Design of Delay Observer-based Controllers for Uncertain Time-lag Systems

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In this paper, the problem of designing observers and observer-based controllers for a class of uncertain systems with input and state time lags is considered. We construct delay-type observers in which both the instantaneous as well as the delayed measurements are utilized. Using feedback control based on the reconstructed state, the behavior of the closed-loop system is investigated. It is established that the uncertain time-lag system with delay observer-based control is asymptotically stable. Expressions for the gain matrices are given based on two linear-matrix inequalities. A numerical example is given to illustrate the theoretical developments.

Keywords: Design; Observer; Controller; Time-delay system; LMI

1 INTRODUCTION

The problem of state reconstruction for linear dynamic systems has received considerable attention for more than three decades. For deterministic systems with known dynamic models, the celebrated results of Leunberger [1] have provided the fundamental thrust upon which the theory of state observers is by now well developed [2]. When the dynamic models have uncertain parameters, the available results are limited to some cases including matched uncertainties [3], dyadic-type uncertainties [4] and norm-bounded uncertainties [5].

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For the case of uncertain dynamical systems with state and/or input delays, it seems that the results are very scattered. Nonlinear and insensitive observers have been developed in [6] for linear systems with mismatched uncertainties and state delays. H_∞-controllers using full-order observers have been studied in [7] when the state and the input delay factors are known. Modal-type observers for time-lag systems have been developed in [8] and [9]. The present work builds on the for-going results and extends them. It deals with the problem of designing observers and observer-based controllers for a class of nominally linear systems with state and input lags. We construct a delay-type observer in which both instantaneous and delayed information is used to recon-struct the state of the system. We then establish that, the uncertain time-lag system with delay observer-based control is asymptotically stable. Expressions for the gain matrices are given based on two linear-matrix inequalities (LMIs). A numerical example is given to illustrate the theoretical developments.

Notations and Facts

In the sequel, we denote by W^t and W^{-1} the transpose and the inverse of any square matrix W. We use $W > 0$ ($W < 0$) to denote a positive- (negative-) definite matrix W. We let I be a unit matrix of appropriate dimensions. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

Fact 1 (Schur Complement)

Given the constant matrices Ω_1, Ω_2, Ω_3 where $\Omega_1 = \Omega_1^t$ and $0 < \Omega_2 = \Omega_2^t$ then $\Omega_1 + \Omega_3^t \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^t \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \text{ or } \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^t & \Omega_1 \end{bmatrix} < 0.$$

Fact 2

For any real matrices Σ_1 and Σ_2 with appropriate dimensions, it follows that

$$\Sigma_1^t \Sigma_2 + \Sigma_2^t \Sigma_1 \leq \alpha \Sigma_1^t \Sigma_1 + \alpha^{-1} \Sigma_2^t \Sigma_2, \quad \alpha > 0.$$

Fact 3

Let Σ_1, Σ_2 and Σ_3 be real constant matrices of compatible dimensions, and $H(t)$ be a real matrix function satisfying $H^t(t)H(t) \leq \psi I;\ \psi > 0$. Then for any $\epsilon > 0$, the following inequality holds,

$$\Sigma_1 \Sigma_2 H(t) \Sigma_3 + \Sigma_3^t H^t(t) \Sigma_2^t \Sigma_1^t \leq \epsilon^{-1} \Sigma_1 \Sigma_2 \Sigma_2^t \Sigma_1^t + \epsilon \psi \Sigma_3^t \Sigma_3.$$

2 PROBLEM DESCRIPTION

In this section, we consider a class of uncertain systems with state and input lags represented by a state space model of the form:

$$\dot{x}(t) = [A_o + \Delta A(t)]x(t) + [B_o + \Delta B(t)]u(t)$$
$$+ [D_o + \Delta D(t)]x(t - \tau) + E_o u(t - \eta), \qquad (1)$$
$$y(t) = [C_o + \Delta C(t)]x(t), \qquad (2)$$
$$x(t) = \psi_1(t) \quad \forall t \in [-\max(\tau, \eta), 0],$$

where $t \in \Re$ is the time, $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the control input; $y(t) \in \Re^p$ is the measured output, and τ, η are known constant scalars representing the amount of delays in the state and at the input of the system, respectively. The matrices $A_o \in \Re^{n \times n}$, $B_o \in \Re^{n \times m}$, $D_o \in \Re^{n \times n}$, $E_o \in \Re^{n \times m}$ and $C_o \in \Re^{p \times n}$ are real constant matrices representing the nominal plant with the pair (A_o, B_o) being controllable, and the pair (A_o, C_o) being observable. The uncertain matrices $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta D(t)$ are assumed to be represented by

$$[\Delta A(t) \ \Delta B(t)] = H_1 F(t)[G_1 \ G_2]; \quad \Delta C(t) = H_2 F(t)G_1; \qquad (3)$$
$$\Delta D(t) = H_3 F(t)G_3; \quad F^t(t)F(t) \leq I. \qquad (4)$$

