Design of Minimum-length, Minimum-phase, Low-group-delay FIR Filter Using Convex Optimization Method

Hamidreza Nazaripouya
University of California, Riverside
Winston Chung Global Energy Center
Riverside, CA, USA
hamidn@ucr.edu

Peter Chu
University of California, Los Angeles
Smart Grid Energy Research Center
Los Angeles, CA, USA
peterchu@g.ucla.edu

Hemanshu Pota
University of New South Wales,Canberra
School of Engineering and Information Technology
Canberra, Australia
h.pota@adfa.edu.au

Rajit Gadh
University of California, Los Angeles
Smart Grid Energy Research Center
Los Angeles, CA, USA
gadh@ucla.edu

Abstract— This paper proposes a new method for optimal design of minimum-length, minimum-phase, low-group-delay FIR filter by employing convex optimization, discrete signal processing (DSP), and polynomial stabilization techniques. The design of a length-N FIR filter is formulated as a convex second-order cone programming (SOCP). In order to design a minimum-phase FIR filter as the necessary condition for having low group delay, the algorithm guarantees that all the filter's zeros are inside the unit circle (minimum-phase). In addition, the quasiconvex optimization problem is developed to minimize the length of minimum-phase, low-group-delay FIR filter. To this end, for a typical low-pass FIR filter, the length of the filter is minimized such that the optimum magnitude response is satisfied, the minimum-phase characteristic is maintained, and the low-group-delay is achieved. The proposed design algorithm only relies on one parameter (cut-off frequency) and the rest of filter parameters are automatically optimized as the trade-off between having minimum-length, minimum-phase, maximum stopband attenuation and low group delay. The effectiveness and performance of proposed approach is demonstrated and compared with other approaches over a set of examples. It is illustrated that this approach converges to the optimal solution in a few iterations.

Key-Words: Group delay, minimum-phase, Finite Impulse Response (FIR) filter, Low pass filter, Convex optimization, Discrete signal processing

I. Introduction

A filter, in practice, is generally implemented through digital computation, and it is used to filter a discrete signal derived by periodic sampling from continuous-time counterpart. To this end, most often, design techniques are based on the discrete-time nature of the signals, and discrete-time filters (digital filters) are deployed. Therefore, the main focus of this paper is on discrete-time finite impulse response (FIR) filters. The design problem of FIR filter is studied more than the one of infinite-duration impulse response (IIR) due to the existence of an optimality theorem for FIR filters, which is meaningful in a wide range of practical situations. In addition, DSP micro-computers generally have arithmetic capabilities that are designed for accumulating sums of products which is a perfect match with FIR filter structure. Therefore, FIR filters have always been one of the main building blocks in digital signal processing considering their high performance in speed, assured stability, and efficient implementations based on the Fast Fourier Transform (FFT) [1] .

So far in the literature, a wide range of FIR filter design techniques have been investigated including design of linear-phase (LP) FIR filters [2], minimum-phase (MP) FIR filters [3], and a general (nonlinear-phase) non-minimum-phase FIR filters where neither of the first two conditions holds [4]. In addition, all of the above filters can be designed with either real or complex coefficients.

Designing the linear-phase FIR filter has attracted significant attention and considerable work has been done in this area [5]. One of the most popular existing method is the Reméz exchange algorithm [6]. The key advantage of linear-phase filters is having constant time delay over the entire band. This feature is useful in certain applications such as data transmission which needs minimum amount of dispersion in the transmission channel to avoid problems such as inter symbol interference (ISI).

Minimum-phase FIR filters are also extensively discussed in the literature. This is because of their ability for a low group delay for a high filter order. They also have lower coefficient sensitivity to quantization errors, and a lower order filter for a given magnitude response, compared with linear-phase filters.

Herrmann and Schuessler [7] initiated a method to transform an equiripple LP FIR filter into an MP FIR filter, while the MP version has the same attenuation characteristics that is found in the modulus squared, but half the degree. Authors in [8], using a change of variables and spectral factorization, could transform FIR filter design problems as a linear or nonlinear convex optimization problem. They solved it globally by interior-point methods. In [9] a non-iterative

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algorithm is suggested to design optimal MP FIR filters with real or complex coefficients based on discrete Hilbert transform. A Newton–Raphson iterative algorithm is used in [10] to extract the minimum-phase spectral factor from a linear phase transfer function. Pei and Lin proposed the real cepstrum to design an arbitrary length minimum-phase finite-impulse response filter from a mixed-phase prototype [11]. A new method of computing the minimum-phase filter and the associated all-pass filter using the QL-factorization is presented in [12], which is an alternative approach for computing the minimum-phase filter in a numerically stable way. Wu, *et al.* propose a direct optimization method for designing low-group-delay FIR filter [13] and show that their method can produce a filter with smaller group delay than that obtained from [8] under the same design criteria. Authors in [14] propose an improved algorithm for the constrained minimal L_p magnitude error design of minimum-phase FIR filters by incorporating the Lawson algorithm with the iterative constrained minimal L_p elliptic error (ICMEE-p) method. In [15] an algorithm is developed for the design of minimum-phase FIR filters with sparse impulse responses. In [16] the authors formulate the MP filter design problem in form of a set of nonlinear equations, which are solved using the Levenberg–Marquardt optimization method.

This paper associates the response delay of FIR filter with filter group delay, and also relates the cost and complexity of the controller to the length of the FIR filter. Thus, the focus of this paper is on a new design for minimum-phase, minimum-length FIR filters with low-group-delay as a solution to achieve lower time delays and lower cost. These features make the proposed filter a good candidate to be utilized in state-of-the-art method for channel coding entities of 4G/5G system [17].

There are several factors that should be considered in designing an optimal, effective, and fast response low-pass FIR filter besides satisfying the cut-off frequency criterion. These factors include 1) length: minimizing the filter length reduces the complexity and cost of the controller. 2) Group delay: a filter with smaller group delay causes less lag between input and output filter signals. 3) Stopband attenuation: a higher stopband attenuation for a specific length of filter leads to a better filtering performance.

