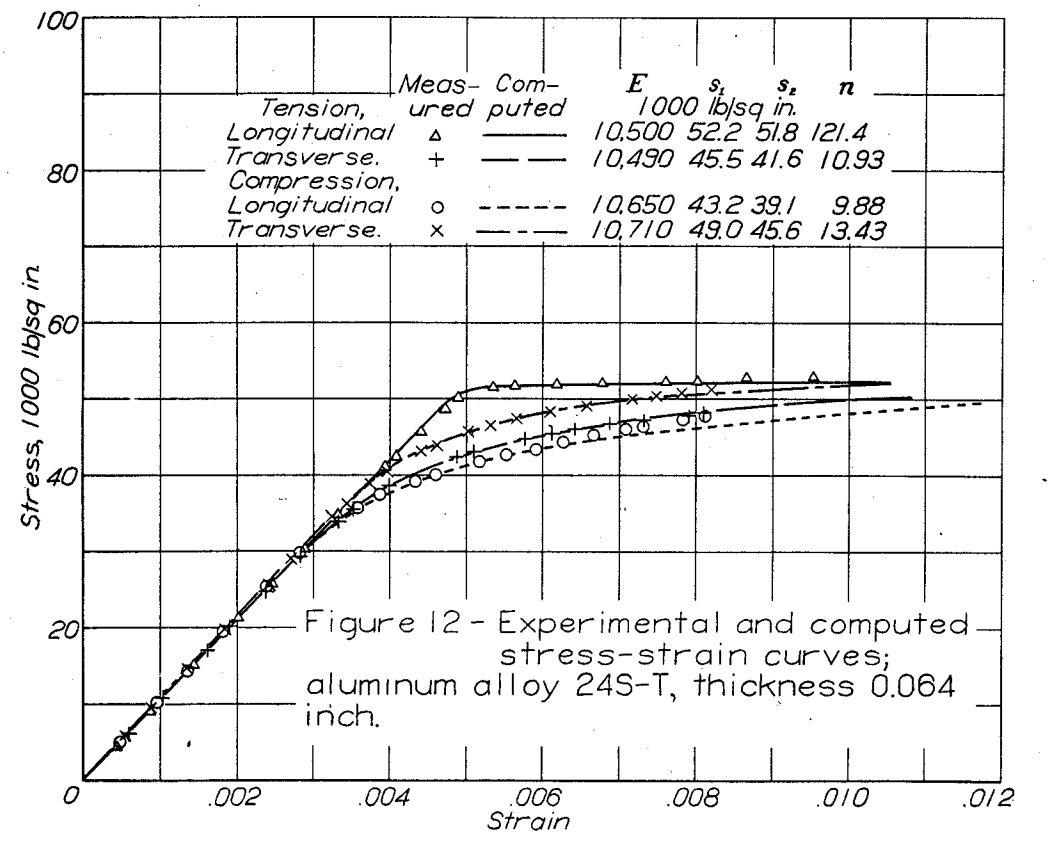


Figure 12 - Experimental and computed stress-strain curves; aluminum alloy 24S-T, thickness 0.064 inch.

