DESIGN OF LINEAR PHASE FIR FILTERS WITH A MAXIMALLY FLAT PASSBAND

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ABSTRACT

The design of linear phase FIR digital filters having symmetric or antisymmetric impulse response is formulated as a constrained minimization problem. The constraints express the maximal flatness of the frequency response at the origin in the case of a lowpass filter or at an arbitrary frequency in the passband in the case of a bandpass filter. The objective function , which is a quadratic form in the filter coefficients , is formed as a convex combination of two objective functions representing the energy of the error between the frequency response of the designed filter and a scaled version of the frequency response of the ideal filter in both the stop and pass bands.

I. INTRODUCTION

Standard approximation routines for the design of Finite Impulse Response (FIR) filters meet the design specifications with minimum degree but in an equiripple sense [1]. From a time domain point of view, minimax filters have the disadvantage of having sidelobes or echoes in the impulse response and ringing in the step response due to their sharp cutoff frequency response. The amplitude of these echoes is proportional to the amplitude of the passband ripples [2]. In those cases where the time domain properties are of prime importance, filters with smooth frequency response are significantly preferred to minimax filters. Smooth frequency response can be achieved by filters with maximally flat (MF) passbands.

Rajagopal and Dutta Roy used Bernstein polynomials for designing maximally flat lowpass (MFLP) filters of even order and symmetric impulse response [3]. Aikawa et. al. designed MFLP filters of odd order [4]. Cooklev and Nishihara used Bernstein approximants which are a generalization of the Bernstein polynomials in designing MF FIR filters [5]. Since attempting to design MF FIR filters by only imposing many derivative constraints at a reference frequency (typically the center of the passband) leads to severe numerical problems, Kaiser and Steiglitz employed a linear programming technique by imposing one derivative constraint at the origin of a lowpass filter and a concavity constraint at many frequency grid points in the passband [6]. Schuessler and Steffen designed MFLP filters by minimizing the energy of the impulse response (which is the same as the integral of the square of the magnitude of the frequency response over all frequencies) subject to several flatness constraints at the origin [7]. Medlin, Adams and Leondes [8] and Stubberud and Leondes [9] designed MFLP filters by minimizing the energy of the error between the desired and designed frequency responses only over the stopbands subject to flatness constraints at the center of the passband (zero frequency).

In this paper linear phase FIR filters with symmetric or antisymmetric impulse response are designed to have a maximally flat passband. The filters are derived by minimizing a quadratic criterion subject to linear constraints. The objective function , which is a quadratic form in the filter coefficients , is formulated as a convex combination of two criteria representing the energy of the error between the frequency response of the actual filter and a scaled version of the desired frequency response in both the pass and stop bands. The constraints express the maximal flatness of the frequency response at zero frequency in the case of a lowpass filter and at an arbitrary frequency in the passband (typically the center frequency) in the case of a bandpass filter. This extends the results reported in [8,9] where the mean squared error is minimized only over the stopband. The simulation results will demonstrate the importance of incorporating the passband in the minimization criterion. Medlin and Kaiser did related work in designing bandpass digital differentiators employing an objective function which is a sum of a quadratic term, a linear term and a constant [10]. Moreover they defined the objective function as a sum - rather than a convex combination - of two criteria corresponding to the pass and stop frequency bands so that the user of the filter does not have the freedom of the tradeoff between more flatness in the passband and better attenuation in the stop band. It is true that recently Stubberud, Awad, Adams and Leondes [11] designed FIR filters by minimizing a criterion which is a convex combination of both criteria, but they did not employ any flatness constraints; instead they constrained the frequency response to be zero at several frequencies in the stop band.

