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# Design of film-substrate single-reflection retarders* 

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The design steps for film-substrate single-reflection retarders are briefly stated and applied to the $\mathrm{SiO}_{2}-\mathrm{Si}$ film-substrate system at wavelength $6328 \AA$. The criterion of minimum-maximum error of the ellipsometric angle $\psi$ is used to choose angle-of-incidence-tunable designs. Use is made of the ( $\phi-d$ ) plane (angle of incidence versus thickness) to determine whether a given film-substrate system with known optical properties and film thickness can operate as a reflection retarder and to determine the associated angles of incidence and retardation angles. This leads to the concept of permissible-thickness bands and forbidden gaps for operation of a film-substrate system as a reflection retarder. Experimental measurements on one of the proposed designs proved the validity of the method.

Index Headings: Films; Reflection; Optical devices; Silicon.

Retarders are commonly constructed by use of the anisotropic property of birefringence exhibited by certain crystals (e.g., quartz) or are based on total internal reflection (the Fresnel rhomb). ${ }^{1}$ In this paper, we consider the design of a different class of retarders, based on external reflection from a film-substrate system. Although these retarders attenuate the incident light to a certain extent, it is expected that such attenuation can be tolerated in many situations. ${ }^{2}$ The film-substrate external-reflection retarder has been suggested recent$1 y^{3}$; here, we elaborate on this interesting device. ${ }^{4}$ These retarders should prove to be particularly useful in wavelength regions (e.g., vacuum uv) where materials of good optical quality necessary for the construction of conventional retarders are not readily available. Although we examine one film-substrate system $\left(\mathrm{SiO}_{2}-\mathrm{Si}\right)$ as an example and assume a single reflection, the considerable amount of control provided by the choice of materials and the number of reflections will make possible designs of greatly improved performance. This subject is currently under investigation.

Operation of a given film-substrate system with known optical properties and film thickness as a reflection retarder is discussed. A simple method to determine the angles of incidence (if any) at which this mode of operation is possible, and the associated retardation angles, is given. The concept of permissible-thickness bands and forbidden gaps for operation of a film-substrate system as a reflection retarder is also presented.

The experimental results obtained on one of the proposed designs are reported. The results confirm the proposed design.

## I. DESIGN PROCEDURE FOR FILM-SUBSTRATE SINGLE-REFLECTION RETARDERS

A specified value of the ellipsometric function $\rho$, the ratio of the complex amplitude-reflection coefficients $R_{p}$ and $R_{s}$ for light polarized parallel ( $p$ ) and perpendicular ( $s$ ) to the plane of incidence, can be realized at a given wavelength by the design of a film-substrate system with known optical constants (see Fig. 1). The design is aimed at finding the least film thickness $d^{\prime}$ and angle of incidence $\phi^{\prime}$ at which the given system has the prespecified value of $\rho$, leading to a particular reflec-tion-type optical device. ${ }^{3}$

In this section, we will outline the design steps of reflection retarders; we will discuss the choice of designs for angle-of-incidence-tunable retarders in Sec. III. We take $\mathrm{SiO}_{2}-\mathrm{Si}$ as an example of a film-substrate system and choose the wavelength of operation $6328 \AA$ for its wide availability. The refractive indices of $\mathrm{SiO}_{2}$ and Si at $6328 \AA$ are $N_{1}=1.46$ and $N_{2}=3.85-j 0.02$, respectively. ${ }^{5}$ The design procedure is general and can be applied at other wavelengths and to other materials; it is also applicable, with minor modification, ${ }^{6}$ to the design of any reflection-type optical device.

We summarize the design steps for a transparent film on an absorbing-substrate single-reflection retarder:
(1) From the desired retardation angle $\Delta$ calculate

$$
\begin{equation*}
\rho=e^{j \Delta} \tag{1}
\end{equation*}
$$

(2) Plot the quantity $1-|X|$ as a function of the angle of incidence $\phi$, by use of the value of $\rho$ obtained in step 1 , where $X$ is the thickness complex-exponential function ${ }^{7}$

$$
\begin{gather*}
X=\frac{-(B-\rho E) \pm\left[(B-\rho E)^{2}-4(C-\rho F)(A-\rho D)\right]^{1 / 2}}{2(C-\rho F)},  \tag{2}\\
A=r_{01 p}, \quad B=r_{12 p}+r_{01 p} r_{01 s} r_{12 s}, \quad C=r_{12 p} r_{01 s} r_{12 s}, \\
D=r_{01 s}, \quad E=r_{12 s}+r_{01 p} r_{12 p} r_{01 s}, \quad F=r_{01 p} r_{12 p} r_{12 s},
\end{gather*}
$$

and $r_{i j_{\nu}}(\nu=p, s)$ are the air-film ( $i j=01$ ) and filmsubstrate ( $i j=12$ ) interface Fresnel reflection coefficients. Figure 2 shows such a design curve, $1-|X|$ versus $\phi$, for a chosen retardation angle $\Delta=30^{\circ}$.


