Sādhanā, Vol. 15, Part 3, November 1990, pp. 133–155. © Printed in India.

Design of linear phase FIR notch filters

TIAN-HU YU, SANJIT K MITRA and HRVOJE BABIC[†]

Signal and Image Processing Laboratory, Department of Electrical and Computer Engineering, University of California at Santa Barbara, Santa Barbara, CA 93106, USA
[†]Electrotechnical Faculty, University of Zagreb, Zagreb, Yugoslavia

Abstract. This paper investigates some approaches for designing one-dimensional linear phase finite-duration impulse-responses (FIR) notch filters, which are based on the modification of several established design techniques of linear phase FIR band-selective filters. Based on extensive design examples and theoretical analysis, formulae have been developed for estimating the length of a linear phase FIR notch filter meeting the given specifications. In addition, the design of two-dimensional linear phase FIR notch filters is briefly considered. Illustrative examples are included.

Keywords. FIR digital filter; digital filter design; digital notch filter.

1. Introduction

The notch filter, which highly attenuates a particular frequency component in the input signal while leaving nearby frequency components relatively unchanged, is used in many applications. For example, the elimination of a sinusoidal interference corrupting a signal, such as the 60 Hz power-line interference in the design of a digital instrumentation system, is typically accomplished with a notch filter tuned to the frequency of the interference. Usually very narrow notch characteristic (width) is desired to filter out the single frequency or sinusoidal interference without distorting the signal. The design of infinite-duration impulse response (IIR) notch filters has been considered by Carney (1963) and later by Hirano *et al* (1974). However, the design of FIR notch filters has not received any attention, even though it has been mentioned by Hamming (1989). The purpose of this paper is to extend a number of well-known techniques for designing linear phase band-selective finite-duration impulse response (FIR) filters to the case of notch filters. First, we exploit the three commonly used design techniques for the linear phase FIR notch (LPFN) filter design: (i) windowed Fourier series approach; (ii) frequency sampling approach, and (iii) optimal FIR filter design approach, based on the Remez exchange algorithm (Rabiner & Gold 1975, pp. 75–183). To investigate the relationships between filter parameters so as to be able to predict the required length of the filter to meet the given specifications, some formulae have been developed based on extensive design examples and some theoretical analyses. Also, we consider the use of some special techniques of the linear phase FIR filter design, such as the frequency response sharpening approach (Kaiser

& Hamming 1977) and interpolated FIR filter technique (Neuvo *et al* 1984) for the design of efficient notch filters. In addition, we include here the design of two-dimensional (2-D) linear phase FIR notch filters. The design of windowed Fourier series and frequency sampling approaches are easily extended to 2-D cases. Besides, a simple approach, based on the use of the concept of the 2-D complementary filter (Mitra & Yu 1986) to convert 1-D LPFN filters to 2-D LPFN filters, is introduced.

2. The linear phase FIR notch filter

2.1 *Basic properties and definitions*

The problem of digital filter design is to make a reasonable choice of the coefficients of the transfer function, so that the response of the resulting filter is a satisfactory approximation to the desired response. Now, the function of the notch filter is to attenuate a particular frequency component highly while leaving other frequency components unaffected. The notch filter is thus essentially a band-stop filter with a very narrow stopband and two passbands. For an ideal notch filter, the notch width $\Delta\omega$ should be zero, the passband magnitudes should be equal to one and the attenuation at the notch frequency ω_n should be infinite as shown in figure 1. In practice, there must be a nonzero transition band and nonzero passband ripples in the frequency response of a notch filter as shown in figure 2. To provide the linear phase, the digital notch filter must be a finite-duration impulse-response filter with a symmetric impulse response. Based on the amplitude response characteristics of linear phase FIR filters (Rabiner & Gold 1975, pp. 75–183) and the amplitude response requirement of digital notch filters, the length N of a digital notch filter must be odd, i.e. $N = 2M + 1$. Now, the frequency response of a linear phase FIR filter can be written as

$$H(\omega) = A(\omega)e^{j\theta(\omega)}, \tag{1}$$

where $A(\omega)$ is a real, even function called the amplitude function to distinguish it from the magnitude function $|H(\omega)|$ and $\theta(\omega)$ is the phase function which is a linear

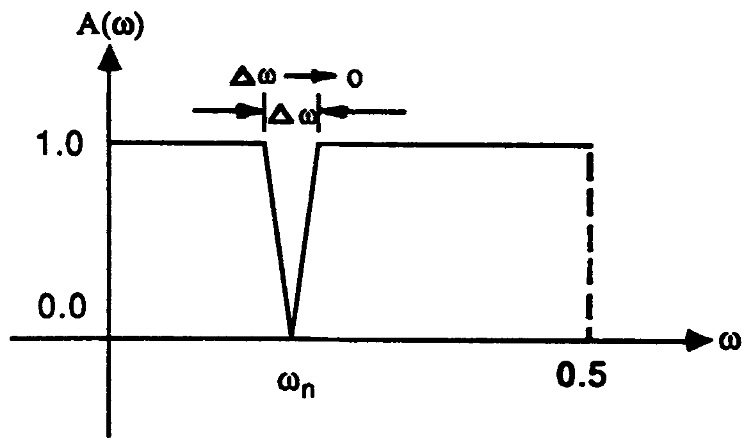

Figure 1. The magnitude response of an ideal notch filter (the amplitude response of an ideal Type 2 LPFN filter).

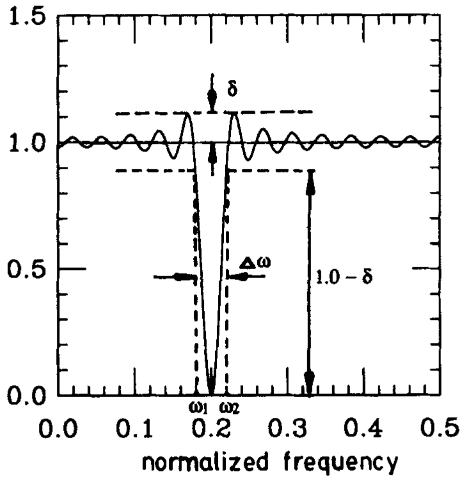

Figure 2. The amplitude response of a Type 2 LPFN filter.

function of $\omega(\theta = \omega M)$. In this case, an LPFN filter is completely characterized by $A(\omega)$ and its impulse response is a noncausal sequence $a(n)$ symmetric around the origin. The causal sequence $h(n)$ for $\theta = \omega M$ is simply given by

$$h(n) = a(n - M). \tag{2}$$

Two types of LPFN filters can be defined. The ideal Type 1 LPFN filter has a 180° phase shift at the notch frequency ω_n, i.e. $A(\omega)$ has opposite signs in the two passbands as shown in figure 3, but the magnitude function $|H(\omega)|$ is the same as illustrated in figure 1. The ideal Type 2 LPFN filter is an exact linear phase filter, as in this case, there is no difference between its magnitude function $|H(\omega)|$ and amplitude function $A(\omega)$ shown in figure 1 for the ideal case.

In practice, we can only obtain an approximation to the ideal LPFN response as shown in figure 2. Here the passband ripple δ is defined as the maximum deviation

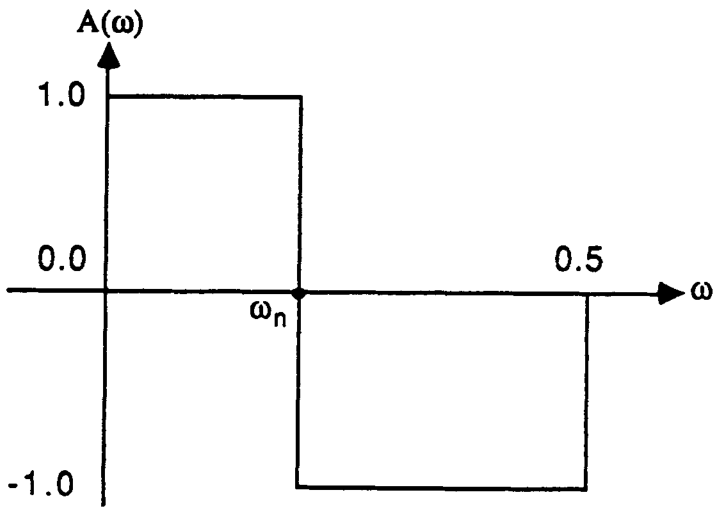

Figure 3. The amplitude response of an ideal Type 1 LPFN filter.

of the amplitude function in passband from the normalized level of unity. The passband edges ω_1 and ω_2 are defined as the angular frequencies where the amplitude function decreases from $1-\delta$. The quanity $\Delta\omega = \omega_2 - \omega_1$ is called the notch width. For the real Type 1 and Type 2 LPFN filters, the amplitude responses in the neighbourhood of the notch frequency ω_n vary approximately as $A(\omega) \approx c \cdot (\omega - \omega_n)$ and $A(\omega) \approx c \cdot (\omega - \omega_n)^2$ respectively.