This paper initiates an approach for an optimal design of minimum-length, low-group-delay FIR filter with maximum stopband attenuation for a given cut-off frequency. The user needs to determine only the level of filtering by defining the low-pass filter cut-off frequency, and the proposed algorithm automatically optimizes the other design parameters of the filter in order to find a low delay, high performance FIR filter, accordingly.

The rest of the paper is organized as follows: in Section II, the FIR filter background and the problem statement are presented. In Section III, the proposed design algorithm including convex optimization, discrete signal processing, and polynomial stabilization techniques is introduced. Design examples for a set of filters and the results comparisons with the alternative approaches are provided in Section IV. Finally, the concluding remarks are given in Section V.

II. FIR Filter Problem description

A length-N FIR filter can be represented by its frequency response as [18]:

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n]e^{-j\omega n} \tag{1}$$

where $h[n], n = 0,1,\dots,N-1$, are real filter coefficients. The frequency response $H(e^{j\omega})$ can be characterized by its magnitude and phase as:

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\varphi(\omega)} \tag{2}$$

where $|H(e^{j\omega})|$ represents the magnitude of $H(e^{j\omega})$, and $\varphi(\omega)$ is the phase. The group delay, $\mathrm{grd}[H(e^{j\omega})]$, is defined as:

$$\tau(\omega) = \mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}(\arg[H(e^{j\omega})]) = -\frac{d\varphi(\omega)}{d\omega} \tag{3}$$

As opposed to the phase delay which is a measure of the time delay of the phase of a sinusoid at frequency ω, group delay is a measure of the time delay of the amplitude envelopes of the various sinusoidal components of the signal. That is, the group delay reflects the time difference from the input pulse envelope peak to the output pulse envelope peak. The idea of designing a filter with low-group-delay is to minimize the time-delay induced by the filter for the real-time signal processing applications.

The problem in this paper is defined as designing a minimum-phase low-pass filter, which has the following specification: 1) maximum attenuation within the stopband, 2) low-group-delay within the passband, and 3) minimum length. This problem can be expressed as (4):

$$\begin{aligned} &\min_{h[n]}\ \alpha_1 N + \alpha_2\varepsilon_1 + \alpha_3\varepsilon_2 \\ &\text{s.t.}\ \ \|H(e^{j\omega})\|_2 \le \varepsilon_1\ for\ \ \omega \ge \omega_c \\ &\mathrm{grd}[H(e^{j\omega})] \le \varepsilon_2\ for\ \omega \le \omega_c \end{aligned} \tag{4}$$

where N is the length of FIR filter, ε_1 and ε_2 are the upper limits for filter magnitude response within the stopband, and filter group delay within the passband, respectively; ω_c represents the cutoff frequency, $\mathrm{grd}[H(e^{j\omega})]$ is the group delay of the $H(e^{j\omega})$ filter, and α_1, α_2 and α_3 are weighting factors.

In (4), the cost function is defined in a way to 1) minimize the filter magnitude at stopband, and thus maximizes the filter stopband attenuation by minimizing ε_1, 2) minimizes the filter passband group-delay by minimizing ε_2 , and minimize the filter length by minimizing N, while there is always a tradeoff between these objectives.

Considering the fact that $H(e^{j\omega})$ should be also a minimum-phase filter, the optimization problem in (4) is not a convex optimization problem. Also because of the nonlinearity of $\mathrm{grd}[H(e^{j\omega})]$ with respect to $h[n]$, the optimization problem defined in (4) may not be solvable directly [19]. Therefore, the original problem should be transformed so that it can be solved.

III. Theory Development

The proposed design algorithm employs three separate strategy including 1) discrete signal processing theorems 2) optimization algorithms, and 3) polynomial root analysis to solve the problem in section II. The discrete signal processing theory is utilized to develop an appropriate FIR filter, in discrete time domain. The optimization algorithm is used to formulate and optimize the parameters of the filter in order to minimize the length, maximize the stopband attenuation and reduce the passband group delay. Finally, the polynomial root analysis theorem helps to include the minimum-phase feature when the optimization problem is developed.

A. Discrete signal processing

One of the key objectives of this paper is to design a filter that has a low-group-delay. For this purpose, the focus of this section is on designing a "minimum-phase" FIR filter, which refers to systems that are causal and stable and that have a causal and stable inverse. The motivation for designing a minimum-phase filter is related to some properties that these type of filters offer. The following theorems summarize these properties [18].

a) The Minimum-phase-Lag Property

Theorem 1 [18]: The causal, stable system that has $|H_{min}(e^{j\omega})|$ as its magnitude response and also has all its zeros and poles inside the unit circle has the minimum-phase-lag function (for $0\leq\omega\leq\pi$) of all the systems having that same magnitude response.

The continuous phase, i.e., $\arg[H(e^{j\omega})]$, of any nonminimum-phase system can be expressed as (5):

$$\arg[H(e^{j\omega})] = \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})] \quad (5)$$

where the continuous phase that would correspond to the principal-value phase of $H(e^{j\omega})$ is the sum of the continuous phase associated with the minimum-phase function ($\arg[H_{min}(e^{j\omega})]$) and the continuous phase of the all-pass system ($\arg[H_{ap}(e^{j\omega})]$).

The continuous-phase of an all-pass system is negative for $0\leq\omega\leq\pi$. Thus, the reflection of zeros of $H_{min}(z)$ from inside the unit circle to conjugate reciprocal locations outside always decreases the (continuous) phase or increases the negative of the phase, which is called the phase-lag function.

b) The Minimum Group-Delay Property

Theorem 2 [18]: Among all the systems that have a given magnitude response $|H_{min}(e^{j\omega})|$, the one that has all its poles and zeros inside the unit circle has the minimum-group-delay. First note that the group delay for the systems ($\mathrm{grd}[H(e^{j\omega})]$) that have the same magnitude response can be calculated as (6):

$$\mathrm{grd}[H(e^{j\omega})] = \mathrm{grd}[H_{min}(e^{j\omega})] + \mathrm{grd}[H_{ap}(e^{j\omega})] \quad (6)$$

The group delay for the minimum-phase system is always less than the group delay for the nonminimum-phase system. This is because, the all-pass system that converts the minimum-phase system into the nonminimum-phase system has a positive group delay. This is a general property of all-pass systems; they always have positive group delay for all ω.

c) The Minimum Energy-Delay Property

Theorem 3 [18]: For any causal, stable sequence $h[n]$ for which $|H(e^{j\omega})| = |H_{min}(e^{j\omega})|$, then $|h[0]| \leq |h_{min}[0]|$.

