Geophys. J. R. astr. Soc. (1973) **34**, 403–419.

Descriptions of the Polarization States of Vector Processes: Applications to ULF Magnetic Fields

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(Received 1973 May 4)

Summary

In recent years a wide variety of methods has been used to describe the polarization characteristics of ultra low frequency (10^{-3} to 1 Hz) magnetic fields. This paper gives a more complete outline of some of the descriptions derived from the spectral matrices of n-variate stochastic processes. The matrices are expanded in three different, standard sets of matrices in order to add some simplification to the interpretation of the polarizations. One set is composed of n^2 trace-orthogonal, hermitean matrices and leads directly to a generalization of the Stokes parameters and the degree of polarization for n-variate processes. The second set is developed from the dyad expansion, which in particular cases is analogous to the spectral decomposition of the matrix. The third set is composed of n commuting idempotent matrices and proves to be the most useful set when the stochastic process is not strictly polarized. Finally, two examples of digital records of ULF magnetic fields are analysed to illustrate some of the limitations of the methods, and to indicate the biases which are inherent in numerical analyses.

Introduction

In the study of ultra-low frequency (ULF, $10^{-3}-1$ Hz) electromagnetic waves both in space and at the Earth's surface, there has arisen a bewildering complexity of analysis techniques for describing the characteristics of the perturbations. The oldest techniques estimate the waves' characteristics such as mean periods, mean amplitudes, and polarization hodograms directly from amplitude-time recordings of the perturbations. The use of hodograms to describe the time-dependent, directional properties, or polarization states of the perturbations is especially cumbersome and restrictive, and the results are sometimes misleading (see e.g. Pope 1964; Paulson 1968; and Fowler *et al.* 1967). Consequently many researchers have been led to develop more sophisticated analysis techniques, some based on direct adaptations of techniques used in optics (see e.g. Paulson *et al.* 1965 and Fowler, Kotick & Elliot 1967) and some based on applications of real multivariate analysis (see e.g. Cummings, O'Sullivan & Coleman 1969). Most of these techniques are rather restricted in applications, and the validity of the approach is sometimes not obvious. An interesting review of some of the above techniques, and many others, is given by McPherron, Russell & Coleman (1972).

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In this paper, I shall try to unify some of the techniques based on the use of the spectral matrix in describing polarization states. With minor exceptions, the discussion deals with n-variate stochastic processes, since this approach makes the techniques much more generalized in application, without adding much more complication. Although motivated by the need for accurate descriptions of ULF phenomena, the descriptions given in this paper have much more generalized applications, and can be used in many fields of research. My hope is that they will be especially useful in the geophysical sciences.

Consider now an n-variate stochastic process

$$\mathbf{x}(t) = [x_1(t), \ldots, x_n(t)]^T, \tag{1}$$

where $x_j(t)$ are the components of $\mathbf{x}(t)$ in an n-dimensional space, and t is time (vectors will be written as column matrices). Superscript 'T' denotes the transpose. We shall assume that the stochastic processes $x_j(t)$ are weakly stationary. That is,

$$\mathscr{E}\{x_j(t)\} = 0 \qquad j = 1, n, \tag{2}$$

where $\mathscr{E}$ denotes the expectation or ensemble average and the covariance moments

$$\phi_{jk} = \mathscr{E}\{x_j(t), x_k(t+u)\} \tag{3}$$

depend only on the interval u. We shall also assume that the second-order moments are adequate for the description of the statistics of the processes.

The spectral matrix $\mathrm{S}(\nu, \Delta f)$ of $\mathbf{x}(t)$ has the elements

$$S_{jk}(\nu, \Delta f) = \int_{\nu-\Delta f}^{\nu+\Delta f} \int_{-\infty}^{\infty} \phi_{jk}(u) \exp(-2\pi i g u)\, du\, dg \qquad (j, k = 1, n). \tag{4}$$

Note that each element is a continuous function of ν (frequency) and Δf (the increment half-width). Equivalently

$$S_{jk}(\nu, \Delta f) = T_{jk}(\nu+\Delta f) - T_{jk}(\nu-\Delta f), \tag{5}$$

where

$$\phi_{jk}(u) = \int_{-\infty}^{\infty} \exp(2\pi i u g)\, T_{jk}(dg). \tag{6}$$

We have assumed that the elements of the spectral distribution matrix $\mathbf{T}$ are absolutely continuous with respect to Lebesque measure. The increments $S_{jk}(\nu, \Delta f)$ form a non-negative, hermitean matrix (see e.g. Hannan 1970, Chapter 2). The spectral matrix $\mathrm{S}(\nu, \Delta f)$ has the minimum set of parameters necessary to characterize the polarization states of the n-variate stochastic process.

Often the term polarization is used to describe the directional properties of a perturbation vector in a real or Euclidean vector space. For example, if we can find an orthogonal basis such that the perturbation vector is always along one axis, then we can consider the perturbation to be linearly polarized or strictly polarized. The other extreme occurs when the powers in all directions are equal and remain so for any choice of the co-ordinate axes. The concept of polarization can be further generalized by substituting a unitary space for the Euclidean space, and we can consider the directional properties or polarization characteristics in the unitary space.

A perturbation vector can be considered to be strictly polarized if through a unitary transformation we can find a new basis such that the perturbation is restricted to one axis (i.e. only one eigenvalue of $\mathbf{S}$ is non-zero). Conversely, a perturbation vector is completely unpolarized if the powers on each axis are equal and remain so under any arbitrary unitary transformation (i.e. $\mathbf{S} = a\mathbf{I}_n$, where a is real and non-negative, and $\mathbf{I}_n$ is the n-square identity matrix).

In describing the polarization states of the perturbation vector, we must choose parameters which have familiar characteristics. Much confusion arises from the fact that many researchers are familiar only with strictly polarized states such as sinusoidal waves, which are completely coherent and polarized, and consequently have perturbation vectors which follow very simple patterns (i.e. ellipses). Seldom does nature provide us with such a pure state, and consequently we must look at the spectral matrix in much more detail.

One procedure which adds some simplifications to the interpretation of the spectral matrix is to expand the matrix in sets of matrices with familiar characteristics. If at least some of the coefficients in the expansion are zero, or there exists a simple relationship between the coefficients, then the interpretation is much simplified. If not, then the perturbation vector is possibly too complicated to merit analysis. In this paper we shall consider three sets for the expansions although there are clearly many more, and in particular cases other sets might prove much more useful than the ones to be discussed.