The initial functions of system (1)–(2) are specified as $x_o \in C([-\tau, 0];$ $\Re^n)$ and $u_o \in C([-\eta, 0]; \Re^m)$.

The problem of interest is to design observer-based controllers for the time-lag uncertain-system (1)–(2) with norm-bounded uncertainties satisfying (3) and (4).

Before proceeding further, the following definitions are given.

DEFINITION 1   *The uncertain time-delay system* (1)–(2) *is said to be robustly stable if the null solution* $x(t) \equiv 0$ *of* (1) *with* $u(t) \equiv 0$ *is globally uniformly asymptotically stable for all admissible uncertainties satisfying* (3) *and* (4).

DEFINITION 2   *The uncertain time-delay system* (1)–(2) *is said to be robustly stabilizable if there exists a feedback controller* $u(t) = K[x(t)]$, *such that the resulting closed-loop system is robustly stable in the sense of Definition* 1.

The design objectives are:

(O1) to reconstruct the state $x(t)$ based on the input and/or output measurements,
(O2) to determine the state-feedback gain matrix such that the closed-loop controlled system is asymptotically stable,
(O3) to provide an efficient procedure to compute the controller and the observer gains.

These objectives should be fulfilled for all admissible uncertainties satisfying (3) and (4).

There are two generic groups of observers that can be used to fulfill objective (O1). Hence, observers can be classified as delay-less observers or delay observers.

1. Delay-less Observers   These observers are observers that depend only on the instantaneous input–output measurements. Examples of such observers can be found in [8–10]. Modal-type observers have been developed by Levy-Ramos and Pearson [8], and Trinh and Aldeen [9] for time-lag systems. Jabbari and Schmitendorf [3] have developed a full state observer for uncertain systems. A delay-less Leunberger-observer for uncertain systems with input and state delays has been developed by Mahmoud and Zribi [10].

2. Delay Observers   These observers are such that the observer state depends on both the instantaneous and the delayed input–output measurements.

Conceptually, the two groups of observers are different at least from the amount of input information for which the second group demands more.

In this paper will concentrate on delay observers for the class of time-lag system (1)–(2). The rationale behind the construction of these observers is the desire to utilize all the available input–output instantaneous and delayed information in estimating the state variables. In fact, the use of delayed state in feedback control design has been shown to provide satisfactory performance [11].

By utilizing all the information at hand (both instantaneous and delayed) we come up with the generalized full-order linear observer:

$$\dot{\hat{x}}(t) = A_o\hat{x}(t) + B_o u(t) + D_o\hat{x}(t - \tau) + E_o u(t - \eta) + L[y(t) - C_o\hat{x}(t)],$$

$$(5)$$

where $L \in \Re^{n \times p}$ is the observer gain matrix to be determined later.

Remark 1 There are two possible special cases of observer (5). The first case is given by:

$$\dot{\hat{x}}(t) = A_o\hat{x}(t) + B_o u(t) + D_o\hat{x}(t - \tau) + L[y(t) - C_o\hat{x}(t)], \qquad (6)$$

where $L \in \Re^{n \times p}$ is the observer gain matrix. This observer uses the delayed state measurements. The second observer is such that:

$$\dot{\hat{x}}(t) = A_o\hat{x}(t) + B_o u(t) + E_o u(t - \eta) + L[y(t) - C_o\hat{x}(t)] \qquad (7)$$

where $L \in \Re^{n \times p}$ is the observer gain matrix. Note that this observer uses the delayed input measurements.

The end result of observer (5) is to develop $\hat{x}(t)$ as a good replica of $x(t)$. Hence it can be used for state-feedback design. A state-feedback controller of the form

$$u(t) = K\hat{x}(t), \qquad (8)$$

where $K \in \Re^{m \times n}$ is the feedback gain matrix will be designed such that the closed-loop controlled system is asymptotically stable.