All the impulse responses whose frequency-response magnitude is equal to $|H_{min}(e^{j\omega})|$ have the same total energy as $h_{min}[n]$, since, by Parseval's theorem,

$$\sum_{n=0}^{\infty}|h[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H(e^{j\omega})\right|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_{min}(e^{j\omega})\right|^2 d\omega = \sum_{n=0}^{\infty}|h_{min}[n]|^2 \quad (7)$$

If we define the partial energy of the impulse response as $E[n] = \sum_{m=0}^{n}|h[m]|^2$ then it can be shown that

$$\sum_{m=0}^{n}|h[m]|^2 \leq \sum_{m=0}^{n}|h_{min}[m]|^2 \quad (8)$$

for all $h[n]$ belonging to the family of systems that have the magnitude response $|H(e^{j\omega})| = |H_{min}(e^{j\omega})|$. According to (8), the partial energy of the minimum-phase system is mostly concentrated around m = 0; i.e., the energy of the minimum-phase system is delayed the least of all systems having the same magnitude response function. For this reason, minimum-phase (lag) systems are also called minimum energy-delay systems, or simply, minimum-delay systems.

In general, the minimum energy delay occurs for the system that has all its zeros inside the unit circle (i.e., the minimum-phase system) and the maximum energy delay occurs for the system that has all its zeros outside the unit circle.

According to the above theorems, the proposed method in this paper formulates the problem in a way to ensure all zeros of the filter are inside the unit circle in order to minimize the delay in real time application. The next section will describe how the optimization algorithm and polynomial root theorem are employed to formulate the optimization problem.

B. Optimization

This section describes how the minimum-length, minimum-phase, low-group-delay filter is designed by solving convex and quasi-convex optimization problems.

According to the definition of group-delay in (3), the first step is to formulate the group-delay based on filter coefficients.

In the amplitude/phase representation:

$$H(e^{j\omega}) = A(\omega)e^{j\varphi(\omega)} \quad (9)$$

where $A(\omega) = |H(e^{j\omega})|$ represents the magnitude of $H(e^{j\omega})$. Differentiating both sides of (9),

$$H'(e^{j\omega}) = A'(\omega)e^{j\varphi(\omega)} + A(\omega)e^{j\varphi(\omega)}\left(j\varphi'(\omega)\right) \quad (10)$$

and dividing (10) by $A(\omega)e^{j\varphi(\omega)}$, since both $A'(\omega)/A(\omega)$ and $\varphi'(\omega)$ are real,

$$\tau(\omega) = Im\left(\frac{H'(e^{j\omega})}{H(e^{j\omega})}\right) \quad (11)$$

If $h[n] \leftrightarrow H(e^{j\omega})$, we can apply the frequency differentiation property of the Fourier transform and get, $-jnh[n] \leftrightarrow H'(e^{j\omega})$.

Denoting Fourier transform of $h[n]$ as $F.T.(h[n])$,

$$\tau(\omega) = -Im\left(\frac{F.T.(-jnh[n])}{F.T.(h[n])}\right) \quad (12)$$

and finally,

$$\tau(\omega) = Real\left(\frac{F.T.(nh[n])}{F.T.(h[n])}\right) \quad (13)$$

It is desired to design the filter with minimum possible group-delay in the passband in order to have a fast response and minimize the gap between input and output signals of the filter. For this purpose, the idea is to force the group delay at ω=0 equal to zero. One can have

$$\tau(0) = Real\left(\frac{F.T.(nh[n])}{F.T.(h[n])}\right)\Bigg|_{\omega=0} = 0 \leftrightarrow \sum_{n=0}^{N-1} nh[n] = 0 \quad (14)$$

Constrain (14) ($\sum_{n=0}^{N-1} nh[n] = 0$) can satisfy the zero group delay for the DC component of the signal.

By expanding (13), the group-delay can be expresses as (15):

$$\tau(\omega) = \frac{\sum_{n=0}^{N-1} nh[n]\, cos(\omega n) \times \sum_{n=0}^{N-1} h[n] \cos(\omega n)}{\sum_{n=0}^{N-1} h[n]e^{-j\omega n} \times \sum_{n=0}^{N-1} h[n]e^{j\omega n}} + \frac{\sum_{n=0}^{N-1} nh[n]\, sin(\omega n) \times \sum_{n=0}^{N-1} h[n]\, sin(\omega n)}{\sum_{n=0}^{N-1} h[n]e^{-j\omega n} \times \sum_{n=0}^{N-1} h[n]e^{j\omega n}} \quad (15)$$

Making the derivation from (15) and performing additional mathematical operations, one can write:

$$\tau'(\omega) = \frac{(-M+N) \times K - L \times G}{K^2} \quad (16)$$

where

$$M = \sum_{n=0}^{N-1} n^2 h[n]\, sin(\omega n) \times \sum_{n=0}^{N-1} h[n]\, cos\omega + \sum_{n=0}^{N-1} nh[n]\, cos(\omega n) \times \sum_{n=0}^{N-1} nh[n]\, sin(\omega n)$$

$$N = \sum_{n=0}^{N-1} n^2 h[n]\, cos(\omega n) \times \sum_{n=0}^{N-1} h[n]\, sin(\omega n) + \sum_{n=0}^{N-1} nh[n]\, sin(\omega n) \times \sum_{n=0}^{N-1} nh[n]\, cos(\omega n)$$

$$L = \sum_{n=0}^{N-1} -jnh[n]e^{-j\omega n} \times \sum_{n=0}^{N-1} h[n]e^{j\omega n} + \sum_{n=0}^{N-1} h[n]e^{-j\omega n} \times \sum_{n=0}^{N-1} jnh[n]e^{j\omega n}$$

$$G = \sum_{n=0}^{N-1} nh[n]\, cos(\omega n) \times \sum_{n=0}^{N-1} h[n]\, cos(\omega n) + \sum_{n=0}^{N-1} nh[n]\, sin(\omega n) \times \sum_{n=0}^{N-1} h[n]\, sin(\omega n)$$

$$K = \sum_{n=0}^{N-1} h[n]e^{-j\omega n} \times \sum_{n=0}^{N-1} h[n]e^{j\omega n}$$

Equation (16) depicts the first derivative of group-delay. By substituting constraint (14) into (16), we obtain:

$$\sum_{n=0}^{N-1} nh[n] = 0 \leftrightarrow \tau'(0) = 0 \quad (17)$$

Equation (17) shows that condition (14) also provides the zero slope for group delay at $\omega = 0$ ($\tau'(0) = 0$), which can further restrict the growth of the group delay in the passband.

