Design and Upgrade of Nonredundant and Redundant Linear Sensor Networks

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The design of sensor networks featuring minimum cost, while satisfying constraints of redundancy only, is examined. For this purpose, the concept of degree of redundancy for measurements and degree of observability for unmeasured variables are merged into one single property, the degree of estimability of a variable. In addition, the concept of estimation efficiency is introduced. Based on these concepts, mathematical programming procedures are presented that allow the design of sensor networks for different degrees of estimability of key variables.

Introduction

In recent years, the scope of data usage has increased from process monitoring, control and production accounting goals, to fault detection and on-line/off-line optimization. As a result, the need for quality and availability of data has increased. Since process measurements contain random errors, data reconciliation has been used to adjust the measurements so that they comply with conservation laws. In this way, redundancy of measurements, through hardware duplication and through software, has been used to correctly assess the value of different variables. As methods to perform data reconciliation and gross error detection proliferate, the need for systematic procedures to design sensor networks has emerged.

Different approaches and driving forces have been used for the design of sensor networks. Vaclavek and Loucka (1976) used graph theory to guarantee variable observability, Kretsovalis and Mah (1987) used a combinatorial search based on the effect of the variance of measurements on the precision of key variables. Madron and Veverka (1992) proposed to classify measured and unmeasured variables of linear systems according to a pre-established criterion of ''required'' and ''nonrequired.'' Unmeasured variables are later ordered from ''hardly measured'' to ''easily measured.'' Madron proposed to use two objective functions: cost and overall precision of the system. By means of matrix decomposition and an elaborate column permutation procedure, suboptimal structures are found. Madron (1992) also presents details of this procedure based on graph theory. The concept of cost-edged graph is introduced and minimum spanning trees of these graphs are used to obtain minimum cost or optimal overall precision sensor networks. However, the method cannot target desired precision levels on individual variables. Ragot et al. (1992) presented a procedure that allows the identification of the set of sensors for which the system becomes observable. Luong et al. (1994) presented a method that provides solutions that feature minimal observability of those variables required for control and high degree of redundancy of variables. They use reliability as means of screening alternatives with equal cost. Maquin et al. (1994) proposed to obtain the location of sensors by inverting the expression that provides the variance of reconciled variables as a function of the variance of measurements. Ali and Narasimhan (1993) proposed to maximize reliability, which is based on sensor failure probability, observability of variables, as well as redundancy. While looking at all networks containing the minimum set of sensors to achieve observability, they propose a Max-Min problem using reliability as the objective function. Another graph oriented procedure was proposed by Meyer et al. (1994) using cost as the objective function and providing solutions featuring networks containing the minimum set of sensors. Lately, Ali and Narasimhan (1995) extended their previous work to redundant networks. Their algorithm uses graph theory to build networks with a specified number of sensors and maximum reliability. Finally, in a recent article, Sen et al. (1998) presented a genetic algorithm that can be applied to design nonredundant sensor networks using different objectives functions.

Departing from graph theory and linear algebra approaches, Bagajewicz (1997) proposed a MINLP problem to

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obtain cost-optimal network structures for linear systems subject to constraints on precision, residual precision and error detectability. Finally, the connection between cost-optimal and maximum precision mathematical programming models was recently established by Bagajewicz and Sánchez (1999).

While all the aforementioned work deals with several aspects of the sensor network design and covers an ample spectrum of goals, often there is no data available to implement these methods. For example, the reliability of a proposed instrument is often unknown or guidelines to pick bounds on precision are many times absent, and so on. Therefore, there is a need of developing design procedures that focus on simple goals.

The purpose of this article is to present a methodology for the grassroots design or the upgrade of a sensor network in a process plant with the goal of achieving a certain degree of observability and/or redundancy for a specific set of variables. The various forms in which a particular variable can be redundant and observable have been hardly explored, the only exception being the work done by Luong et al. (1994) and Maquin et al. (1994).

This article discusses the goals of sensor network design and/or upgrade first. Then, the concept of degree of estimability and its connections to observability and redundancy are introduced. In the next section, the relation between degree of estimability, spanning trees, and cutsets of the process graph is established. Following, a new method that allows the design with estimability goals is presented. Several examples are included throughout the text.

Upgrade and/or Design Goals

When it is necessary to have data available, the options are: measure the variable directly, or if it results less expensive, measure other variables that will allow the calculation of the desired value using model equations. When this idea is extended to a large system, several objectives can be identified:

- If the value of *all* variables is of interest, then the objective becomes to design a system for which all unmeasured variables are observable and all measured variables are nonredundant. However, if only the value of key variables is required, some may remain unobservable. Thus, a design goal should be that of observability in only the set of variables of interest.

- If some reassurance is desired that data for a variable will remain observable in the presence of sensor failures, then a certain degree of redundancy is required. Although this issue can also be addressed in light of the concept of reliability (Ali and Narasimhan, 1993), this angle of analysis is omitted in this article.

In addition, restrictions on measuring certain variables can be imposed due to, for example, space limitations, lack of proper access for calibration, and so on. In this article, cost is excluded from this list, as this issue will be handled by the design procedure itself. In other cases, the measurement of some variables may be made compulsory. Aside from economical reasons, reasons for such compulsory choice can be related to ease and/or small frequency of maintenance, or even political, as many production accounting personnel may feel uneasy about not measuring certain variables that they consider of importance. These fears may not be unfounded, as, for example, one very reliable instrument may fail less often than, for example, a set of two instruments.

Estimability

A generalized definition of observability was attempted by Ali and Narasimhan (1993, 1995) to denote as observable any variable measured or unmeasured for which an estimate can be produced. The new name of estimability is preferred to avoid confusion and reserve the name observable to unmeasured variables, as it has become popular in the literature. Thus, the definition of estimability is formally presented.

Definition. A variable i is estimable if it is measured, or unmeasured but observable.

Degree of Estimability

In this section the concepts of degree of observability and redundancy are reviewed. In addition the concept of degree of estimability is introduced. It unifies previous definitions in a single one, making the distinction of measured and unmeasured unnecessary.

The concepts of degree of observability and degree of redundancy have been introduced by Luong et al. (1994). For convenience, they are presented in this article in a slightly different, but equivalent, manner.

Observability and redundancy

The concepts of observability and redundancy in linear systems are well known. A nonmeasured variable is observable if it can be calculated in at least one way from the measurements. A measurement is redundant if it can also be calculated in at least one way from the remaining measurements. Finally, a measurement of a variable is nonredundant if after removing this measurement, the variable is unobservable.

Redundancy is, therefore, a desirable property of a system because in the case when an instrument fails, its variable can be estimated through balances. Moreover, if the number of different balances that can be used increases, there will be additional ways to calculate the variable. We say that the reliability of such a system and the precision of the estimation increase. There is, therefore, a need to distinguish these different levels of redundancy.

Degree of Redundancy of a System. *Definition.* A system has a degree of redundancy k, when, at least, k linearly independent balance equations can be written using measured variables only.

Let us represent the material balances around the n_U process units as

$$Dx = 0 \tag{1}$$

where D is a $(n_U \times n)$ matrix and n stands for the number of process streams. By using simple rearrangement of columns in matrix D, the system can be rewritten in the following way

$$\begin{bmatrix} D_U & D_M \end{bmatrix} \begin{bmatrix} x_U \\ x_M \end{bmatrix} = 0 \tag{2}$$

where x_U corresponds to unmeasured flow rates and x_M stands for the measured ones.