The first set we shall discuss is composed of n^2, trace-orthogonal, hermitean matrices, and leads to a generalization of the Stokes parameters for n-variate processes. It is surprising that these parameters are seldom used in describing ULF waves, especially since they lead to a simple and computationally efficient parameter for describing the degree of polarization. The second set is developed from the dyad expansion, which in particular cases is equivalent to the spectral decomposition of the matrix. The third set is composed of n commuting, idempotent matrices and appears to be one generalization of the expansion used by Fowler *et al.* (1967) for the case $n = 2$ (see also Born & Wolf 1959). In the spectral and idempotent expansions, the matrices in the sets are non-negative and hermitean, all coefficients of the expansion are real and non-negative, and consequently each element of the set is a representation of an n-variate stochastic process. Specifically this means that $\mathbf{S}$ is expanded in the form (henceforth the dependence of ν and Δf is implied and will be written only in specific cases)

$$\mathbf{S} = \sum_{l=1}^{n} a_l\,{}^l\mathbf{S} \tag{7}$$

where

$$a_l\,{}^lS_{jk} = \int_{\nu-\Delta f}^{\nu+\Delta f} \int_{-\infty}^{\infty} {}^l\theta_{jk}(u) \exp(-2\pi i g u)\, du\, dg, \tag{8}$$

$${}^l\theta_{jk}(u) = \mathscr{E}\{{}^lx_j(t), {}^lx_k(t+u)\} \tag{9}$$

$$\mathscr{E}\{{}^lx_j(t), {}^mx_k(t+u)\} = 0 \qquad (l \neq m), \tag{10}$$

$$\mathbf{x}(t) = \sum_{l=1}^{n} {}^l\mathbf{x}(t), \tag{11}$$

and

$${}^l\mathbf{x}(t) = [{}^lx_1(t), {}^lx_2(t) \ldots {}^lx_n(t)]^T. \tag{12}$$

Equations (8) and (9) indicate that each matrix ${}^l\mathbf{S}$ must be non-negative and hermitean, and each coefficient a_l must be non-negative.

Expansion of S on a basis of trace-orthogonal matrices

The first expansion of S that we shall consider has the form

$$\mathbf{S} = \sum_{l=0}^{n^2-1} a_l \mathbf{U}_l, \tag{13}$$

where the n-square matrices $\mathbf{U}_l$ are hermitean, $\operatorname{tr} \mathbf{U}_l \mathbf{U}_k = \delta_{lk}$ (δ is the Kronecker delta), and consequently

$$a_l = \operatorname{tr}(\mathbf{S}\mathbf{U}_l). \tag{14}$$

If we choose $\mathbf{U}_0 = n^{-\frac{1}{2}} \mathbf{I}_n$, then $\operatorname{tr} \mathbf{U}_l (l \neq 0) = 0$ is required to meet the orthogonality condition. If $n = 2$, a complete set of $\mathbf{U}_l$ are given by the four matrices

$$\left.\begin{aligned} &\mathbf{U}_0 = 2^{-\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U}_1 = 2^{-\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{U}_2 = 2^{-\frac{1}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ &\mathbf{U}_3 = 2^{-\frac{1}{2}} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}. \end{aligned}\right\} \tag{15}$$

The matrices $\mathbf{U}_1$, $\mathbf{U}_2$, and $\mathbf{U}_3$ are clearly recognizable as the Pauli spin matrices. Fano (1957) has indicated that the coefficients $a_l(n = 2)$ in equation (13) are actually the Stokes parameters of the stochastic process. The matrices (15) were also used by Wiener (1930) in his discussions on harmonic analysis, but he did not note the connection between the expansion and the Stokes parameters.

If $n = 3$, a complete set of $\mathbf{U}_l$ are given by the nine matrices

$$\left.\begin{aligned} &\mathbf{U}_0 = 3^{-\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{U}_1 = 2^{-\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{U}_2 = 6^{-\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ &\mathbf{U}_3 = 2^{-\frac{1}{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{U}_4 = 2^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{U}_5 = 2^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\ &\mathbf{U}_6 = 2^{-\frac{1}{2}} \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{U}_7 = 2^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad \mathbf{U}_8 = 2^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}. \end{aligned}\right\} \tag{16}$$

The coefficients of the expansion, which we might call the Stokes parameters for the interval $\nu - \Delta f$ to $\nu + \Delta f$, are

$$\left.\begin{aligned} a_0 &= 3^{-\frac{1}{2}}(S_{11} + S_{22} + S_{33}), \\ a_1 &= 2^{-\frac{1}{2}}(S_{11} - S_{33}), \\ a_2 &= 6^{-\frac{1}{2}}(S_{11} - 2S_{22} + S_{33}), \\ a_3 &= 2^{-\frac{1}{2}}(S_{12} + S_{12}{}^*), \quad a_4 = 2^{-\frac{1}{2}}(S_{13} + S_{13}{}^*), \quad a_5 = 2^{-\frac{1}{2}}(S_{23} + S_{23}{}^*) \\ a_6 &= i^{-1}\, 2^{-\frac{1}{2}}(S_{12}{}^* - S_{12}), \quad a_7 = i^{-1}\, 2^{-\frac{1}{2}}(S_{13}{}^* - S_{13}), \\ a_8 &= i^{-1}\, 2^{-\frac{1}{2}}(S_{23}{}^* - S_{23}). \end{aligned}\right\} \tag{17}$$

where * denotes the complex conjugate.

An alternative basis for the algebra of 3-square hermitean matrices has been constructed by Roman (1959) using the Kemmer algebra (Kemmer 1943). This basis is also composed of a set of nine hermitean matrices, but the matrices are not pairwise trace-orthogonal. Consequently the coefficients of the expansion, which Roman calls the generalized Stokes parameters, differ from those given in equation (17). However, the choice of the coefficients of the trace-orthogonal set for the Stokes parameters seems to be more consistent. For example, from (14), $a_0 = n^{-\frac{1}{2}} \operatorname{tr} \mathbf{S}$ for arbitrary n, but this is not the case in Roman's basis where $a_0 = S_{22}$.

Trace-orthogonal sets for arbitrary n can be constructed by trial. However, some of these sets can be found in descriptions of elementary particles (see e.g. Gell-Mann 1962 for $n = 3$), or in group theory where they are the infinitesimal operators of the unitary groups $U(n)$.

In analogy with the procedure followed in optics for the case $n = 2$, we can construct a polarization vector $\mathbf{p} = [a_1, a_2 \dots a_{n^2-1}]^T$ and define the degree of polarization P (for arbitrary n) by the relation

$$P^2 = p^2/[(n-1)\,a_0{}^2] = \operatorname{tr} \mathbf{A}^2/[(n-1)\,a_0{}^2] \tag{18}$$

where

$$a_0 = n^{-\frac{1}{2}} \operatorname{tr} \mathbf{S} \quad \text{and} \quad \mathbf{A} = \sum_{l=1}^{n^2-1} a_l \mathbf{U}_l.$$

If $P = 1{\cdot}0$ then the stochastic process is considered to be completely polarized. More will be said about this in the discussions of the dyad and idempotent expansions.

With the possible exceptions of the case $n = 2$ or the cases where $P = 1{\cdot}0$ or $0{\cdot}0$, interpretation of the polarization states from the trace-orthogonal expansion is very difficult. In most instances it is probably easier to use the dyad and idempotent expansions.