Although constraint (14) reduces the low frequency group delay significantly, another condition is required to obtain the low-group-delay FIR filter. Minimum-phase characteristic can guarantee the minimum-group-delay among all the systems that have the same magnitude response. Next, a helpful theorem to define the optimization problem for designing a minimum-phase filter is given.

Theorem 4 [20]: Let $B_0, B_1, \dots, B_n$ be real scalars (with $B_1, \dots, B_n$ not all zero) and consider the affine family of monic polynomials

$$P = \left\{ z^n + a_1 z^{n-1} + \dots + a_{n-1}z + a_n \mid B_0 + \sum_{j=1}^{n} B_j a_j = 0, a_i \in \mathbb{R} \right\} \quad (18)$$

By defining the optimization problem

$$\rho^* := \inf_{p \in P} \rho(p) \quad (19)$$

where $\rho(p)$ denotes the root radius of a polynomial p and is defined as (20),

$$\rho(p) = max\{|z| \mid p(z) = 0, z \in \mathbb{C}\} \quad (20)$$

The optimization problem in (19) has a globally optimal solution of the form

$$p^*(z) = (z-\gamma)^{n-k}(z+\gamma)^k \in P \quad (21)$$

for some integer k with $0 \le k \le n$, where $\rho^* = \gamma$.

The theorem describes that for the maximum root radius (γ) equal to zero, all the coefficient of polynomial need to be equal to zero $a_1 = \dots = a_{n-1} = a_n$ which leads to $p(z) = z^n$. Therefore, since problem in Theorem 4 is a convex optimization problem, by minimizing the coefficients of the polynomial one can move towards the zero root radius. In other words, the radius of polynomial roots will be reduced by minimizing the polynomial coefficient ($a_1, \dots, a_{n-1}, a_n$) while $B_0 + \sum_{j=1}^{n} B_j a_j = 0$. This approach will help to push polynomial roots inside the unit circle, which is the requirement of the minimum-phase filter.

On the other hand, equation (22) represents the group delay contribution of a zero $(1 - re^{j\theta}e^{-j\omega})$ in the filter transfer function where r is the radius and θ is the angle of the zero in the z-plane,

$$\text{grd}\left[1 - re^{j\theta}e^{-j\omega}\right] = \frac{r^2 - r\cos(\omega-\theta)}{1 + r^2 - 2r\cos(\omega-\theta)} \quad (22)$$

The maximum amount of group delay for each zero happens when $\omega - \theta = \pi$ thus:

$$max(\text{grd}\left[1 - re^{j\theta}e^{-j\omega}\right]) = \frac{r}{1+r} \quad (23)$$

Therefore, minimizing the polynomial coefficient and consequently reducing the radius of the polynomial roots (r), the group delay eventually decreases which is desirable.

In summary, considering the FIR filter described by $H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n]e^{-j\omega n}$ while design variables are coefficients $h[n]$, we explore the problem of the design of a low-pass filter, with the following specifications

1) For $\omega = 0$ the magnitude of the filter should be equal to one ($H(e^{j0}) = \sum_{n=0}^{N-1} h[n] = 1$).
2) For $\omega_c \le \omega \le \pi$ the magnitude of the filter should be minimized: ($\min_{h[n]} \left|H(e^{j\omega})\right| = \min_{h[n]} \left\|\sum_{n=0}^{N-1} h[n]e^{-j\omega n}\right\|$).
3) For $\omega = 0$ the group delay should be zero to have no delay for DC component and minimum delay in low frequency ($\sum_{n=0}^{N-1} nh[n] = 0$).

4) The coefficient of the filter should be minimized to guarantee the minimum-phase filter with minimum-group-delay ($\min h[n]\ for\ n = 1, \dots, (N-1)$).

To this end, the optimization problem can be expressed as follows:

Optimization Problem 1

Optimization Variable: $h[n]$

$$\min_{h[n]} \left\| \sum_{n=0}^{N-1} h[n] e^{-j\omega n} \right\| + \alpha \left\| \sum_{n=1}^{N-1} h[n] \right\| \qquad \omega_c \le \omega \le \pi$$

subject to:

$$\sum_{n=0}^{N-1} h[n] = 1 \qquad (24)$$

$$\sum_{n=0}^{N-1} n h[n] = 0$$

where α is a weighting factor which is changed in a loop to make sure that all roots are placed inside the unit circle for a minimum-phase filter.

Problem (24) can be expressed as the second-order cone programming (SOCP) by defining some new parameters as follows:

$$\begin{aligned} \min_{h[n]}\ & (t_1 + \alpha t_2) \\ \text{s.t.}\ & \|A_k x\|_2 \le t_1 \\ & \|Bx\|_2 \le t_2 \\ & Cx = d \end{aligned} \qquad (25)$$

where

$$x = [h[0] \quad h[1] \quad \dots \quad h[N-1]]^T$$

$$A_k = [e^{-j\omega_k \cdot 0} \quad e^{-j\omega_k \cdot 1} \quad \dots \quad e^{-j\omega_k \cdot (N-1)}]$$

$\omega_c \le \omega_1 \le \omega_2 \le \cdots \le \omega_k \le \pi,\ k = 1, \dots, M.$ A rule of thumb for choosing M, $M \approx 15N$, is recommended in [21]

$$B = [0 \quad 1 \quad 1 \quad \dots \quad 1]_{1\times N}$$

$$C = \begin{bmatrix} 0 & 1 & 2 & \cdots & N-1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{2\times N}$$

$$d = [0 \quad 1]^T$$

As it was shown in (25), a very special form of the semi-infinite constraints appears in the filter design problems. General semi-infinite convex optimization is a well-developed field that allows for a solution with no great theoretical or practical difficulty.