FIG. 1. The film-substrate system: ambient (0), film (1) of thickness $d$, and substrate (2).


FIG. 2. Design curve, $1-|X|$ vs $\phi$, for a $\mathrm{SiO}_{2}-\mathrm{Si}$ film-substrate single-reflection retarder with retardation angle $\Delta=30^{\circ}$ at $6328 \AA$.
(3) Find the angle of incidence $\phi^{\prime}$ at which the obtained curve intersects the $\phi$ axis. ${ }^{8}$ This angle is the required angle of incidence. For the chosen example, we have $\phi^{\prime}=82.53^{\circ}$.
(4) Calculate the angle (argument) $\alpha$ of the thickness complex-exponential function $X$ at $\phi^{\prime}$, using Eq. (2). Also, check that the magnitude of $X$ is unity.
(5) Calculate the film thickness $d^{\prime}$ necessary for the operation of the film-substrate system as an exact reflection retarder, by use of ${ }^{7}$

$$
\begin{equation*}
d^{\prime}=(\alpha / 2 \pi) D_{\phi^{\prime}}+m D_{\phi^{\prime}}, \tag{3}
\end{equation*}
$$

where

$$
D_{\phi^{\prime}}=\frac{1}{2} \lambda\left[N_{1}^{2}-N_{0}^{2} \sin ^{2} \phi^{\prime}\right]^{-1 / 2} .
$$

$D_{\phi^{\prime}}$ is the film-thickness period evaluated at $\phi^{\prime}$, and $m$ is an integer that can be chosen to obtain the film thickness in the required range. For the retardation angle $\Delta=30^{\circ}$, we found that the least film thickness equals $1084.5 \AA$, and the film thickness period $D_{\phi^{\prime}}=2952.4 \AA$.

## II. DESIGN RESULTS FOR Si-SiO ${ }_{2}$ SYSTEM AT $6328 \AA$

The design procedure of Sec. I has been carried out for different values of the retardation angle $\Delta$, to cover the whole range $-180^{\circ}<\Delta<+180^{\circ}$. The angle of incidence $\phi^{\prime}$ and the film thickness $d^{\prime}$ were obtained for each value of $\Delta$. In addition, for each retardation angle $\Delta$, the film-thickness period $D_{\phi}$, and the reflectances $\mathcal{R}_{p}$ and $\mathbb{R}_{s}$ were calculated. Table I lists the results. The values of $\Delta=0^{\circ}$ and $\Delta= \pm 180^{\circ}$ were excluded from the calculations, and are not given in Table I, because $\Delta=0^{\circ}$ occurs at all film thicknesses at grazing incidence ( $\phi=90^{\circ}$ ) and $\Delta= \pm 180^{\circ}$ occurs at all film thicknesses at perpendicular incidence ( $\phi=0^{\circ}$ ).

Figures 3-5 represent the design results. Figure 3 gives the angle of incidence $\phi^{\prime}$ for each retardation angle $\Delta$. It shows two branches, one for $\phi^{\prime}$ values associated with positive retardation angles $B_{\phi^{\prime}}^{+}$, and the other for $\phi^{\prime}$ values associated with negative retardation angles $B_{\Phi^{\prime}}^{-}$. From Fig. 3, it is clear that the two branches, $B_{\phi^{\prime}}^{+}$and $B_{\phi^{\prime}}^{-}$, are almost mirror images of one another. ${ }^{9}$ Also, Fig. 3 shows that the angle of incidence $\phi^{\prime}$ approaches $90^{\circ}\left(0^{\circ}\right)$ as the retardation angle $\Delta$ approaches $0^{\circ}\left( \pm 180^{\circ}\right)$, as expected.