In some applications, more than one notch frequency may be required. Since there are no substantial differences between the designs LPFN filters with a single notch frequency and multiple notch frequencies, in this paper we consider only the design of LPFN filters with a single notch frequency. However, the proposed approaches can be easily extended to the design of LPFN filters with more than one notch frequency.

There is only type of 2-D LPFN filter, which is the counterpart of the Type 2 1-D LPFN filters, as the amplitude function $A(\omega_1, \omega_2)$ of a 2-D LPFN filter must have the same sign in all passbands.

2.2 *General techniques for the design of LPFN filters*

The well-known approaches to linear-phase band selective FIR filter design are the windowed Fourier series approach, the frequency sampling approach, and the equiripple optimization technique. For the extension of these methods to the design of LPFN filters, it is necessary to develop (i) the relationship between the linear phase band selective FIR filter and Type 1 LPFN filter, and (ii) the relationship between the very narrow band bandpass filter, the so-called tone filter and the Type 2 LPFN filter with a very narrow stopband.

A very simple relationship between FIR lowpass filters and Type 1 LPFN filters is considered first. Since the ideal Type 1 LPFN filter is the amplitude shifted version of an ideal lowpass filter, if $A_n(\omega)$ is the amplitude function of a Type 1 LPFN filter and $A_l(\omega)$ is the amplitude function of the associated lowpass filter, the two amplitude functions are related as

$$A_n(\omega) = \pm [2A_l(\omega) - 1]. \tag{3}$$

Note that the notch frequency ω_n is in the transition band of the lowpass filter and ideally, $A_l(\omega_n) = 0{\cdot}5$. Based on this relation, some design approaches of FIR lowpass filters are easily extended to that of Type 1 LPFN filters. From (3) the noncausal impulse responses of the two filters are thus related as

$$a_n(n) = \pm [2a_l(n) - \delta(n)]. \tag{4}$$

The second relationship is based on the concept of the complementary filter. The complementary filter transfer function $G(z)$ of a linear phase FIR filter transfer function $H(z)$ is defined by (Golden 1973)

$$H(z) = z^{-(N-1)/2} - G(z), \tag{5}$$

where $N-1$, the order of $H(z)$, is assumed to be even. Note that the zero phase response $B(\omega)$ of $G(z)$ is complementary with respect to unity with the amplitude function $A(\omega)$ of $H(z)$, i.e.

$$A(\omega) = 1 - B(\omega). \tag{6}$$

Thus, if $A(\omega)$ is a lowpass FIR filter, $B(\omega)$ is a highpass filter with the same transition band and with passband and stopband ripples interchanged. However, if $H(z)$ is a Type 2 LPFN filter with a notch frequency at ω_n, $G(z)$ is an extremely narrow-band bandpass filter called a tone filter and vice-versa. Therefore the design of a notch filter can be executed as the design of a tone filter and then converted using the above equation.

3. Windowed Fourier series design approach

As the frequency response of a linear phase FIR filter, $H(\omega)$, is a periodic function of ω with a period 2π, the corresponding impulse response is given by the Fourier series coefficients of $H(\omega)$, which is of infinite length. The basic idea here is to arrive at an approximate version of $H(\omega)$ by truncating and modifying the infinite impulse response to a finite one with a window function. The most frequently used window function for the FIR filter design is the Kaiser window (Hamming 1989) which is also used here for the LPFN filter design.

3.1 *Type 1 LPFN filter design*

It is quite straightforward to design Type 1 LPFN filters using the Fourier series approach. Basically there are two steps to implement this approach. In the first step, the Fourier series coefficients $a_i(n)$, from the given ideal amplitude function $A_i(\omega)$ with the cutoff frequency $\omega_c = \omega_n$, are derived and then the coefficients are weighted by a selected window function $w(n)$,

$$a(n) = a_i(n)\cdot w(n). \tag{7}$$

The amplitude response function is given by

$$A(\omega) = (1/2\pi)\int_{-\pi}^{\pi} A_i(\theta)\, W(\omega - \theta)\,\mathrm{d}\theta, \tag{8}$$

where the $W(\omega)$ is the spectrum of the window sequence. In the second step, the impulse response of a Type 1 LPFN filter are obtained from the above lowpass counterpart using (4).

As the desired notch frequency ω_n determines the cutoff frequency ω_c of the ideal lowpass model, the passband ripples are obtained by the selection of the window parameters. These determine the transition band width $\Delta\omega$ and consequently the required filter length The filter attenuation is obtained by a choice of the parameter α of the Kaiser window given by the formula (Hamming 1989)

$$\alpha = \begin{cases} 0{\cdot}1102(A - 8{\cdot}7), & 50 \leqslant A, \\ 0{\cdot}5842(A - 21)^{0{\cdot}4} + 0{\cdot}07886(A - 21), & 21 < A < 50, \\ 0, & A \leqslant 21, \end{cases} \tag{9}$$

where A is the required ripple δ of the lowpass filter in decibels:

$$A = -20\log_{10}(\delta). \tag{10}$$

Kaiser also developed the relation between attenuation A and lowpass filter length N:

$$N \approx (A - 7{\cdot}95) \cdot 2\pi / 14{\cdot}36 \Delta\omega. \tag{11}$$

Using the relationship between the lowpass filters and Type 1 LPFN filters given by (3), the formula in (11) is modified for Type 1 LPFN filter design as:

$$N \approx (A - 1{\cdot}93) \cdot 2\pi / 14{\cdot}36 \Delta\omega. \tag{12}$$

The 6 dB difference comes from the fact that the passband level of the Type 1 LPFN filter is two times smaller than that of its lowpass model while the ripple is the same.

The magnitude response of a Type 1 notch filter designed using the Kaiser window ($\alpha = 4$) with a notch frequency of $0{\cdot}4\pi$ is shown in figure 4.

It should be noted that with the windowed Fourier series approach, the attenuation at the notch frequency may not be infinite, since the actual notch frequency may not be strictly located at the required position.

3·2 *Type 2 LPFN filter design*

In the window method of design of the Type 2 LPFN filter, we can consider it as a complement of an FIR tone filter. As an ideal tone filter, we can take a very narrow rectangular frequency response $B_i(\omega)$ centred at $\pm\omega_n$ and having a width $\Delta_i\omega$ which approaches zero. The convolution integral of (8) becomes

$$B(\omega) = \frac{1}{2\pi} \int_{-\omega_n - (\Delta_i\omega/2)}^{-\omega_n + (\Delta_i\omega/2)} B_i(\theta)\, W(\omega - \theta) \mathrm{d}\theta + \\ + \frac{1}{2\pi} \int_{+\omega_n - (\Delta_i\omega/2)}^{+\omega_n + (\Delta_i\omega/2)} B_i(\theta)\, W(\omega - \theta) \mathrm{d}\theta. \tag{13}$$

In case the term $\Delta_i\omega$ is small enough, the integrands can be taken as constants and we may write

$$B(\omega) \approx (\Delta_i\omega/2\pi)[B_i(\omega_n)\, W(\omega - \omega_n) + B_i(-\omega_n)\, W(\omega + \omega_n)]. \tag{14}$$

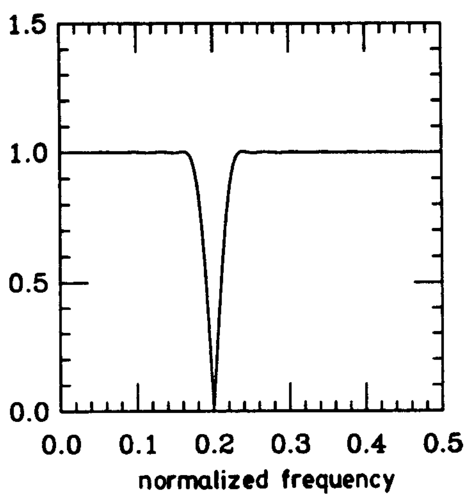

Figure 4. The magnitude response of the Type 1 LPFN filter designed using the windowed Fourier series approach.