3 STABILITY OF THE CLOSED-LOOP SYSTEM

We start by defining the state error $e(t)$ such that:

$$e(t) = x(t) - \hat{x}(t). \tag{9}$$

The error dynamics depends on the type of observer being used. An augmented system can be formed in the (x, e) space in terms of the composite state vector

$$x_t(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \in \Re^{2n}. \tag{10}$$

Note that (8)–(10) imply that

$$u(t) = K_t x_t(t), \tag{11}$$

where $K_t = [K \quad -K] \in \Re^{m \times 2n}$. By differentiating (9), making use of system (1)–(2) in addition to the observer dynamics (5) and manipulating, we can cast the augmented system into the general form:

$$\dot{x}_t(t) = [A_t + \Delta A_t(t)]x_t(t) + [B_t + \Delta B_t(t)]u(t)$$
$$+ [D_t + \Delta D_t(t)]x_t(t - \tau) + E_t u(t - \eta), \tag{12}$$

where

$$A_t = \begin{bmatrix} A_o & 0 \\ 0 & A_o - LC_o \end{bmatrix}, \quad B_t = \begin{bmatrix} B_o \\ 0 \end{bmatrix}, \tag{13}$$

$$D_t = \begin{bmatrix} D_o & 0 \\ 0 & D_o \end{bmatrix}, \quad E_t = \begin{bmatrix} E_o \\ 0 \end{bmatrix}, \tag{14}$$

$$\Delta A_t = \begin{bmatrix} \Delta A(t) & 0 \\ \Delta A(t) - L\Delta C(t) & 0 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_1 - LH_2 \end{bmatrix} F(t)[G_1 \quad 0], \tag{15}$$

$$\Delta B_t = \begin{bmatrix} \Delta B(t) \\ \Delta B(t) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_1 \end{bmatrix} F(t)G_2, \tag{16}$$

$$\Delta D_t = \begin{bmatrix} \Delta D(t) & 0 \\ \Delta D(t) & 0 \end{bmatrix} = \begin{bmatrix} H_3 \\ H_3 \end{bmatrix} F(t)[G_3 \quad 0]. \tag{17}$$

In order to investigate the stability behavior of the uncertain time-lag system (12), we use the Lyapunov–Krasovskii functional approach [12] and introduce the following storage functional:

$$V(x_t) = x_t^t(t) W_1 x_t(t) + \int_{-\tau}^{0} x_t^t(t + \alpha) W_2 x_t(t + \alpha) \, d\alpha$$

$$+ \int_{-\eta}^{0} x_t^t(t + \beta) W_3 x_t(t + \beta) \, d\beta, \tag{18}$$

where the weighting matrices are such that $0 < W_1 = W_1^t \in \Re^{2n \times 2n}$, $0 < W_2 = W_2^t \in \Re^{2n \times 2n}$ and $0 < W_3 = W_3^t \in \Re^{2n \times 2n}$. Note that, $V(x_t) > 0$ for all $x_t(t) \neq 0$ and $V(x_t) = 0$ for $x_t(t) = 0$. Let

$$W_1 = \begin{bmatrix} W_{11} & 0 \\ 0 & W_{12} \end{bmatrix}, \quad W_2 = \begin{bmatrix} W_{21} & 0 \\ 0 & W_{22} \end{bmatrix}, \quad W_3 = \begin{bmatrix} W_{31} & 0 \\ 0 & W_{32} \end{bmatrix}.$$

$$\tag{19}$$

The following theorem summarizes the main result.

THEOREM 1 *Consider the system (1)–(2) satisfying (3) and (4) and the linear observer (5). Then, given τ and η, the closed-loop controlled system (12) is robustly stabilizable if there exist matrices $0 < P = P^t$, $0 < Q = Q^t$, $0 < S_j = S_j^t$ $(j = 1, \ldots, 5)$ and scalars $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$ satisfying the following LMIs:*

$$\begin{bmatrix} A_0 P + P A_0^t + \Pi_1 & \vdots & P & \sqrt{3} B_0 & E_0 & D_0 \\ \cdots & & \cdots & \cdots & \cdots & \cdots \\ P & \vdots & -S_5^{-1} & 0 & 0 & 0 \\ \sqrt{3} B_0^t & \vdots & 0 & -S_1 & 0 & 0 \\ E_0^t & \vdots & 0 & 0 & -S_4^{-1} & 0 \\ D_0^t & \vdots & 0 & 0 & 0 & -S_2 \end{bmatrix} < 0, \tag{20a}$$