Figure 4 also shows two branches for the $d^{\prime}-\Delta$ curve, namely $B_{d^{\prime}}^{+}$and $B_{d^{\prime}}^{-}$, corresponding to film thicknesses $d^{\prime}$ that give positive and negative retardations $\Delta$ in the ranges $0^{\circ}<\Delta<+180^{\circ}$ and $-180^{\circ}<\Delta<0^{\circ}$, respectively. The discontinuity of tine $d^{\prime}-\Delta$ curve at $\Delta=0^{\circ}$ is not unexpected, because the condition, $\Delta=0$, can be realized at any value of the film thickness. It is important to note that the two branches, $B_{d^{\prime}}^{+}$, and $B_{d^{+}}^{-}$, exist in two isolated

TABLE I. This table gives, for each value of the retardation angle $\Delta$, the angle of incident $\phi^{\prime}$ and the least film thickness $d^{\prime}$ necessary for operation of the $\mathrm{SiO}_{2}-\mathrm{Si}$ film-substrate system as an exact reflection retarder at $6328 \AA$ and the associated film-thickness period $D_{\phi^{\prime}}$ and reflectances $\mathbb{R}_{\phi}$ and $\mathbb{R}_{s}$ (note that $\left.\mathbb{R}_{p}=\mathbb{R}_{s}\right){ }^{2}$

| $\Delta$ (deg) | $\phi^{\prime}(\mathrm{deg})$ | $d^{\prime}\left(\begin{array}{l}\text { ( }\end{array}\right)$ | $D_{\phi^{\prime}}(\AA)$ | $\mathcal{O}_{p}$ | $R_{s}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | 87.506 | 1087.9 | 2971.9 | 0.8191 | 0.8191 |
| 20 | 85.016 | 1086.6 | 2964.5 | 0.6728 | 0.6728 |
| 30 | 82.530 | 1084.5 | 2952.4 | 0.5555 | 0.5555 |
| 40 | 80.048 | 1081.6 | 2935. 8 | 0.4621 | 0.4621 |
| 50 | 77.562 | 1077.9 | 2915.2 | 0.3880 | 0.3380 |
| 60 | 75.063 | 1073.5 | 2890.7 | 0.3291 | 0.3291 |
| 70 | 72.534 | 1068.5 | 2862.6 | 0.2822 | 0.2822 |
| 80 | 69.953 | 1062.7 | 2831. 0 | 0.2448 | 0.2448 |
| 90 | 67.290 | 1056.3 | 2795.9 | 0.2148 | 0.2148 |
| 100 | 64.505 | 1049.1 | 2757.2 | 0.1907 | 0.1907 |
| 110 | 61.547 | 1041.1 | 2714.5 | 0.1711 | 0.1711 |
| 120 | 58.341 | 1032.1 | 2667.3 | 0.1551 | 0.1551 |
| 130 | 54.779 | 1022.0 | 2614.8 | 0.1420 | 0.1420 |
| 140 | 50.690 | 1010.5 | 2555.5 | 0.1311 | 0.1311 |
| 150 | 45.779 | 997.1 | 2487.4 | 0.1220 | 0.1220 |
| 160 | 39.441 | 981.0 | 2406.9 | 0.1141 | 0.1141 |
| 170 | 30.035 | 960.9 | 2306.9 | 0.1069 | 0.1069 |
| -10 | 87.512 | 1880.1 | 2971.9 | 0.8192 | 0.8192 |
| -20 | 85. 028 | 1874.1 | 2964.5 | 0.6730 | 0.6730 |
| -30 | 82.547 | 1864.2 | 2952.5 | 0.5558 | 0.5558 |
| -40 | 80.070 | 1850.7 | 2936.0 | 0.4624 | 0.4624 |
| $-50$ | 77.590 | 1833.8 | 2915.4 | 0.3882 | 0.3882 |
| -60 | 75. 097 | 1813.8 | 2891.0 | 0.3293 | 0.3293 |
| -70 | 72.573 | 1790.9 | 2863.0 | 0.2825 | 0.2825 |
| -80 | 69.997 | 1765.1 | 2831.5 | 0.2451 | 0.2451 |
| -90 | 67. 340 | 1736.6 | 2796.6 | 0.2151 | 0.2151 |
| -100 | 64.561 | 1705.3 | 2758. 0 | 0.1909 | 0.1909 |
| -110 | 61.608 | 1670.7 | 2715.4 | 0.1713 | 0.1713 |
| $-120$ | 58.407 | 1632.7 | 2668.3 | 0.1553 | 0.1553 |
| $-130$ | 54.850 | 1590.3 | 2615.8 | 0.1422 | 0.1422 |
| $-140$ | 50.768 | 1542.7 | 2556.6 | 0.1314 | 0.1314 |
| -150 | 45.862 | 1488.2 | 2488.5 | 0.1222 | 0.1222 |
| -160 | 39.528 | 1423.8 | 2408.0 | 0.1143 | 0.1143 |
| -170 | 30.122 | 1343.9 | 2307.7 | 0.1071 | 0.1071 |