II. THE MAXIMAL FLATNESS CONSTRAINTS

(A) FIR FILTERS WITH SYMMETRIC IMPULSE RESPONSE

The frequency response of a linear phase FIR filter ,i.e., one with a symmetric impulse response is [12] :

$$H(e^{j\omega}) = e^{-j\omega(\frac{N-1}{2})} H_a(\omega) \tag{1}$$

where the amplitude function $H_a(\omega)$ can be expressed as [1] :

$$H_a(\omega) = s^T(\omega)x \tag{2}$$

and where the vectors x and $s(\omega)$ are given below for the cases of odd and even length N of the real finite impulse response h(n) :

$$x^T = \begin{cases} [h(0.5(N-1)) \quad 2h(0.5(N-1)-1) \quad \ldots \quad 2h(1) \quad 2h(0)] & \text{N odd} \\ [2h(0.5N-1) \quad 2h(0.5N-2) \quad \ldots \quad 2h(1) \quad 2h(0)] & \text{N even} \end{cases} \tag{3}$$

and

[1] The superscript T is used to denote the transpose of a matrix or vector.

$$s^T(\omega) = \begin{cases} [1 \quad \cos(\omega) \quad \cos(2\omega) \quad \ldots \quad \cos(0.5(N-1)\omega)] & \text{N odd} \\ [\cos(0.5\omega) \quad \cos(1.5\omega) \quad \cos(2.5\omega) \quad \ldots \quad \cos((0.5N - 0.5)\,\omega)] & \text{N even} \end{cases} \qquad (4)$$

Since FIR filters with symmetric impulse responses will be used for designing lowpass filters with maximally flat passband , the following flatness conditions will be applied at zero frequency :

$$\left.\frac{d^k H_a(\omega)}{d\omega^k}\right|_{\omega=0} = 0 \quad , \text{ k integer} \qquad (5)$$

From Eq(2) we have :

$$\frac{d^k H_a(\omega)}{d\omega^k} = \left(\frac{d^k s(\omega)}{d\omega^k}\right)^T x \quad . \qquad (6)$$

It can be shown that condition (5) is satisfied for all odd order derivatives. From Eq(4) the even order derivatives are given by :

$$\left.\frac{d^k s(\omega)}{d\omega^k}\right|_{\omega=0} = \begin{cases} (-1)^{0.5k}[0 \quad 1 \quad 2^k \quad \ldots \quad (0.5(N-1))^k]^T & \text{N odd} \\ (-1)^{0.5k}[(0.5)^k \quad (1.5)^k \quad (2.5)^k \quad \ldots \quad (0.5(N-1))^k]^T & \text{N even} \end{cases} \quad , k = 2,4,6,\ldots \qquad (7)$$

The following normalization condition will always be imposed :

$$\left.H_a(\omega)\right|_{\omega=0} = 1.0 \qquad (8)$$

From Eqs(2) and (4), the above condition can be expressed as :

$$(1 \quad 1 \quad 1 \quad \ldots \quad 1)x = 1 \qquad (9)$$

Constraint (9) and (r-1) constraints of the form of Eq(5) for k = 2, 4, ... , 2(r-1) can be compactly expressed as :

$$Cx = K \qquad (10)$$

where matrix C and vector K are given by :

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2^2 & 3^2 & \ldots & (0.5(N-1))^2 \\ 0 & 1 & 2^4 & 3^4 & \ldots & (0.5(N-1))^4 \\ . & . & . & . & \ldots & . \\ . & . & . & . & \ldots. & . \\ 0 & 1 & 2^{2(r-1)} & 3^{2(r-1)} & \ldots & (0.5(N-1))^{2(r-1)} \end{bmatrix} \quad \text{, N odd} \qquad (11a)$$

$$C = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 3^2 & 5^2 & \ldots & (N-1)^2 \\ 1 & 3^4 & 5^4 & \ldots & (N-1)^4 \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ 1 & 3^{2(r-1)} & 5^{2(r-1)} & \ldots & (N-1)^{2(r-1)} \end{bmatrix} \quad \text{, N even} \qquad (11b)$$

$$K = (1 \quad 0 \quad 0 \quad \ldots \quad 0)^T \qquad (12)$$

Matrix C is r x m and vector K is r-dimensional where :

$$m = \begin{cases} 0.5(N+1) & \text{N odd} \\ 0.5\text{N} & \text{N even} \end{cases} \qquad (13)$$

and $r \le m$.