Using $B_i(\omega_n) = B_i(-\omega_n)$ and the value $B(\omega_n)$, we normalize the tone filter response as

$$B(\omega)/B(\omega_n) = [W(\omega - \omega_n) + W(\omega + \omega_n)]/[W(0) + W(2\omega_n)], \tag{15}$$

from which it can be seen that the tone filter response is equal to the shifted window spectra. The notch filter response $A(\omega)$ follows from (6):

$$A(\omega) = 1 - [W(\omega - \omega_n) + W(\omega + \omega_n)]/[W(0) + W(2\omega_n)]. \tag{16}$$

The impulse response of the notch filter is then

$$a_n(n) = \delta(n) - 2bw(n)\cos(\omega_n n), \tag{17}$$

where $b = 1/[W(0) + W(2\omega_n)]$.

As the filter frequency response is a shifted window spectrum, the passband ripple δ and notch width $\Delta\omega$ should be equal to the window ripple and window width. Thus, the design of a Type 2 LPFN filter can be reduced to the determination of the window with the main lobe width equal to $\Delta\omega$ and the window ripple equal to δ. The overlapping of the ripples of $W(\omega - \omega_n)$ and $W(\omega + \omega_n)$ can be neglected with windows having roll-offs larger than 6 dB/octave as for example in the Kaiser window.

To develop a formula suitable for the design of Type 2 LPFN filters using the Kaiser window, we have designed a large number of filters with α ranging from 0·0 to 9·9 and with filter lengths varying from 21 to 99. The numerically calculated relationship between α and the attenuation A is presented graphically in figure 5a and has been approximated by a formula similar to (9) using the curve fitting technique and is as given below

$$\alpha = \begin{cases} 0{\cdot}1284(A + 0{\cdot}4551), & 50{\cdot}1943 < A, \\ 0{\cdot}1517(A - 4{\cdot}7071), & 20{\cdot}5278 < A \leqslant 50{\cdot}1943, \\ 0{\cdot}7798(A - 12{\cdot}9164)^{0{\cdot}4} + & 12{\cdot}9164 < A \leqslant 20{\cdot}5278, \\ \quad + 0{\cdot}0842(A - 12{\cdot}9164), & \\ 0, & A \leqslant 12{\cdot}9164. \end{cases} \tag{18}$$

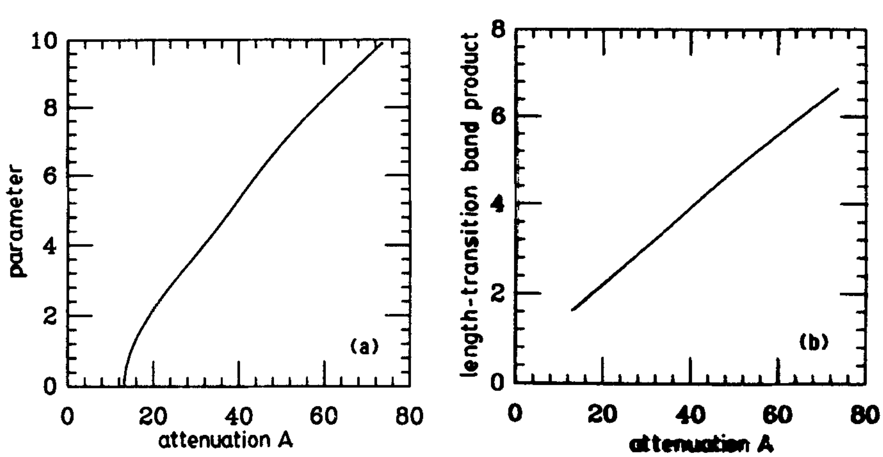

Figure 5. (a) The relationship between the parameter α and the passband attenuation A of the Type 2 LPFN filter designed using the Kaiser window. (b) The relationship between $N\Delta\omega$ and A – using the Kaiser window.

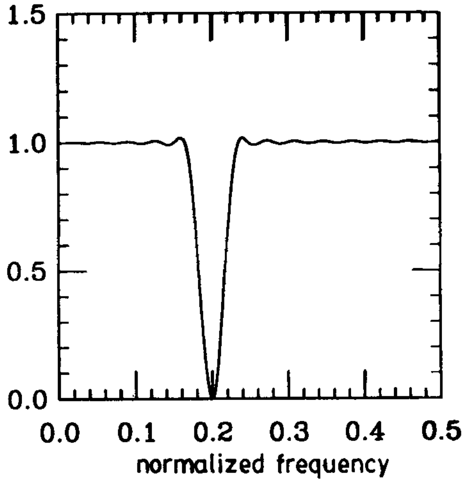

Figure 6. The magnitude response of the Type 2 LPFN filter designed using the windowed Fourier series approach.

The relation between the notch width $\Delta\omega$ and attenuation A is given in figure 5b. Because of the linearity of the above relation, a practical formula for the required length can be derived from the plot and is given by

$$N \cong (A + 6{\cdot}35)/11{\cdot}87\,\Delta\omega. \tag{19}$$

A design example using the above method with $\omega_n = 0{\cdot}4\pi$ is shown in figure 6.

4. Frequency sampling design approach

The most straightforward approach to the linear phase FIR filter design is the frequency sampling method. After a design is completed, the amplitude response $A(\omega)$ is specified at N typically equidistant points or frequency samples. These samples A_k give coefficients in the recursive FIR filter realization (Rabiner & Schafer 1971), which becomes computationally efficient if coefficients A_k are zeros or ones. Therefore, a typical design of a lowpass filter consists of choosing samples $A_k = 1$ in the passband $(0, \omega_p)$, $A_k = 0$ in the stopband (ω_s, π) and the optimization of samples $0 < A_k < 1$ in the transition band (ω_p, ω_s). A small number of transition samples ensures efficient realization and a low order optimization problem, but introduces some design inflexibility and does not yield as good results as that obtained when all N samples are optimized.

The optimization based on adjustment of all samples results in an optimal filter. In such and any other design approaches giving satisfactory frequency responses, the sampling of the resulting frequency response will give coefficients A_k for a recursive realization. However, with all $A_k \neq 0$ and $A_k \neq 1$, in most cases there are no sufficient justifications in favour of the recursive realization. On the other hand, if the nonrecursive realization is preferred, on the basis of the $N = 2M + 1$ known coefficients A_k, the coefficients $a(n)$ of the filter impulse response can be obtained by solving $M + 1$ equations, or by computing the inverse discrete Fourier transform. In the computation, the realness and symmetry of both responses $a(n)$ and A_k can be exploited.

4.1 *Type 1 LPFN filter design*

The design of Type 1 LPFN filter is again based on the design of a lowpass (LP) filter model. The conventional approach assumes equally spaced samples A_k satisfying

$$A_k = \begin{cases} 1, 0 \leqslant k \leqslant P, & \omega_p = 2\pi P/N, \\ 0, S \leqslant k \leqslant M, & \omega_s = 2\pi S/N, \end{cases} \tag{20}$$

with a transition band given by

$$\Delta\omega = 2\pi Q/N = 2\pi(S - P)/N, \tag{21}$$

where $P, S, Q, N = 2M + 1$ are integers. This shows that parameters ω_p, ω_s and $\Delta\omega$ are discrete values determined by the number N, or by the size of the bin $(2\pi/N)$.

In the transition band of width Q bins, we then have $Q - 1$ transition samples $(0 < A_k < 1)$. The amplitude response $A_l(\omega)$ of the filter is given by samples A_k (Rabiner & Schafer 1971)

$$A(\omega) = A_0\Phi(\omega, 0) + \sum_{k=1}^{M} A_k\Phi(\omega, k), \tag{22}$$

where $\Phi(\omega, k)$ is a set of interpolation functions

$$\Phi(\omega, 0) = \sin(N\omega/2)/N\sin(\omega/2) \tag{23}$$

and

$$\Phi(\omega, k) = \frac{\sin[N(\omega - 2\pi k/N)/2]}{N\sin[(\omega - 2\pi k/N)/2]} + \frac{\sin[N(\omega + 2\pi k/N)/2]}{N\sin[(\omega + 2\pi k/N)/2]}. \tag{24}$$

For the above LPFN filter model the response becomes

$$A_l(\omega) = A_p(\omega) + \sum_{k=P+1}^{S-1} A_k\Phi(\omega, k), \tag{25}$$

where $A_p(\omega)$ represents the known contribution of unity samples $k \in [0, P]$ and of unknown transition samples $k \in (P, S)$ while zero samples do not contribute.