${ }^{2}$ By combining a number of reflections (e.g., two or three), we can use the results of this table to generate new multiplereflection retarders that maintain the parallelism or collinearity of the incident and emergent beams.


FIG. 3. Angle of incidence $\phi^{\prime}$ (in degrees) vs retardation angle $\Delta$ (in degrees) for exact operation as reflection retarders, for $\mathrm{Si}-\mathrm{SiO}_{2}$ system at $6328 \AA$.
(nonoverlapping) thickness ranges $933 \leq d^{\prime} \leq 1088 \AA$ and $1232 \leq d^{\prime} \leq 1883 \AA$, respectively.

In Fig. 5 , the two branches $B_{\mathfrak{A}}^{+}$and $B_{\mathfrak{R}}^{-}$give the reflectances $\mathfrak{R}_{p}=\mathfrak{R}_{s}=\mathfrak{R}$ for the positive and negative ranges of $\Delta$, respectively. Each branch is almost a mirror


FIG. 4. Least film thickness $d^{\prime}$ (in angstroms) vs retardation angle $\Delta$ (in degrees) for exact operation as reflection retarders for $\mathrm{Si}-\mathrm{SiO}_{2}$ system at $6328 \AA$.


FIG. 5. The $p$ and $s$ reflectances, $\mathscr{R}_{p}(\Delta)$ and $\mathscr{R}_{s}(0)$ vs retardation angle $\Delta$ (in degrees) for exact $\mathrm{SiO}_{2}-\mathrm{Si}$ film-substrate sin-gle-reflection retarders at 6328 A.
image of the other. The reflectances asymptotically approach unity at $\Delta=0^{\circ}$, which should take place at grazing incidence.

## III. ANGLE-OF-INCIDENCE-TUNABLE RETARDERS

The difference between the smallest and largest film thicknesses required for operation of the $\mathrm{Si}-\mathrm{SiO}_{2}$ system as an exact reflection retarder over the range $0^{\circ}<\Delta<+180^{\circ}$ at $6328 \AA$ is approximately $150 \AA$ (see Fig. 4). This suggests that an angle-of-incidencetunable retarder is feasible, by choice of one of the designs for positive retardations $\Delta$ in Table I. (Because this difference of film thickness is relatively small, an error in oxidizing the silicon wafer to the desired thickness of the $\mathrm{SiO}_{2}$ film will only change the




FIG. 6. Performance curves for $\mathrm{SiO}_{2}$-Si film-substrate sin-gle-reflection retarder of thickness $d=1010 \AA$ at $6328 \AA$.
Left: ellipsometric angle $\psi$ (in degrees) vs angle of incidence $\phi$ (in degrees). Middle: ellipsometric angle $\Delta$ (in degrees) vs angle of incidence $\phi$ (in degrees). This is the retarder's tuning curve. Right: $p$ and $s$ reflectances $\mathbb{R}_{p}(\Delta)$ and $\mathbb{R}_{s}(0)$ vs angle of incidence $\phi$ (in degrees), respectively.
angle of incidence $\phi^{\prime}$ at which the system operates as an exact retarder.) In fact, any of the designs in Table I will operate as an angle-of-incidence-tunable reflection retarder with an acceptable error over a wide range of $\phi$. It seems logical to consider a film whose thickness is the average ( $d=1010 \AA$ ) between the upper ( $d=1088 \AA$ ) and lower ( $d=933 \AA$ ) thickness limits for an angle-of-incidence-tunable film-substrate retarder. The performance curves of this design are shown in Fig. 6. The angle of incidence at which this design operates as an exact retarder is $\phi^{\prime}=50.69^{\circ}$, with retardation angle $\Delta=140.0^{\circ}$.