Depending on the topology of the process and the location of sensors, unmeasured variables are classified into observable and unobservable. Measured variables are divided into redundant and nonredundant. Variable classification is therefore an essential tool for data reconciliation and the design of monitoring systems. Following one existing classification strategy (Madron, 1992), linear combinations of rows and column rearrangements are performed to obtain the following system, where D is already in its canonical form

$$\begin{array}{c} \begin{array}{cccc} \text{Observable} & \text{Unobservable} & \text{Redundant} & \text{Non-redundant} \end{array} \\ \left[\begin{array}{cccc} I & 0 & -G_{RO} & -G_{NRO} \\ 0 & G_{UO} & G_{RUO} & G_{NRUO} \\ 0 & 0 & G_R & 0 \end{array}\right] \\ \begin{array}{cc} \underbrace{\hphantom{I \quad G_{UO}}}_{\text{Unmeasured}} & \underbrace{\hphantom{G_{RUO} \quad G_{NRUO}}}_{\text{Measured}} \end{array} \end{array} \begin{bmatrix} x_O \\ x_{UO} \\ x_R \\ x_{NR} \end{bmatrix} = 0 \qquad (3)$$

Rewriting system 3 only for observable and measured variables, the following expressions are obtained

$$x_O = G_{RO} x_R + G_{NRO} x_{NR} \qquad (4)$$

$$G_R x_R = 0 \qquad (5)$$

A third equation involving unobservable variables can be written. However, this equation is irrelevant for the analysis that follows.

The system degree of redundancy can now be given a mathematical interpretation. First, note that matrix G_R is of full row rank. If this was not the case, there would be one balance that can be written as a linear combination of the others. However, by construction, D has as many rows as units and, thus, all balances are linearly independent. Thus, since the degree of redundancy is the number of *l.i.* balance equations that can be written with measured quantities, it is equal to the rank of G_R.

In principle, this concept implies that a system has poor redundancy when too many measurements are nonredundant. However, this concept does not fully encompass the richness of the different types of system redundancies that one can find. For example, systems can have the same number of measurements and the same number of units, but different degrees of redundancy. The following example illustrates this.

Consider the system of three units depicted in Figure 1a. Assume all streams are measured. Thus, it contains three units and four measurements resulting in a degree of redundancy of 3. Consider now the system in Figure 1b, and assume that all but S_5 and S_6 are measured. As in the case of Figure 1a, this system contains three units and four measurements. However, its degree of redundancy is 1. If the set of measured streams is $x_M = \{S_1, S_2, S_4, S_5\}$, the degree of redundancy is also one, but if $x_M = \{S_1, S_2, S_3, S_6\}$, then the system degree of redundancy is 2.

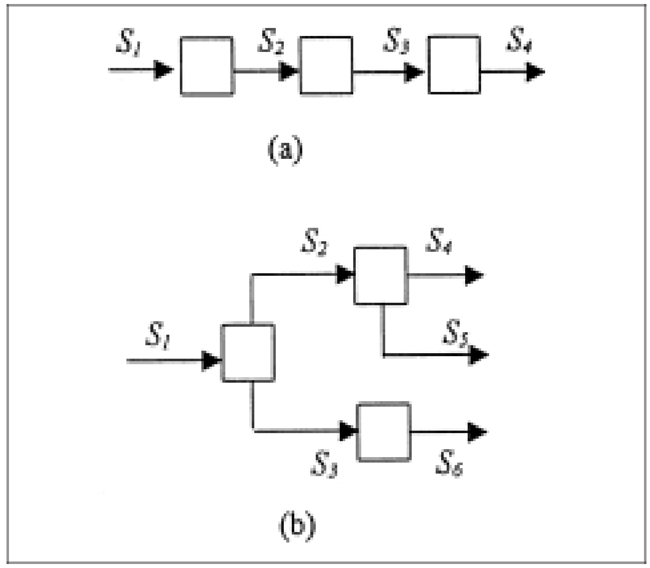

Figure 1. System degree of redundancy.

The degree of redundancy of a system is thus a condition that reflects how effectively a certain number of measurements is distributed throughout the system. For example, two systems can have the same number of measurements but a different system degree of redundancy, as in one case, too many variables are redundant and in the other too many are nonredundant. However, not much connection has been found between the degree of redundancy of the system and specific goals, such as reliability, gross error detectability, resilience, or the availability of precision under sensor failure, other than the fuzzy statement of saying that the more redundancy, the better. For this reason, the attributes of specific variables at specific locations will be analyzed in the rest of the article.

Degree of Observability of Variables. For convenience, we denote $A(p)$ as the set of all possible combinations of p measurements. We call $A_j(p)$, the jth element (combination) of this set. We are now ready for the following definition:

Definition. An unmeasured variable i has the degree of observability O_i if: {it remains observable after the elimination of any combination $A_j(O_i - 1) \in A(O_i - 1)$} AND {it becomes unobservable when at least one set $A_j(O_i) \in A(O_i)$ is eliminated}.

Consider the system of Figure 2, if $x_M = \{S_1, S_2\}$, then variable S_6 has the degree of observability $O_6 = 1$. In turn, if $x_M = \{S_1, S_2, S_3\}$ then $O_6 = 2$, because elimination of one measurement at a time (S_1, S_2 or S_3) would not make it unobservable. However, a deletion of any of the following two sets: (S_1, S_2) (S_1, S_3), would render it unobservable. Note, however, that the elimination of the set (S_2, S_3) would not make S_6 unobservable.

Degree of Redundancy of Variables. For convenience, we denote $B(p, S_i)$ as the set of all possible combinations of p

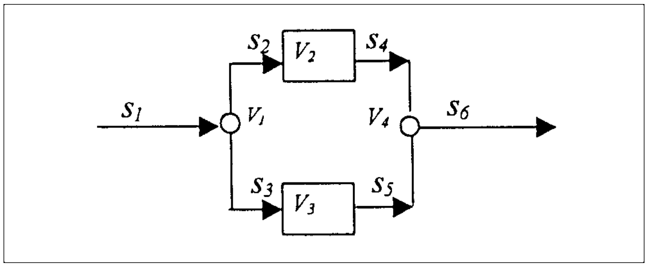

Figure 2. Concept of degree of observability.

measured variables, not including the measured stream S_i. We call $B_j(p, S_i)$ the jth element (combination) of this set. The following definition is then presented.

Definition. A redundant measured variable i has degree of redundancy R_i if: {it remains redundant after the elimination of any combination $B_j(R_i - 1, S_i) \in B(R_i - 1, S_i)$} AND {it becomes nonredundant when at least one set $B_j(R_i, S_i) \in B(R_i, S_i)$ is eliminated}.

Remark. Accordingly, the degree of redundancy of a nonredundant measurement is zero.

In the system of Figure 2, for $x_M = \{S_1, S_2\}$, variable S_1 has a degree of redundancy $R_1 = 0$, because it is already nonredundant. If S_3 is measured, $x_M = \{S_1, S_2, S_3\}$, then S_1 has degree of redundancy $R_1 = 1$ because it is sufficient to eliminate S_2 or S_3 to make it nonredundant.

A redundant measurement is such that the variable becomes observable when the measurement is eliminated. A nonredundant variable, in turn, becomes unobservable if its measurement is eliminated. Thus, if a variable has degree of redundancy R_i, the elimination of its measurement will make it a variable of degree of observability $O_i = R_i$.