Lemma 1 [8]: The semi-infinite inequality constraint

$$g_i(x,\omega) \le t_i \text{ for all } \omega \in [\omega_c, \pi] \qquad (26)$$

can be expressed as the ordinary inequality constraint

$$h_i(x) = \sup_{\omega\in[\omega_c\, \pi]} g_i(x,\omega) \le t_i \qquad (27)$$

It is easily verified that h_i is a convex function of *x*, since for each ω, $g_i(x,\omega)$ is convex in *x*. On the other hand, h_i is often nondifferentiable, even if the functions g_i are differentiable. Several methods such as bundle methods, ellipsoid methods, or cutting plane methods for general (nondifferentiable) convex optimization are proposed that can be used to solve the semi-infinite constraints in (25). These methods require an efficient approach to evaluate h_i and a subgradient at any *x*. This involves computing a frequency υ for which $g_i(x,\upsilon) = h_i(x)$. It is also possible to solve some magnitude filter design problems exactly, by transforming the semi-infinite constraints into (finite-dimensional) constraints that involve linear matrix inequalities. The semi-infinite constraints can also be approximated in a very straightforward way by sampling or discretizing frequency. We choose a set of frequencies $\omega_c \le \omega_1 \le \omega_2 \le \cdots \le \omega_M \le \pi$ often uniformly or logarithmically spaced, and replace the semi-infinite inequality constraint (28)

$$g_i(x,\omega) \le t_i \text{ for all } \omega \in [\omega_c, \pi] \qquad (28)$$

with the set of *M* ordinary inequality constraints

$$g_i(x,\omega_k) \le t_i \text{ for all } k = 1, \dots, M \qquad (29)$$

Note that sampling preserves convexity. When *M* is sufficiently large, discretization yields a good approximation of the semi-infinite programming (SIP). A standard rule of thumb is to choose $M \approx 15N$.

There are still two more parameters, *N* and ω_c, that need to be designed in order to have a desired low pass FIR filter, where *N* is the length of the FIR filter and ω_c is the cut-off frequency.

Theorem 5*:* The length of an FIR filter is a quasi-convex function of its coefficients [22].

Lemma 2: By defining $f(x) = min\{k \mid x_{k+1} = \cdots = x_N = 0\}$, the sublevel sets of *f* are affine sets and convex: $\{x \mid f(x) \le k\} = \{x \mid x_{k+1} = \cdots = x_N = 0\}$. This means that *f* is a quasi-convex function, and optimization problem is a quasi-convex optimization problem.

Hence, the problem of finding the minimum-length FIR filter given cut-off frequency ω_c can be expressed as:

Optimization Problem 2

minimize *N*

subject to (30)

$$\|A_k x(N)\|_2 \le 0.7079\ (-3\ dB) \qquad k \in I,$$

where

$$I = \{k | \omega_k \ge \omega_c\}$$

The quasi-convex problem shown in (30) can be solved using bisection on *N*. Each iteration of the bisection involves solving an SOCP feasibility problem.

In summary, the proposed algorithm is an iterative scheme with two loops: 1) the outer quasi-convex optimization problem loop which adjusts *N* using bisection method and 2) the inner SOCP convex optimization problem loop which adjusts α. Once the minimum-phase and cuff-off frequency criteria are met, the proposed FIR filter is obtained. Fig. 1 shows the pseudo code of the algorithm, which describes the design process of the proposed filter.

Filter Design Algorithm

Set the cut-off frequency (ω_c)
$n_{bot} = 1$
n_{top} = maximum feasible filter order
while $(n_{top} - n_{bot} > 1)$ **do**
  $N = ceil\left(\frac{(n_{top} - n_{bot})}{2}\right)$
  while $\rho(H(z)) > 1$ **do**
    Increase α
    Solve SOCP in (24)
  end
  if $\|A_k x\|_2 \leq 0.7079$
    $n_{top} = N$
  else
    $n_{bot} = N$
  end
end

Fig. 1. Designing algorithm of the proposed filter

IV. Design Examples and Result

In this section, the proposed technique is employed to design a set of exemplifying low-pass FIR filters with different specifications. The results are compared with the ones obtained from alternative design techniques.

Design 1: the objective of this example is to design the minimum-length, low-group-delay low-pass FIR filter with maximum attenuation at stopband. Without loss of generality the cut-off frequency is selected as $\omega_c = 0.05\,\pi$. In this example, the design process, including solving the convex and quasi-convex optimization problems, takes 3.18 second to converge. Satisfying the selected criterion for cut-off frequency, the quasi-convex optimization minimizes the length of filter to 52. The obtained cut-off frequency is 0.0501π, which is matched with the design parameter and the small deviation is due to discretizing the convex optimization problem.

In order to compare the results, the optimization method proposed by Wu, *et al.* for designing a low-group delay FIR Filter [13], is utilized to design the same filter with the same cut-off frequency. To this end, the parameters of the Wu filter are selected as follows: the length of the filter is 52 to be identical with the filter designed by the proposed approach, the passband edge is equal to $\omega_p = 0.0485\,\pi$, the stopband edge is $\omega_s = 0.057\,\pi$, the weighting factor α is 0.00001 based on the authors suggestion. It takes 16.6 seconds for the proposed algorithm in [13] to converge to the optimal solution, which is almost five times of the convergence time of the proposed approach in this paper. Fig. 2 and Fig. 3 depicts the magnitude and phase response of the both proposed filter in this paper and Wu filter, respectively. In Fig. 2 although the cut-off frequency of both filters are aligned, the attenuation of the proposed filter is considerably higher at stopband meaning higher level of the filtering.

Fig. 4 shows the group delay of both filters. It can be seen that at the passband the group delay of the proposed approach is lower than its counterpart desined by algorithm in [17].