The optimum transition sample values can be obtained by a procedure which for notch filter minimizes the error or ripple δ in both passbands or for the LP model minimizes

$$\delta = \max_{\omega \in [0, \omega_p] U [\omega_s, \pi]} |E(\omega)|. \tag{26}$$

The error function is

$$E(\omega) = \begin{cases} A_l(\omega) - 1, & 0 \leqslant \omega \leqslant \omega_p, \\ A_l(\omega), & \omega_s \leqslant \omega \leqslant \pi. \end{cases} \tag{27}$$

However, an additional condition

$$A_l(\omega_n) = 0{\cdot}5 \tag{28}$$

should be included to ensure the zero transmission at the notch frequency.

As the amplitude response is expressed as a linear function in (25), the linear

Table 1. Relationship between the sample number Q and the passband attenuation A.

Q	2	3	4	5	6	7	8	
A_1	24	42	56	70	85	100	114	dB
A_2	13	27	43	54	71	83	98	dB

programming approach is a suitable optimization method. There are well established procedures for digital filter design based on linear programming (Land & Powell 1973).

To run the optimization procedures for the notch filter design, it is very useful to estimate a priori, the filter length N and number $(Q-1)$ of transition samples, from the designed filter specifications, such as the bandwidth $\Delta\omega$ and the attenuation A. To estimate Q from the desired attenuation, table 1 can be used. This table is based on optimal transition samples for a half-band filter (Babic & Dobrenic 1987). After Q is selected, the required length N can be estimated from the specified bandwidth $\Delta\omega$ using

$$N \geqslant \Delta\omega/2\pi Q. \tag{29}$$

The passband ripple δ and its decibel equivalent $A = -20\log(\delta)$ obtained by the sample optimization depend on the position of the notch frequency within a bin of the sample grid. The best results are obtained in the symmetrical cases:

(i) ω_n is close to the middle point between samples

$$\omega_c = (2L+1)\pi/N, \quad L = 0, 1, \ldots M, \tag{30}$$

when Q is odd;
(ii) ω_n is close to the sample position

$$\omega_c = (2L)\pi/N, \quad L = 0, 1, \ldots M, \tag{31}$$

when Q is even.

For symmetric case $\omega_n = \omega_c$, the passband ripple will be A_1, as given in table 1. Generally, for $\omega_n \neq \omega_c$, the passband ripple will be lower, i.e. $A = A_1 - A_c$. The difference A_c depends on the offset $|\omega_n - \omega_c|$ of the notch frequency from ω_c given by the above equations and is approximately

$$A_c \approx 5{\cdot}75N|\omega_n - \omega_c|. \tag{32}$$

Equation (32) is valid for $|\omega_n - \omega_c| < \pi/2N$. Thus, the design should start with the first estimate of Q and N using table 1 and (29) to satisfy required A and $\Delta\omega$, and then, corrrected if necessary from the following consideration. The parameter Q is an integer and table 1 offers few available values for the attenuation. Therefore, we usually have an attenuation margin from specified attenuation. This margin can be taken as a difference A_c, which determines the tolerable notch offset, i.e.

$$|\omega_n - \omega_c| < A_c/5{\cdot}75N. \tag{33}$$

If the notch offset is not within the above tolerance, a slight increase of N will change ω_c and L, so that the offset can be made small enough to ensure a moderate reduction

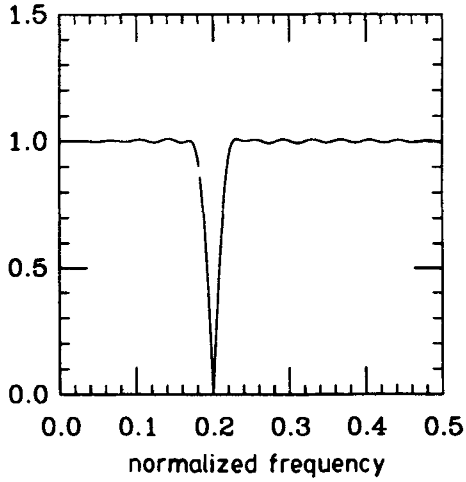

Figure 7. The magnitude response of the Type 1 LPFN filter designed using the frequency sampling approach.

of the attenuation from A_1, and at least fulfill the requirement in (33). When the attenuation margin is too small, one can increase the number of the transition samples by one, i.e. replace Q with $Q+1$ and start from a higher A_1.

Now having estimated the proper Q, N and L, the locations ω_s and ω_p of the frequency samples at band edges in (20) follow from:

$$S = (2L+Q)/2, \qquad P = (2L-Q)/2, \tag{34}$$

$$S = (2L+1+Q)/2, \quad P = (2L+1-Q)/2, \tag{35}$$

for Q even and odd respectively. By this approach, all parameters ω_n, P, S required for optimization procedure are known.

The response of a design example with two transition samples ($Q = 3$) is given in figure 7.

4·2 *Type 2 LPFN filter design*

The Type 2 LPFN filter design is based on the tone filter design. Here the frequency samples are zero in both stopbands

$$A_k = \begin{cases} 0, & 0 \leqslant k \leqslant P, \\ 0, & S \leqslant k \leqslant M, \end{cases} \tag{36}$$

and $Q-1$ samples in the passband $0 < A_k < 1$ are to be optimized.

The amplitude response given by the specified frequency samples is

$$A_t(\omega) = \sum_{k=P+1}^{S-1} A_k \Phi(\omega, k). \tag{37}$$

The additional constraint for a normalized tone filter is

$$A_t(\omega_n) = 1, \tag{38}$$

to ensure that there is a zero transmission at ω_n, for the corresponding notch filter.

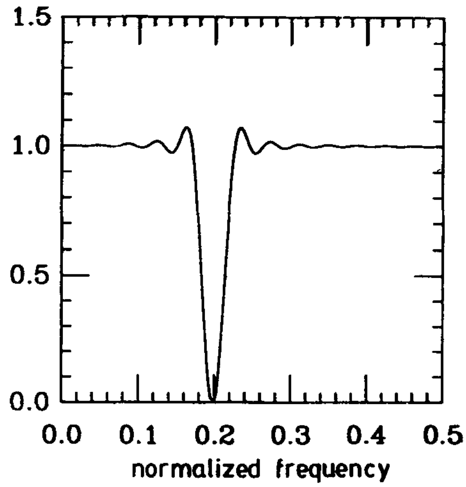

Figure 8. The magnitude response of the Type 2 LPFN filter designed using the frequency sampling approach.

The optimum passband samples are obtained by minimizing the ripple in (26) in both stopbands. The error function is

$$E(\omega) = A(\omega); \quad 0 \leqslant \omega \leqslant \omega_p, \omega_s \leqslant \omega \leqslant \pi. \tag{39}$$

The passband samples show symmetry when $\omega_n = L(2\pi/N)$ or $(L + 1/2)2\pi/N$, but generally they form an asymmetric sequence.

For the start of the optimization procedure, it is good to estimate values for Q and N. The attenuation A_2 in table 1 can be used for the estimation of Q. Table 1 is based on the properties of the low order minimax windows which can be viewed as tone filters with $\omega_n = 0$. These windows are obtained earlier by the linear programming (Babic & Dobrenic 1983). After Q is selected, the value of N follows from (29).

There is no significant influence of the notch offset $(\omega_n - \omega_c)$ on the ripple, thus table 1 is sufficient for the estimation of Q.

The offset can cause the appearance of one additional zero in the immediate vicinity of the notch. This zero is not of practical importance for $|\omega_n - \omega_c| < \pi/2N$. However, it can be eliminated with an additional condition in the optimization procedure, namely

$$A'(\omega_n) = \frac{dA(\omega)}{d\omega}\bigg|_{\omega=\omega_n} = 0, \tag{40}$$

which together with (31) ensures the exact maximum (tone filter) or a double zero at ω_n (notch filter).

The design example with two samples ($Q = 3$) is shown in figure 8.