For example, if for Figure 2, $x_M = \{S_1, S_2, S_3\}$, then the elimination of S_1 makes it a nonmeasured variable with degree of observability $O_1 = 1$, because it is enough to eliminate S_2 or S_3 to make it unobservable.

Remark. From the above discussion, the system degree of redundancy is directly the number of rows of G_R, and is not related to the number and degree of redundancy of other measured variables.

Degree of Estimability of Variables. Definition. A variable i (measured or not) has a degree of estimability E_i if: {it remains estimable after the elimination of any combination $A_j(E_i - 1) \in A(E_i - 1)$} AND {it becomes unobservable when at least one set $A_j(E_i) \in A(E_i)$ is eliminated}.

To illustrate the above definition, consider the process graph in Figure 3 and assume all flow rates are measured. In this case $B(1, S_2) = \{(S_1), (S_3), (S_4)\}$. Therefore, stream S_2 has a degree of redundancy $R_2 = 1$, because just the elimination of S_1 makes S_2 nonredundant. Since the elimination of S_1 makes S_2 nonredundant, and the elimination of either S_3 or S_4 does not alter its redundancy status, its degree of estimability is larger than one. Consequently, it is necessary to evaluate the elements of the set $A(2) = \{(S_1, S_2), (S_1, S_3), (S_1, S_4), (S_2, S_3), (S_2, S_4), (S_3, S_4)\}$. From this analysis, it can be easily seen that the elimination of (S_1, S_2) makes S_2 unobservable, thus $E_2 = 2$.

In a similar way, it can be shown that $R_3 = 2$, because if all the elements of $B(1, S_3) = \{(S_1), (S_2), (S_4)\}$ are individually eliminated, variable S_3 stays redundant, but becomes nonredundant if the elements (S_1, S_4) or (S_2, S_4) from $B(2, S_3) = \{(S_1, S_2), (S_1, S_4), (S_2, S_4)\}$ are deleted. The inspection of $A(2)$ and $A(3) = \{(S_1, S_2, S_3), (S_1, S_2, S_4), (S_2, S_3, S_4)\}$ helps to conclude that $E_3 = 3$, because the elimination of (S_2, S_3, S_4) renders S_3 unobservable.

The following are properties that follow naturally from the definition.

Property. The degree of estimability of a nonmeasured variable is equal to the degree of observability.

Property. The degree of estimability of a measured redundant variable is its degree of redundancy plus one.

Property. A measured nonredundant variable has a degree of redundancy zero and degree of estimability one.

Property. When the degree of estimability of a variable is larger than or equal to one, then the number of sensors that need to fail to render the variable unobservable is equal to its degree of estimability.

The property follows from the definition, but it has deep connections with the concept of reliability. In addition, it is independent of its original status of measured or unmeasured.

Property. A system where all variables have a degree of estimability equal to one is a system of nonredundant measured variables and observable unmeasured ones.

Proof. If a variable is measured and has degree of estimability one, then it is nonredundant. If in turn, it is unmeasured, it is observable (because its estimability is one) and its observability is given by nonredundant variables.

Property. If the degree of estimability of an unmeasured variable E_j is larger than one, then the entries in row j of G_{NRO} are all zero.

Proof. If any nonredundant measurement is used to estimate variable j, then the elimination of this variable will make S_j unobservable.

Graph Theory and Estimability

We now connect the concept of spanning tree and cutset to estimability. The Appendix presents a short overview of graph theory.

Property. A system where all variables have degree of estimability $E_i = 1$ corresponds to a system where all unmeasured streams are given by a spanning tree.

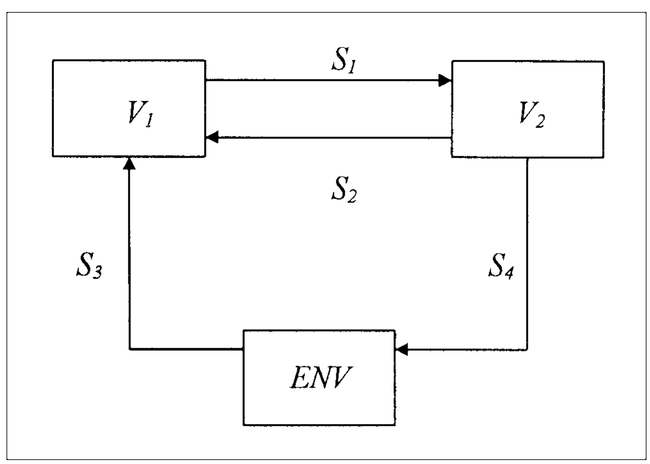

Figure 3. Reduced process graph.

Proof. If all variables have $E_1 = 1$, this implies that all unmeasured variables are observable and measured variables are nonredundant. If all unmeasured variables are observable, then I_D contains all of them, and therefore, as all the measured variables are nonredundant, there is no row with zeros corresponding to the columns of the unmeasured variables. Thus, I_D corresponds to all unmeasured variables that form a spanning tree.

Adding one measurement to a system can have a different effect on the estimability of the rest of the variables, depending on which measurement is added.

Consider the system of Figure 2 and assume the measured variables are given by $x_M = \{S_1, S_2\}$. The set of unmeasured variables is a spanning tree, as it can be observed in Figure 4. The degree of estimability of all variables is one, as they are all observable, and x_M is a set of nonredundant measurements. The canonical form of the incidence matrix is

$$\begin{array}{c} \begin{array}{cccccc} S3 & S4 & S5 & S6 & S1 & S2 \end{array} \\ \left[\begin{array}{cccc:cc} 1 & & & & -1 & 1 \\ & 1 & & & & -1 \\ & & 1 & & -1 & 1 \\ & & & 1 & -1 & \end{array}\right] \end{array} \tag{7}$$

Assume now that S_4 becomes measured. Then, the new canonical form is

$$\begin{array}{c} \begin{array}{cccccc} S3 & S5 & S6 & S4 & S2 & S1 \end{array} \\ \left[\begin{array}{ccc:cc:c} 1 & & & & 1 & -1 \\ & 1 & & & 1 & -1 \\ & & 1 & & & -1 \\ \hdashline & & & 1 & -1 & \end{array}\right] \end{array} \tag{8}$$

The set $\{S_2, S_4\}$ is now redundant, whereas S_1 remains nonredundant. However, S_3, S_5, and S_6 have still a degree of estimability $E_3 = E_5 = E_6 = 1$, because it is enough to delete S_1 to make any one of them unobservable. In turn, S_2 and S_4 increased their degree of estimability to $E_2 = E_4 = 2$.

Consider now that instead of S_4, S_5 is measured. Then, the new canonical matrix is:

$$\begin{array}{c} \begin{array}{cccccc} S3 & S4 & S6 & S5 & S1 & S2 \end{array} \\ \left[\begin{array}{ccc:ccc} 1 & & & & -1 & 1 \\ & 1 & & & & -1 \\ & & 1 & & -1 & \\ \hdashline & & & 1 & -1 & 1 \end{array}\right] \end{array} \tag{9}$$

The set $\{S_1, S_2, S_5\}$ is now redundant, and their degree of estimability is $E_1 = E_2 = E_5 = 2$. There are no nonredundant variables and the degrees of estimability of all the unmeasured variables is $E_3 = E_4 = E_6 = 2$.

We now turn our attention to the relation between cutsets and estimability.

Definition. A cutset is estimable if it includes no more than one unmeasured stream.