Expansions of S on dyad bases

Consider a complete set of n vectors $\mathbf{u}_l (l = 1, n)$ which are pairwise orthonormal (i.e. $\mathbf{u}_l^+ \mathbf{u}_k = \delta_{lk}$ where the superscript $+$ denotes the hermitean adjoint). From these vectors we can construct n^2 matrices $\mathbf{E}_{lk} = \mathbf{u}_l \mathbf{u}_k^+$ (the subscripts lk identify the matrix and not its elements). Since the set $\mathbf{u}_l$ was assumed to be complete, the set of n-square matrices $\mathbf{E}_{lk}$ can be used as a basis for the algebra of n-square matrices. Thus we can expand $\mathbf{S}$ in the form:

$$\mathbf{S} = \sum_{l,\,k=1}^{n} a_{lk} \mathbf{E}_{lk} \tag{19}$$

where

$$a_{lk} = \mathbf{u}_l^+ \mathbf{S} \mathbf{u}_k.$$

The matrices $\mathbf{E}_{ll}$ are idempotent (i.e. $\mathbf{E}_{ll}^2 = \mathbf{E}_{ll}$) have rank one, are hermitean, and consequently are also non-negative and hermitean. The matrices $\mathbf{E}_{lk} (l \neq k)$ are not hermitean.

If the vectors $\mathbf{u}_l$ are the eigenvectors of $\mathbf{S}$ (or any matrix which has distinct eigenvalues and commutes with $\mathbf{S}$), then $a_{lk} = 0$ if $l \neq k$, and $a_{lk} = \lambda_k$ if $l = k$, where λ_k is the appropriate eigenvalue of $\mathbf{S}$ (i.e. $\mathbf{S}\mathbf{u}_k = \lambda_k \mathbf{u}_k$). Consequently

$$\mathbf{S} = \sum_{l=1}^{n} \lambda_l \mathbf{E}_{ll}. \tag{20}$$

Since the eigenvalues λ_l are non-negative, each element $\lambda_l \mathbf{E}_{ll}$ can be considered to be the spectral matrix of an n-variate stochastic process (see equations (7)–(12)). In addition, the matrices $\mathbf{E}_{ll}$ are idempotent and disjoint $\left(\text{i.e. } \mathbf{E}_{ll}\mathbf{E}_{kk} = 0 \ (k \neq l), \sum_{l=1}^{n} \mathbf{E}_{ll} = \mathbf{I}_n\right)$ and consequently expansion (20) is identical to the spectral decomposition of the matrix $\mathbf{S}$ if $\lambda_1 \neq \lambda_2 \ldots \neq \lambda_n$ (see e.g. Mal'Cev 1963, Chapter 5).

Obviously many other sets for the expansion of $\mathbf{S}$ can be constructed with elements $\mathbf{A}_l = \sum_{k=1}^{n} a_{lk}\mathbf{E}_{kk}$ $(l = 1, n)$ such that the $\mathbf{A}_l$ constitute a set of linearly independent n-square matrices. Not all these sets will be particularly useful.

If only one eigenvalue of S is non-zero, all the principal minors except the diagonal elements are identically zero, and we can consider the process to be strictly polarized. Conversely, if all the eigenvalues are equal $(\lambda_1 = \lambda_2 \ldots = \lambda_n)$, $\mathbf{S}$ is invariant under any unitary transformation, and we can consider the process to have maximum randomness or to be completely unpolarized. These features suggest that the eigenvalues of $\mathbf{S}$ also give us some indication of the degree of polarization, and that we should choose an appropriate function $f(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with the limiting values $f = 0$ if all eigenvalues are equal and $f = 1{\cdot}0$ if only one eigenvalue is non-zero. Such a function can be derived from the definition of the degree of polarization given by the Stokes parameters (equation 18). Since $a_0^2 = n^{-1}(\operatorname{tr}\mathbf{S})^2$ and $\operatorname{tr}\mathbf{A}^2$ are invariant under any unitary transformation of $\mathbf{S}$, the degree of polarization P is not dependent on the particular choice of co-ordinates (which should be the case, otherwise the meaning would be ambiguous). If $\mathbf{S}_\lambda =$

$$\mathbf{U}^+\mathbf{S}\mathbf{U} = \begin{bmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \lambda_2 & \vdots \\ 0 & \ldots & \lambda_n \end{bmatrix} = \operatorname{diag}[\lambda_1, \lambda_2 \ldots \lambda_n],$$

where $\mathbf{U}$ is an appropriate unitary matrix, then we have

$$\begin{aligned} \mathbf{A} &= \mathbf{S}_\lambda - n^{-\frac{1}{2}} a_0 \mathbf{I}_n \\ &= \operatorname{diag}[\lambda_1 - n^{-1}\operatorname{tr}\mathbf{S}, \lambda_2 - n^{-1}\operatorname{tr}\mathbf{S}, \ldots, \lambda_n - n^{-1}\operatorname{tr}\mathbf{S}]. \end{aligned} \tag{21}$$

Thus

$$p^2 = \operatorname{tr}\mathbf{A}^2 = \sum_{l=1}^{n} (\lambda_l - (\lambda_1 + \lambda_2 + \ldots + \lambda_n)n)^2. \tag{22}$$

Equation (22) can be reduced to

$$p^2 = \frac{1}{2n} \sum_{l,\, k=1}^{n} (\lambda_l - \lambda_k)^2. \tag{23}$$

Thus the degree of polarization P is given by the relation

$$P^2 = \frac{1}{2(n-1)\operatorname{tr}\mathbf{S}} \sum_{l,\, k=1}^{n} (\lambda_l - \lambda_k)^2. \tag{24}$$

The definition of the degree of polarization given by equation (24) fits the desired characteristics for $f(\lambda_1, \lambda_2 \ldots \lambda_n)$ in that if all the eigenvalues are equal, then $P = 0{\cdot}0$, and if only one is non-zero then $P = 1{\cdot}0$. In analogy with the variance ellipsoid of real multivariate analysis, P might be considered to be a measure of the eccentricity of a hyperellipsoid in a unitary space, with n axes of lengths $\lambda_1, \lambda_2 \ldots \lambda_n$. When $P = 0{\cdot}0$ the hyperellipsoid becomes a hypersphere, and when $P = 1{\cdot}0$ the hyperellipsoid becomes a line.

Expansion of S in non-disjoint idempotent matrices

In the preceding discussion we found that any hermitean non-negative matrix can be written as a linear combination of disjoint idempotent matrices, each of rank one, and furthermore all the coefficients in the expansion are non-negative. **S** can also be expanded in sets of non-disjoint idempotent matrices. One such expansion which is particularly simple is

$$\mathbf{S} = \sum_{l=1}^{n} a_l \mathbf{D}_l, \tag{25}$$

where each n-square matrix $\mathbf{D}_l$ is idempotent, and has rank $(\mathbf{D}_l) = l$, and $\mathbf{D}_l \mathbf{D}_k = \mathbf{D}_k$ $(k \leqslant l)$. If the coefficients a_l are chosen to be non-negative, then this expansion is unique. Since $\mathbf{D}_l \mathbf{D}_k = \mathbf{D}_k$, each element of the set commutes with all other elements and the set $\mathbf{D}_l$ has many of the characteristics of an Abelian group. Note, however, that the matrices $\mathbf{D}_l$ $(l \neq n)$ have no inverses. One more useful point is that any linear combination of the matrices $\mathbf{D}_l$ has rank less than or equal to the largest subscript l in the combination.

