DESIGN OF MULTIDIMENSIONAL NON-SEPARABLE REGULAR FILTER BANKS AND WAVELETS *

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ABSTRACT

The design of multidimensional non-separable wavelets based on iterated filter banks is investigated. To obtain regularity of the wavelet, a maximum number of zeros is put at aliasing frequencies in the lowpass filter. Two approaches are pursued. A direct one designs non-separable perfect reconstruction filter banks based on cascade structures and with prescribed zeros both analytically (small cases) and numerically (larger cases). A second, indirect method maps biorthogonal one-dimensional banks with high regularity into multidimensional banks using the McClellan transformation. A number of properties relevant to perfect reconstruction and zero locations are shown in this case. Design examples are given in all cases, and the testing of regularity is discussed, together with a fast algorithm to compute iterates.

1. INTRODUCTION

In the last few years, the emerging wavelet theory, its connection to multirate filter banks and a possible impact on image and video coding have caused quite a stir in the applied mathematics and signal processing communities. Daubechies in [1] showed that by iterating filter banks one can obtain continuous wavelet bases (assuming the lowpass filter is *regular*). In the one-dimensional case, there already exist a number of techniques to design filters with an appropriate degree of regularity (see, for example, [1, 2]).

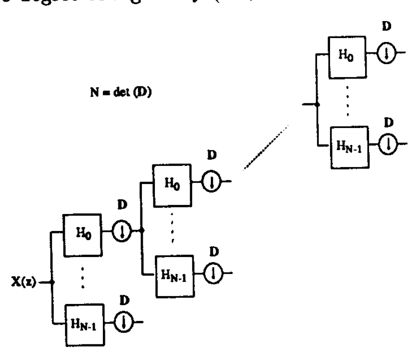

Figure 1: Iterated filter bank. If the lowpass filter is regular, this construction leads to a continuous wavelet basis.

The field of multidimensional wavelets and associated filter banks is, however, quite young. In more than one dimension sampling is described by a lattice and its corresponding basis (matrix). Thus, when using the method of iterated filter banks (see Figure 1), one has to deal with taking powers of matrices instead of scalars, that is, the iterated filter becomes

$$H_\omega^{(i)}(\omega) = \prod_{k=0}^{i-1} H_\omega((\mathbf{D}^t)^k \omega) \qquad i = 1, 2, \ldots, \qquad (1)$$

where $\mathbf{D}$ is the sampling matrix, $H_\omega(\omega)$ is the lowpass filter that we iterate, and $H(\mathbf{z}) = H_\omega(\omega)$ for $\mathbf{z}$ on the unit hyper-circles. But while different matrices can represent the same sampling lattice, taking their powers can lead to vastly different behavior of iterated filters. Therefore, although there have been some initial results on the design of irreducible wavelet bases [3, 4], a number of questions still remain open.

In one dimension, Daubechies gave a sufficient condition for a filter to be regular (the existence of a continuous wavelet basis is then guaranteed), namely it must possess a certain number of zeros at the aliasing frequency π (for the case of sampling by 2). In multiple dimensions one would like to follow the same approach, *i.e.* to impose a zero of an order m at multidimensional aliasing frequencies $2\pi(\mathbf{D}^t)^{-1}\mathbf{n}$. For example, in the quincunx case (non-separable sampling by two in two dimensions), the following partial derivatives have to equal zero

$$\frac{\partial^{k-1} H_\omega(\omega_1, \omega_2)}{\partial^l \omega_1 \partial^{k-l-1} \omega_2} \Big|_{(\pi,\pi)} = 0, \qquad (2)$$

for $k = 1, \ldots, m-1$, $l = 0, \ldots, k-1$. The difficult task is precisely how to achieve the above requirement and at the same time have a perfect reconstruction system. In what follows we investigate two approaches; (a) structurally impose the perfect reconstruction property by the use of cascade structures and then try to either algebraically or numerically impose a zero of a sufficiently high order at aliasing frequencies and (b) use transformations from one-dimensional into multidimensional filters that would preserve both perfect reconstruction and any number of zeros that the starting filters might possess. We also discuss methods for checking regularity of iterated filters.

2. DIRECT DESIGN

From the filter bank point of view, one of the most obvious design approaches is to use cascade structures [4], since in that case one can automatically guarantee perfect reconstruction and some other properties such as orthogonality and linear phase. Unfortunately, imposing a zero of a certain order becomes a non-trivial problem in multiple dimensions, and thus algebraic solutions can be obtained just for very small size filters.