This definition relates to the estimability of the variables in the cutset. If all the variables in the cutset are measured, then all the variables are redundant, with a degree of redundancy of at least one. Since the variables could participate in other cutsets where all streams are measured, then the degree of redundancy can be higher. If one variable is unmeasured, then it is observable, and all the measured variables are nonredundant if this is the only cutset involving them. Otherwise, they may be redundant if they participate in other cutsets. Thus, an estimable cutset is a cutset in which all the variables have a degree of estimability *of at least one.*

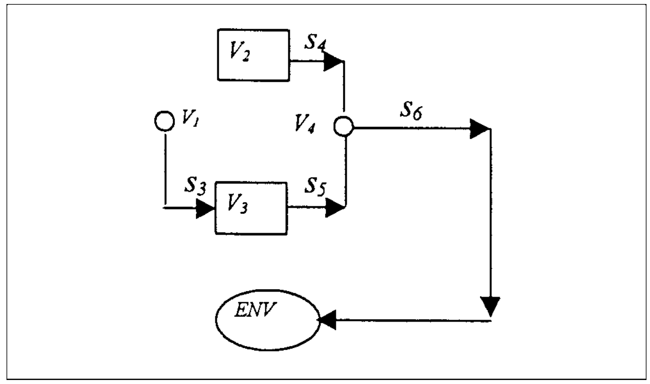

Figure 4. One spanning tree of the system of Figure 2.

Definition. An estimable cutset is redundant if all its streams are measured.

Property. The number of all cutsets containing a variable S_k is equal to the number of material balances that can be written involving variable S_k.

This property is self-evident.

Property. A bound on the maximum possible degree of estimability of an unmeasured variable S_k is given by the maximum number of cutsets containing S_k.

Proof. Consider all the cutsets containing variable S_k. A nonestimable cutset does not contribute to the estimation of S_k. In fact the estimability of S_k is due only to estimable cutsets. Assume now that S_k belongs to n estimable cutsets $C_{k,1}, C_{k,2}, \ldots, C_{k,n}$. Since S_k is unmeasured, then $C_{k,1} \cap C_{k,2} \cap \ldots \cap C_{k,n} = S_k$. Therefore, the elimination of a measurement belonging to $C_{k,1}$ makes this cutset nonestimable, but it has no effect on the observability of S_k, since it will still be guaranteed by the rest of the estimable cutsets. In fact, one variable per cutset will have to be eliminated to render S_k unobservable. Thus, S_k has a degree of estimability n. However, if the intersection of any subset of estimable cutsets contains at least one more variable in addition to S_k, *then* these subset of cutsets can be rendered nonestimable by just eliminating any variable belonging to such an intersection. Thus, if S_k is not measured, the maximum possible degree of estimability cannot be larger than the number of estimable cutsets.

Property. A bound on the maximum possible degree of estimability of a measured variable S_k is given by the number of redundant cutsets containing S_k plus one.

Proof. Assume first that out of the n estimable cutsets, the first m are redundant. Consider again the case that $C_{k,1} \cap C_{k,2} \cap \ldots \cap C_{k,n} = S_k$. Then, the elimination of a measurement belonging to $C_{k,i} (i \leq m)$ makes this cutset nonredundant. The elimination of S_k has no effect on the observability

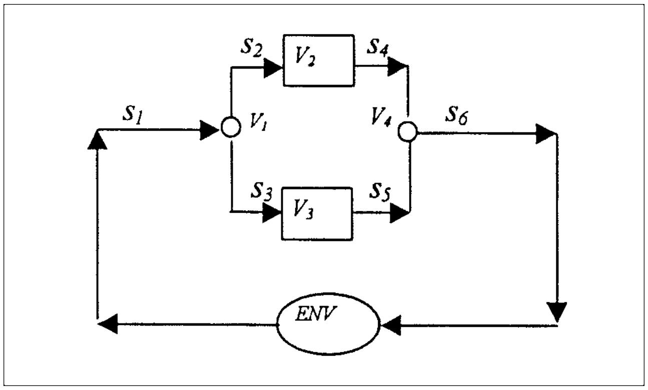

Figure 5. Augmented graph corresponding to the flowsheet of Figure 2.

of this variable, since the rest of the measurements of this cutset will still guarantee it. In fact, two variables per redundant cutset will have to be eliminated to render S_k unobservable. In general, if the measurement of S_k is eliminated, then the m redundant cutsets will become nonredundant, and, to make S_k unobservable, one additional measurement per redundant cutset will have to be eliminated.

Property. The degree of estimability of an unmeasured variable S_k, whose estimable cutsets have only S_k as an intersection, is equal to the number of these estimable cutsets. If the variable is measured, then its degree of estimability is the number of the estimable and redundant cutsets plus one.

This property is actually a corollary of the two previous properties.

To illustrate the above properties, consider the variable S_1 in the system of Figure 5 that represents the augmented graph of the process in Figure 2. All the cutsets containing S_1 are: $C_{1,1} = \{S_1, S_6\}$, $C_{1,2} = \{S_1, S_2, S_3\}$, $C_{1,3} = \{S_1, S_2, S_5\}$, $C_{1,4} = \{S_1, S_4, S_3\}$, $C_{1,5} = \{S_1, S_4, S_5\}$. These are obtained using linear combinations of the fundamental cutsets included in the canonical form of the incidence matrix (see the Appendix for details). Assume all variables, except S_1, are measured. This set of measurements should render all the cutsets containing S_1 estimable cutsets. In particular, a deletion of one variable per cutset will certainly make S_1 unobservable. However, the minimum number of variables is even smaller. The elimination of the measurements in variables S_2, S_4, and S_6 will render S_1 unobservable. Thus, in this case, the degree of estimability of S_1 is $E_1 = 3$, lower than the bound of five given by the number of estimable cutsets. However, if $x_M = \{S_2, S_3, S_6\}$, then only $C_{1,1}$ and $C_{1,2}$ are estimable cutsets. Thus, the bound on the degree of estimability of S_1 is two, and, indeed, this is the minimum number of variables required to render it unobservable.

If, in the same example, all variables are measured, then all cutsets are redundant, and the bound on the degree of estimability is six. However, in reality, the degree of estimability is four. Elimination of S_1, S_2, S_4, and S_6 is enough to render S_1 unobservable. However, if $x_M = \{S_1, S_2, S_3, S_6\}$, then only $C_{1,1}$ and $C_{1,2}$ are redundant cutsets. Now, the bound is three, which is indeed the degree of estimability of S_1.

Efficiency of Estimability

We now explore a concept that will be of fundamental importance when designing sensor networks. We are interested in the ability of the sensor network to provide the value of a certain variable, even when a certain number of sensors fail regardless of position. In this sense, the degree of estimability of a variable indicates the number of sensors that should fail to render it unobservable.

As was described above, there is an upper bound for the degree of estimability of a variable that is related to the number of estimable cutsets in which it participates. Let us consider that the degree of estimability of an unmeasured variable S_k is equal to the number of estimable cutsets; therefore, their only intersection is S_k. Assume now that no other cutset exists that could be made estimable, such that its intersection with the existing estimable cutsets is only variable S_k. In such a case, the maximum efficiency of the installed measurements has been obtained. Any other cutset made estimable will increase the number of measurements, but it will not increase the estimability of the variable. Now, this efficiency will be formally defined.