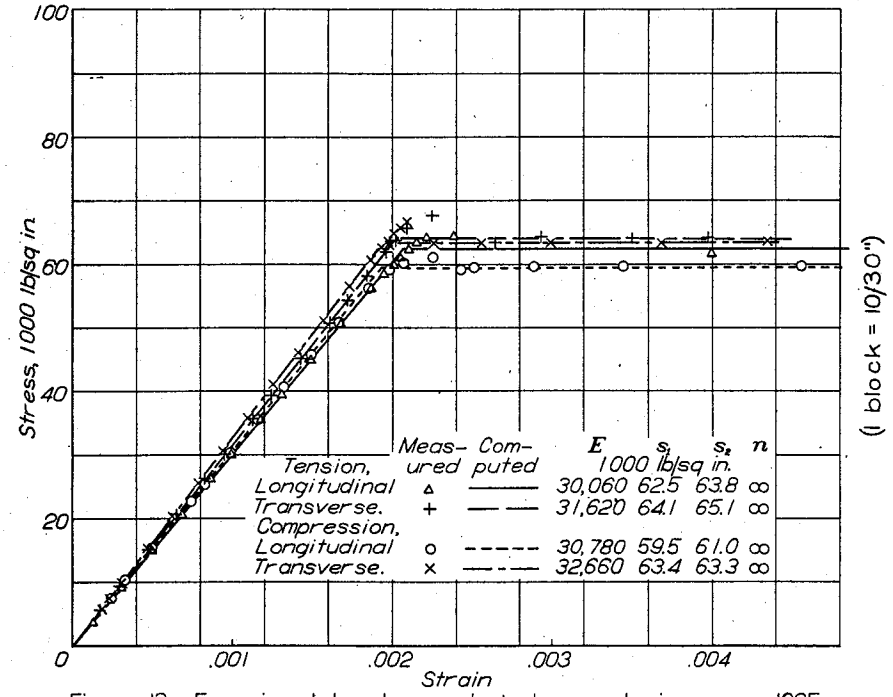


Figure 13.- Experimental and computed stress-strain curves, 1025 carbon-steel, thickness=0.054".

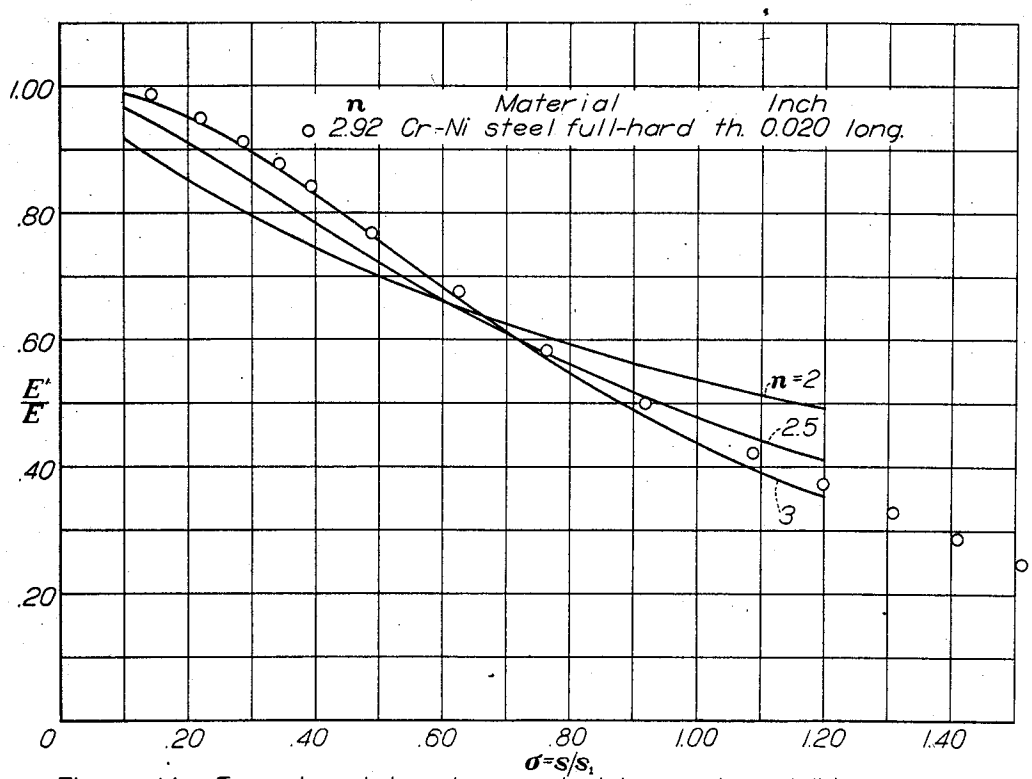


Figure 14.- Experimental and computed tangent moduli in compression, $2<n<3$.