$$\alpha_2 G_2^t G_2 > 0, \tag{20b}$$

$$W_{21} - \alpha_3 G_3^t G_3 > 0, \tag{20c}$$

$$W_{21} + W_{31} + \alpha_1 G_1^t G_1 > 0, \tag{20d}$$

and

$$\begin{bmatrix} \widehat{A}Q + Q\widehat{A}^t + \Phi + QR_2Q & \vdots & QK^t & Q & D_o \\ \cdots & & \cdots & \cdots & \cdots \\ KQ & \vdots & -S_1^{-1} & 0 & 0 \\ Q & \vdots & 0 & -R_1^{-1} & 0 \\ D_o^t & \vdots & 0 & 0 & -S_3 \end{bmatrix} < 0, \quad (21)$$

where

$$\Pi_1 = \alpha_1^{-1} H_1 H_1^t + \alpha_2^{-1} H_1 H_1^t + \alpha_3^{-1} H_3 H_3^t, \quad (22)$$

$$\varphi = H_{12}^t (H_{12} H_{12}^t)^{-1} \text{ with } H_{12} = \alpha_1^{-1} H_2 H_1^t, \quad (23)$$

$$\widehat{A} = A_o - \Pi_1 \varphi C_o - 2\Pi_1 \varphi \theta \varphi^t B_o K + 2B_o K, \quad (24)$$

$$R_2 = 2K^t B_o^t \varphi C_o + 2C_o^t \varphi^t B_o K + 4K^t B_o^t \varphi \theta \varphi^t B_o K, \quad (25)$$

$$\Phi = -\Pi_1 + \Pi_1 \varphi \theta \varphi^t \Pi_1, \quad \theta = \alpha_1^{-1} H_2 H_2^t. \quad (26)$$

Moreover, the observer and controller gain matrices are given by

$$K = S_1^{-1} B_o^t P^{-1}, \quad (27)$$

$$L = (-2QK^t B_o^t + \Pi_1)\varphi. \quad (28)$$

Proof By differentiating (18) along the solutions of (12) using (13)–(17), the Lyapunov derivative $\dot{V}(x_t)$ can be expressed as:

$$\dot{V}(x_t) < Z^t(t)\Omega_t Z(t), \quad (29)$$

where the extended state vector $Z(t) = [x_t^t(t) \ x_t^t(t - \tau) \ x_t^t(t - \eta)]^t$, and

$$\Omega_t = \begin{bmatrix} \Sigma_1 & \Sigma_2 & \vdots & W_{11}D_o & 0 & \vdots & W_{11}E_oK & -W_{11}E_oK \\ \Sigma_2^t & \Sigma_3 & & 0 & W_{12}D_o & & 0 & 0 \\ \cdots & & & \cdots & & & \cdots & \\ D_o^t W_{11} & 0 & \vdots & -W_{21} + \alpha_3 G_3^t G_3 & 0 & \vdots & & 0 \\ 0 & D_o^t W_{12} & & 0 & -W_{22} & & & \\ \cdots & & & \cdots & & & \cdots & \\ K^t E_o^t W_{11} & 0 & \vdots & & 0 & \vdots & -W_{31} & 0 \\ -K^t E_o^t W_{11} & 0 & & & & & 0 & -W_{32} \end{bmatrix},$$

$$(30)$$

with

$$\begin{aligned}
\Sigma_1 &= W_{11}A_o + A_o^t W_{11} + K^t B_o^t W_{11} + W_{11}B_o K + W_{21} + W_{31} \\
&\quad + \alpha_1 G_1^t G_1 + \alpha_2 K^t G_2^t G_2 K + W_{11}\Pi_1 W_{11},
\end{aligned} \tag{31}$$

$$\begin{aligned}
\Sigma_2 &= -W_{11}B_o K + W_{11}[\alpha_1^{-1}H_1(H_1^t - H_2^t L^t) + \alpha_2^{-1}H_1 H_1^t \\
&\quad + \alpha_3^{-1}H_3 H_3^t]W_{12} - \alpha_2 K^t G_2^t G_2 K,
\end{aligned} \tag{32}$$

$$\begin{aligned}
\Sigma_3 &= W_{12}(A_o - LC_o) + (A_o - LC_o)^t W_{12} + W_{22} + W_{32} \\
&\quad + \alpha_2 K^t G_2^t G_2 K + W_{12}\Pi_3 W_{12},
\end{aligned} \tag{33}$$

$$\Pi_3 = \alpha_1^{-1}(H_1 - LH_2)(H_1^t - H_2^t L^t) + \alpha_2^{-1}H_1 H_1^t + \alpha_3^{-1}H_3 H_3^t. \tag{34}$$