Definition. The efficiency of estimability of a variable is given by the ratio of the degree of estimability and the maximum degree of estimability.

For example, consider in Figure 5 the case where S_1 is unmeasured and $x_M = \{S_2, S_3, S_6\}$. Then, only two estimable cutsets exist, namely $C_{1,1} = \{S_1, S_6\}$, $C_{1,2} = \{S_1, S_2, S_3\}$. The intersection of these two cutsets is only S_1, and, therefore, its degree of estimability is two, and the efficiency is one. If we make the cutset $C_{1,5}$ estimable, the degree of estimability of S_1 is three. In other words, the efficiency of the estimability can go as low as $3/5$.

Thus, when a variable has an efficiency of estimability one, no unnecessary estimable cutsets have been used. In other words, any additional cutset made estimable by the addition of measurements may contribute to other goals, such as the increase of precision, but will not affect the degree of estimability.

We are now in a position to discuss another type of efficiency. Consider the case in which a certain unmeasured variable has a certain degree of estimability. Furthermore, assume that there exist cutsets whose intersection with all the estimable cutsets that contain S_k is $\{S_k\}$. Thus, one can eliminate the measurements of one estimable cutset and introduce measurements in another of these nonintersecting cutsets to make it estimable, and, therefore, maintain the same degree of estimability. This exchange of cutsets can continue until the minimum number of measurements is used. This motivates the following definition.

Definition. The minimum estimation cardinality of degree k of a nonmeasured variable is the smallest number of measurements that are needed to obtain a degree of estimability k for the variable.

The minimum cardinality can thus be obtained by finding a combination of k cutsets whose intersection is variable S_k only, and whose union has the minimum number of variables. Once the cutsets that include the variable in question are known, this is a simple task. All is needed to do is to enumerate the cutsets in increasing order of cardinality, pick the first k sets, and count all measurements involved. The efficiency defined as the quotient between the actual number of measurements involved and the minimum number can also be defined.

Consider again in Figure 5 the case where S_1 is unmeasured and $x_M = \{S_2, S_3\}$. Then, only one estimable cutset exists, namely $C_{1,2} = \{S_1, S_2, S_3\}$, and the degree of estimability of S_1 is $E_1 = 1$. However, since $C_{1,1} \cap C_{1,2} = S_1$, then the set $x_M = \{S_6\}$ can achieve the same degree of estimability, and, therefore, the minimum estimation cardinality of S_1 is one.

Design for Estimability

If, for each variable z_i, there is only one potential measuring device with associated cost c_i, then the total cost is given by

$$C(q) = \sum_{\forall i} c_i q_i \tag{10}$$

where q is a vector of binary variables defined by

$$q_i = \begin{cases} 1 & \text{if } z_i \text{ is measured} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Then, the design of the sensor network is an optimization problem that can be written as

$$\begin{aligned} &\text{Min} \ \sum_{\forall i} c_i q_i \\ &\text{s.t.} \\ &\quad E_k(q) \geq E_k^* \quad \forall k \in I_s \\ &\quad q_i = 0, 1 \quad \forall i \end{aligned} \tag{12}$$

where E_i is the degree of estimability of variable i, and E_k^* is the minimum degree of estimability imposed. The inequality in the constraint of estimability is essential, as sometimes, in order to achieve a certain degree of estimability in one variable, a larger degree of estimability than necessary may be required in others. If, instead of inequalities, equalities are used, then the problem may be overly constrained and becomes infeasible.

A work of caution should be included regarding the above model. Even though a variable can have a high degree of estimability, its value may be obtained through differences of large numbers and, therefore, the precision of such variables may not be satisfactory. Such shortcomings come from the fact that estimability cannot directly replace precision constraints, as it cannot replace reliability constraints either. However, in the context of the above model, one can increase the degree of estimability required for variables that exhibit inadequate precision. This will force the activation of more cutsets, and the consequent improvement of precision.

Different types of problems arise depending on the degrees of estimability required for the variables. All these aspects will be explored next. In addition, if for some reason a measurement in a certain variable should be forbidden (because of safety, space and other constraints), the corresponding binary variable q_i can be *a priori* set to zero. Similarly, if the variable is to be compulsory measured, then the corresponding binary variable q_i can be set *a priori* to one. This is actually one case of instrumentation upgrade in which the existing instrumentation is not changed and only additions of new instrumentation are considered.

Several approaches have been proposed to address this problem. In principle, the problem is MINLP, but it has the added inconvenience that the functions $E_i(q)$ cannot be explicitly represented using expressions in terms of q, so one has to determine estimability by inspection procedures. To circumvent this obstacle, one can perform a tree search procedure of the type presented by Bagajewicz (1997). Meyer et al. (1994) attempted a similar tree search. These tree-searching algorithms can, of course, be improved in terms of computational speed. This is, however, not the objective of this article, which concentrates on the development of the conceptual aspects of these problems.

Luong et al. (1994) and Meyer et al. (1994) presented a strategy based on the identification of cycles. However, cycles are directly related to cutsets, as a cutset that contains a certain variable also contains one variable from each cycle that includes this variable. Therefore, the procedures are closely related.

In this article the power of spanning trees and cutsets to develop procedures to solve Eq. 12 is explored. Furthermore, various types of networks obtained from applying different estimability constraints are analyzed. In particular some networks that have been presented in the literature will be shown to be specific cases of Eq. 12.

Minimal networks

In this section we present networks, which are called minimal networks, that feature the minimum possible number of sensors. These networks arise from requesting estimability of order one for *all* the variables. Indeed, consider a network where all the variables have estimability of order one. The deletion of one measurement will cause the loss of observability of at least one unmeasured variable, as it will make that variable unobservable.

It was shown in the Appendix that the identity part of the canonical matrix corresponds to a set of variables that form a spanning tree. It was also shown that a system where all variables have degree of estimability $E_i = 1$ corresponds to a system where all unmeasured streams are given by a spanning tree. Thus, the task of designing such a system consists of determining what spanning tree has the largest cost. Once that is picked, the rest of the variables will be nonredundant, and will carry the lowest cost. In the special case of equal cost, any spanning tree suffices.

Direct inspection, however, is costly. A given graph can have a large number of spanning trees. This number is given by the determination of DD^T. A procedure to obtain all these spanning trees is given by Chen (1971). For example, the flowsheet of Figure 2 has 10 spanning trees, but the number grows very rapidly to millions for a fairly normal flowsheet.

The following algorithm, which avoids the enumeration of all spanning trees, was presented by Madron (1992), and is based on the notion of minimum spanning tree (Even, 1979):

(1) Pick the edge with the largest cost. This is the first edge of the tree.

(2) From all the remaining streams that form a tree when added to the existing tree, pick the one with the largest cost.