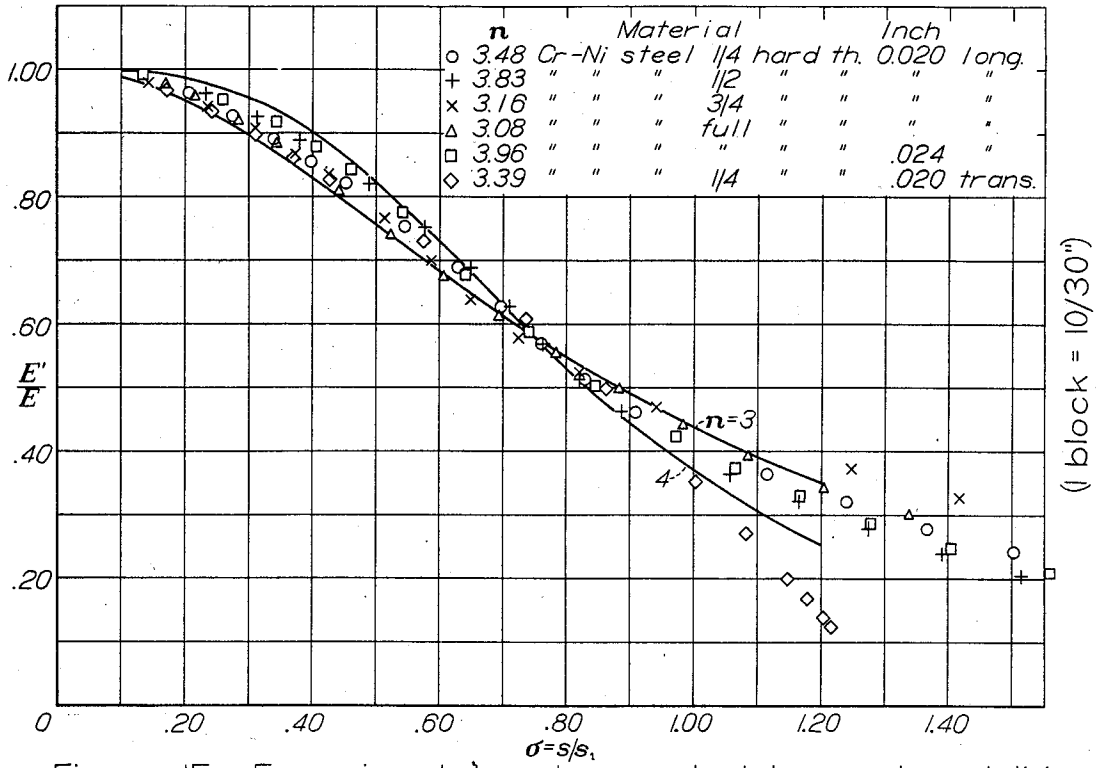


Figure 15.- Experimental and computed tangent moduli in compression, $3<n<4$.