The set of matrices $\mathbf{D}_l$ can be constructed from the set $\mathbf{E}_{ll}$ (equation 20). However, we shall go about it in a slightly different fashion. Since any element $\mathbf{D}_l$ commutes with all other elements, the matrices $\mathbf{D}_l$ have a set of eigenvectors in common. If $\mathbf{U}$ is the unitary matrix formed from these eigenvectors, then $\mathbf{U}^+ \mathbf{S} \mathbf{U}$ must be diagonal. The n diagonal elements give us the set of equations

$$\sum_{l=k}^{n} a_l = \lambda_k \qquad (k = 1, n). \tag{26}$$

From (26) by successive subtractions we obtain the set of equations

$$a_l = \lambda_l - \lambda_{l+1} \qquad (\lambda_{n+1} = 0, l = 1, n). \tag{27}$$

Inspection of (27) shows that if we wish all the coefficients a_l to be non-negative we must choose the eigenvalues such that $\lambda_1 \geqslant \lambda_2 \ldots \geqslant \lambda_n$. Consequently

$$\mathbf{D}_l = \mathbf{U} \operatorname{diag} [\mathbf{I}_l, \mathbf{O}_{n-l}] \mathbf{U}^+, \tag{28}$$

where $\mathbf{U}^+ \mathbf{S} \mathbf{U} = \operatorname{diag} [\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n]$, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_n$, and $\mathbf{O}_{n-l}$ is the $(n-l)$-square null matrix. We have thus expanded our stochastic process in a set of n uncorrelated stochastic processes ${}^l\mathbf{x}$ $(l = 1, n)$, each with the degree of polarization

$$P_l^2 = \frac{n-l}{nl-l}. \tag{29}$$

${}^n\mathbf{x}$ is unpolarized $(P_n = 0)$ and ${}^1\mathbf{x}$ is strictly polarized $(P_1 = 1{\cdot}0)$. All principal minors of $\mathbf{D}_1$, except the diagonal elements, are zero and consequently all the multivariate coherencies of ${}^1\mathbf{x}(t)$ are equal to unity (e.g. the bivariate coherencies $D_{lk}{}^2/D_{ll} D_{kk} = 1{\cdot}0$). In most cases it is this strictly polarized or pure state which is interesting to researchers.

The total power in the intervals $\nu - \Delta f$ to $\nu + \Delta f$ in each of the stochastic processes ${}^l\mathbf{x}$ is $\operatorname{tr}(a_l \mathbf{D}_l) = l(\lambda_l - \lambda_{l+1})$ and consequently a convenient measure of the relative amounts of power in each is given by the relations

$$R_l = \frac{l(\lambda_l - \lambda_{l+1})}{\operatorname{tr} \mathbf{S}}, \qquad (\lambda_{n+1} = 0) \tag{30}$$

where

$$0 \leqslant R_l \leqslant 1{\cdot}0, \qquad \sum_{l=1}^{n} R_l = 1{\cdot}0.$$

For example, if $n = 2$

$$\left.\begin{aligned} \mathbf{U}^+\,\mathbf{SU} &= \lambda_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\lambda_1 - \lambda_2) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ \mathbf{D}_1 &= \mathbf{U} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{U}^+, \qquad P^2 = {R_1}^2 = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}. \end{aligned}\right\} \tag{31}$$

This expansion is identical to that given by Born & Wolf (1959) and Fowler *et al.* (1967, equation 3).

If $n = 3$, the expansion has the form

$$\left.\begin{aligned} &\mathbf{U}^+\,\mathbf{SU} = \lambda_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (\lambda_2 - \lambda_3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\lambda_1 - \lambda_2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ &\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3, \quad {P_3}^2 = 0{\cdot}0, \quad {P_2}^2 = 1/4, \quad {P_1}^2 = 1{\cdot}0, \\ &R_3 = \frac{3\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}, \quad R_2 = \frac{2(\lambda_2 - \lambda_3)}{\lambda_1 + \lambda_2 + \lambda_3}, \quad R_1 = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}, \\ &\text{and} \\ &P^2 = \frac{1}{2} \frac{(\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2 + \lambda_3)^2}. \end{aligned}\right\} \tag{32}$$

Further simplicity in interpreting $\mathbf{D}_1$ can be found by diagonalizing the real symmetric part of $\mathbf{D}_1$ (denoted by $R_e\,\mathbf{D}_1$). Consider first the submatrix $\mathbf{F}$ composed of the elements $F_{lk} = J_{lk}$ $(l, k = 1, 3)$, where

$$\mathbf{J} = \mathbf{R}^T\,\mathbf{D}_1\,\mathbf{R} \quad \text{and} \quad \mathbf{R}^T\,R_e\,\mathbf{D}_1\,R = \text{diag}\,[J_{11}, J_{22} \ldots J_{nn}].$$

It is possible to show that since $F_{ll}\,F_{kk} - {F_{lk}}^2 = 0$, at least one of F_{ll} must be zero. From this starting point we can prove by induction that $\mathbf{J}$ can have at most two non-zero diagonal elements. Thus we can always find an orthogonal transformation $\mathbf{R}$ such that

$$\left.\begin{aligned} &\mathbf{R}^T\,\mathbf{D}_1\,\mathbf{R} = \text{diag}\,[\mathbf{J}, \mathbf{O}_{n-2}], \\ &\mathbf{J} = \begin{bmatrix} J_{11} & iJ_{12} \\ -iJ_{12} & J_{22} \end{bmatrix}, \quad \text{and } |\mathbf{J}| = 0. \end{aligned}\right\} \tag{33}$$

The matrix $\mathbf{J}$ is identical to that for an elliptically polarized wave (see e.g. Fowler *et al.* 1967). The eigenvector corresponding to the larger of J_{11} and J_{22} is along the principal axis of the ellipse, and the ellipticity is J_{12}/J_{ll}, where J_{ll} is the larger of J_{11} and J_{22}. The sign of the ellipticity gives the sense of rotation. In practice the concept of an elliptically polarized wave is meaningful only if ${}^1\mathbf{x}(t)$ is quasi-monochromatic, i.e. $\lim_{\Delta f \to 0} \big(\lambda_1(\nu, \Delta f) - \lambda_2(\nu, \Delta f)\big)/\Delta f$ has a local maximum at a frequency ν_0 with a peak width $\Delta\nu$ such that $\Delta\nu/\nu_0 \ll 1$. The concept of an elliptically polarized wave is also difficult to interpret if $n > 3$.

All the characteristics of $\mathbf{D}_1$ apply to the matrices $\mathbf{E}_{ll}$ (equation (20)), implying that we can also interpret any arbitrary perturbation-vector as the incoherent superposition of n elliptically polarized waves.

Discussion of the case $n = 3$

In the analysis of ULF phenomena one is normally concerned only with perturbations with three spatial components, so an expanded discussion of the case $n = 3$ is in order.