(B) FIR FILTERS WITH ANTISYMMETRIC IMPULSE RESPONSE

The frequency response of an FIR filter having an antisymmetric impulse response is [12] :

$$H(e^{j\omega}) = je^{-j\omega(\frac{N-1}{2})} H_a(\omega) \tag{14}$$

where the amplitude function $H_a(\omega)$ is given by Eq(2) and where the vectors x and s(ω) are given below for the cases of odd and even length N of h(n) :

$$x^T = \begin{cases} [2h(0.5(N-3)) \quad 2h(0.5(N-5)) \quad \ldots \quad 2h(0)] & \text{, N odd} \\ [2h(0.5N-1) \quad 2h(0.5N-2) \quad \ldots \quad 2h(0)] & \text{, N even} \end{cases} \tag{15}$$

and

$$s^T = \begin{cases} [\sin(\omega) \quad \sin(2\omega) \quad \ldots \quad \sin(0.5(N-1)\omega)] & \text{, N odd} \\ [\sin(0.5\omega) \quad \sin(1.5\omega) \quad \ldots \quad \sin(0.5(N-1)\omega)] & \text{, N even} \end{cases} \tag{16}$$

Since FIR filters with antisymmetric impulse response will be used for designing bandpass filters with a maximally flat passband, the following flatness conditions will be applied at an arbitrary frequency ω_m in the passband (typically the center frequency) :

$$\left. \frac{d^k H_a(\omega)}{d\omega^k} \right|_{\omega=\omega_m} = 0 \qquad , k = 1 , 2 , \ldots \tag{17}$$

In addition to the above flatness conditions the normalization condition ,

$$\left. H_a(\omega) \right|_{\omega=\omega_m} = 1 \ , \tag{18}$$

will be imposed .

This normalization condition and (r - 1) flatness conditions of the form of (17) can be compactly expressed as Eq (10) where vector K is given by (12) and matrix C is given by :

$$C = \begin{bmatrix} \sin(\omega_m) & \sin(2\omega_m) & \cdots & \sin(0.5(N-1)\omega_m) \\ \cos(\omega_m) & 2\cos(2\omega_m) & \cdots & (0.5(N-1))\cos(0.5(N-1)\omega_m) \\ \sin(\omega_m) & 2^2\sin(2\omega_m) & \cdots & (0.5(N-1))^2\sin(0.5(N-1)\omega_m) \\ \cos(\omega_m) & 2^3\cos(2\omega_m) & \cdots & (0.5(N-1))^3\cos(0.5(N-1)\omega_m) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad \text{, N odd} \tag{19a}$$

$$C = \begin{bmatrix} \sin(0.5\omega_m) & \sin(1.5\omega_m) & \cdots & \sin(0.5(N-1)\omega_m) \\ \cos(0.5\omega_m) & 3\cos(1.5\omega_m) & \cdots & (N-1)\cos(0.5(N-1)\omega_m) \\ \sin(0.5\omega_m) & 3^2\sin(1.5\omega_m) & \cdots & (N-1)^2\sin(0.5(N-1)\omega_m) \\ \cos(0.5\omega_m) & 3^3\cos(1.5\omega_m) & \cdots & (N-1)^3\cos(0.5(N-1)\omega_m) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad \text{, N even} \tag{19b}$$

The above matrix C is r x q and vector K of (12) is r-dimensional where :

$$q = \begin{cases} 0.5(N-1) & \text{for N odd} \\ 0.5N & \text{for N even} \end{cases} \tag{20}$$

and $r \le q$.

III. THE MEAN SQUARED ERROR CRITERION

The error criterion will be derived as a quadratic form expressing the energy between the frequency response of the designed filter and a scaled version of the desired frequency response. Both pass and stop bands will be taken into account. The treatment in this section will be divided into two subsections, one for a lowpass filter having a symmetric impulse response and one for a bandpass filter having antisymmetric impulse response.