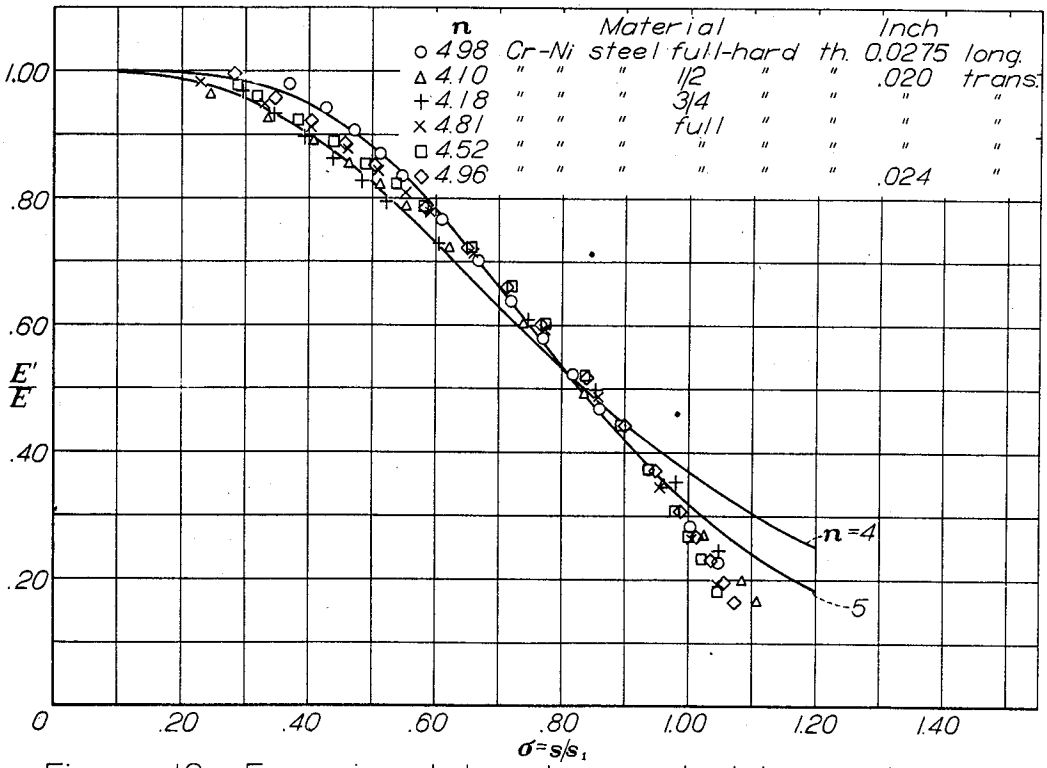


Figure 16.- Experimental and computed tangent moduli in compression, $4<n<5$.

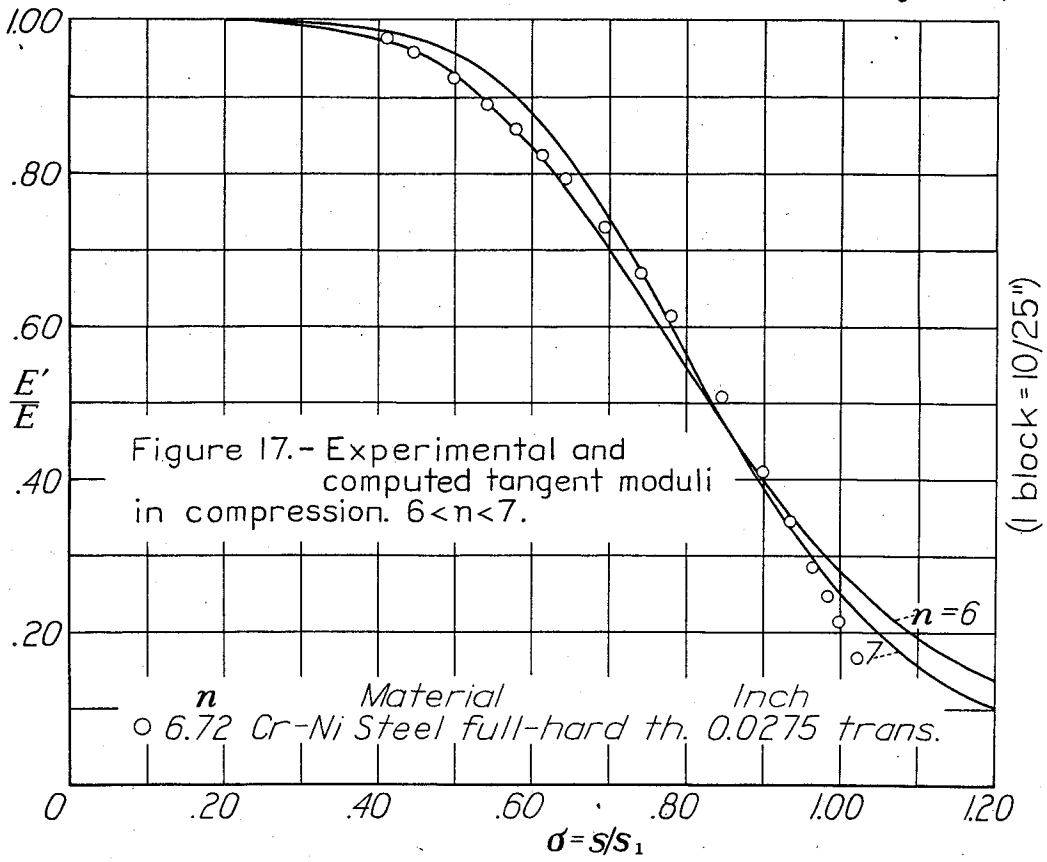


Figure 17.- Experimental and computed tangent moduli in compression. $6 < n < 7$.