Equation (33) indicates that in general the strictly polarized process is restricted to a plane determined by the real eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$ corresponding to the eigenvalues J_{11} and J_{22}. If ${}^1\mathbf{x}(t)$ is not linearly polarized (i.e. $J_{11}, J_{22} \neq 0{\cdot}0$), then the third eigenvector $\mathbf{r}_3$, which is perpendicular to the plane, can be obtained directly from the imaginary part of $\mathbf{D}_1$ by noting that

$$\mathbf{R}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{R}^T = \begin{bmatrix} 0 & r_{33} & -r_{32} \\ -r_{33} & 0 & r_{31} \\ r_{32} & -r_{31} & 0 \end{bmatrix} \tag{34}$$

where

$$\mathbf{r}_3 = [r_{31}, r_{32}, r_{33}]^T, \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}.$$

Equation (34) is simply a representation of the vector product of $\mathbf{r}_1$ and $\mathbf{r}_2$, and the matrix on the right-hand side is the 3-square antisymmetric matrix representation of the 3-dimensional vector $\mathbf{r}_3$ (see, e.g. Jeffreys 1963, Chapter 1). Equation (33) indicates that $\mathbf{r}_3$ is also an eigenvector of $\mathbf{D}_1$, but this does not imply that it is an eigenvector of $\mathbf{D}_2$ and S, because $\mathbf{r}_3$ corresponds an eigenvalue (zero) for which $\mathbf{D}_1$ has a two-fold degeneracy. If $R_2 = R_3 = 0{\cdot}0$, and the wave is known to be polarized transverse to the direction of propagation, then $\mathbf{r}_3$ is parallel or antiparallel to the wave-normal direction. Means (1972) has discussed this case in some detail.

More general conditions for the existence of a *preferred* plane of polarization are that $R_2/R_1 \ll 1{\cdot}0$ or that one eigenvalue of $R_e\,\mathbf{D}_2$ be zero (i.e. there exists a real vector $\mathbf{r}$ such that $R_e\,\mathbf{D}_2\,\mathbf{r} = \mathbf{0}$). If $R_e\,\mathbf{D}_2\,\mathbf{r} = \mathbf{0}$, then it is a simple matter to show that since $\mathbf{D}_1$ and $\mathbf{D}_2$ commute, $\mathbf{D}_1\,\mathbf{r} = \mathbf{D}_2\,\mathbf{r} = \mathbf{0}$, and

$$\mathbf{R}^T\,\mathbf{D}_2\,\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{35}$$

where $\mathbf{R}$ is the orthogonal matrix from equation (33). Thus

$$\mathbf{R}^T\,\mathbf{S}\mathbf{R} = \mathrm{diag}\,[\mathbf{K}, \lambda_3], \tag{36}$$

where λ_3 is the minimum eigenvalue of $\mathbf{S}$, and

$$\mathbf{K} = \lambda_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\lambda_1 - \lambda_2)\begin{bmatrix} J_{11} & iJ_{12} \\ -iJ_{12} & J_{22} \end{bmatrix}. \tag{37}$$

The structure of the matrix on the right-hand side of (36) implies that the concept of a preferred plane is meaningful in this case, because the wave can be considered to be the incoherent superposition of a plane wave and a completely unpolarized wave. The condition $R_2/R \ll 1{\cdot}0$ is trivial because (37) reduces to

$$\mathbf{K} \approx \lambda_1 \begin{bmatrix} J_{11} & iJ_{12} \\ -iJ_{12} & J_{22} \end{bmatrix}.$$

Examples

In this section I would like to give some examples of the numerical computation of the parameters discussed in the previous section. First one should note that particular care must be used in interpreting the numerical estimates of the polarization ratio P and the ratios R, since the expectation and variance of these parameters is intimately related to the actual process of analysis. For example, if no smoothing of the spectral estimates is used, $P = R_1 = 1{\cdot}0$ and $R_2 = R_3 = \dots R_n = 0{\cdot}0$ independent of the statistical nature of the multivariate stochastic process (see Jenkins & Watts 1968 for a discussion of the analogous case for the bivariate coherencies).

To interpret the parameters accurately we must find some method to determine the bias and variance of the elements of the estimated spectral matrix and the parameters determined from this matrix. In the examples that follow, the estimated spectral matrix $\mathbf{S}_E(\nu, \Delta f)$ is computed from

$$\mathbf{S}_E(\nu, \Delta f) = \frac{1}{N} \sum_{k=l-m}^{l+m} \mathbf{T}(k) \tag{38}$$

where

$$\mathbf{T}(l) = \frac{1}{N} \mathbf{z}(l)\, \mathbf{z}^+(l),$$

$$\mathbf{z}(l) = \sum_{t=0}^{N-1} \exp(-2\pi i l t)\, \mathbf{y}(t),$$

$$l = 0, 1, \dots \frac{N}{2} - 1, \quad \nu = \frac{l}{N\Delta t}, \quad \Delta f = \frac{2m+1}{2N\Delta t}, \quad \mathbf{y}(t)$$

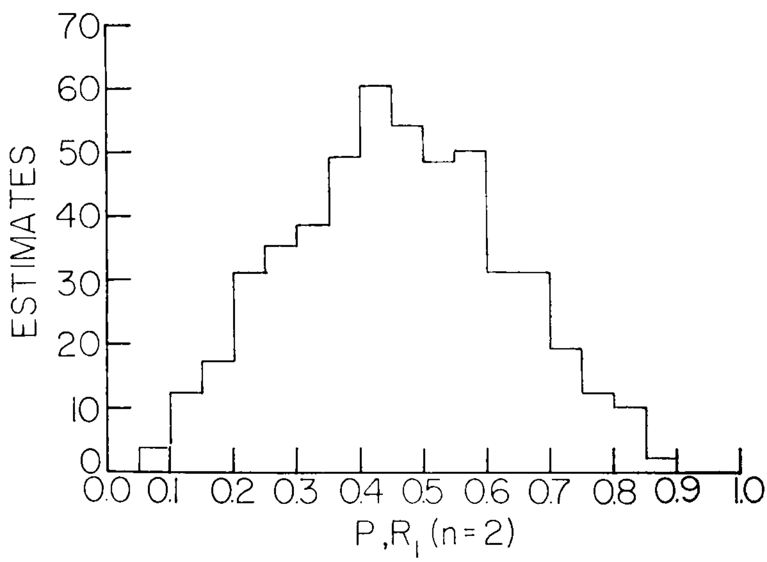

FIG. 1. Histogram of the values of $P(n = 2)$ and R_1 which were calculated using a spectral window with 7 degrees of freedom ($k = 7$).

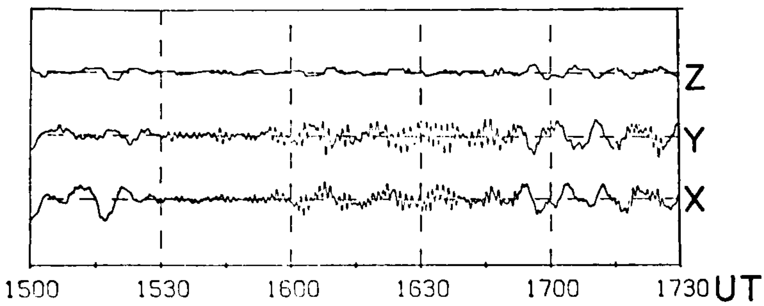

FIG. 2. Fluxgate recordings of geomagnetic micropulsations. The sensitivity is $12 \cdot 0\ \gamma$ (10^{-5} gauss) per dashed line. The time scale is marked in hours of universal time. X is magnetic north, Y is magnetic east, and Z is vertical (downward). These recordings have been detrended with a 1–20 mHz digital filter.