Expansion of Ω_t leads to

$$\begin{bmatrix} \Xi_{11} & \Xi_{21} \\ \Xi_{21}^t & \Xi_{31} \end{bmatrix} < 0, \tag{35}$$

where

$$\Xi_{11} = \Sigma_1 + W_{11}\Pi_2 W_{11}, \tag{36}$$

$$\Xi_{21} = \Sigma_2, \tag{37}$$

$$\Xi_{31} = \Sigma_3 + W_{12}D_o W_{22}^{-1}D_o^t W_{12}, \tag{38}$$

$$\Pi_2 = E_o(KW_{31}^{-1}K^t)E_o^t + E_o(KW_{32}^{-1}K^t)E_o^t + D_o S_2^{-1}D_o^t. \tag{39}$$

A sufficient condition to satisfy (35) is that

$$\Xi_{11} < 0, \quad \Xi_{31} < 0, \quad \Xi_{21} = 0. \tag{40}$$

Choosing K as given by (27), and letting $S_1 = \alpha_2 G_2^t G_2$ and $S_5 = W_{21} + W_{31} + \alpha_1 G_1^t G_1$, the inequality $\Xi_{11} < 0$ reduces to

$$W_{11}A_o + A_o^t W_{11} + 3W_{11}B_o S_1^{-1}B_o^t W_{11} + S_5 + W_{11}(\Pi_1 + \Pi_2)W_{11} < 0. \tag{41}$$

Pre-multiplying and post-multiplying the above inequality by W_{11}^{-1}, and letting $P = W_{11}^{-1}$, $S_2 = W_{21} - \alpha_3 G_3^t G_3$, $S_4 = K(W_{31}^{-1} + W_{32}^{-1})K^t$, we obtain

$$A_o P + PA_o^t + \Pi_1 + PS_5 P + 3B_o S_1^{-1}B_o^t + E_o S_4 E_o^t + D_o S_2^{-1}D_o^t < 0 \tag{42}$$

which when arranged using the Schur complement formula yields block form (20a).

From (37), using (32) and setting $\Xi_{21} = 0$, yields

$$-2W_{11}B_oK + W_{11}\Pi_1 W_{12} - \alpha_1^{-1} W_{11}H_1 H_2^t L^t W_{12} = 0. \qquad (43)$$

By letting $Q = W_{12}^{-1}$, and pre-multiplying the above inequality by W_{11}^{-1}, and post-multiplying the result by W_{12}^{-1}, one obtains,

$$-2B_o KQ + \Pi_1 - \alpha_1^{-1} H_1 H_2^t L^t = 0 \qquad (44)$$

or equivalently stated,

$$LH_{12} = -2QK^t B_o^t + \Pi_1. \qquad (45)$$

Assuming that $H_{12}H_{12}^t$ is nonsingular, and letting $\varphi = H_{12}^t(H_{12}H_{12}^t)^{-1}$, one directly obtains L as given by (28).

Finally considering the inequality $\Xi_{31} < 0$. With $R_1 = W_{32} + W_{22}$, starting from (38) and using (33), (22)–(26), we get

$$W_{12}\widehat{A} + \widehat{A}^t W_{12} + W_{12}\Phi W_{12} + R_1 + R_2 + K^t S_1 K$$
$$+ W_{12}D_o W_{22}^{-1} D_o^t W_{12} < 0. \qquad (46)$$

Pre-multiplying and post-multiplying the above inequality by W_{12}^{-1}, one obtains,

$$\widehat{A}Q + Q\widehat{A}^t + \Phi + QR_1Q + QR_2Q + QK^t S_1 KQ + D_o S_3^{-1} D_o^t < 0. \qquad (47)$$

Inequality (47) can be easily put in the block form (21) by using the Schur Complement formula.

Remark 2 To implement Theorem 1, we apply the following procedure:

(1) Read in the data $A_o, B_o, \dots, H_1, \dots$
(2) Solve inequality (20) for P, then compute the gain matrix K from (27).