The recursive realization for Type 2 LPFN filter is very efficient because of only few required IIR sections. For the notch filter realization based on (5), the tap in the middle of the delay cascade will give the required delayed signal.

5. Optimal LPFN filter design

As in all optimization problems, the criterion for optimality must be stated first in designing an optimal LPFN filter. It could be easily assumed that the optimal LPFN filter should have an infinite attenuation at the notch frequency and the smallest passband ripple with a prescribed notch width. The criterion minimizing the maximum error over a set of frequency bands is called a Chebyshev approximation (Rabiner & Gold 1975, pp. 75–183). Filters that have the minimum value of the maximum error exhibit equiripple behaviour over the set of frequency bands in their frequency response. McClellan *et al* (1973) have developed a useful program based on a direct application of the Chebyshev approximation theory to the linear phase FIR filter design problem, and using the Remez exchange algorithm. In the section, we use this program (to be called the MPR program after the authors McClellan, Parks & Rabiner) to design the optimal LPFN filters.

5·1 *Type 1 optimal LPFN filter design*

The main function of the MPR program is to achieve equiripple behaviour over the set of frequency bands, such as stopbands and passbands for band selective linear phase FIR filters. It can also be extended to design Type 1 LPFN filters with equal passband ripple behaviour in two ways: one is based on an equiripple lowpass model and (3), and the other on the design of a filter with the notch frequency at the centre of the transition band and the passbands bracketing the transition band having opposite signs. There is no substantial difference between these two approaches, and the designed filters have an equiripple characteristic, but in most cases with a finite attenuation at the notch frequency.

We use the prefilter-equalizer concept (Adams & Willson 1983) to develop optimal LPFN filters. An optimal LPFN filter design can be achieved in two parts: the realization of the infinite attenuation at the notch frequency with a prefilter and the design of the corresponding equalizer which is cascaded with the prefilter to achieve the passband equiripple characteristic.

The simplest transfer function suitable for a prefilter which sets a zero amplitude response at the notch frequency is

$$H_p(z) = (1 - 2\cos\omega_n z^{-1} + z^{-2})^r, \tag{41}$$

where ω_n is the notch frequency and r is an integer. Once the prefilter has been established, the MPR program is then used to design the equalizer, so that the cascade of the prefilter and the equalizer provides equiripple passbands with opposite signs and an infinite attenuation at the notch frequency. Note that in the strict sense, the linear phase does not hold in Type 1 LPFN filters in which r is odd, due to a constant phase difference of $r\cdot\pi$ between the two passbands. Even though any odd r can be used to design the Type 1 LPFN filter, the notch width increases with increasing r. Therefore, $r = 1$ is preferable for the design of the narrow notch width Type 1 LPFN filter.

A number of formulae are available for estimating the filter length N to meet given specifications in designing an optimal linear phase band-selective FIR filter (Rabiner 1973; Bellanger 1984). These formulae can be easily extended to the design of Type 1 LPFN filters with equiripple passbands. However, they do not work well on the design of Type 1 LPFN filters, since they are derived by data fitting for large ranges of transition bandwidth and ripple ratio, but LPFN filters usually have quite narrow notch widths and equal passband ripples.

In order to obtain a better approximate relationship between the Type 1 LPFN filter parameters, we have designed a large number of Type 1 optimal LPFN filters

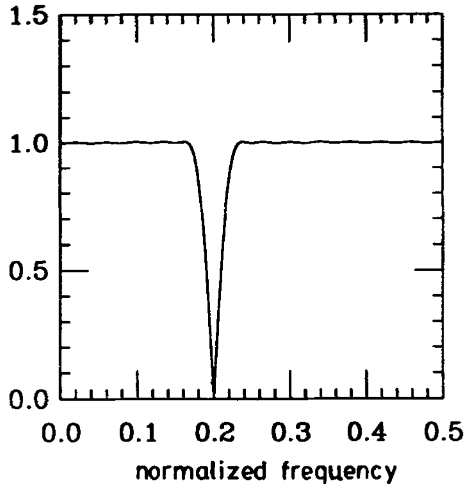

Figure 9. The magnitude response of the designed optimal Type 1 LPFN filter.

with notch width $\Delta\omega$ ranging from 0·01 to 0·1, and with passband attenuation A varying from 30 to 75 dB. The formula was assumed to be of the form

$$N \approx [k_1 A^3 + k_2 A^2 + k_3 A + k_4]/(\Delta\omega) + k_5. \tag{42}$$

The derivation of five coefficients $k_1, \ldots, k_5$ were based on the least mean squares (LMS) approximation method and are given by

$$k_1 = -4{\cdot}17739 \cdot 10^{-6}, \quad k_2 = 6{\cdot}60403 \cdot 10^{-4}, \quad k_3 = 3{\cdot}42442 \cdot 10^{-2},$$

$$k_4 = 0{\cdot}326081, \quad k_5 = -0{\cdot}8721.$$

A design example is shown in figure 9.

5.2 *Type 2 optimal LPFN filter design*

For an optimal Type 2 LPFN filter, r in (41) is even. We restrict our attention to $r = 2$ as here also the notch width increases with increasing r. There are two ways to establish the prefilter. One is to use the prefilter as shown in (41) with $r = 1$ and the designed equalizer contributing another first-order zero at the notch frequency. The other is to establish a prefilter transfer function with $r = 2$. The equalizer is then designed using the MPR program.

The minimax optimization procedure for tone filters and notch filters centred at ω_n are based on the following criterion, which is also the basis for the design of the Dolph–Chebyshev window. To design the optimal tone and notch filters according to (15) and (16), we consider the design of a Dolph–Chebyshev window centred at $\pm\omega_n$. The relationship among the passband ripple δ, transition band $\Delta\omega$ and the filter length N of a Type 2 LPFN filter is expected to be the same as that of a Dolph–Chebyshev window. The exact relationship between the parameters of the Dolph–Chebyshev window has been introduced as follows (Ballanger 1984),

$$\delta = [ch\{(N-1) \cdot ch^{-1}(x_0)\}]^{-1}, \tag{43}$$

where $1/x_0 = \cos\{(\Delta\omega)\pi/2\}$. This equation is strictly valid for the optimal Type 2

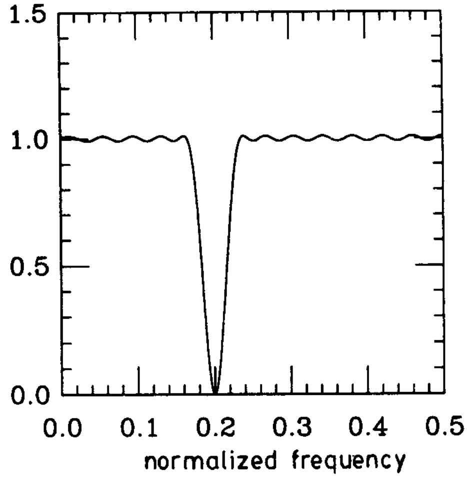

Figure 10. The magnitude response of the designed optimal Type 2 LPFN filter.

LPFN filter with the notch frequency $\omega_n = 0$. Based on extensive investigation, we have found that the relationship between the parameters δ, $\Delta\omega$ and N of a Type 2 optimal LPFN filter is relatively insensitive to the positions of notch frequencies, and following (25), we get

$$N \approx ch^{-1}(1/\delta)/ch^{-1}(x_0) + 1, \tag{44}$$

which can be used to estimate the length of Type 2 optimal LPFN filters.

A design example is shown in figure 10.

6. Special techniques for the design of LPFN filters

In addition to techniques mentioned above, there are some special linear phase FIR filter design techniques, such as the response sharpening of linear phase FIR filters, and the interpolated FIR filters, which can be used in the LPFN filter design directly or indirectly to yield LPFN filters with more desirable characteristics.