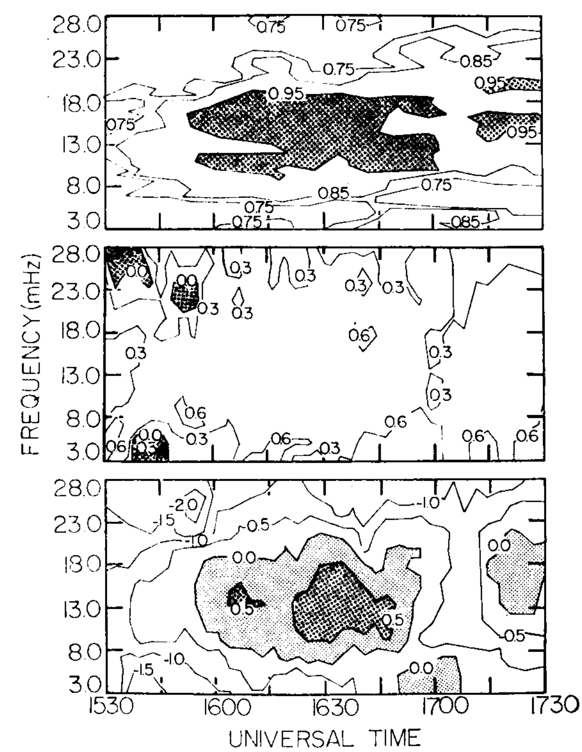

FIG. 3. Contour-plots of the frequency-dependent and time-dependent polarization characteristics of the pulsations in the X and Y components shown in Fig. 2. Top to bottom: $P(n = 2)$, ellipticity (J_{12}/J_{11}), and $\log_{10} (\lambda_1 - \lambda_2)$ (see equation (30)). λ_1 and λ_2 are in units of γ^2. The data window has 7 degrees of freedom and the corresponding width $(2\Delta f)$ is $\sim 7 \cdot 0$ mHz.

is the sampled vector time series, N is the number of sample points, and Δt is the sample interval. In certain cases $k\mathbf{S}_E$ has a complex Wishart distribution of dimension n, degrees of freedom $k = 2m+1$ (see Goodman 1963; Parzen 1969; and Hannan 1970 for more details). Although the distribution of $\mathbf{S}_E$ can be approximated, it still remains a formidable task to estimate the distribution functions of many of the parameters discussed in this paper. Almost all are non-linear functions of random variables. Probably the simplest approach is to make specific assumptions about the statistical properties of the n-variate stochastic process, and then by numerical methods generate empirical tables of the distribution functions of the various parameters.

Fig. 1 shows a histogram of values of $P(n = 2)$ which were obtained using this simple technique. The values of P were computed from two, uncorrelated, gaussian processes which were produced by a computer random number generator. In this case the spectral window had 7 degrees of freedom ($k = 7$), and obviously the estimates are highly biased. The mean values of P and R_1 are near $0{\cdot}4 \sim 0{\cdot}5$. However, 95 per cent of the estimates are below 0·75, and there are no estimates greater than 0·90. In using the value of P as a criterion for the existence of strictly polarized waves it is probably safe to choose P, $R_1 > 0{\cdot}90$. Conversely, to make an accurate estimate of $2\lambda_2$ and $(\lambda_1 - \lambda_2)$ in data where one suspects P to be less than 0·90 will require a data window with more degrees of freedom, leading to a loss in the spectral resolution.

To illustrate the usefulness of the descriptions of the polarization states, I have chosen two examples of 3-component time series of magnetic fluctuations which were recorded by a ground-based fluxgate magnetometer-system. The first recording (Fig. 2) shows examples of $Pc4$ geomagnetic micropulsations (see Jacobs 1970 for a description of the nomenclature), with the pulsations beginning at approximately 1530 UT and continuing with variable amplitude to 1730 UT. In this example, there is little power in the Z component and consequently a simplified analysis of only the X and Y components ($n = 2$) should be adequate.

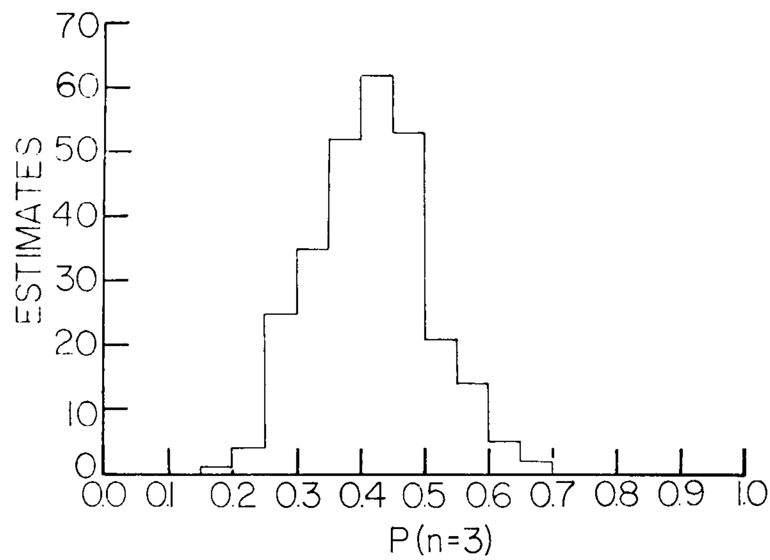

FIG. 4. Histogram of the values of $P(n = 3)$ which were calculated using a spectral window with 7 degrees of freedom.

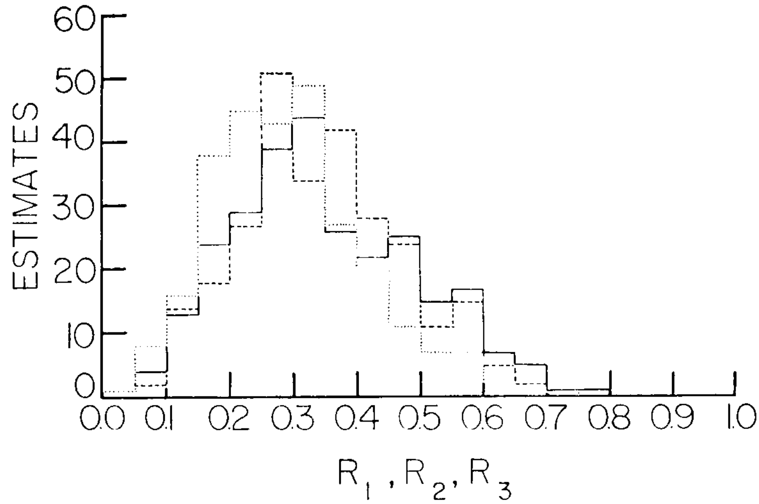

FIG. 5. Histograms of the values of R_1 (dotted line), R_2 (solid line), and R_3 (dashed line) which were calculated using a spectral window with 7 degrees of freedom.