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2.1. Two-Dimensional Regular Filters

8-Tap Filter: In [4], the authors examined the quincunx case with the sampling matrix $\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (note $\mathbf{D}^2 = 2\mathbf{I}$). An orthogonal filter was constructed, based on the following cascade

$$\mathbf{H}_p(z_1, z_2) = \mathbf{R}_0 \cdot \mathbf{D}_1\mathbf{R}_1\mathbf{D}_2\mathbf{R}_2, \quad (3)$$

where $\mathbf{H}_p$ denotes the polyphase matrix and

$$\mathbf{D}_i = \begin{pmatrix} 1 & \\ & z_i^{-1} \end{pmatrix}, \quad \mathbf{R}_i = \begin{pmatrix} 1 & a_i \\ -a_i & 1 \end{pmatrix}. \quad (4)$$

The filters obtained have 8 taps arranged in three rows $(2, 4, 2)$. After imposing a second-order zero at (π, π) on the lowpass filter, the following two solutions were found

$$a_0 = \mp\sqrt{3} \quad a_1 = \mp\sqrt{3} \quad a_2 = 2 \pm \sqrt{3}, \quad (5)$$
$$a_0 = \pm\sqrt{3} \quad a_1 = 0 \quad a_2 = 2 \pm \sqrt{3}. \quad (6)$$

The first solution as given in (5) led to an orthogonal filter conjectured to be regular, while the one given in (6), interestingly enough, is the same as the famous Daubechies' D4 filter (within scaling) [1].

24-Tap Filter: For larger size filters, obtaining algebraic solutions becomes a very demanding task. However, numerical approaches are possible. Thus, we extend the cascade in (3) as follows

$$\mathbf{H}_p(z_1, z_2) = \mathbf{R}_0 \cdot \mathbf{D}_1\mathbf{R}_1\mathbf{D}_2\mathbf{R}_2 \cdot \mathbf{D}_1\mathbf{R}_3\mathbf{D}_2\mathbf{R}_4 \cdot \mathbf{D}_1\mathbf{R}_5. \quad (7)$$

The filters obtained have 24 taps arranged in 6 rows $(2, 4, 6, 6, 4, 2)$. After imposing a third-order zero at (π, π) on the lowpass filter from (7), two numerical solutions are obtained and are given in Table 1. Comparing this filter to the one obtained in [4] from (5), one can observe that its higher order zero reflects in a better rate of convergence of the largest first-order difference (see Table 2). Figure 2 gives the eighth iteration of the lowpass filter.

a_i	Solution 1	Solution 2
a_0	0.18086073	−0.14101995
a_1	−0.07356250	0.25065223
a_2	−0.35310838	−0.27860678
a_3	−0.16178988	−0.23216639
a_4	0.19127283	−2.80190711
a_5	1.52618074	−0.90189581

Table 1: Two solutions yielding a lowpass filter with a third-order zero at (π, π) using the cascade (7).

Iteration number	Largest first order difference	Rate of convergence
2	1.00396460	
4	0.61660280	1.62822
6	0.35251753	1.74914
8	0.21656604	1.62776
10	0.12829728	1.68800

Table 2: The successive largest first-order differences for the filter obtained using the second solution given in Table 1, computed on the rectangular grid.

Note here that we also obtained an algebraic solution leading to a one-dimensional filter. This solution requires

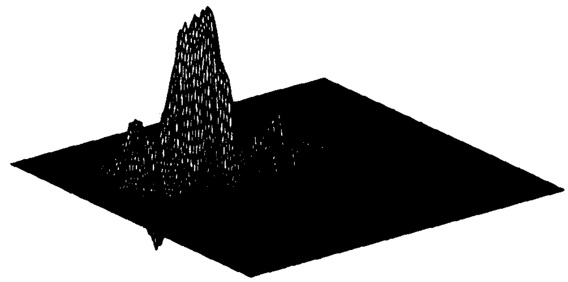

Figure 2: Eighth iteration of the filter obtained using the second solution given in Table 1.

$a_0 = a_2 = a_4 = 0$ and

$$a_1 = \frac{72 + 9\sqrt{40} \mp \sqrt{3}(11 + \sqrt{40})\sqrt{5 + \sqrt{160}}}{27},$$
$$a_3 = \pm\frac{(-7 + \sqrt{40})\sqrt{5 + \sqrt{160}}}{3^{\frac{3}{2}}},$$
$$a_5 = \pm\frac{\sqrt{5 + \sqrt{160}}}{\sqrt{3}}.$$

Similarly to the previous case, this choice of coefficients leads to the one-dimensional regular filter of size 6, that is Daubechies' D6 filter [1] (within scaling).