(A) LOWPASS FILTER (WITH SYMMETRIC IMPULSE RESPONSE)

The amplitude of the ideal low-pass frequency response is :

$$H_d(\omega) = \begin{cases} 1 & 0 \le \omega \le \omega_p \\ 0 & \omega_s \le \omega \le \pi \end{cases} \tag{21}$$

where ω_p and ω_s are the cutoff frequencies of the pass and stop bands respectively.
The weighted squared error measure in the stop band is :

$$E_s = \int_{\omega_s}^{\pi} W(\omega) e_s^2(\omega) d\omega \tag{22}$$

where $W(\omega)$ is a positive weighting function and $e_s(\omega)$, the stop band error , is :

$$e_s(\omega) = H_a(\omega) - H_d(\omega) = H_a(\omega) \tag{23}$$

Substituting Eqs(23) and (2) into (22) we get :

$$E_s = x^T P_s x \tag{24}$$

where

$$P_s = \int_{\omega_s}^{\pi} W(\omega) s(\omega) s^T(\omega) d\omega \tag{25}$$

The weighted squared error measure in the pass band is :

$$E_p = \int_{0}^{\omega_p} W(\omega) e_p^2(\omega) d\omega \tag{26}$$

where $e_p(\omega)$, the pass band error , is defined as :

$$e_p(\omega) = H_a(\omega) - \gamma H_d(\omega). \tag{27}$$

In the above equation γ is a scale factor defined as :

$$\gamma = \frac{H_a(\omega_0)}{H_d(\omega_0)} \tag{28}$$

where ω_0 is a reference frequency usually taken as the frequency corresponding to the maximum of $H_d(\omega)$. Substituting Eqs(27),(28) and (2) into Eq(26), we get :

$$E_p = x^T P_p x \tag{29}$$

where

$$P_p = \int_0^{\omega_p} W(\omega) \left[s(\omega) - \frac{H_d(\omega)}{H_d(\omega_0)} s(\omega_0) \right] \left[s(\omega) - \frac{H_d(\omega)}{H_d(\omega_0)} s(\omega_0) \right]^T d\omega \tag{30}$$

Actually the idea of introducing the scale factor γ was used in [13] so that the pass band mean squared error can be expressed as the quadratic form of Eq(29) .

Taking the reference frequency $\omega_0 = 0$ and the weighting function $W(\omega)=1$ (for mathematical tractability) and using Eqs(4) and (21), it can be shown that the elements of the symmetric matrices P_s and P_p of Eqs(25) and (30) are as given below :

(i) For N odd :

$$\left(P_s\right)_{r,c} = \begin{cases} \pi - \omega_s & r = c = 1 \\ \frac{\pi}{2} - \frac{\omega_s}{2} - \frac{\sin(2(r-1)\omega_s)}{4(r-1)} & r = c \neq 1 \\ -0.5\left[\frac{\sin((r+c-2)\omega_s)}{(r+c-2)} + \frac{\sin((r-c)\omega_s)}{(r-c)}\right] & r \neq c \end{cases} \tag{31}$$

$$\left(P_p\right)\Big|_{r,c} = \begin{cases} 0 & r = 1 \text{ or } c = 1 \\ \frac{3}{2}\omega_p + \frac{\sin(2(r-1)\omega_p)}{4(r-1)} - 2\frac{\sin((r-1)\omega_p)}{(r-1)} & r = c \neq 1 \\ \omega_p + \frac{\sin((r+c-2)\omega_p)}{2(r+c-2)} + \frac{\sin((r-c)\omega_p)}{2(r-c)} - \frac{\sin((r-1)\omega_p)}{(r-1)} - \frac{\sin((c-1)\omega_p)}{(c-1)} & r \neq 1, c \neq 1, r \neq c \end{cases} \tag{32}$$