As can be appreciated from Table I, the difference between the smallest and largest film thicknesses required for the system to operate as exact reflection retarders over the range $-180^{\circ}<\Delta<0^{\circ}$ is relatively large, $650 \AA$. Therefore, there is no suitable film thickness at which the system can operate as an angle-of-incidence-tunable retarder, over a wide range of $\phi$, with a reasonable error.

By study of the behavior of all of the designs, we can select some that are more suitable than others as angle-of-incidence-tunable retarders. ${ }^{10}$ The $\psi-\phi$ curves for each of the designs of Table I, with $0^{\circ}<\Delta<+180^{\circ}$, are shown, collectively, in Fig. 7. In Fig. 8, only the envelope of the corresponding $\Delta-\phi$ tuning curves is shown, because the difference of $\Delta$ between any two successive tuning curves is small. All of the curves meet at $\phi=0^{\circ}$ and $\phi=90^{\circ}$. The closeness of these tuning curves, Fig. 8, means that almost the same angle-


FIG. 7. Ellipsometric angle $\psi$ (in degrees) versus the angle of incidence $\phi$ (in degrees) for $\mathrm{Si}_{\mathrm{SiO}}^{2}$ exact reflection retarder designs with retardation angles $\Delta=10^{\circ}-170^{\circ}$ with a step of $10^{\circ}$ at $6328 \AA$.


FIG. 8. The retardation angle $\Delta$ (in degrees) vs the angle of incidence $\phi$ (in degrees) for $\mathrm{Si}-\mathrm{SiO}_{2}$ exact reflection retarder designs with retardation angles $\Delta=10^{\circ}$ and $170^{\circ}$ at $6328 \AA$.
of-incidence-tunable retardation will be obtained for the designs listed in Table I with $0^{\circ}<\Delta<+180^{\circ}$. Therefore, a small error of film thickness can be tolerated.

The choice of an angle-of-incidence-tunable design, for a certain range of angle of incidence, can be based on the criterion of minimum-maximum error (MME) of the ellipsometric angle $\psi$. This error ( $\psi-45^{\circ}$ ) should not exceed a prespecified value. [ $\psi$ is related to the ratio of the $p$ and $s$ reflectances $\mathcal{R}_{p}$ and $\mathcal{R}_{s}$ by $\left.\psi=\tan ^{-1}\left(\mathcal{R}_{p} / \mathcal{R}_{s}\right)^{1 / 2}\right]$. We determined the film thickness values that lead to $\psi-\mathrm{MME}$ over the angle of incidence ranges $0^{\circ}-45^{\circ}, 45^{\circ}-90^{\circ}$, and $0^{\circ}-90^{\circ}$. The results are summarized in Table II. The performance curves of the designs of Table $\Pi 1$ are shown in Figs. 9 and 10. Figure 9 shows, from left to right, the $\psi-\phi$ curve, the $\Delta-\phi$ tuning curve, and the reflectances $\mathcal{R}_{p^{-}}$and $\mathcal{R}_{s}-\phi$ curves for $d=981.01 \AA$, respectively. Figure 10 shows the corresponding curves for $d=1041.1 \AA$.

Other criteria for screening the angle-of-incidencetunable retarder designs such as the rate of change of the ellipsometric angles $\psi$ and $\Delta$ with respect to the film thickness $d$ and angle of incidence $\phi, \partial \psi / \partial d, \partial \Delta / \partial d$;

TABLE II. Angle-of-incidence-tunable $\mathrm{Si}-\mathrm{SiO}_{2}$ retarders at $6328 \AA$ that achieve minimum-maximum error of $\psi$ over selected angle-of-incidence ranges $0^{\circ}-45^{\circ}, 45^{\circ}-90^{\circ}$, and $0^{\circ}-90^{\circ}$.

| range of $\phi$ <br> (degrees) | $d^{\prime}$ <br> (angstroms) | maximum error in $\psi$ <br> $\left(\psi-45^{\circ}\right)$ <br> (degrees) | $\phi^{\prime}$ <br> (degrees) |
| :---: | :--- | :--- | :--- |
| $0-45$ | 981.01 | 0.61055 | 39.441 |
| $45-90$ | 1041.1 | 2.13 | 61.547 |
| $0-90$ | 1041.1 | 2.16 | 61.547 |



FIG. 9. Performance curves, as in Fig. 6, but for $d=981.01 \AA$.
$\partial \psi / \partial \phi$, and $\partial \Delta / \partial \phi$, respectively, are being investigated. Also under investigation is the effect of other choices of film and substrate materials as well as the use of two or three reflections to maintain the parallelism or collinearity between the light beams incident on and emergent from the retarder.