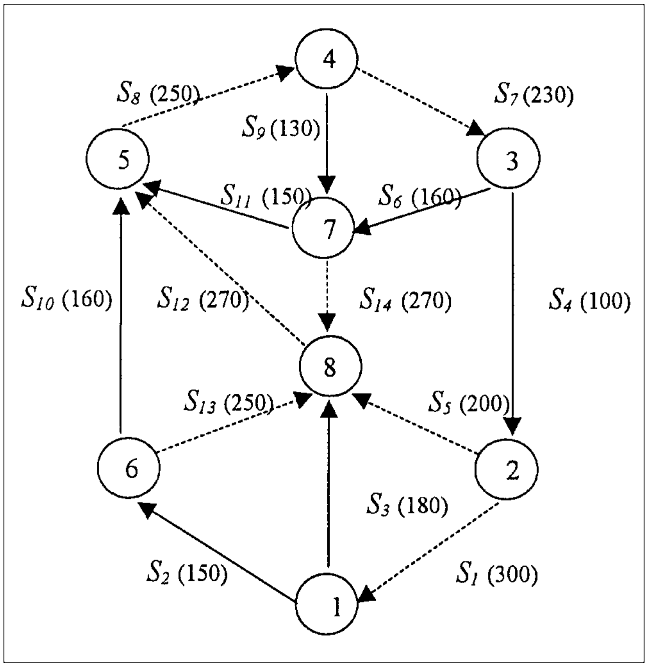

Figure 6. Simplified hydrodealkylation of toluene (Douglas, 1988).

(3) Repeat step 2 until n_U streams have been picked.

Example. Let us consider the simplified process of hydrodealkylation (HDA) of toluene (Douglas, 1988) shown in Figure 6. Instrumentation costs are assumed to be

$$c = [300\ \ 150\ \ 180\ \ 100\ \ 200\ \ 160\ \ 230\ \ 250\ \ 130\ \ 160\ \ 150\ \ 270\ \ 250\ \ 270]$$

and they are indicated between parenthesis after the label of the stream.

By applying the above procedure, the set of unmeasured and measured variables are the following

$$x_U = [S_1\ \ S_5\ \ S_7\ \ S_8\ \ S_{12}\ \ S_{13}\ \ S_{14}]$$
$$x_M = [S_2\ \ S_3\ \ S_4\ \ S_6\ \ S_9\ \ S_{10}\ \ S_{11}]$$

As can be seen from Figure 6, the unmeasured variables form a spanning tree of the undirected graph that corresponds to the more expensive sensors. The canonical form of the incidence matrix is the following

$$D' = \begin{array}{c} \begin{array}{cccccccccccccc} S_1 & S_5 & S_7 & S_8 & S_{12} & S_{13} & S_{14} & S_4 & S_9 & S_{10} & S_{11} & S_2 & S_6 & S_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Subminimal networks

When only certain variables are of interest, at least estimability of order one should be requested for these variables, whereas the rest of the variables can remain unobservable. These types of networks are here called subminimal because less than the minimum number of sensors will be needed. Note, however, that this may not be true if the estimability of an order larger than one is requested in a subset of variables. This minimum cost sensor network problem is stated as follows

$$\begin{aligned} &\text{Min} \ \sum_i c_i q_i \\ &\text{s.t.} \\ &\quad E_k(q) \geq 1 \quad \forall k \in I_s \\ &\quad q_i = (0,1) \quad \forall i \end{aligned} \tag{13}$$

where I_s is the set of variables with a required degree of estimability.

The term minimal is thus reserved to relate to networks in which the minimum number of sensors is installed to achieve the lowest possible estimability of *all* variables.

Feasibility Analysis for Estimability Constraints. To complement the tree-type enumeration procedure proposed by Bagajewicz (1997) to solve these problems, a simple procedure has been developed to check the feasibility of estimability constraints and thus save computation time.

Consider that the vector q is given. The following procedure checks the feasibility of constraint $E_k(q) \geq E_k^*$:

(1) Identify all the estimable cutsets for variable k. These cutsets are obtained from the list of cutsets by choosing those cutsets that contain variable k, and the rest of the variables in the cutset are measured variables. Assume the number of these cutsets is s and the total number of variables involved in all these cutsets is b, excluding k.

(2) Construct the $(s \times b)$ matrix M_k by including all the cutsets. The entries of these rows are binary values $(1,0)$.

(3) Determine the minimum number of measurements (z) the deletion of which makes k an unobservable variable. This may be accomplished by solving the following problem

$$\begin{aligned} &\text{Min} \ \sum_1^b \alpha_{pk} \\ &\text{s.t.} \\ &\quad \sum_1^b \alpha_{pk} m_{pk} \geq \underline{1} \end{aligned} \tag{14}$$

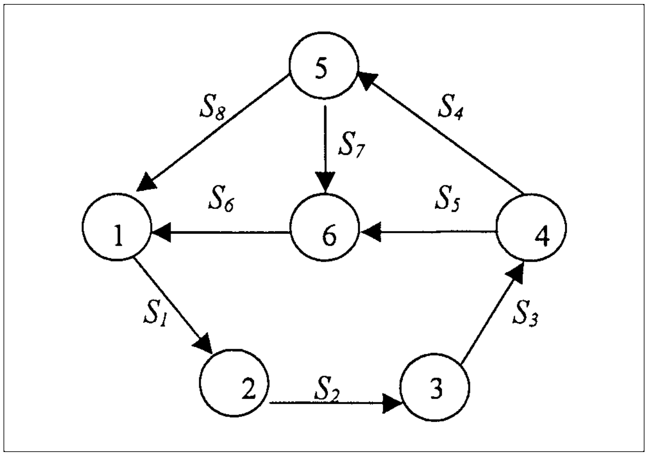

Figure 7. Simplified ammonia plant network.

where b is the total number of variables that participate in the s estimable cutsets excluding k; α_{pk} is the binary variable $(0,1)$ ($\alpha_{pk}=1$ implies measurement p participates in the set of measurements the deletion of which makes k unobservable. In contrast, $\alpha_{pk}=0$ means that measurement p has not been deleted to make k unobservable); m_{pk} is the pth column of the $(s\times b)$ estimable cutset matrix M_k; $\underline{1}$ is the $(s\times 1)$ vector of ones.

The problem is solved by inspecting combinations of an increasing number of measurements until the constraint of problem 14 is satisfied.

(4) Determine the degree of estimability of variable k. If k is unmeasured, its degree of estimability is z, but, if it is measured, its degree of estimability is $E_k = z+1$.

We now illustrate this procedure to check feasibility using the simplified ammonia network included in Figure 7. All cutsets for this network are extracted from Ali and Narasimhan (1995) and presented here in Table 1.

Let us consider that $E_3 \geq 1$ and $E_5 \geq 1$ are the constraints of problem 13. If $q=(1,2,4)$ is under analysis, matrices M_3 and M_5 are the following

$$\begin{array}{cccccccc} & S_1 & S_2 & S_4 & S_5 & S_6 & S_7 & S_8 \end{array}$$
$$M_3=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccccccc} & S_1 & S_2 & S_3 & S_4 & S_6 & S_7 & S_8 \end{array}$$
$$M_5=\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Table 1. All Cutsets of Ammonia Plant Network

No.	Streams	No.	Streams	No.	Streams
1	$S_5\ S_6\ S_7$	7	$S_1\ S_6\ S_8$	13	$S_1\ S_4\ S_5$
2	$S_1\ S_2$	8	$S_1\ S_4\ S_6\ S_7$	14	$S_2\ S_4\ S_6\ S_7$
3	$S_1\ S_3$	9	$S_1\ S_5\ S_7\ S_8$	15	$S_3\ S_4\ S_6\ S_7$
4	$S_2\ S_3$	10	$S_2\ S_4\ S_5$	16	$S_3\ S_4\ S_5$
5	$S_2\ S_6\ S_8$	11	$S_4\ S_7\ S_8$	17	$S_2\ S_5\ S_7\ S_8$
6	$S_3\ S_6\ S_8$	12	$S_4\ S_5\ S_6\ S_8$	18	$S_3\ S_5\ S_7\ S_8$

It can be seen from matrix M_3 that the deletion of only one measurement at a time does not make S_3 unobservable and, thus, does not satisfy the constraint of problem 14. However, the simultaneous deletion of measurements 1 and 2, that is, $\alpha_1=1,\ \alpha_2=1,\ \alpha_u=0,\ \forall u=3,\ \ldots,\ 7$), satisfy the constraint of problem 14, that is

$$\sum_1^b \alpha_p m_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so the degree of estimability of variable 3 is two for this set of instruments.