(3) Solve inequality (21) for Q, then compute the gain matrix L from (28).

(4) Simulate the performance of system (12) when controller (11) is used.

COROLLARY 1   *Consider the uncertain time-lag system (1)–(2) with observer (6) and controller (11). The resulting observer-based controlled system is asymptotically stable with controller gain matrix K as given by (27) and observer gain matrix L such that:*

$$L = (-2QK^t B_o^t + \Pi_1 + E_o S_4 E_o^t)\varphi \tag{48}$$

where the matrix $0 < Q = Q^t$ satisfies:

$$
\begin{bmatrix}
\widehat{A}Q + Q\widehat{A}^t + \Phi + QR_2Q & \vdots & QK^t & Q & D_o & E_o \\
\cdots & & \cdots & \cdots & \cdots & \cdots \\
KQ & \vdots & -S_1^{-1} & 0 & 0 & 0 \\
Q & \vdots & 0 & -R_1^{-1} & 0 & 0 \\
D_o^t & \vdots & 0 & 0 & -S_3 & 0 \\
E_o^t & \vdots & 0 & 0 & 0 & -S_4^{-1}
\end{bmatrix} < 0.
\tag{49}
$$

Proof   The proof follows parallel lines to the proof of Theorem 1 with

$$\Xi_{21} = \Sigma_2 + W_{11}E_o S_4 E_o^t W_{12}, \tag{50}$$

$$\Xi_{31} = \Sigma_3 + W_{12}E_o S_4 E_o^t W_{12} + W_{12}D_o W_{22}^{-1} D_o^t W_{12}, \tag{51}$$

and all the other quantities remain unchanged.

COROLLARY 2   *Consider the uncertain time-lag system (1)–(2) with observer (7) and controller (11). The resulting observer-based controlled system is asymptotically stable with controller gain matrix K as given by (27) and observer gain matrix L expressed as:*

$$L = (-2QK^t B_o^t + \Pi_1 + D_o S_2^{-1} D_o^t)\varphi \tag{52}$$

where the matrix $0 < Q = Q^t$ *satisfies:*

$$
\begin{bmatrix}
\widehat{A}Q + Q\widehat{A}^t + \Phi + QR_2Q & \vdots & QK^t & Q & D_o \\
\cdots & & \cdots & \cdots & \cdots \\
KQ & \vdots & -S_1^{-1} & 0 & 0 \\
Q & \vdots & 0 & -R_1^{-1} & 0 \\
D_o^t & \vdots & 0 & 0 & -S_2
\end{bmatrix} < 0. \qquad (53)
$$

Proof The proof follows parallel lines to the proof of Theorem 1 using

$$\Xi_{21} = \Sigma_2 + W_{11}D_oS_2^{-1}D_o^tW_{12}, \qquad (54)$$

$$\Xi_{31} = \Sigma_3 + W_{12}D_oS_2^{-1}D_o^tW_{12}, \qquad (55)$$

and all the other quantities remain unchanged.

Remark 3 In view of Theorem 1, the results of Corollaries 1 and 2 are expected due to two reasons:

(1) The controller (11) remains the same, hence K must be the same in all cases.
(2) The change takes place in the expression of L as result of changing the observer structure. More importantly, it is noted that the more information used to process the observer dynamics, the lesser will be the effort to determine the observer gain.

4 SIMULATION EXAMPLE

Consider the following third-order system:

$$
\begin{aligned}
\dot{x}(t) &= [A_o + \Delta A(t)]x(t) + [B_o + \Delta B(t)]u(t) \\
&\quad + [D_o + \Delta D(t)]x(t - \tau) + E_o u(t - \eta), \\
y(t) &= [C_o + \Delta C(t)]x(t),
\end{aligned}
$$

with

$$[\Delta A(t) \quad \Delta B(t)] = H_1 F(t)[G_1 \quad G_2], \quad \Delta C(t) = H_2 F(t) G_1,$$
$$\Delta D(t) = H_3 F(t) G_3,$$

where

$$A_o = \begin{bmatrix} -3 & 2 & 0 \\ 1 & -2 & -1 \\ 3 & 0 & -6 \end{bmatrix}; \quad B_o = \begin{bmatrix} 1 & 1 \\ 4 & 3 \\ 2 & -1 \end{bmatrix}; \quad C_o = [1 \quad 0 \quad 0];$$

$$D_o = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad E_o = \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.7 \\ 0.6 & 0.2 \end{bmatrix};$$