The zero/pole placment of the filters are shown in Fig. 5. This figure ilustrtates that the filter desined by the proposed algorithm is also minmium-phase (all zeros are inside the unit circle) which gauarntees the minimum-group-delay among all the filters with the same magnitude response. While the filter designed by algorithm in [13] does not hold this condidtion.

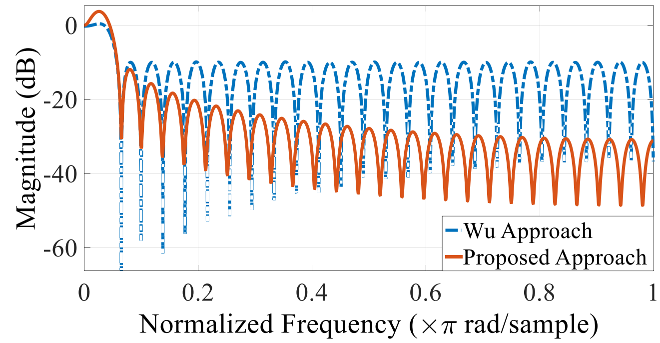

Fig. 2. Magnitude response of the proposed filter and the Wu filter

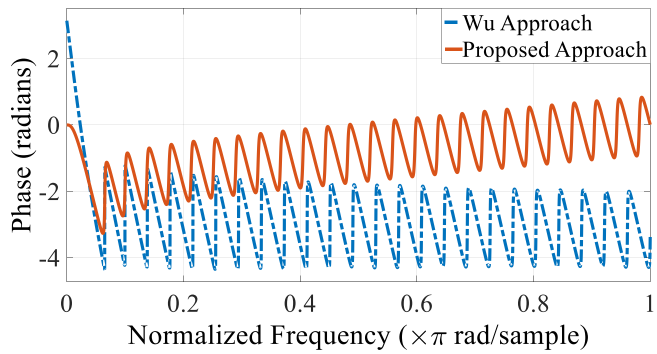

Fig. 3. Phase diagram of the proposed filter and the Wu filter

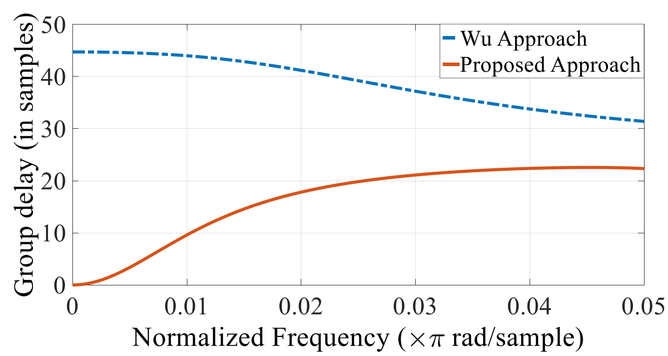

Fig. 4 Group delay comparison of the proposed filter, and the Wu filter

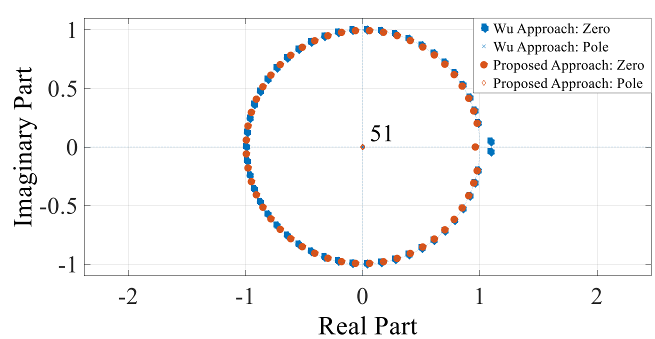

Fig. 5. Zero/pole placement of the proposed filter and the Wu filter

Design 2: In this example, a low-pass FIR filter with narrow passband is designed. To this end, the cut-off frequency is considered as $\omega_c = 0.006\,\pi$. The minimum-length, low-group-delay FIR filter is designed by solving the optimization problems (24) and (30). In this example, the time of convergence for the proposed algorithm is 28 minutes and 58 seconds. The minimum length of the filter obtained from quasi-convex optimization problem that meets the selected cut-off frequency is 433. The magnitude and phase of the proposed filter is shown in Fig. 6 and Fig. 7, respectively.

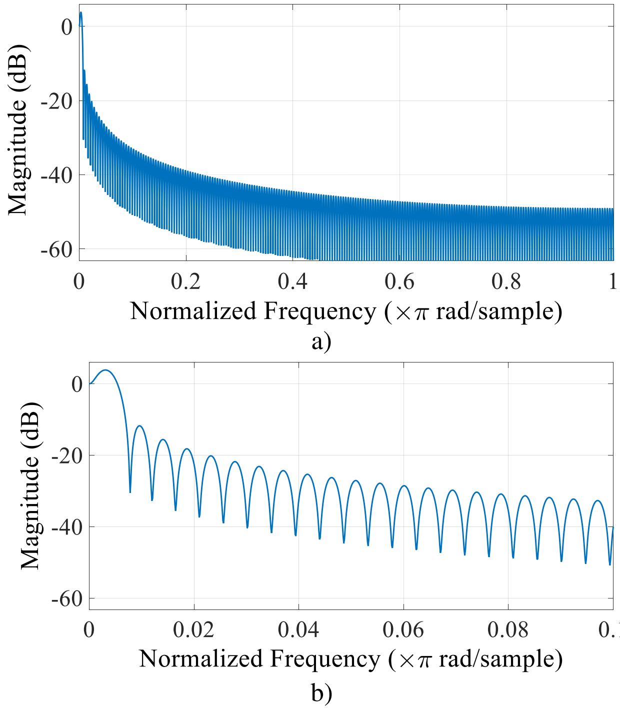

Fig. 6. Magnitude response of the proposed filter a) the magnitude response in dB b) detailed plot of the magnitude response