2.2. Three-Dimensional Regular Filters

In three dimensions, one can follow the same approach. Thus, for the FCO sampling case (non-separable sampling by two in three dimensions) and matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

with $\mathbf{D}^3 = 2\mathbf{I}$, the following cascade is used

$$\mathbf{H}_p(z_1, z_2, z_3) = \mathbf{R}_0 \cdot \mathbf{D}_1\mathbf{R}_1\mathbf{D}_2\mathbf{R}_2\mathbf{D}_3\mathbf{R}_3, \quad (8)$$

with the same notation as before. Imposing a second-order zero at (π, π, π) leads to one of the possible solutions as follows

$$a_0 = -a_1 = -2 - \sqrt{3}, \quad a_2 = -2 + \sqrt{3}, \quad a_3 = \sqrt{3}.$$

An interesting observation at this point is that these values are the same as those in (5,6), suggesting a possible transformation of a one-dimensional into a two-dimensional into a three-dimensional regular filter. However, the authors have not yet been able to establish the relationship.

Setting $a_2 = -a_1$, a different set of solutions is obtained,

$$a_1 = \pm\sqrt{7 \pm 4\sqrt{3}},$$

where four combinations are possible, and

$$a_0 = \frac{-4 + 13a_1 - a_1^3}{2}, \quad a_3 = a_1(3 - 2\sqrt{3}).$$

Another useful cascade is given in [4], which allows construction of an n-dimensional linear phase solution from the $(n-1)$-dimensional one, for the two-channel non-separable case. A useful feature of this cascade is, that the smallest size filters (the first block in the cascade) are a general solution (which is usually not the case with multidimensional solutions due to the fact that factorizations theorems are lacking). Based on this cascade highly regular synthesis filters can be constructed as has been already observed in [3, 4] for two-dimensional diamond shaped filters. In [5], a three-dimensional perfect reconstruction linear phase filter pair is constructed using the above cascade and is used for processing of digital video. In three dimensions, highly regular filters are obtained by convolving the following filter

$$H(z_1, z_2, z_3) = 6 + z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_3 + z_3^{-1}, \quad (9)$$

a number of times with itself.

3. ONE TO MULTIDIMENSIONAL TRANSFORMATIONS

Another way to approach the design problem is to use transformations of one-dimensional filters into multidimensional ones in such a way that [4]

1. perfect reconstruction is preserved (in order to have a valid subband coding system) and
2. zeros at aliasing frequencies are preserved (necessary but not sufficient for regularity).

We will discuss two approaches. The first method involves using filters with separable polyphase components (see, for example, [6]). The second one is to use the McClellan transformation.

3.1. Separable Polyphase Components

In this approach, the polyphase components of a multidimensional filter are obtained from the polyphase components of a prototype one-dimensional filter. We will concentrate on the two-dimensional case, but the same analysis can be carried out in more than two dimensions. Thus each polyphase component can be expressed as

$$H_i(z_1, z_2) = H_i(z_1) H_i(z_2).$$

The first advantage of this method is that the implementation is very cheap (due to separable polyphase components). Then the zeros at aliasing frequencies carry over. To show this let us first express the two-dimensional filter using the one-dimensional one as follows

$$H(z_1, z_2) = \frac{1}{2} \sum_{i=0}^{1} H((-1)^i \sqrt{z_1 z_2})\, H((-1)^i \sqrt{\frac{z_1}{z_2}}).$$

Then it is easy to see that if the one-dimensional filter has a zero of order N at π, *i.e.* if it can be written as

$$H(z) = (1 + z^{-1})^N\, P(z),$$

then the two-dimensional one can be expressed in the following fashion

$$H(z_1, z_2) = \frac{1}{2} \sum_{i=0}^{1} A_i(z_1, z_2)^N\, B_i(z_1, z_2), \quad (10)$$