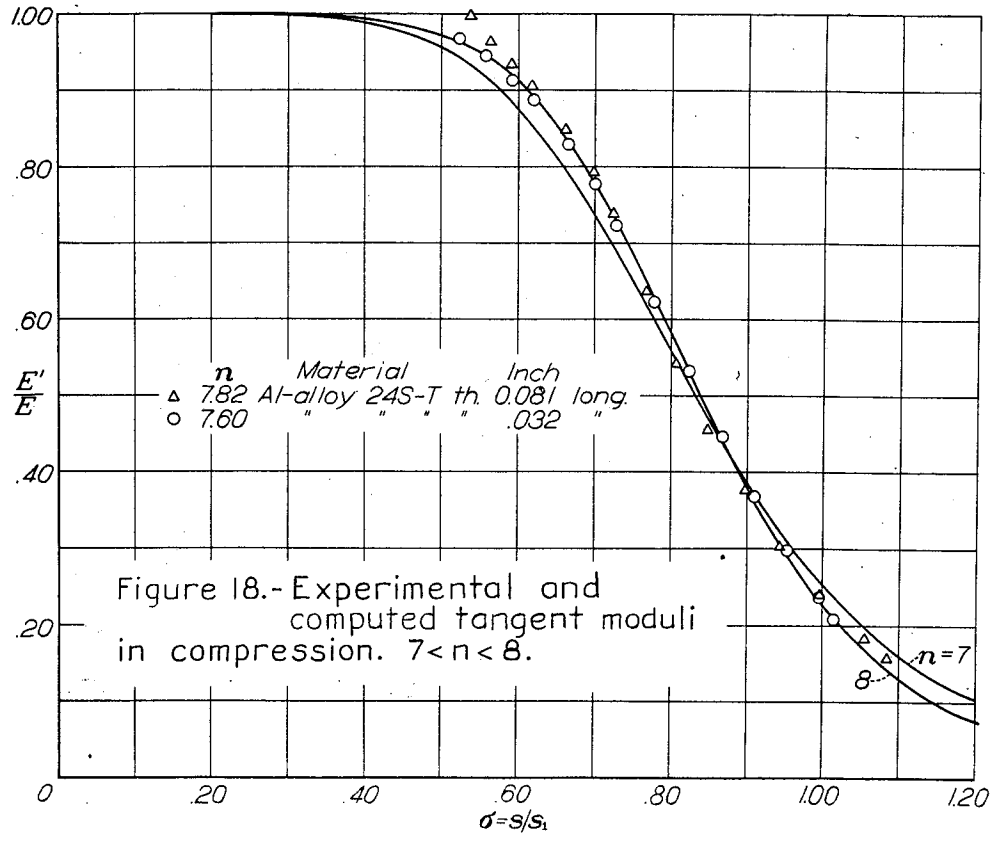


Figure 18.- Experimental and computed tangent moduli in compression. $7 < n < 8$.