6.1 *Sharpening the response of the LPFN filter*

The performance of a linear-phase FIR filter can be improved using a filter sharpening approach (Kaiser & Hamming 1977) which suitably combines the results of several passes through the same filter. In this approach, the amplitude response $H_{\text{new}}(\omega)$ of the new filter is related to the amplitude response $H_{\text{old}}(\omega)$ of the old filter through the following relation, called the amplitude change function

$$H_{\text{new}}(\omega) = H_{\text{old}}^{n+1}(\omega) \sum_{k=0}^{m} \frac{(n+k)!}{n!k!} [1 - H_{\text{old}}(\omega)]^k, \tag{45}$$

which has an nth order tangency at zero, an mth order tangency at unity and passes through (0, 0), (1, 1). For example, for $n = 0$ and $m = 1$, the function is

$$H_{\text{new}}(\omega) = H_{\text{old}}(\omega) \cdot [2 - H_{\text{old}}(\omega)], \tag{46}$$

more commonly known as the twicing function (Kaiser & Hamming 1977) and for $n = 1$, $m = 1$, the function is

$$H_{\text{new}}(\omega) = H^2_{\text{old}}(\omega)\cdot[3 - 2H_{\text{old}}(\omega)]. \tag{47}$$

This technique can also be extended to improve the performance of LPFN filters, by reducing the passband ripple of LPFN filters with modified amplitude change functions. In the case of Type 1 LPFN filters, the amplitude change function should have tangencies at ± 1 and pass through points $(-1, -1)$, $(1, 1)$ and $(0, 0)$, which can be readily derived from that for the band-selective filters. For example, an amplitude change function for Type 1 LPFN filters can be obtained from (48) as

$$H_{\text{new}}(\omega) = [3H_{\text{old}}(\omega) - H^3_{\text{old}}(\omega)]/2. \tag{48}$$

As the stopband ripple is almost doubled, the function of (46) is not attractive for sharpening the response of band-selective filters. However, for Type 2 LPFN filters, the amplitude change function is required to have a tangency only at unity and to pass through points $(0, 0)$ and $(1, 1)$. Thus, the twicing function is appropriate to sharpening the response of Type 2 LPFN filters.

For the Type 1 LPFN filters, in the case of $n = m = 1$, and if the passband ripple δ of the old filter is relatively small, the passband ripple of the new filter is approximately $0{\cdot}75\,\delta^2$, and the new filter length is almost three times that of the old one. If we use the windowed Fourier series approach with the Kaiser window design, the length of the new filter can be expressed as

$$N_{\text{new}} \approx 3[2\pi(-20\log_{10}\delta - 1{\cdot}93)/14{\cdot}36\Delta\omega]. \tag{49}$$

However, based on the established empirical formula, to meet the new passband ripple, the suitable length should be

$$N_s \approx 2\pi(-20\log_{10}0{\cdot}75\delta^2 - 1{\cdot}93)/14{\cdot}36\Delta\omega. \tag{50}$$

The efficiency of the sharpening design can be measured by the ratio N_s/N_{new} as

$$N_s/N_{\text{new}} \approx (2/3)[1 + \{4{\cdot}428/(-60\log_{10}\delta - 5{\cdot}79)\}]. \tag{51}$$

When δ is small, this expression approaches $\frac{2}{3}$ from above. For the Type 2 LPFN filters, there are no stopband ripples to be taken into account, and we can use the twicing function as the amplitude change function. We found that by normalizing, the passband ripple of the new LPFN filter is slightly more than half of the square of the old one and the order of the new filter is twice that of the old one. We can calculate the sharpening efficiency based on the above information as follows: suppose that the passband ripple of the old Type 2 optimal LPFN filter is δ which satisfies (44) and the filter length is N, then the passband ripple δ' of the new filter with the twicing function would be:

$$\delta' = (\delta^2/2)/(1 - \delta^2/2) = 1/(2/\delta^2 - 1). \tag{52}$$

Using (44) and some algebraic manipulation, the above equation can be expressed as

$$\delta' = 1/ch\{2(N-1)ch^{-1}(x_0)\}, \tag{53}$$

where $1/x_0 = \cos\{(\Delta\omega)\pi/2\}$.

Note that the transition band width of the new LPFN filter is the same as that of the old one and the length of the new LPFN filter is $(2N - 1)$. Thus, we can conclude that the passband ripple δ' of the new filter with the twicing function is exactly equal to that of the directly designed one with the same length $(2N - 1)$ and the same transition band, i.e. in this case, we achieve 100% efficiency with the sharpening technique. With sharpening technique, a high-order Type 2 optimal LPFN filter can be designed and implemented based on the low-order one very efficiently.

6·2 *Interpolated LPFN filtering*

In the interpolated FIR (IFIR) filter design approach, a cascade of two FIR sections is used to meet the given specifications, one generating a sparse set of impulse values with every Lth sample being non-zero and the other performing the interpolation (Neuvo *et al* 1984). The latter can often be implemented with only a few simpler arithmetic operations. The implementation of IFIR filters results in a reduction of multipliers along with reduction of the coefficient sensitivities and output roundoff noise levels. Since the IFIR filtering approach is based on the smoothness of the impulse response, it works very well on narrow band FIR filters. As an extremely narrow bandpass filter, the FIR tone filter can be very efficiently designed and implemented with the IFIR filtering approach. The proposed steps for the Type 2 LPFN IFIR filtering are the following:

(1) Design a Type 2 LPFN filter with a transition band L times the given one and a notch frequency ω_d which has such a relation to the given notch frequency ω_n, that ω_n must be one of the following L frequencies:

$$(1/L)\cdot[\pi(k-1)+\omega_d], \quad k = 1, \ldots, L. \tag{54}$$

(2) Develop the complementary tone filter of the designed LPFN filter.
(3) Use the interpolation filter to eliminate $L - 1$ frequency peaks except the one at the desired notch frequency.
(4) The complementary filter of the interpolation tone filter is the final result.

A design example is shown in figure 11.

An efficient realization of tone and notch filters can be implemented with the running sum structures (Babic *et al* 1985).

7. Techniques for the design of 2-D LPFN filters

The design of the 2-D LPFN filter is philosophically identical to that of its 1-D counterpart with windowed Fourier series and frequency sampling approaches. The definition of 1-D Type 2 optimal LPFN filters can be extended to the 2-D case. Because the alternation theorem can not be readily applied in the the two-dimensional case, the MPR approach can not be used for the design of 2-D optimal LPFN filters. Linear programming techniques may be used to solve for the coefficients of an optimal 2-D LPFN filter, but it is very time-consuming. In this section, we restrict our discussion to some simple and efficient approaches for the design of 2-D LPFN filters.

7.1 *2-D LPFN filter design by windowing*

As in the case of the design of 1-D Type 2 LPFN filters (§ 3·2), a frequency shifted

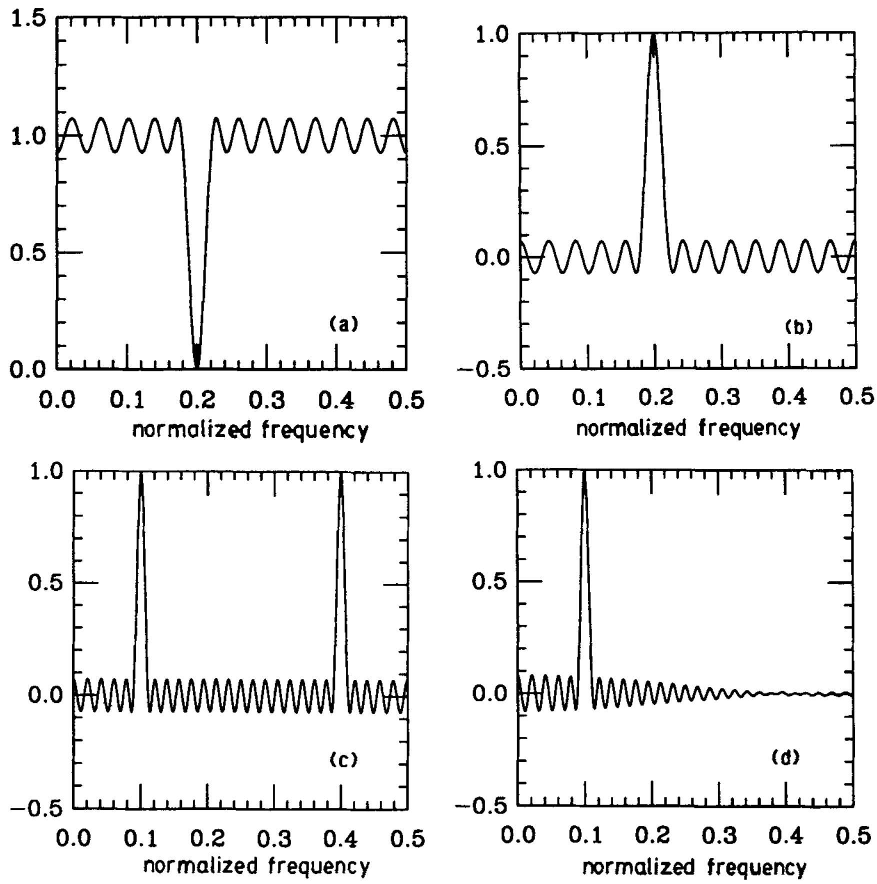

Figure 11. (a) The amplitude response of the designed Type 2 prototype LPFN filter. (b) The amplitude response of the tone filter–complementary filter of the prototype filter. (c) The amplitude response of the interpolator section. (d) The amplitude response of the IFIR tone filter.