(ii) For N even

$$\left(P_s\right)_{r,c} = \begin{cases} \frac{\pi}{2} - \frac{\omega_s}{2} - \frac{\sin((2r-1)\omega_s)}{2(2r-1)} & r = c \\ -0.5\left[\frac{\sin((r+c-1)\omega_s)}{(r+c-1)} + \frac{\sin((r-c)\omega_s)}{(r-c)}\right] & r \neq c \end{cases} \tag{33}$$

$$\left(P_p\right)_{r,c} = \begin{cases} \frac{3}{2}\omega_p + \frac{\sin((2r-1)\omega_p)}{2(2r-1)} - 2\frac{\sin((r-0.5)\omega_p)}{(r-0.5)} & r = c \\ \omega_p + \frac{\sin((r+c-1)\omega_p)}{2(r+c-1)} + \frac{\sin((r-c)\omega_p)}{2(r-c)} - \frac{\sin((r-0.5)\omega_p)}{(r-0.5)} - \frac{\sin((c-0.5)\omega_p)}{(c-0.5)} & r \neq c \end{cases} \tag{34}$$

In the above equations the indices are given by r,c = 1 , ... , m where m is defined by Eq(13). From Eqs(25) and (30), it is evident that the matrices P_s and P_p are at least positive semidefinite. Actually they are positive definite except for the case of odd N where matrix P_p becomes positive semidefinite since its first row and column are identically zero as can be seen from Eq(32).

The total mean squared error , E , will be taken as a convex combination of the mean squared errors over the stop and pass bands. More specifically, we define :

$$E = \alpha E_s + (1-\alpha)E_p \quad (35)$$

where the parameter α lies in the interval $0 \le \alpha \le 1$. Substituting Eqs(24) and (29) into (35), we get :

$$E = x^T P x \quad (36)$$

where

$$P = \alpha P_s + (1-\alpha)P_p \quad (37)$$

The symmetric matrix P is positive definite except in the extreme case of $\alpha = 0$ and N is an odd number where P becomes a positive semidefinite matrix . Therefore we restrict α to the interval $0 < \alpha \le 1$.

(B) BANDPASS FILTER (WITH ANTISYMMETRIC IMPULSE RESPONSE)

The amplitude of the ideal bandpass frequency response is :

$$H_d(\omega) = \begin{cases} 1 & \omega_{p_1} \le \omega \le \omega_{p_2} \\ 0 & 0 \le \omega \le \omega_{s_1} \text{ and } \omega_{s_2} \le \omega \le \pi \end{cases} \quad (38)$$

where ω_{p_1} and ω_{p_2} are the cutoff frequencies of the passband, and ω_{s_1} and ω_{s_2} are the cutoff frequencies of the stopband.

The weighted squared error measure in the stopband is given by Eq (24) where matrix P_S is defined by :

$$P_s = \int_0^{\omega_{s_1}} W(\omega)s(\omega)s^T(\omega)d\omega + \int_{\omega_{s_2}}^{\pi} W(\omega)s(\omega)s^T(\omega)d\omega \quad (39)$$

The weighted squared error measure in the passband is given by Eq (29) where matrix P_p is defined by :

$$P_p = \int_{\omega_{p_1}}^{\omega_{p_2}} W(\omega)\left[s(\omega) - \frac{H_d(\omega)}{H_d(\omega_0)}s(\omega_0)\right]\left[s(\omega) - \frac{H_d(\omega)}{H_d(\omega_0)}s(\omega_0)\right]^T d\omega \quad (40)$$

Taking the weighting function $W(\omega) = 1$, leaving ω_0 as an arbitrary reference frequency in the passband (typically the center frequency), and using Eqs (16) and (38), it can be shown that the elements of the symmetric matrices P_S and P_p of Eqs (39) and (40) are as given below where r,c = 1 , ... , q with q being given by Eq (20).