## IV. OPERATION OF A GIVEN FILM-SUBSTRATE SYSTEM WITH KNOWN OPTICAL CONSTANTS AND FILM THICKNESS AS AN EXACT REFLECTION RETARDER

The unit circle $U$ in the complex $\rho$ plane is the locus of all possible reflection retarders for any film-substrate system. If the constant-thickness contour (CTC) for a given value of film thickness intersects the unit circle $U$, then at the angles of incidence at the intersection points the given film-substrate system will operate as an exact reflection retarder with different retardations. Figure 11 (right) shows the unit circle $U$ and the CTC's $A$ and $B$ for film thicknesses $d=18700 \AA$ and $d=3000 \AA$, respectively, for the $\mathrm{Si}-\mathrm{SiO}_{2}$ system at 6328 Å.

Instead of plotting the CTC in the complex- $\rho$ plane and determining its intersection points with the unit circle $U$, we can easily obtain the possible angles of incidence $\phi$ for operation as an exact reflection retarder and the associated retardation angles $\Delta$, by use of the $\phi-d$ plane, Fig. 11(left). First, we draw the images of both the unit circle $U$ and the CTC in the $\phi-d$ plane. From their intersection points, we find the values of $\phi$. By straightforward calculations, we get the values of $\Delta$. If the two images do not intersect one another, the exact reflec-tion-retardation mode can not be associated with this film thickness.

To get the image of the unit circle $U$ in the $\phi-d$ plane, $U^{\prime}$, we calculate the values of $\phi^{\prime}$ and $d^{\prime}$ over the ranges $0^{\circ}<\Delta<+180^{\circ}$ and $-180^{\circ}<\Delta<0^{\circ}$, and plot the results in the $\phi-d$ plane. (Such calculations have already been made for the $\mathrm{Si}-\mathrm{SiO}_{2}$ system in Sec. II at $6328 \AA$.) In Fig. 11 (left) the image of the upper half of the unit


FIG. 10. Performance curves, as in Fig. 6, but for $d=1041.1 \AA$.


FIG. 11. (Right) The CTC's $A$ and $B$ for film thicknesses $d=18700 \AA$ and $d=3000 \AA$, respectively, in the complex- $\rho$ plane for $\mathrm{Si}-\mathrm{SiO}_{2}$ system at $6328 \AA$. Curve $U$, the unit circle, is the locus of all possible reflection retarders. (Left) The images $A^{\prime}, B^{\prime}$, and $U^{\prime}$ in the $\phi-d$ plane of the CTC's $A$ and $B$, and the unit circle $U$ of the complex- $\rho$ plane, respectively, for $\mathrm{Si}-\mathrm{SiO}_{2}$ system at 6328 A.
circle $U$ is the bottom branch of $U^{\prime}$, and the image of the lower half is the upper branch. Because the vertical lines at $\phi=0^{\circ}$ and $\phi=90^{\circ}$ in the $\phi-d$ plane are the images of only two points on the unit circle $U, \rho=-1$ and $\rho=+1$, respectively, the image of the entire unit circle