window is a linear phase FIR tone filter and its complementary filter is an LPFN filter. We can get the 2-D LPFN filter as the complementary filter of the resulting 2-D tone filter. Suppose that 2-D window coefficients are $w(n_1, n_2)$ with a filter support of size $N_1 \times n_2$, where $N_1 = 2M_1 + 1$, $N_2 = 2M_2 + 1$, the amplitude response of this window is

$$W(\omega_s, \omega_p) = \sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} w(n_1, n_2) \cos(\omega_s n_1) \cos(\omega_p n_2). \tag{55}$$

The normalized coefficients $t(n_1, n_2)$ of the tone filter are

$$t(n_1, n_2) = w(n_1, n_2) \cdot b \cdot \cos(n_1 \omega_{n1}) \cos(n_2 \omega_{n2}), \tag{56}$$

where b is the normalized coefficient $1/b = W(0,0) + W(2\omega_{n1}, 0) + W(0, 2\omega_{n2}) + W(2\omega_{n1}, 2\omega_{n2})$. The coefficients $a(n_1, n_2)$ of the resulting 2-D LPFN filter are

$$a(n_1, n_2) = \delta(n_1, n_2) - t(n_1, n_2). \tag{57}$$

The effect of the window in the frequency domain is to smooth the frequency response of the ideal filter function, and the choice of the 2-D window function is governed by the following requirements. The window should satisfy the linear phase relation, i.e. the coefficients of the window should possess the symmetric property. The mainlobe width of the window frequency response should be small so that the transition bandwidth of the designed filter frequency response could be small. A small sidelobe amplitude of the window frequency response would be preferable resulting in small ripples in the passbands. Since these requirements are the same as the requirements for a 1-D window function, 1-D windows are often used as a basis for generating 2-D windows. Generally, there are three methods which can be used to produce 2-D windows from their 1-D counterparts. The first one forms a 2-D window with a square region of support by taking the outer product of two 1-D windows. The second forms a 2-D window by sampling circularly rotated 1-D continuous window function, the resulting 2-D windows have nearly circular regions of support. In the third one (Yu & Mitra 1985), a 2-D window is obtained from a 1-D window by the frequency transformation approach, in which case the frequency response of the resulting 2-D window is more circularly symmetric than that obtained using the second approach.

7.2 *2-D* LPFN *filter design from its 1-D counterparts*

It is very hard to find a straightforward way to design a 2-D LPFN filter from its 1-D counterparts. However, we develop an indirect approach by using a 2-D linear phase FIR tone filter as a bridge. The simplest way to obtain a 2-D linear phase FIR tone filter is to cascade two 1-D linear phase FIR tone filters. Thus the complementary filter of a 2-D linear phase FIR tone filter would be a 2-D LPFN filter. Generally, there are four steps in designing a 2-D LPFN filter with the notch at (ω_1, ω_2), from its 1-D counterparts, which can be described as follows:

(1) Design two 1-D Type 2 LPFN filters which notches as ω_1 and ω_2, respectively, using the techniques mentioned in the previous sections. The lengths of these 1-D filters are $2N_1 + 1$, and $2N_2 + 1$ respectively.
(2) Using (18), we obtain two complementary filters of the filters designed in step 1.
(3) By cascading the two designed 1-D tone filters, we get a 2-D linear phase FIR tone filter with the tone frequency at (ω_1, ω_2), the transfer function of which can be expressed as $H_t(z_1, z_2)$.
(4) The 2-D LPFN filter can then be obtained as the complementary filter of $H_t(z_1, z_2)$. using the following equation:

$$H(z_1, z_2) = z_1^{N_1} z_1^{N_2} - H_t(z_1, z_2). \tag{58}$$

An example of such a design technique is shown in figures 12 and 13.

8. Concluding remarks

We have discussed a variety of LPFN filter design approaches. All of these approaches are either extensions and/or modifications of well-known linear phase band-selective FIR filter design techniques. As can be seen from the design examples shown in table 2, the optimal linear phase design approach yields better results. In particular, notch filters with an exact notch frequency, equiripple passbands and full band linear phase

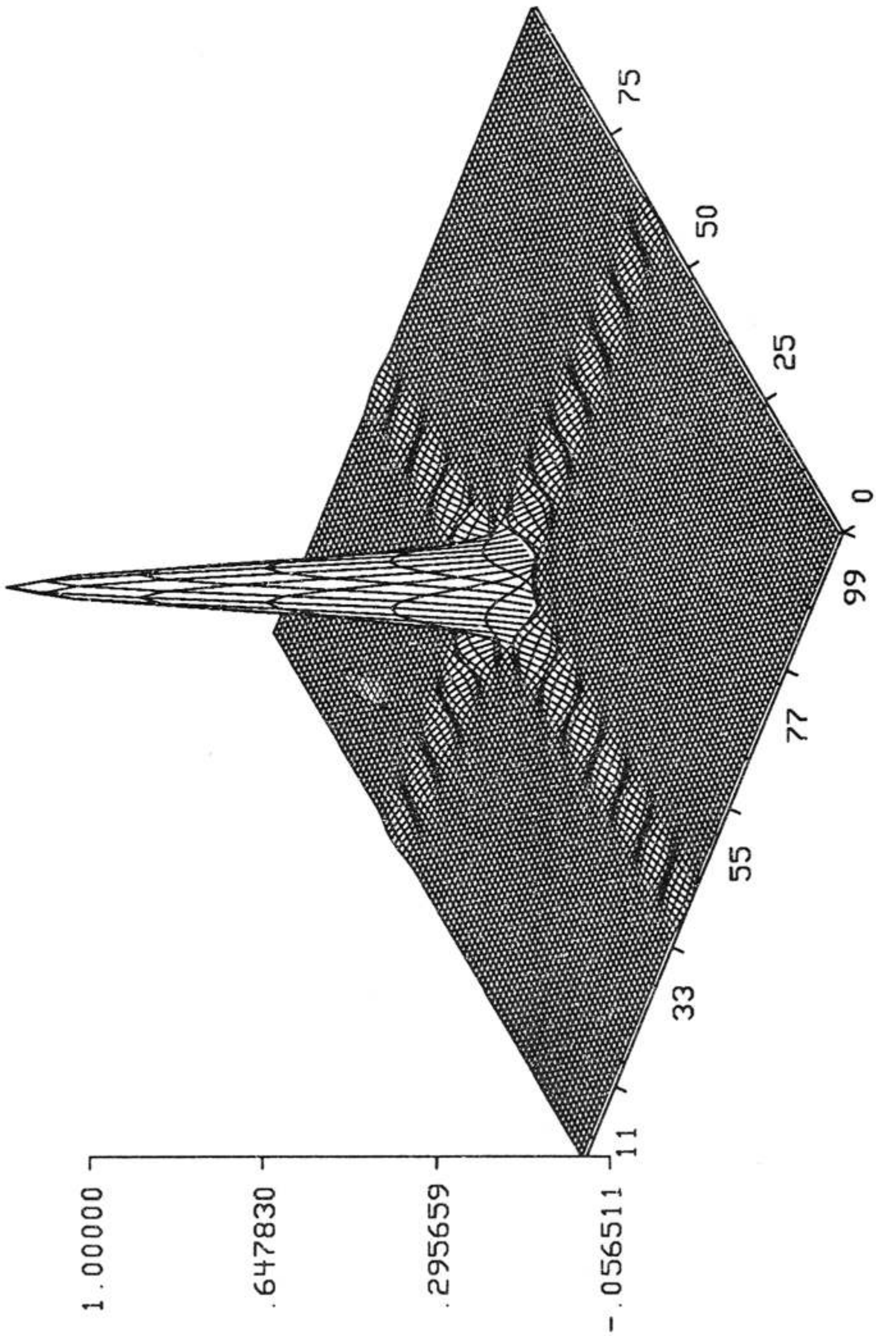

Figure 12. A 2-D tone filter.