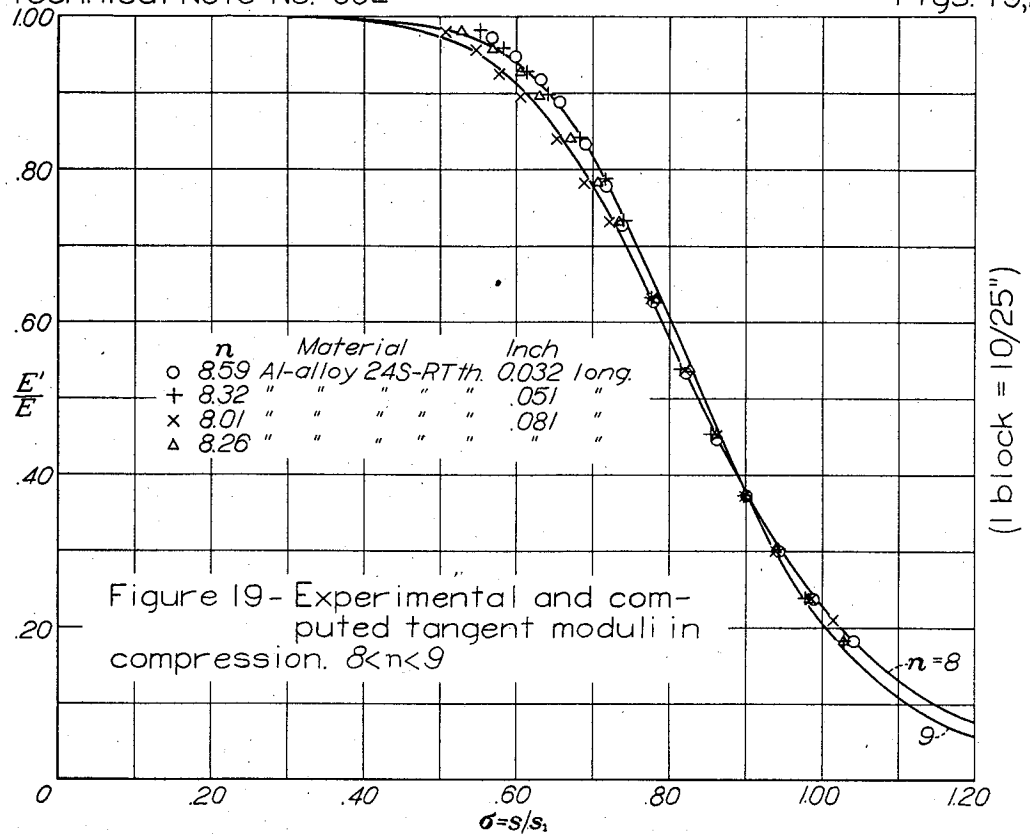


Figure 19- Experimental and computed tangent moduli in compression. $8<n<9$

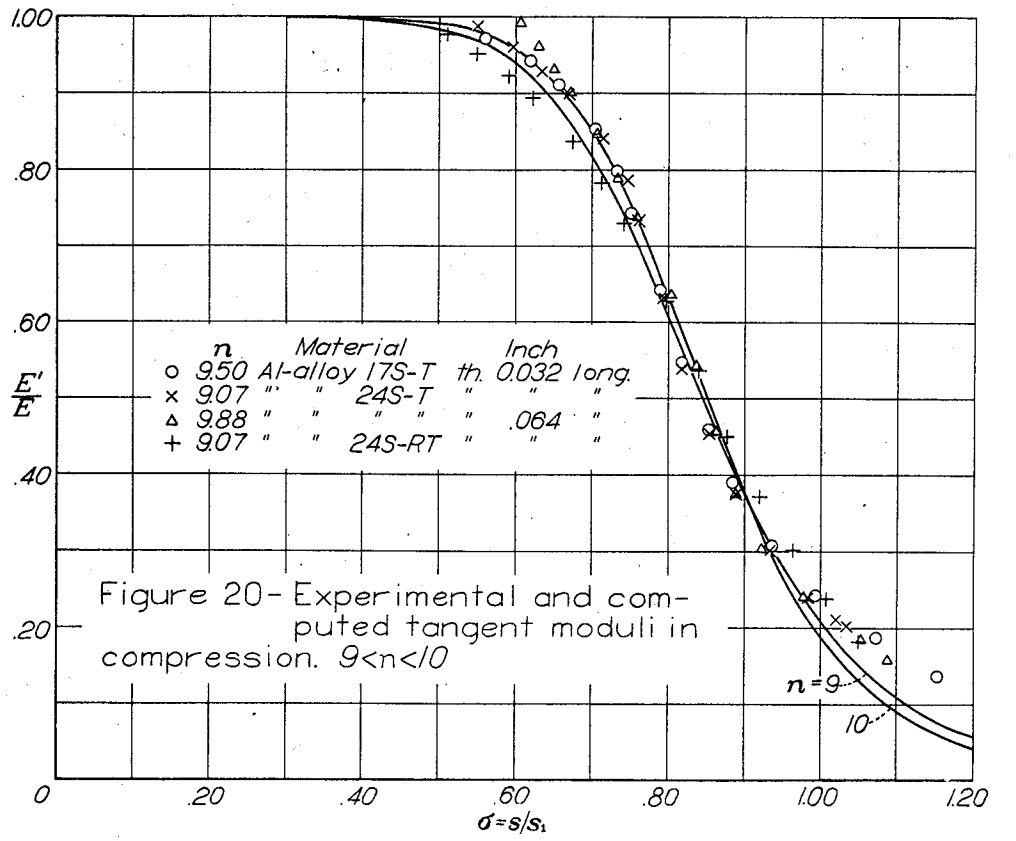