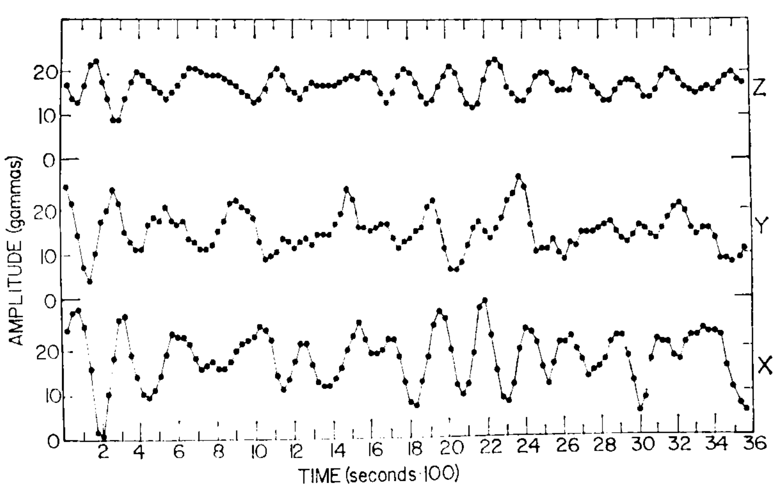

FIG. 6. Fluxgate recordings of geomagnetic micropulsations. The solid circles depict the sample points for the digital data. These recordings have been detrended with a 1–20 mHz digital filter.

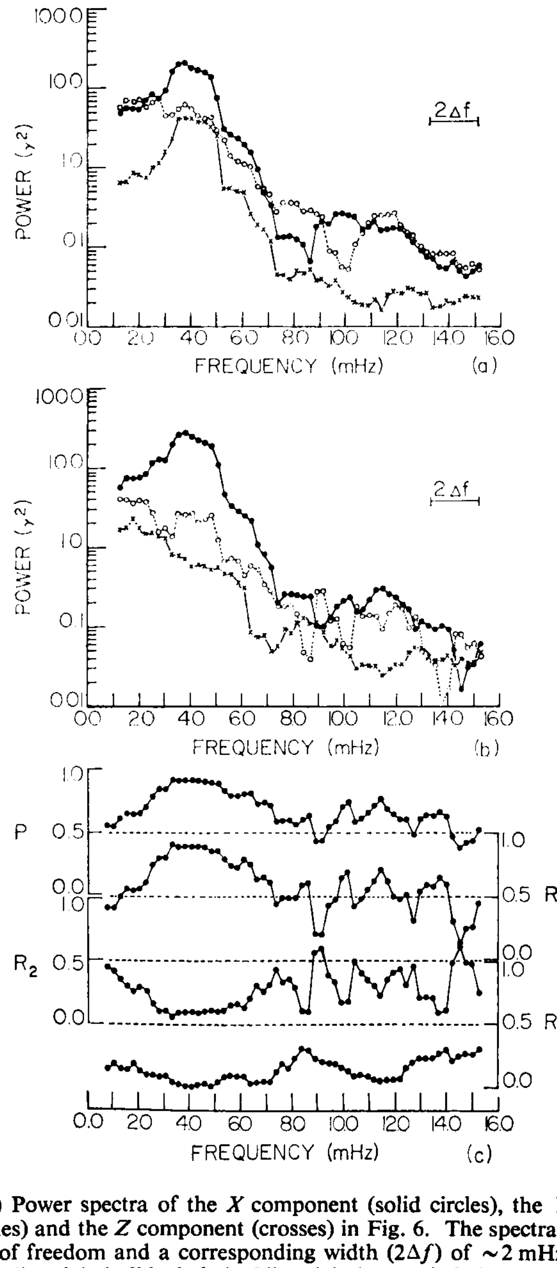

FIG. 7. (a) Power spectra of the X component (solid circles), the Y component (open circles) and the Z component (crosses) in Fig. 6. The spectral window has 7 degrees of freedom and a corresponding width ($2\Delta f$) of ~ 2 mHz. (b) Power spectra of $(\lambda_1-\lambda_2)$ (solid circles), $2(\lambda_2-\lambda_3)$ (open circles), and $3\lambda_3$ (crosses). (c) $P(n = 3)$, R_1, R_2 and R_3 for the recordings depicted in Fig. 6.

Table 1

The transformed matrices $\mathbf{D}_1(\nu = 4{\cdot}2\ mHz)$ *and* $\mathbf{D}_2(\nu = 4{\cdot}2\ mHz)$.

$$P = 0{\cdot}91,\ R_1 = 0{\cdot}90,\ R_2 = 0{\cdot}08,\ R_3 = 0{\cdot}02$$

$$\mathbf{R}_a = \begin{bmatrix} 0{\cdot}94 & -0{\cdot}34 & 0{\cdot}04 \\ 0{\cdot}23 & 0{\cdot}72 & 0{\cdot}65 \\ -0{\cdot}25 & -0{\cdot}60 & 0{\cdot}76 \end{bmatrix} \qquad \mathbf{R}_b = \begin{bmatrix} 0{\cdot}47 & 0{\cdot}89 & 0{\cdot}16 \\ 0{\cdot}87 & 0{\cdot}42 & -0{\cdot}27 \\ 0{\cdot}16 & -0{\cdot}27 & 0{\cdot}95 \end{bmatrix}$$

$$\mathbf{R}_a^T\,\mathbf{D}_1\,\mathbf{R}_a = \begin{bmatrix} 0{\cdot}75 & 0{\cdot}43i & 0{\cdot}0 \\ -0{\cdot}43i & 0{\cdot}25 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \end{bmatrix} \qquad \mathbf{R}_b^T\,\mathbf{D}_2\,\mathbf{R}_b = \begin{bmatrix} 1{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}71 & -0{\cdot}45i \\ 0{\cdot}0 & 0{\cdot}45i & 0{\cdot}29 \end{bmatrix}$$

Fig. 3 depicts the frequency-dependent and time-dependent changes in the polarization characteristics of pulsations in the X and Y components. The contour plot of the polarized power $(\lambda_1-\lambda_2)$ shows distinct spectral peaks in the $Pc4$ spectral band, with the largest peaks occurring near 1610, 1635, and 1720 UT. The centre frequency in the peaks changes from $\sim$13 mHz at 1630 UT to $\sim$16 mHz at 1730 UT. The contour plots of the ellipticity (J_{12}/J_{ll}) and $P(n = 2)$ indicate that P is greater than 0·95 for these pulsations and that the ellipticity is very stable with values between $+0{\cdot}3$ and $+0{\cdot}6$. In this example, a positive ellipticity indicates counter-clockwise polarization in the $X-Y$ plane, when viewed in the Z direction. All these features are typical of $Pc4$'s occurring in the morning hours in northern latitudes at the Earth's surface. One further point of interest is the correlation of the stable ellipticities with large values of P. Apparently when P is less than 0·75, the ellipticity is much more variable.

When we consider three-variate $(n = 3)$ stochastic processes, the interpretation of the polarization states becomes much more complicated. Figs 4 and 5 give histograms of the values of $P(n = 3)$, R_1, R_2, and R_3 which were derived from computer-generated gaussian processes. In this case, the mean value of P is approximately 0·4–0·45, but the variance appears to be somewhat less than for $P(n = 2)$. In using the value of P or R_1 to indicate the presence of a strictly polarized wave, the requirement that P, $R_1 > 0{\cdot}8$ should be adequate.