(i) For N odd

$$(P_s)_{r,r} = 0.5\pi + 0.5(\omega_{s_1} - \omega_{s_2}) - \frac{1}{4r}\left[\sin(2r\omega_{s_1}) - \sin(2r\omega_{s_2})\right] \quad (41a)$$

$$(P_s)_{r,c} = \frac{1}{2(r-c)}\left[\sin((r-c)\omega_{s_1}) - \sin((r-c)\omega_{s_2})\right] - \frac{1}{2(r+c)}\left[\sin((r+c)\omega_{s_1}) - \sin((r+c)\omega_{s_2})\right]$$

, r ≠ c (41b)

$$\left(P_p\right)_{r,r} = \frac{1}{2}\left(\omega_{p_2} - \omega_{p_1}\right) - \frac{1}{4r}[\sin\left(2r\omega_{p_2}\right) - \sin\left(2r\omega_{p_1}\right)]$$

$$+\frac{2}{r}\sin\left(r\omega_0\right)[\cos\left(r\omega_{p_2}\right) - \cos\left(r\omega_{p_1}\right)] + \left(\omega_{p_2} - \omega_{p_1}\right)\sin^2\left(r\omega_0\right)$$

(42a)

$$\left(P_p\right)_{r,c} = \frac{1}{2(r-c)}\left[\sin\left((r-c)\omega_{p_2}\right) - \sin\left((r-c)\omega_{p_1}\right)\right] - \frac{1}{2(r+c)}\left[\sin\left((r+c)\omega_{p_2}\right) - \sin\left((r+c)\omega_{p_1}\right)\right]$$

$$+\frac{1}{r}\sin(c\omega_0)\left[\cos(r\omega_{p_2}) - \cos(r\omega_{p_1})\right] + \frac{1}{c}\sin(r\omega_0)\left[\cos(c\omega_{p_2}) - \cos(c\omega_{p_1})\right]$$

$$+\left(\omega_{p_2} - \omega_{p_1}\right)\sin(r\omega_0)\sin(c\omega_0)$$

, r ≠ c (42b)

(ii) For N even

$$(P_s)_{r,r} = 0.5\pi + 0.5(\omega_{s_1} - \omega_{s_2}) - \frac{1}{2(2r-1)}\left[\sin((2r-1)\omega_{s_1}) - \sin((2r-1)\omega_{s_2})\right] \quad (43a)$$

$$(P_s)_{r,c} = \frac{1}{2(r-c)}\left[\sin((r-c)\omega_{s_1}) - \sin((r-c)\omega_{s_2})\right]$$

$$-\frac{1}{2(r+c-1)}\left[\sin((r+c-1)\omega_{s_1}) - \sin((r+c-1)\omega_{s_2})\right]$$

, r ≠ c (43b)

$$\left(P_p\right)_{r,r} = \frac{1}{2}(\omega_{p_2} - \omega_{p_1}) - \frac{1}{2(2r-1)}\left[\sin\left((2r-1)\omega_{p_2}\right) - \sin\left((2r-1)\omega_{p_1}\right)\right]$$

$$+\frac{2}{(r-0.5)}\sin((r-0.5)\omega_0)\left[\cos((r-0.5)\omega_{p_2}) - \cos((r-0.5)\omega_{p_1})\right] \quad (44a)$$

$$+(\omega_{p_2} - \omega_{p_1})\sin^2((r-0.5)\omega_0)$$

$$\left(P_p\right)_{r,c} = \frac{1}{2(r-c)}[\sin((r-c)\omega_{p_2}) - \sin((r-c)\omega_{p_1})]$$
$$- \frac{1}{2(r+c-1)}[\sin((r+c-1)\omega_{p_2}) - \sin((r+c-1)\omega_{p_1})]$$
$$+ \frac{1}{(r-0.5)}\sin((c-0.5)\omega_0)[\cos((r-0.5)\omega_{p_2}) - \cos((r-0.5)\omega_{p_1})] \quad , r \neq c \quad (44b)$$
$$+ \frac{1}{(c-0.5)}\sin((r-0.5)\omega_0)[\cos((c-0.5)\omega_{p_2}) - \cos((c-0.5)\omega_{p_1})]$$
$$+ (\omega_{p_2} - \omega_{p_1})\sin((r-0.5)\omega_0)\sin((c-0.5)\omega_0)$$

The total mean squared error is given by (36) where matrix P is given by (37).