Figure 20- Experimental and computed tangent moduli in compression. $9<n<10$

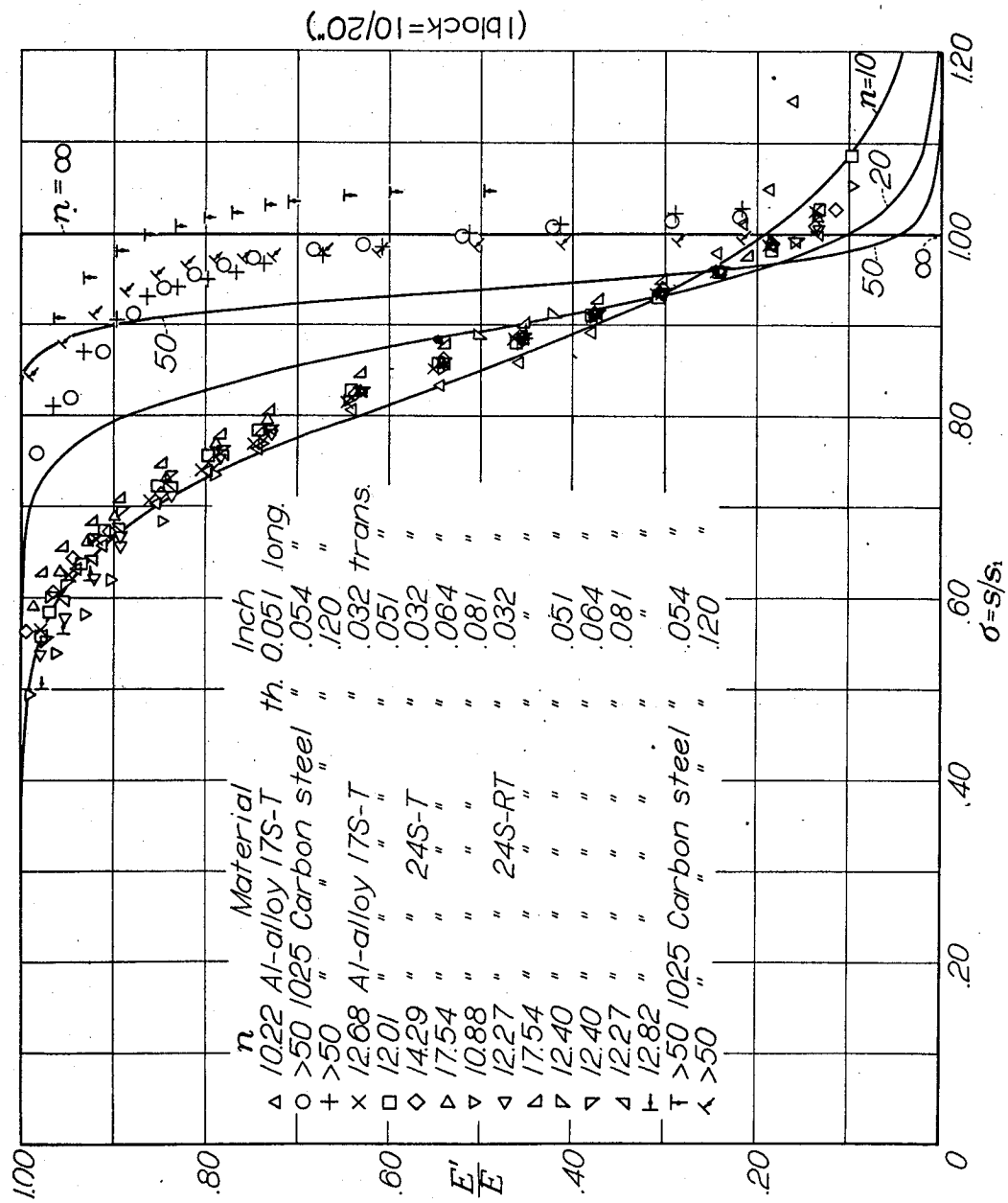


Figure 21.- Experimental and computed tangent moduli in compression. n>10.