The second set of data to be analysed is depicted in Fig. 6. The most prominent waves in these plots are $Pc5$ geomagnetic micropulsations with a mean frequency of about 4·2 mHz (period = 240 s). The pulsations are evident in all three components, especially at 0 s and 1800 s. The plots of the power spectra Fig. 7(a) show marked

Table 2

The transformed matrices $\mathbf{D}_1(\nu = 12{\cdot}2mHz)$ *and* $\mathbf{D}_2(\nu = 12{\cdot}2\ mHz)$

$$P = 0{\cdot}62,\ R_1 = 0{\cdot}48,\ R_2 = 0{\cdot}43,\ R_3 = 0{\cdot}08$$

$$\mathbf{R}_a = \begin{bmatrix} -0{\cdot}50 & 0{\cdot}86 & 0{\cdot}07 \\ 0{\cdot}85 & 0{\cdot}47 & 0{\cdot}23 \\ -0{\cdot}16 & -0{\cdot}18 & 0{\cdot}97 \end{bmatrix} \qquad \mathbf{R}_b = \begin{bmatrix} 0{\cdot}98 & -0{\cdot}10 & 0{\cdot}19 \\ 0{\cdot}05 & 0{\cdot}97 & 0{\cdot}26 \\ -0{\cdot}21 & -0{\cdot}24 & 0{\cdot}95 \end{bmatrix}$$

$$\mathbf{R}_a^T\,\mathbf{D}_1\,\mathbf{R}_a = \begin{bmatrix} 0{\cdot}60 & -0{\cdot}49i & 0{\cdot}0 \\ 0{\cdot}49i & 0{\cdot}40 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \end{bmatrix} \qquad \mathbf{R}_b^T\,\mathbf{D}_2\,\mathbf{R}_b = \begin{bmatrix} 1{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}99 & -0{\cdot}10i \\ 0{\cdot}0 & 0{\cdot}10i & 0{\cdot}01 \end{bmatrix}$$

spectral peaks for both the X and Z components, indicating that a quasi-monochromatic interpretation of the polarization characteristics would be useful. There are no other significant peaks in the three spectra.

The power spectra of ${}^1\mathbf{x}(t)$, ${}^2\mathbf{x}(t)$, and ${}^3\mathbf{x}(t)$ are plotted in Fig. 7(b). The spectrum of ${}^1\mathbf{x}(t)$, i.e. $(\lambda_1(\nu, \Delta f)-\lambda_2(\nu, \Delta f))$ shows a distinct peak centred at 4·2 mHz, and although the pulsations depicted in Fig. 6 may appear to be rather irregular, the plots in Fig. 7(c) show that they are highly polarized. Both $P(n = 3)$ and R_1 exceed 0·90 in the region of the spectral peak. Another feature which is evident in Fig. 7(c) is that the variances of P, R_1, R_2 and R_3 are large when P is less than 0·7.

Table 1 shows the computed matrices $\mathbf{D}_1$ ($\nu = 4{\cdot}2$ mHz) and $\mathbf{D}_2$ ($\nu = 4{\cdot}2$ mHz), transformed such that the real off-diagonal terms are zero (see equation (33)). The corresponding orthogonal transformations are $\mathbf{R}_a$ and $\mathbf{R}_b$, respectively. Inspection of $\mathbf{R}_a$ and $\mathbf{R}_a{}^T\mathbf{D}_1\mathbf{R}_a$ indicates that ${}^1\mathbf{x}(t)$ is highly elliptical with an ellipticity of $+0{\cdot}60$, and that the principal axis is very close to the X direction (note that a unit vector in the X direction has components $[1, 0, 0]^T$ and in the Y direction $[0, 1, 0]^T$. $\mathbf{R}_b$ and $\mathbf{R}_b{}^T\mathbf{D}_2\mathbf{R}_b$ show that ${}^2\mathbf{x}(t)$ is restricted largely to the $X-Y$ plane, although ${}^2\mathbf{x}(t)$ is definitely not a simple plane wave. In any case, R_2 is very small and consequently $\mathbf{D}_2$ probably has little physical significance.

Table 2 gives the matrices $\mathbf{R}_a{}^T\mathbf{D}_1(\nu = 12{\cdot}2\text{ mHz})\mathbf{R}_a$ and $\mathbf{R}_b{}^T\mathbf{D}_2(\nu = 12{\cdot}2\text{ mHz})\mathbf{R}_b$. At this frequency, one diagonal element of $\mathbf{R}_b\mathbf{D}_2\mathbf{R}_b$ is almost zero, and consequently both $\mathbf{D}_1$ and $\mathbf{D}_2$ have real eigenvectors which are approximately parallel (see $\mathbf{R}_a$ and $\mathbf{R}_b$ in Table 2). In this example the eigenvectors are directed in the Z direction as would be expected from an inspection of the spectra in Fig. 7(a). Thus at this frequency, the waves can be considered to be plane polarized, and restricted to the $X-Y$ plane (see equations (35), (36) and (37)).

Summary

The expansions of the spectral matrix which are given in this paper can lead to more simplified and objective descriptions of the polarization states of vector processes. In selecting strictly polarized waves, the degree of polarization P can be determined directly from the Stokes parameters, and then only those waves for which $P \approx 1{\cdot}0$ can be chosen for analysis. The interpretation of the polarization states of these waves is simplified by diagonalizing the real part of $\mathbf{S}$, and if the wave is quasi-monochromatic, the parameters of the polarization ellipse can be computed directly from the elements of the transformed matrix. Conversely, if $n = 3$ and only the vector perpendicular to the plane of an elliptically polarized wave is required, then this vector can be computed directly from $\mathbf{S}$ without first transforming the matrix.

Interpretation of polarization states when P is less than 1·0 is more difficult, and in these cases it is probably easiest to expand $\mathbf{S}$ in the set of commuting, idempotent matrices. Although the general form of the idempotent expansion can be very complicated, interpretation of the cases $n = 2$ and $n = 3$ can be considerably simplified by diagonalizing the real parts of the idempotent matrices. By following this procedure, the information in $\mathbf{D}_1$ can always be represented by a two-square matrix. This procedure also gives an indication of whether $a_1\mathbf{D}_1 + a_2\mathbf{D}_2 (n = 3)$ can be considered to be the matrix of a plane wave.

The descriptions of the polarization states which are outlined in this paper should be particularly useful in analysing the frequency-dependent polarization characteristics of irregular waveforms, or waves accompanied by broad band noise. Some examples of ULF magnetic fluctuations for which the descriptions should be very useful are continuous emissions, which may in some cases have structured components (see, e.g. Jacobs 1970), Pi2 micropulsations accompanying geomagnetic substorms, and the irregular *Pc*5's which occur during the recovery phase of substorms (see, e.g., McPherron *et al.* 1972). Objective descriptions of the polarization states of these waves will certainly lead to a better understanding of the physical processes which are involved, will allow more direct comparisons of the data with pertinent theories, and will simplify the comparison of data obtained through different experiments.

Acknowledgments

Funds for the computer-analysis were obtained through NASA grant NGR 05-009-081. I also wish to thank the National Research Council of Canada for financial assistance through a post-doctoral fellowship.

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