$$H_1 = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & -0.4 \\ 0.1 & 0.2 \end{bmatrix}; \quad H_2 = [0.1 \quad -0.3]; \quad H_3 = \begin{bmatrix} -1.0 & 0.5 \\ 0.3 & 0.5 \\ 0 & 1.0 \end{bmatrix};$$

$$G_1 = \begin{bmatrix} 1.0 & 0.5 & -1.0 \\ 0.8 & -1.0 & 0.2 \end{bmatrix}; \quad G_2 = \begin{bmatrix} -1.0 & -0.5 \\ -1.0 & 0.7 \end{bmatrix};$$

$$G_3 = \begin{bmatrix} 0.4 & 0.4 & 0.6 \\ -0.5 & 0.6 & -0.7 \end{bmatrix}; \quad F(t) = \begin{bmatrix} 0.5\sin(2t) & 0.3\sin(t) \\ -0.4\sin(t) & 0.6\sin(3t) \end{bmatrix}.$$

The amount of delays in the state and at the input of the system are such that $\tau = 0.4$ and $\eta = 0.2$.

It is desired to design an observer-based controller for this uncertain system. The results of Theorem 1 are used to design the observer-based controller.

Selecting the scalars α_1, α_2 and α_3 such that $\alpha_1 = 0.0001$, $\alpha_2 = 10$ and $\alpha_3 = 1$, and then solving the matrix inequality (20) by using the LMI-Control Toolbox [13] gives:

$$P = \begin{bmatrix} 1729.9 & 605.2 & 629.4 \\ 605.2 & 1951.2 & 168.0 \\ 629.4 & 168.0 & 981.5 \end{bmatrix}.$$

Using (27), the controller gain K is found to be:

$$K = \begin{bmatrix} 0.0002 & 0.0084 & -0.0008 \\ 0.0003 & 0.0139 & -0.0016 \end{bmatrix}.$$

Then, solving inequality (21) by using the **LMI**-Control Toolbox yields:

$$Q = \begin{bmatrix} 380.2 & 42.3 & 101.7 \\ 42.3 & 413.2 & -40.2 \\ 101.7 & -40.2 & 208.2 \end{bmatrix}.$$

Using (28), the observer gain L is found to be:

$$L = \begin{bmatrix} -1.1454 \\ 1.4073 \\ -0.5684 \end{bmatrix}.$$

The performance of system (12) is then simulated using **Matlab**. The state trajectories are shown in Figs. 1–3. The error trajectories are shown in Figs. 4–6. The trajectories of the controller are shown in Figs. 7–8. From the figures it can be seen that the state and error
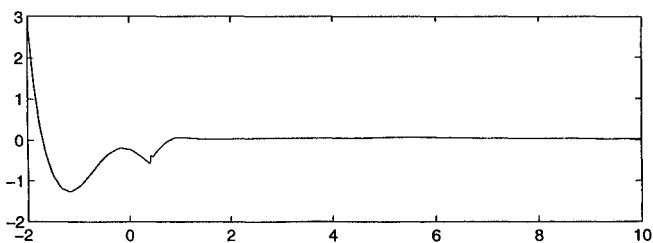
FIGURE 1   Trajectory of state one versus time.
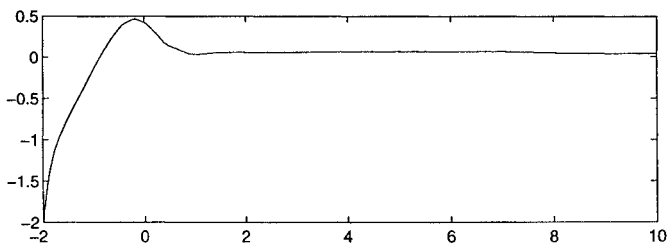
FIGURE 2   Trajectory of state two versus time.
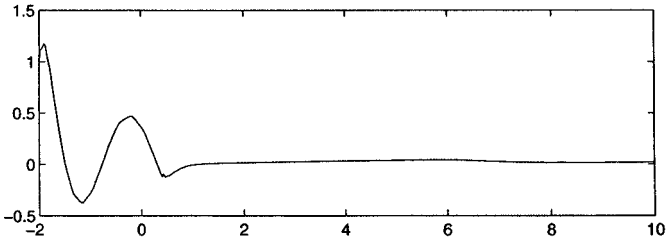
FIGURE 3   Trajectory of state three versus time.
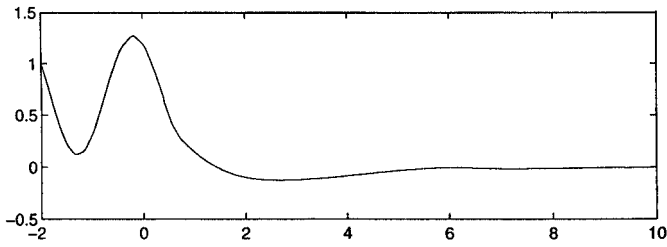
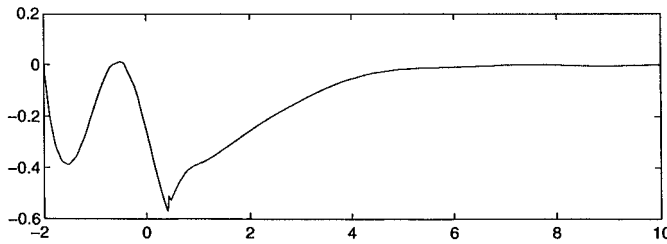FIGURE 4   Trajectory of error one versus time.