409.910

1217.125


FIG. 12. Permissible-thickness bands and forbidden-thickness gaps for operation of the $\mathrm{Si}-\mathrm{SiO}_{2}$ system as an exact reflection retarder at $6328 \AA$.
in the $\rho$ plane is a closed curve $U^{\prime}$ in the $\phi-d$ plane.
The image of any CTC in the reduced thickness range $0 \leq d<D_{\phi}$ is obtained very simply by subtracting from the given film thickness $d=$ const (assumed greater than $D_{0}$ ) the appropriate multiple of the film-thickness period $D_{\phi}$, at each angle of incidence $\phi$ in the range $0^{\circ} \leq \phi \leq 90^{\circ}$. Curve $A^{\prime}$ in Fig. 11(left) is the image of the CTC $A$ in Fig. 11 (right), $d=18700 \AA$. The images $U^{\prime}$ and $A^{\prime}$ of both the unit circle $U$ and the CTC $A$ intersect one another at the points $U_{1}, U_{2}, U_{3}, U_{4}, U_{5}$, and $U_{6}$. The corresponding angles of incidence ${ }^{11}$ are $\phi_{1}^{\prime}=9.813^{\circ}$, $\phi_{2}^{\prime}=18.375^{\circ}, \phi_{3}^{\prime}=43.875^{\circ}, \phi_{4}^{\prime}=48.75^{\circ}, \phi_{5}^{\prime}=69.375^{\circ}$, and $\phi_{6}^{\prime}=80.125^{\circ}$. From these angles of incidence and the given film thickness ( $d=18700 \AA$ ) we can get the associated retardation angle and the $p$ and $s$ reflectances by direct calculation. This gives the retardation angles $\Delta_{1}=-179.15^{\circ}, \Delta_{2}=176.78^{\circ}, \Delta_{3}=-153.59^{\circ}, \Delta_{4}=144.13^{\circ}$, $\Delta_{5}=-82.37^{\circ}$, and $\Delta_{6}=39.69^{\circ}$ and the reflectances $\mathfrak{R}$ $\left(\mathcal{R}_{p}=\mathcal{R}_{s}=\mathbb{R}\right)$ are $\mathcal{R}_{1}=0.100, \mathcal{R}_{2}=0.102, \mathcal{R}_{3}=0.119$, $\mathfrak{R}_{4}=0.127, \mathfrak{R}_{5}=0.237$, and $\mathfrak{R}_{6}=0.465$.

The CTC for $d=3000 \AA$ and its image, curves $B$ and $B^{\prime}$ in Fig. 11, represent the case where the CTC and the unit circle ( $B$ and $U$ in the complex $\rho$ plane) and consequently their images ( $B^{\prime}$ and $U^{\prime}$ in the $\phi-d$ plane) do not intersect one another. ${ }^{12}$ Therefore, at this film thickness, the system cannot operate as an exact reflection retarder at any angle of incidence. This implies the existence of ranges of film thickness for which the mode of operation as an exact reflection retarder is possible, separated by gaps for which it is not. For the $\mathrm{Si}-\mathrm{SiO}_{2}$ system at $6328 \AA$ the permissible-thickness bands are shown in Table III in addition to the sign of the possible retardation angles $\Delta$ in each band. Table IV gives the forbidden-thickness gaps and their widths. These tables are obtained by adding the film thickness periods $D_{0} \circ=2167.125 \AA$ and $D_{90} \circ=2974.340 \AA$ to the corresponding minimum $d_{\text {min }}^{-}=1232 \AA$ and $d_{\text {min }}^{+}=933 \AA$ (at $\phi=0^{\circ}$ ) and maximum $d_{\max }^{-}=1883 \AA$ and $d_{\max }^{+}=1088 \AA$ (at $\phi=90^{\circ}$ ) least thickness for exact retarder operation with negative and positive retardations, respectively. ${ }^{13}$ By use of Table III or IV, we can immediately determine if at a certain film thickness the film-substrate system can operate as an exact reflection retarder, or not. Figure 12 shows a schematic diagram of the per-missible-thickness bands and the forbidden-thickness gaps.

TABLE III. Permissible-thickness bands and their widths (in angstroms) and the retardation sign for operation of the $\mathrm{Si}-\mathrm{SiO}_{2}$ system as an exact reflection retarder at $6328 \AA$.

| $d_{\min }(\AA)$ | $d_{\max }(\AA)$ | bandwidth $(\AA)$ | retardation sign |
| :--- | :--- | :--- | :--- |
| 933 | 1089 | 156 | + |
| 1232 | 1883 | 651 | - |
| 3100.125 | 3399.125 | 299 | + |
| 3399.125 | 4063.340 | 664.215 | $\pm$ |
| 4063.340 | 4857.340 | 794 | - |
| 5267.250 | 5566.250 | 299 | + |
| 5566.250 | 7037.680 | 1471.430 | $\pm$ |
| 7037.680 | 7434.375 | 396.695 | - |
| 7434.375 | $\infty$ | $\infty$ | $\pm$ |

TABLE IV. Forbidden-thickness gaps and their bandwidths (in angstroms) for operation of the $\mathrm{Si}-\mathrm{SiO}_{2}$ system as an exact reflection retarder at $6328 \AA$.