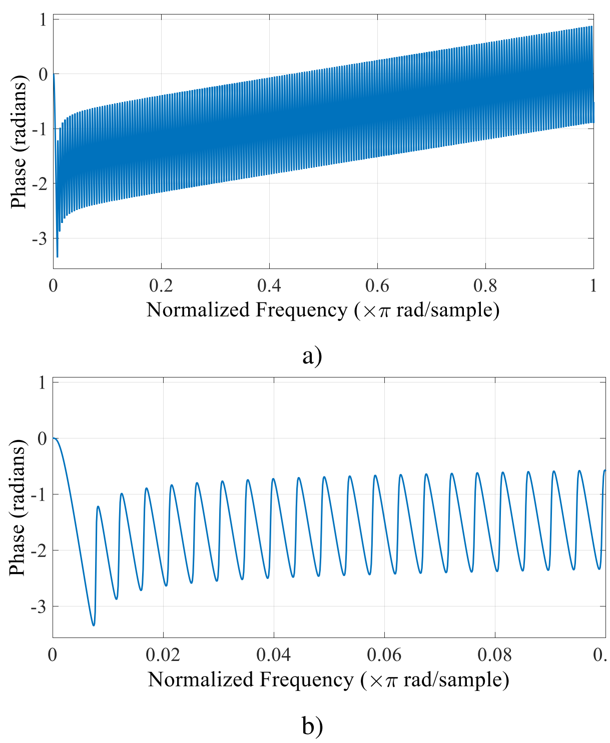

Fig. 7. Phase diagram of the proposed filter a) the Phase response in radians b) detailed plot of the Phase response

It can be seen that the cut-off frequency ($0.00609\ \pi$) matches with the design criterion. The phase response in Fig. 7 can be uniquely specified based on mangnitude response for this filter due to minimum-phase feature. Fig. 8 shows the group delay of the filter and although the length of the filter is 433, the maximum delay reaches to 191.8 samples. the zero/pole placment of the filter is shown in Fig. 9 and it can be seen that all poles and zeros are inside the unit circle.

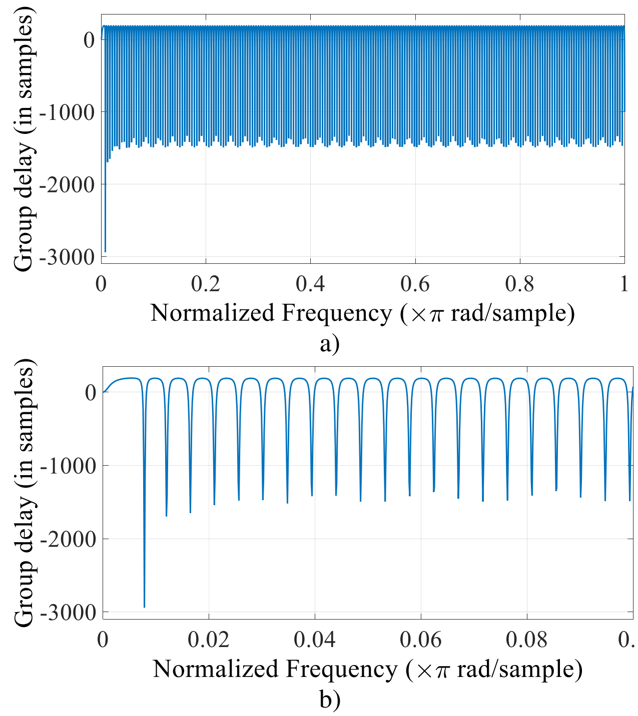

Fig. 8. Group delay diagram of the proposed filter a) the group delay response in samples b) detailed plot of the group delay response

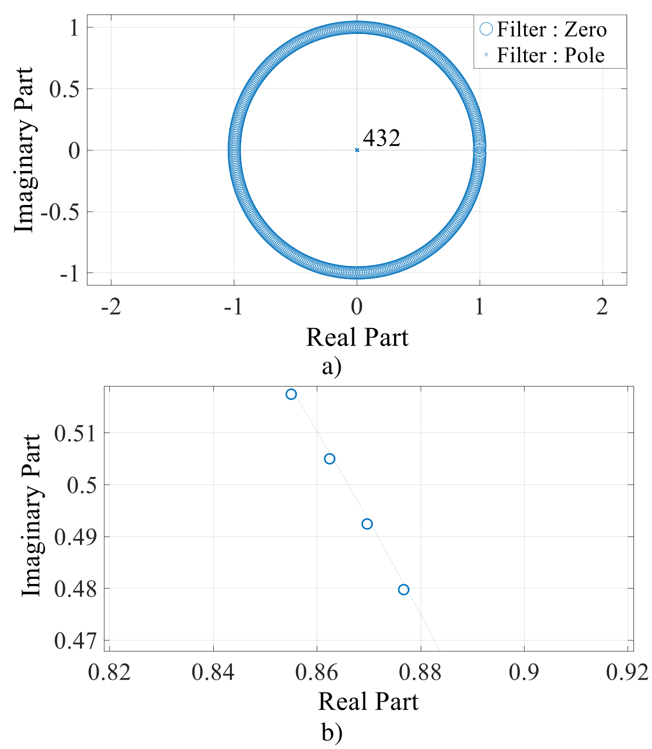

Fig. 9. Zero/pole placement of the proposed filter a) the zero/pole placement b) detailed plot of the zero/pole placement

However, this design example is not solvable by the method proposed in [13] as the defined linearly perturbed optimization problem, due to high dimensionality, does not converge, and thus the approach is not able to approximate the global solution of the original FIR filter design optimization problem. Instead, in order to compare and demonstrate the effectiveness of the proposed approach the filter with the same passband magnitude response is designed using the method proposed by Herrmann and Schuessler for MP FIR filter designing.

It should be noted that in general designing the Herrmann and Schuessler filter is not as easy as designing the filter using the proposed approach. The proposed approach only relies on one parameter and the rest are optimized automatically to minimize the length of filter and maximize the attenuation. While the Herrmann and Schuessler filter requires four design parameters including the passband edge, stopband edge, passband ripple, and stopband ripple. Therefore, one of the advantages of the proposed approach over traditional FIR filter design approaches is simplicity in design. In this experiment to achieve the same passband magnitude response, the passband edge of Herrmann and Schuessler filter is designed to be $\omega_p = 0.005370\,\pi$, the stopband edge is $\omega_s = 0.007568\,\pi$, the passband ripple is 0.29183, and stopband ripple is 0.05702.