with

$$A_i(z_1, z_2) = 1 \pm \sqrt{z_1 z_2} \pm \sqrt{\frac{z_1}{z_2}} + z_1,$$

$$B_i(z_1, z_2) = P(\pm\sqrt{z_1 z_2})\, P(\pm\sqrt{\frac{z_1}{z_2}}).$$

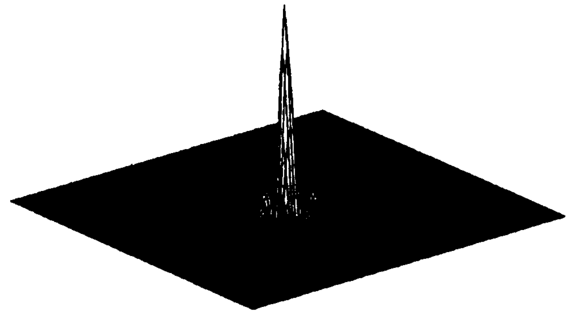

Figure 3: Fourth iteration of the filter conjectured to be regular. It is obtained using the McClellan transformation of the filter given in (11).

Now observe that $A_i(-1,-1) = 0$ (where $(z_1, z_2) = (-1,-1)$ is (z_1, z_2) at the aliasing frequency (π, π)). Then using (2) it is obvious that all the partial derivatives of order N and less are going to be zero, since upon differentiating (10), every term will possess $A_i(z_1, z_2)^k$ where k is at least 1 and at most N. This in turn means that a one-dimensional zero of order N will produce a two-dimensional zero of the same order.

The problem with this approach, however, is that the perfect reconstruction property is preserved only for filters with allpass polyphase components. To prove this statement one just needs to prove that if the filter bank is orthogonal then the polyphase components obtained in this fashion have to be allpass. But the previous statement is equivalent to the polyphase components satisfying the so-called power complementary (PC) property (see, for example, [4]). Thus, we assume that in one dimension, the polyphase components satisfy the PC property. Writing the PC property in two dimensions and using the assumption, one obtains that for it to hold the polyphase components have to be allpass (leading to trivial two-tap or IIR filters).

Iteration number	Largest first order difference	Rate of convergence
2	0.95161612	
4	0.53629625	1.77442
6	0.24966172	2.14809
8	0.10269017	2.43121

Table 3: The successive largest first-order differences for the filter obtained using the McClellan transformation of the length-19 one-dimensional filter (11).

3.2. McClellan Transformation

This transformation is well suited for the design of multidimensional filters since it leads to very efficient implementation. It transforms one-dimensional zero-phase filters into multidimensional zero-phase filters. Recently, the McClellan transformation has been recognized as a way to build multidimensional [7] as well as regular filter banks [3, 4]. To be more specific, if a filter is linear-phase, then it can be written as

$$H(z_1, \ldots, z_n) = d \cdot H_s(\frac{z_1 + z_1^{-1}}{2}, \ldots, \frac{z_n + z_n^{-1}}{2}),$$

where d denotes a pure delay. The trick is to substitute the one-dimensional kernel $K(z) = (z + z^{-1})/2$ by a multidimensional kernel $K(z_1, \ldots, z_n)$. As long as this latter kernel is zero-phase, the filter will be linear phase [7].

Let us first prove that perfect reconstruction is preserved. As before we will concentrate on the two-dimensional, quincunx, case. It is known that in one dimension the following is the form of a perfect reconstruction linear phase pair

$$H_0(z) = z^{-k} H_{s_0}(K(z)) \qquad H_1 = z^{-l} H_{s_1}(K(z)),$$

where k, l cannot be both odd or both even at the same time. Suppose that k is even and l odd. Then the polyphase components of the filters can be expressed as follows

$$H_{00}(z^2) = z^{-k} H_{s_{00}}(K(z)),\ H_{01}(z^2) = z^{-k+1} H_{s_{01}}(K(z)),$$

$$H_{10}(z^2) = z^{-l} H_{s_{10}}(K(z)),\ H_{11}(z^2) = z^{-l+1} H_{s_{11}}(K(z)).$$

Thus the determinant of the polyphase matrix is

$$\det \mathbf{H}_p(z^2) = z^{-(k+l-1)}(P(K(z)) - Q(K(z))),$$

where $P(K(z)) = H_{s_{00}}(K(z))H_{s_{11}}(K(z))$, $Q(K(z)) = H_{s_{01}}(K(z))H_{s_{10}}(K(z))$. Since perfect reconstruction holds then $P(K(z)) = K_1 + R(K(z))$, $Q(K(z)) = K_2 + R(K(z))$. Defining $K(z_1, z_2) = (z_1 + z_1^{-1} + z_2 + z_2^{-1})/4$ and the two-dimensional filters by their polyphase components as $H_{00}(z_1 z_2, z_1/z_2) = z_1^{-k} H_{s_{00}}(K(z_1, z_2))$ (and the other ones similarly), one can see that the determinant of the polyphase matrix reduces to $(K_1 + K_2) z_1^{-(k+l-1)}$, that is, perfect reconstruction is preserved.