For variable 5, the deletion of measurement 4 only allows the constraint in 14 to be satisfied and renders this variable unobservable, so the degree of estimability of variable 5 is one. Consequently, the set of instruments, represented by vector q, is feasible.

Although the procedure is combinatorial, the nature of the design problem avoids a significant increase in the number of combinations. As the design follows a minimum cost criterion, the number b of measurements involved in the s estimable cutsets for a variable k are low.

Example of a subminimal network design

Consider again the hydrodealkylation process (Figure 6) with the instrumentation costs provided earlier. Assume that a degree of estimability of one is first required for streams S_1, S_8 and S_9, so a subminimal sensor network design is accomplished. For this case, the minimum cost solution is 590, which corresponds to the installation of sensors in streams $[S_4\ S_5\ S_6\ S_9]$. For this case, matrices M_1, M_8, and M_9 are the following

$$\begin{array}{ccccccccccccc} S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} & S_{11} & S_{12} & S_{13} & S_{14} \end{array}$$
$$M_1=\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{array}{ccccccccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_9 & S_{10} & S_{11} & S_{12} & S_{13} & S_{14} \end{array}$$
$$M_8=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$M_9=\varnothing$$

It is easy to see that the constraint of problem 14 is satisfied.

General networks

General networks arise when the required degree of estimability of some variables is greater than one. Estimability constraints may be imposed on some or all variables. The general sensor network design problem is stated by Eq. 12. The feasibility of constraints is checked using the procedure described previously.

Example of a Network Design Requiring Higher Degrees of Estimability. If a larger degree of estimability is required for streams S_1, S_8, and S_9 of the hydrodealkylation process, higher instrumentation cost will result. For example, if the degree of estimability lower bounds are $E_1^*=2$, $E_8^*=3$, $E_9^*=1$, the feasible set of instruments corresponding to the minimum cost solution is $x_M=[S_1\ S_5\ S_7\ S_8\ S_9\ S_{11}\ S_{14}]$, which has a cost of 1,530. The corresponding matrices of estimable cutsets for this example are

$$\begin{array}{c} \\ M_1 = \end{array}\begin{array}{c} \begin{array}{ccccccccccccc} S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} & S_{11} & S_{12} & S_{13} & S_{14} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ M_8 = \end{array}\begin{array}{c} \begin{array}{ccccccccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_9 & S_{10} & S_{11} & S_{12} & S_{13} & S_{14} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ M_9 = \end{array}\begin{array}{c} \begin{array}{ccccccccccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_{10} & S_{11} & S_{12} & S_{13} & S_{14} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

These matrices show that constraints for variables S_1 and S_8 are satisfied as equalities, in contrast, variable S_9 has a degree of estimability of 2, which is higher than its lower bound.

Design for Estimability Efficiency

If instrumentation costs are not available, a good goal for mass sensor network design is the selection of the minimum number of sensors that fulfill estimability constraints for key variables. This kind of sensor network design problem may be posed as follows

$$\begin{aligned} &\text{Min} \quad N_s \\ &\text{s.t.} \\ &\quad E_k \geq E_k^* \quad \forall k \in I_s \end{aligned} \tag{15}$$

where N_s is the number of sensors of the network.

The tree type enumeration strategy with stopping criteria proposed by Bagajewicz (1997) is applied to solve the MINLP problem. The estimability constraints are checked using the procedure described above. Although the stopping criterion avoids the enumeration to be exhaustive, this procedure is still not efficient for large-scale systems. As the conceptual development of the problem is the focus of this article, the implementation of numerical efficient strategies will be considered as an extension of this work in future articles.

Consider the sensor network design for the simplified process flowsheet of ammonia production (Figure 7). Assume that the estimability constraints are the following $E_2^* = 2$, $E_5^* = 3$. These constraints are satisfied when a minimum number of five instruments is installed. Three alternative sets of instruments fulfill the estimability requirements: $[S_1\ S_4\ S_5\ S_6\ S_7]$, $[S_2\ S_4\ S_5\ S_6\ S_7]$, and $[S_3\ S_4\ S_5\ S_6\ S_7]$.

Different runs were performed for the hydrodealkylation process, considering different estimability constraints. For each case, the lower estimability bounds on streams, the minimum number of sensors, and the solution set of instruments are presented in Table 2.

It can be seen from results of Cases 1 and 2 that a lower number of sensors are required by decreasing the estimability bounds for the same sets of streams. Obviously, the tendency is that the minimum number of instruments increases when greater requirements of estimability are imposed.

Table 2. HDA Process: Results for the Minimum Number of Sensors Problem

Case	Streams	E_i^*	n_s	Solution
1	S_1	2	7	$S_1\ S_5\ S_7\ S_8\ S_9\ S_{11}\ S_{14}$
	S_8	3		
	S_9	1		
2	S_1	2	5	$S_1\ S_5\ S_6\ S_8\ S_9$
	S_8	2		
	S_9	2		
3	S_7	2	9	$S_1\ S_2\ S_3\ S_7\ S_9\ S_{10}\ S_{11}\ S_{12}\ S_{13}$
	S_{13}	3		$S_2\ S_3\ S_5\ S_6\ S_9\ S_{10}\ S_{11}\ S_{12}\ S_{13}$
	S_{11}	2		

Compulsory Measurements and Upgrade

It is usually mandatory to install instruments on some streams due to control, balance accounting, or safety requirements. This situation is taken into account by setting *a priori* the corresponding binary variable q_i to one. Thus, the design of minimum-cost sensor networks subject to estimability and location constraints is stated as follows

$$\begin{aligned} &\text{Min} \quad \sum_{\forall i} c_i\, q_i \\ &\text{s.t.} \\ &\quad E_k(q) \geq E_k^* \quad \forall k \in I_s \\ &\quad q_j = 1 \qquad \forall j \in I_m \\ &\quad q_t = 0,1 \quad \forall t \in I_r \end{aligned} \tag{16}$$

where I_m contains all variables that should be measured.

A similar problem arises when the currently installed sets of instruments does not fulfill the estimability requirements, so it is necessary to incorporate others. The formulation of the optimization problem is as before. The solution involves the same set of instruments, but the objective function value is lower, because the cost of the already existing instruments is zero.

Consider the case where two flowmeters are already installed on streams S_1 and S_4 of the ammonia process flowsheet. Assume again that the estimability constraints are $E_2^* = 2$, $E_5^* = 3$. These requirements are not satisfied with the initial set of instruments, so the location of new instruments is proposed by solving problem 16. Instrumentation costs are given by the vector $c = [0\ 300\ 300\ 0\ 220\ 280\ 250\ 250]$, where the already located sensors have a zero cost. The solution indicates that constraints may be fulfilled by incorporating sensors on streams $[S_5\ S_6\ S_7]$. The optimal cost is 750.