Fig. 10 shows the magnitude response of the FIR Filters designed by the proposed approach and Herrmann and Schuessler approach. Although both filters has the same passband magnitude response, the attenuation of proposed approach in high frequency is considerably higher which leads to smoother output.

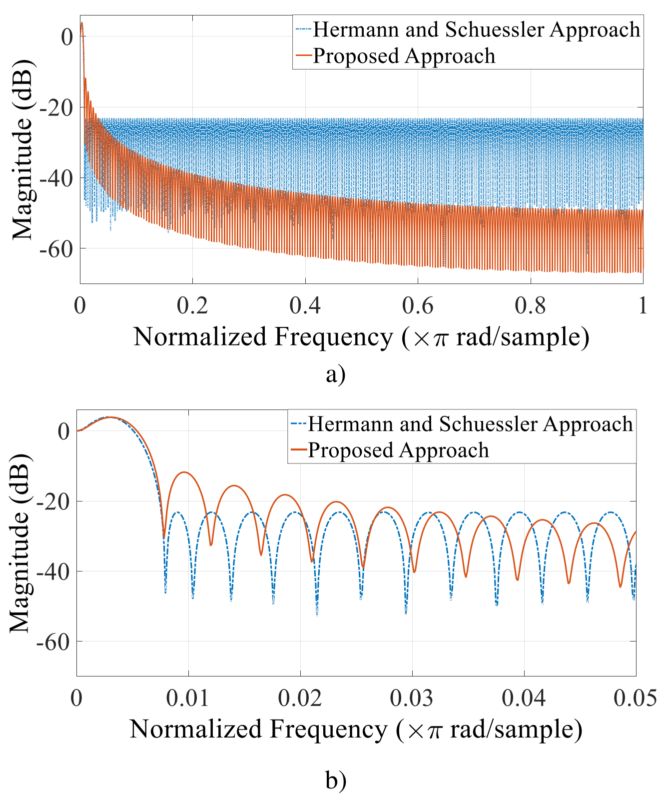

Fig. 10 Magnitude response comparison of the proposed filter and Herrmann and Schuessler filter while the passband magnitude response are the same a) the magnitude response in dB b) detailed plot of the magnitude response

The plot in Fig. 11 validates that the group delay of the proposed filter is lower than Herrmann and Schuessler filter with the same passband magnitude response.

In order to show the effect of the cut-off frequency on the designed low-pass filter characteristics, the FIR filter is redesigned with (a) $\omega_c = 0.008\,\pi$, (b) $\omega_c = 0.004\,\pi$, and (c) $\omega_c = 0.012\,\pi$. The specifications of each filter are shown in Table 1. The results show that by reducing the cut-off frequency, the narrower passband is achieved, although, it costs higher order of filter. For instance, the optimum length of filter satisfying the cut-off frequency constraint ($\omega_c = 0.004\,\pi$) is 650, which means a higher group delay of 288.2 samples. The design result for $\omega_c = 0.012\,\pi$ shows that with the wider passband, the optimum length of the filter decreases to 216, which means less filter complexity.

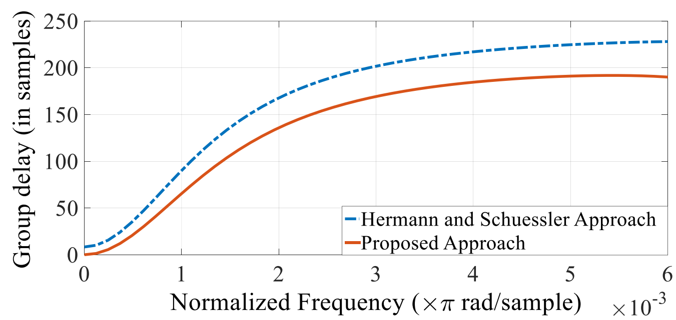

Fig. 11. Group delay comparison of the proposed filter, and Herrmann and Schuessler filter with the passband magnitude response are the same.

Table 1: FIR filter specifications with different cut-off frequency

Designed Cut-off frequency (ω_c)	Filter Length	Obtained ω_c	Maximum Group-delay (samples)
$\mathbf{0.004\,\pi}$	650	$0.00402\,\pi$	288.2
$\mathbf{0.006\,\pi}$	433	$0.00609\,\pi$	191.8
$\mathbf{0.008\,\pi}$	326	$0.0081\,\pi$	145
$\mathbf{0.012\,\pi}$	216	$0.0121\,\pi$	95.4

The simulation results validate the performance of the proposed approach and its simplicity where one can design the low-group delay, low-pass filter with just choosing one parameter (cut-off frequency) and find the optimum solution in a trade-off between the length of the filter and stopband attenuation of the filter. The higher cut-off frequency means lower group delay and less stopband attenuation, and the lower cut-off frequency means higher group delay and more stopband attenuation. Therefore, the filter can be designed for a wide range of cut-off frequencies in a short time offline, and then the user can easily adjust the level of filtering in real time by changing the cut-off frequency while the rest of filter parameters has been optimized by the proposed algorithm.

The future steps in the direction of this research is to apply the proposed algorithm in different domains including but not limited to the state-of-the-art channel coding in 4G/5G systems, for example by replacing the component codes of MIMO-based turbo-like coded systems.

V. Conclusion

In this paper a new methodology for designing a fast response FIR filter is proposed. The proposed method designs the minimum-phase, minimum-length, low-group-delay FIR filter by employing the convex and quasi-convex optimization methods. The polynomial root analysis theorem helps with the optimization problem to include minimum-phase feature of the filter. The minimum-phase, low-group-delay feature of the filter reduces the filter time delay and the minimum-length feature of the filter simplifies the filter structure and reduces the cost of implementation. Another advantage of the proposed technique is to adjust the level of filtering directly by setting up the cut-off frequency while the other parameters of the filter are optimized automatically by the algorithm. The design performance of the proposed approach is validated over different design cases. The obtained results and comparisons have illustrated that the proposed technique provides lower group delay compared with the alternative design techniques. In addition, the proposed design algorithm depends on fewer design parameters and converges faster compared to the alternative design techniques discussed in the paper.

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