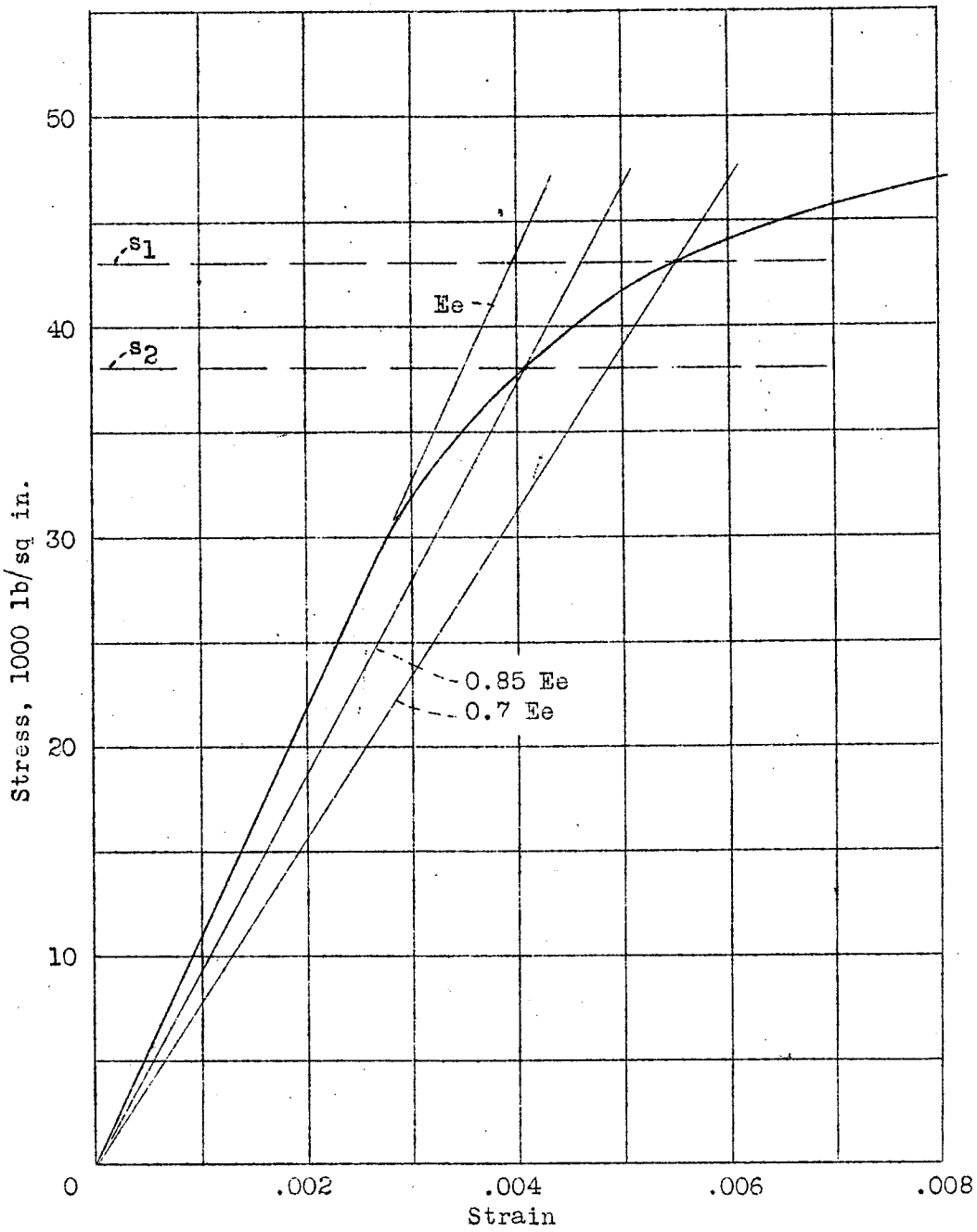


Figure 22.- Compressive stress-strain curve.