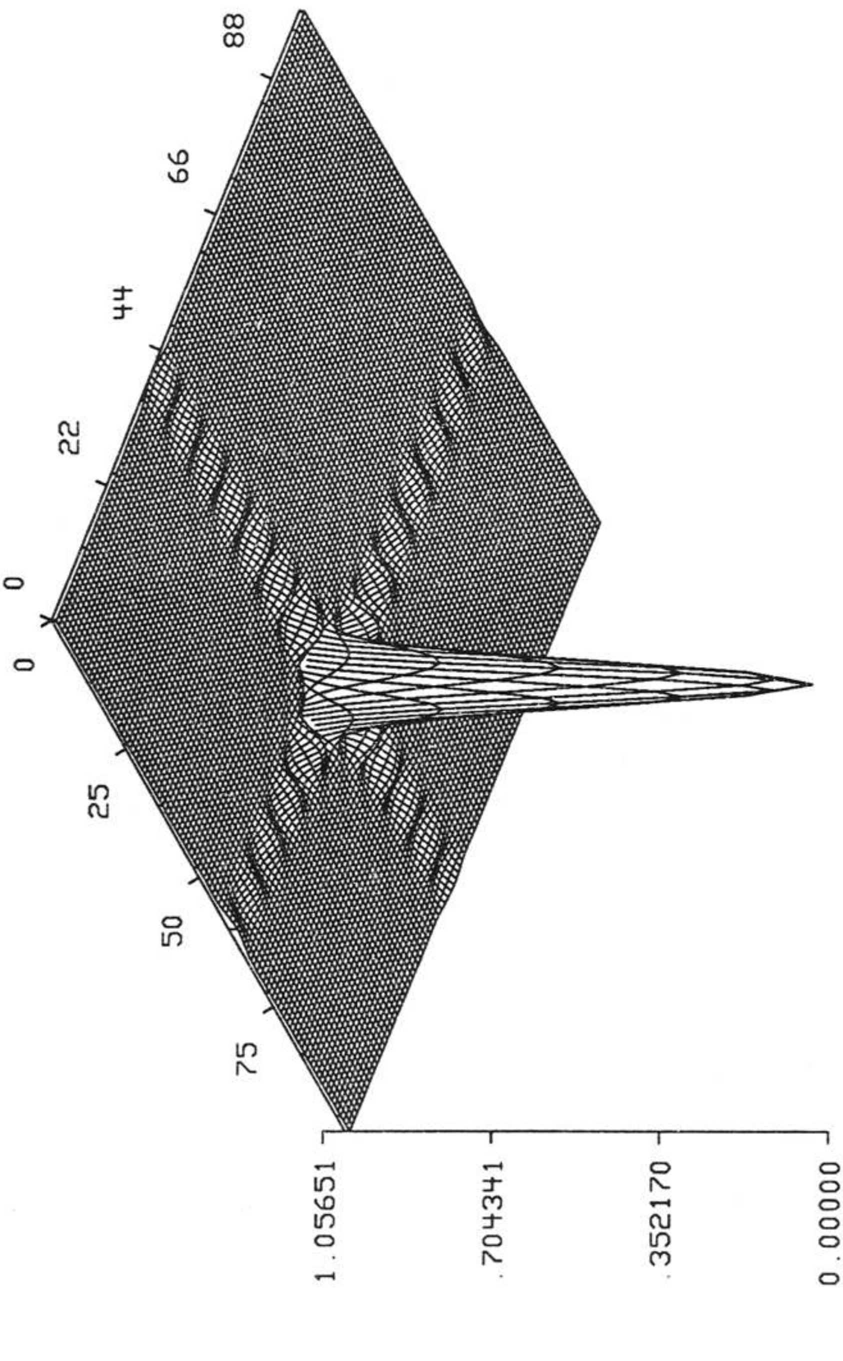

Figure 13. A 2-D LPFN filter.

Table 2. Comparison of LPFN filters designed using different approaches.

	Approach					
	Fourier series		Frequency sampling (using linear programming)		Optimal design	
Specifications	Type 1	Type 2	Type 1	Type 2	Type 1	Type 2
Passband ripple in dB	49·77365	34·79025	41·97141	26·22172	53·69482	39·66895
Band-width of the notch	$6{\cdot}68 \times 10^{-2}$	$6{\cdot}70 \times 10^{-2}$	$5{\cdot}25 \times 10^{-2}$	$5{\cdot}50 \times 10^{-2}$	$6{\cdot}66 \times 10^{-2}$	$6{\cdot}67 \times 10^{-2}$
Magnitude at the notch	$1{\cdot}2 \times 10^{-4}$	0·0	0·0	0·0	0·0	0·0

can be obtained with the Type 2 LPFN optimal filter design. On the other hand, the windowed Fourier series approach is simpler and straightforward to apply and leads to quite acceptable design results. Based on extensive design examples and some theoretical analyses, several design formulae have been developed for estimating the filter length to meet given frequency specifications. Sharpening and interpolated FIR filtering approaches work well on the design of the Type 2 LPFN filter. In fact, sharpening provides us with an alternative way to design and implement the Type 2 optimal LFPN filters from their lower-order counterparts quite efficiently. In addition, we have briefly discussed the design of 2-D LPFN filters.

The authors would like to thank Drs Tapio Saramaki and Ganesh Rajan for their assistance in programming. This work was supported in part by a University of California MICRO grant with matching supports from Rockwell International and ESL Inc.

References

Adam J W, Willson A N Jr 1983 A new approach to FIR digital filters with fewer multipliers and reduced sensitivity. *IEEE Trans. Circuits Syst.* CAS-30: 277–283

Babic H, Dobrenic D 1983 Low order minimax windows. *Proc. 1983 IEEE Int. Symp. on Circuits and Systems* pp. 86–89

Babic H, Dobrenic D 1987 A simple frequency sampling FIR filter design. *Proc. IASTED Symp. on Applied Control, Filtering and Signal Processing*, Geneva, Switzerland

Babic H, Rajan G, Mitra S K 1985 Generation of lowpass and bandpass window sequences by cascaded running sums. *Proc. IASTED Symp. on Applied Signal Processing and Digital Filtering*

Bellanger M 1984 *Digital processing of signals* (New York: John Wiley & Sons)

Carney R 1963 Design of a digital notch filter with tracking requirements. *IEEE Trans. Space Electron. Telem.* SET-9: 109–114

Golden R M 1973 Digital filters. In *Modern filter theory and design* (eds) GC Temes, S K Mitra (New York: John Wiley & Sons) Chap. 12

Hamming R W 1989 *Digital filters* (Englewood Cliffs, NJ: Prentice-Hall) Chap. 6

Hirano K, Nishimura S, Mitra S K 1974 Design of digital notch filters. *IEEE Trans. Circuits Syst.* CAS-21: 540–546

Kaiser J F, Hamming R W 1977 Sharpening the response of a symmetric non-cursive filter by multiple use of the same filter. *IEEE Trans. Acoust. Speech, Signal Process.* ASSP-25: 415–422

Land A, Powell S 1973 *Fortran codes for mathematical programming* (New York: John Wiley & Sons)

McClellan J H, Parks T W, Rabiner L R 1973 A computer program for designing optimum FIR linear phase digital filters. *IEEE Trans. Audio Electroacoust.* AU-21: 506–526

Mitra S K, Yu T-H 1986 Complementary two-dimensional digital filters. *Proc. IEEE* 74: 229–230

Neuvo Y, Dong C-Y, Mitra S K 1984 Interpolated finite impulse response filters. *IEEE Trans. Acoust. Speech, Signal Process.* ASSP-32: 563–570

Rabiner L R 1973 Approximate design relationships for lowpass FIR digital filters. *IEEE Trans. Audio Electroacoust.* AU-12: 456–460

Rabiner L R, Gold B 1975 *Theory and applications of digital processing* (Englewood Cliffs, NJ: Prentice-Hall)

Rabiner L R, Schafer R W 1971 Recursive and nonrecursive realizations of digital filters designed by frequency sampling techniques. *IEEE Trans. Audio Electroacoust.* AU-19: 200–207

Yu T H, Mitra S K 1985 A new two-dimensional window. *IEEE Trans. Acoust. Speech, Signal Process.* ASSP-33: 1058–1061