IV. THE CONSTRAINED OPTIMIZATION PROBLEM

The linear system of Eq (10) is typically underdetermined since the number of constraints r is less than the length of vector x (m in case of a symmetric h(n) and q in case of an antisymmetric h(n)). Matrix C of Eq (11) has a full row rank since it is a submatrix of the first r rows of an m x m Vandermonde matrix [14]. Heuristically, matrix C of Eq(19) has a full row rank for $0 < \omega < \pi$. Consequently the linear system (10) is always consistent and has a family of solutions whenever r < m in the first case or r < q in the second case. To select the member of this family which minimizes the mean squared error of Eq (36) we apply the Lagrange multipliers technique for minimizing (36) subject to (10) to get the following unique solution [8] :

$$x = P^{-1}C^T\left(CP^{-1}C^T\right)^{-1}K \quad (45)$$

The uniqueness of the solution follows from the assumption that matrix C has a full row rank and matrix P^{-1} is nonsingular and consequently $\left(CP^{-1}C^T\right)$ is nonsingular [15].

V. EXAMPLE

A lowpass FIR filter having a symmetric impulse response of length N = 33 and cutoff frequency $\omega_p = \omega_s = 0.3\pi$ radians/sample is designed . The number of constraints used in Eq(10) is r = 3 , i.e., the first 5 derivatives of the amplitude of the frequency response at $\omega = 0$ are zero. Fig 1 shows the amplitude of the frequency response for 3 values of the parameter α of Eq(35), namely α = 1.0, 0.8, 0.01 . As expected, as the parameter α decreases the pass band frequency response improves with a corresponding deterioration in the stop band. In the extreme case of α =1 the mean squared error is minimized only in the stop band as in the work reported in [8,9]. Instead of considering the other extreme case of $\alpha = 0$ where matrix P of Eq(37) becomes singular we considered $\alpha = 0.01$ where the mean squared error is minimized mainly in the pass band.

To examine the effect of changing the bandwidth of the filter , the same lowpass filter was redesigned using $\omega_p = \omega_s = 0.4\pi$ rad/sample and the corresponding frequency response is plotted in Fig 2. The relatively poor passband behavior for $\alpha = 1$ demonstrates to the importance of incorporating both pass and stop bands in the minimization criterion of Eq (35). The maximal flatness constraint of Eq (5) is not sufficient for providing a smooth passband.
To examine the effect of the number of maximal flatness constraints on the frequency response, the filter of Fig (1) for the case of $\alpha = 1$ is redesigned for r = 2 in addition to r = 3. Fig (3) shows the frequency response for the 2 values of r. For a smaller number of constraints the solution space of Eq(10) has a higher dimension and consequently there is more freedom to minimize the objective function , E , of Eq(36) resulting in an improved stopband frequency response (since $\alpha = 1$) .

VI. CONCLUSION

A class of maximally flat linear phase FIR filters is derived by minimizing an objective function expressing the energy of the error between the frequency responses of the desired and designed filters in both the pass and stopbands subject to maximal flatness constraints on the passband. The method is applied for designing lowpass filters having symmetric impulse response and bandpass filters having antisymmetric impulse response. This technique offers the filter designer the freedom of the tradeoff between the flatness of the frequency response in the passband and the high attenuation in the stopband.

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Figure Captions

Fig 1 : The frequency response of the lowpass filter with cutoff frequency = 0.3π and r = 3 .
(solid line : $\alpha = 1$, dotted line : $\alpha = 0.8$, dashed line : $\alpha = 0.01$)

Fig 2 : The frequency response of the lowpass filter with cutoff frequency = 0.4π and r = 3 .
(solid line : $\alpha = 1$, dotted line : $\alpha = 0.8$, dashed line : $\alpha = 0.01$)

Fig 3 : The frequency response of the lowpass filter with cutoff frequency = 0.3π and alpha = 1.0 .
(solid line : r = 3 , dotted line : r = 2)