To show that zeros carry over, simply express $(1 + z^{-1})^2$ as $2(1 + K(z))$. Then it becomes obvious that a zero of order $2N$ at π in one dimension, will map into a zero of order $2N$ at (π, π) in two dimensions.

As an example, we design a two-dimensional regular filter bank starting with the one-dimensional one from [2]

$$H(z) = (1 + z^{-1})^{10}\ z^{-4}(0.474823 - 0.654174(z + z^{-1}) + 0.364721(z+z^{-1})^2 - 0.095712(z+z^{-1})^3 + 0.01(z+z^{-1})^4), \tag{11}$$

where the filter is given in the form convenient for further transformation. After applying the McClellan transformation a two-dimensional filter is obtained. Its fourth iteration is given in Figure 3 and the first-order differences in Table 3.

4. REGULARITY TESTING

Regularity of one-dimensional filters is by now well understood, and several procedures are available to find the degree of regularity of a given filter and its associated wavelet [1, 8]. Except in cases where the wavelet can be expressed in closed form (*e.g.* Meyer and Battle-Lemarié wavelets), it is often necessary to evaluate the iterated filter given by (1). In one dimension, (1) translates into

$$H^{(i)}(z) = H(z) \cdot H(z^2) \cdot \ldots \cdot H(z^{2^{i-1}}). \tag{12}$$

The behavior of this filter as i becomes large is a good indication of the regularity of the wavelet (for a precise discussion of this relation, see [8]).

In multiple dimensions, regularity testing is substantially more difficult [3], and evaluation of the iterated filter might be the only approach available. We indicate a simple algorithm that speeds up the computation of $H^{(i)}(z)$. It is first derived in one dimension, and then the extension to multiple dimensions is discussed. There are many different ways to compute (12), one of the efficient ones being

$$H^{(i)}(z) = H(z) \cdot H^{(i-1)}(z^2). \tag{13}$$

A direct computation of this polynomial product would use $L \cdot 2L^{(i-1)}$ operations, where L is the filter length and $L^{(i-1)} = (2^{i-1} - 1)(L-1) + 1$ is the length of $H^{(i-1)}(z)$. Expanding $H(z)$ into its polyphase components ($H(z) = H_0(z^2) + z^{-1}H_1(z^2)$) allows one to compute two products in z^2 with a complexity of $2 \cdot (L/2 \cdot L^{(i-1)})$, achieving a reduction by 2. Because the complexity of computing $H^{(i-1)}(z)$ is comparable to this last step, the evaluation of $H^{(i)}(z)$ takes $O[2^i L^2]$ operations, while the non-optimized approaches can take up to $O[2^{2i} L^2]$ operations. If FFT's are used to compute individual convolutions, complexity can be further improved, but it is advantageous to use the following recursion

$$H^{(i)}(z) = H^{(i-1)}(z) \cdot H(z^{2^{i-1}}). \tag{14}$$

Then, it can be shown that the complexity of (14) (using both sparsity of products and FFT's), is $O[2^{i+1} L \log L]$.

In multiple dimensions, the relations (12)-(14) hold, with z^2 replaced by $\mathbf{z}^{\mathbf{D}}$ where $\mathbf{D}$ is the subsampling matrix and the matrix exponential is appropriately defined [4]. In the particular case considered in the previous sections, where $\mathbf{D}^2 = 2\mathbf{I}$, it is easier to consider the filter $H^{(2)}(z_1, z_2) = H(z_1, z_2)H(z_1 z_2, z_1/z_2)$ and iterate it with respect to $\mathbf{D}^2$ since this leads to a separable implementation of the suggested algorithms. In the above case, one is actually computing the even iterates $H^{(2i)}(z_1, z_2)$.

5. CONCLUSION

New results on the design of multidimensional irreducible wavelet bases were presented. Two different approaches were investigated, the first one using cascade structures for building perfect reconstruction filter banks, and the second one designing multidimensional filters from one-dimensional ones. Regularity testing was discussed and a fast algorithm for computing iterated filters was given.

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