For the hydrodealkylation process, the examples of upgrading shown in Table 3 were prepared. Cases 1 and 2 have the same estimability constraints, but the number of already installed sensors is different. The flow rate on stream S_{12} is considered unmeasured for Case 2. Nevertheless, this flowmeter has a cost of 270, and the optimal solution of Case 2 is only 180 more expensive than the optimum value of Case 1. The optimal solution of Case 3 has also been obtained for the design of minimum number of sensors in the previous section.

Table 3. HDA Process: Results for the Upgrading Design Problem

Case	Streams	E_i^*	Cost	Streams with Installed Sensors	Streams with New Sensors
1	S_1	2	1,080	S_2 S_{11} S_{12}	S_3 S_5 S_6 S_8 S_9 S_{10}
	S_8	3			
	S_9	1			
2	S_1	2	1,260	S_2 S_{11}	S_3 S_5 S_7 S_8 S_9 S_{14}
	S_8	3			
	S_9	1			
3	S_7	2	1,350	S_2 S_{11}	S_3 S_5 S_6 S_9 S_{10} S_{12} S_{13}
	S_{13}	3			
	S_{11}	2			

Conclusions

This article presented algorithms to design different types of sensor networks. The concept of degree of estimability of variables has been introduced to merge the concepts of degree of observability and degree of redundancy. In addition, some properties of sensor networks such as estimable cutsets and the efficiency of estimability of variables have been introduced. These concepts were later used to discuss the design of subminimal and general networks, and to introduce a new class of sensor networks where cost is not optimized and the only concern is the goal of estimability with maximum efficiency. The article fulfills thus a task of presenting procedures to design redundant and nonredundant sensor networks amenable to be used when very little information is provided about the cost of sensors or about sensor network robustness goals such as reliability, error detectability, precision, residual precision, and resilience.

Notation

c = vector of instrument costs
C = total cost of instrumentation
C_k = cutset including variable k
D' = canonical representation of D
G_i = matrices from the Gauss Jordan factorization of D
I = identity matrix
I_m = set of compulsory measurements
m = number of redundant cutsets
$M = (s \times b)$ matrix of estimable cutsets
n_u = number of units
S_i = stream i
Vi = unit i
x = vector of state variables

Subscripts

NR = nonredundant
O = observable
R = redundant
UO = unobservable

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Appendix

A few definitions and properties of graphs are presented in this Appendix. Some relationships to process flowsheets are briefly reviewed.

Augmented graph

Definition. Given a flowsheet the augmented graph is a graph that is obtained by adding the environment as another node of the graph.

For example, the augmented graph corresponding to Figure 2 is given in Figure 5.

Remark. In an augmented graph every stream connects nodes and all nodes are connected by streams.

Definition. The Augmented Incidence Matrix is obtained by adding one row representing the environmental node to the incidence matrix.

Spanning trees

Definition. A tree of a graph is a set of connected edges (streams) that does not form a cycle.

Definition. A spanning tree of a graph is a tree that connects all vertices (units) of the graph.

Property. The identity part of the canonical representation of matrix D, that is, D', corresponds to a set of variables that form a spanning tree.

Proof. Note first that we refer to a canonical form that does not distinguish measured from unmeasured, that is, $D' = [I_D \;\; D']$. The identity part of D' is I_D, and it corresponds to a tree because there are no cycles and, since it has a nonzero value in every row, it is connected to all vertices (units).

Cutsets

Definition. Given a graph, a cutset is defined as the set of edges (streams) that, when eliminated, separate the graph in exactly two disjoint subgraphs.

Consider, for example, the system of Figure 5. The set $\{S_1, S_6\}$ is a cutset. When these streams are eliminated, two subgraphs are left, the original flowsheet (Figure 2) and the environmental node. The set $\{S_2, S_4\}$ is another cutset. It separates all the nodes from V_2. Finally, the set $\{S_2, S_5\}$ is not a cutset, because it does not separate the graph into two sets, that is, after the elimination of these two streams, all the units of the graph remain connected. In turn, the set $\{S_1, S_3, S_5, S_6\}$ is not a cutset because the elimination of all these streams leaves three disjoint sets of units, $\{ENV\}$, $\{V_3\}$ and $\{V_1, V_2, V_4\}$. In fact, this set is a union of two cutsets, $\{S_1, S_6\}$ and $\{S_3, S_5\}$.

Remark. A cutset corresponds to a set of variables with which a material balance involving a certain number of units can be written.

This was already pointed out by Kretsovalis and Mah (1987) in the context of process systems. A cutset is, by construction, a set of streams that connects two subsystems of the graph. Thus, since no other stream is leaving or entering the subsystems, aside from the ones of the cutset, the sum of all the flows of the cutset should be equal to zero. This is a material balance.

Fundamental cutsets

Consider the canonical form of the incidence matrix $D' = [I_D \;\; D^*]$. This matrix contains as many rows as units are in the system. Thus, it represents as many linear independent balance equations as it is possible to write in the system. In other words, it contains n_u cutsets, that are called fundamental cutsets.

Definition. The non-zero entries of each row of the canonical matrix represent the fundamental cutsets of the system.

Determination of cutsets

Since the rows of the canonical matrix represent all the linearly independent cutsets of the system, then they are the base of a linear space of vectors representing linear combinations of balance equations. Since a cutset is a balance equation, then all cutsets are included in this space. However, cutsets are represented by entries in each position of the balance that are restricted to $+1$, -1, or 0 and they leave only two disjoint subgraphs after the elimination of the variables of the cutset.

Thus, to find all the cutsets we resort to the following procedure:

(1) Create the cutset list by putting all fundamental cutsets in the list. Set the counter $k = 2$.

(2) Create all linear combinations of k fundamental cutsets, restricting the coefficients of such combinations to the numbers 1 and -1. These linear combinations have to be such that absolute values of the entries of the resulting vector are binary $(0,1)$. If the result has any other existing cutset as a subset, eliminate this result. Otherwise, include it in the list of cutsets.

(3) $k = k + 1$.

(4) If $k < n_U$ perform step (2) (n_U is the number of units).

Thus, to find all the cutsets that contain a specific variable, step (1) is modified to include only the row that contains the variable of interest. A canonical form of the incidence matrix can always be constructed such that the variable of interest has only one nonzero entry in its corresponding column, that is, it is included in I_D. Other methods to obtain cutsets exist, for example, Tsukiyama and Verma (1980) and Fong and Buzacott (1987). The problem should not be underestimated, as the size of a graph may render some methods impractical.

Consider the system of Figure 2. Assume that all the cutsets containing stream S_3 are to be determined. Then we start from (7). The following combinations of rows have to be explored: $(1+/-2)$, $(1+/-3)$, $(1+/-4)$, $(1+/-2+/-3)$, $(1+/-2+/-4)$, $(1+/-3+/-4)$.

The fundamental cutset containing S3 is the first row in the cutset list, that is, $C_{1,1} = \{S_1, S_2, S_3\}$.

The other members of the cutset line are the results of the following successful linear operations:

- Addition of row 2 to $C_{1,1}$ to obtain a new cutset $C_{1,2} = \{S_1, S_3, S_4\}$.
- Subtraction of row 3 from $C_{1,1}$ to obtain $C_{1,3} = \{S_3, S_5\}$.
- Subtraction of row 4 from $C_{1,1}$ to obtain $C_{1,4} = \{S_3, S_2, S_6\}$.

Subtraction of row 4 from the addition of rows 1 and 2 to obtain a new cutset $C_{1,5} = \{S_3, S_4, S_6\}$.

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