| $d_{\min }(\AA)$ | $d_{\max }(\AA)$ | bandwidth $(\AA)$ |
| :--- | :---: | :---: |
| 0 | 933 | 933 |
| 1089.000 | 1232.000 | 143.000 |
| 1883.000 | 3100.125 | 1217.125 |
| 4857.340 | 5267.250 | 409.910 |

## V. EXPERIMENTAL

To verify the operation of one design of a film-substrate reflection retarder, the ellipsometric angles $\psi$ and $\Delta$ were measured with a conventional ellipsometer, by use of a commercial oxidized silicon wafer of a nominal film thickness $d=1000 \AA$ at wavelength $6328 \AA$. On the basis of the published data ${ }^{5}$ for the optical constants of Si and $\mathrm{SiO}_{2}$, and the nominal film thickness, the system was expected to work as a reflection retarder with retardation angle $\Delta \simeq 150^{\circ}$ at $\phi \simeq 45.78^{\circ}$ (for exact operation the film thickness should be $d=997 \AA \AA$ ), see Table I. From our measurements on this $\mathrm{Si}-\mathrm{SiO}_{2}$ system at $\lambda=6328 \AA$, we found, at $\phi=46.5^{\circ}$, that $\psi=46.916^{\circ}$ and $\Delta=149.974^{\circ}$. The difference between $\tan \psi$ and unity is 0.068 , which is of the same order of magnitude as the relative transmittance of a typical compensator used in ellipsometers. The error of $\Delta$ ( $0.026^{\circ}$ ) is also reasonable. The measurement thus confirms the suggested design.

## VI. COMPUTER PROGRAMS

Computer programs written in Fortran IV Language for carrying out the design procedure and obtaining the performance data are available from the authors.

[^0]results of the film-substrate single-reflection retarder designs has shown that the mirror symmetry of the two branches $B^{+}$and $B^{-}$occurs exactly only when the substrate is totally transparent. Deviation from exact symmetry increases with substrate absorption.
${ }^{10}$ Angle-of-incidence-tunable retarders are those with the least thickness. By adding multiples of $D_{\phi}$, the performance of the retarder gets worse, refer to Fig. 11(left).
${ }^{11}$ An alternative procedure would be to add the appropriate mul-
tiple of $D_{\phi}$, at each $\phi$, to the image of the unit circle and to obtain the intersection points of this vertically translated image with the straight line $d=$ const. In general, the first method is more convenient.
${ }^{12}$ The existence of exact reflection-retardation modes depends, for a particular system at a given wavelength, on the film thickness only.
${ }^{13} d_{\min }$ and $d_{\max }$ are obtained by extrapolation to $\Delta=0^{\circ}$ and $\Delta= \pm 180^{\circ}$ of the branches shown in Fig. 4.

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    ${ }^{\dagger}$ Also with the Div. of Hematology, Dept. of Internal Medicine, College of Medicine, University of Nebraska Medical Center, Omaha, Neb. 68105.
    ${ }^{1}$ J. M. Bennett and H. E. Bennett, in Handbook of Optics, edited by W. G. Driscoll and W. Vaughan (McGraw-Hill, New York, 1975).
    ${ }^{2}$ The magnitude of the attenuation is independent of the incident polarization.
    ${ }^{3}$ R. M. A. Azzam, A.-R. M. Zaghloul, and N. M. Bashara, J. Opt. Soc. Am. 65, 252 (1975).
    ${ }^{4}$ Note that, whereas the use of a film-substrate system as a reflection retarder is new, its use as a reflection polarizer is well known. See, for example, M. Ruiz-Urbieta and E. M. Sparrow, J. Opt. Soc. Am. 62, 1188 (1972) and other papers in this series.
    ${ }^{5}$ Ellipsometric Tables of the Si-SiO ${ }_{2}$ System for Mercury and He-Ne Laser Spectral Lines, edited by G. Gergely (Akademiai Kiado, Budapest, 1971).
    ${ }^{6}$ In step 1 , instead of using $\rho=e^{j \Delta}$, we use the general form $\rho=\tan \psi e^{j \Delta}$; steps $2-5$ remain unchanged.
    ${ }^{7}$ The derivation of Eqs. (2) and (3) is given in Ref. 3.
    ${ }^{8}$ Alternatively, the angle of incidence $\phi^{\prime}$ can be obtained by any numerical method (e.g., successive bisection) to find the root of the equation $|X|=1$.
    ${ }^{9}$ Our investigation of the effect of substrate absorption on the