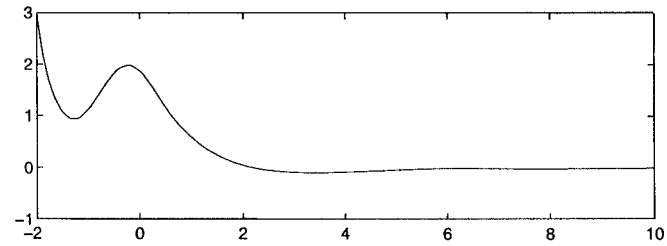FIGURE 5   Trajectory of error two versus time.

FIGURE 6   Trajectory of error three versus time.
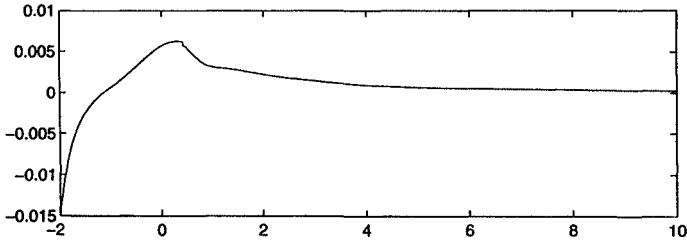
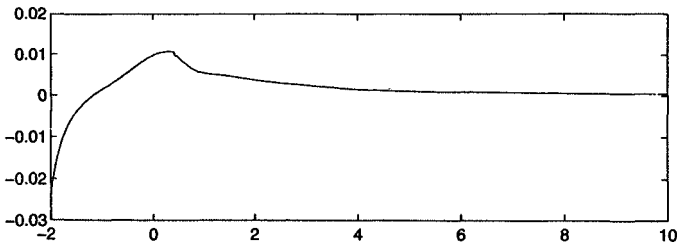FIGURE 7   Trajectory of input (component 1) versus time.

FIGURE 8   Trajectory of input (component 2) versus time.

trajectories converge to zero asymptotically. Hence, it can be con-
cluded that the proposed observer and observer-based controller
work quite well for a class of uncertain systems with input and state
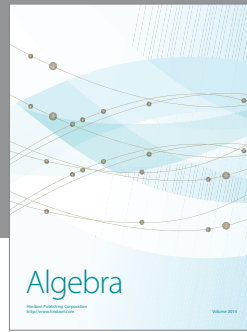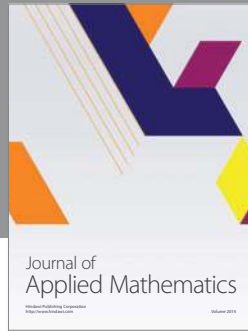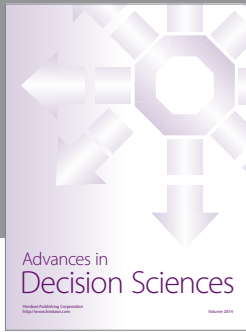time lags.

5   CONCLUSIONS

This paper has developed a delay-type observer-based controller for a
class of uncertain time-lag systems. It is established that the closed-
loop controlled system is asymptotically stable. The controller and
observer gains are determined by solving two LMIs. Simulation results
are given to support the theoretical developments.

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