Design of multidirectional multiscroll chaotic attractors based on fractional differential systems via switching control

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This paper proposes a saturated function series approach for generating multiscroll chaotic attractors from the fractional differential systems, including one-directional (1-D) n-scroll, two-directional (2-D) $n \times m$-grid scroll, and three-directional (3-D) $n \times m \times l$-grid scroll chaotic attractors. Our theoretical analysis shows that all scrolls are located around the equilibria corresponding to the saturated plateaus of the saturated function series on a line in the 1-D case, a plane in the 2-D case, and a three-dimensional space in the 3-D case, respectively. In particular, each saturated plateau corresponds to a unique equilibrium and its unique scroll of the whole attractor. In addition, the number of scrolls is equal to the number of saturated plateaus in the saturated function series. Finally, some underlying dynamical mechanisms are then further investigated for the fractional differential multiscroll systems. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2401061]

In 1695, Leibniz wrote a letter to L'Hôspital asking whether or not the meaning of derivatives with integer orders could be naturally generalized to the derivatives with noninteger orders. L'Hôspital felt somewhat curious about this question and then asked a simple question as a reply: "What if the order will be 1/2?" In a re-reply letter on September 30 of the same year, Leibniz anticipated: "It will lead to a paradox, from which one day useful consequences will be drawn." This special date, September 30, 1695, is then regarded as the exact birthday of fractional calculus. Over the past few centuries, the theories of fractional calculus (fractional derivatives and fractional integrals) had attained the significant development, primarily contributed to the pure, not applied, mathematicians. Until recently, some applied scientists and engineers have realized that such fractional differential equations indeed provide a natural framework for various kinds of real-world questions, such as viscoelastic systems and electrode-electrolyte polarization.[1–12] This paper extends the saturated function series approach from the classical differential equations to fractional differential equations for creating various multiscroll attractors, including 1-D n-grid, 2-D $n \times m$-grid scroll, and 3-D $n \times m \times l$-grid scroll attractors. These scrolls are located around the equilibria corresponding to the saturated plateaus of the saturated function series on a line in 1-D case, a plane in the 2-D case, and a three-dimensional space in the 3-D case, respectively. In particular, each saturated plateau corresponds to a unique equilibrium and its unique scroll of the whole attractor. Moreover, the number of scrolls is also equal to the number of saturated plateaus in the saturated function series.

I. INTRODUCTION

The fractional differential systems have received increased attention from various research fields over the past few decades.[1–15] On the one hand, more and more fractional differential systems recently have been applied to the interdisciplinary fields, including model acoustics and thermal systems, rheology and modeling of materials and mechanical systems, signal processing and systems identification, control and robotics, etc.[1] On the other hand, many real-world systems modeled by fractional calculus also display rich fractional dynamical behaviors, such as viscoelastic systems,[2] colored noise,[3] boundary layer effects in ducts,[4] electromagnetic waves,[5] fractional kinetics,[6–8] and electrode-electrolyte polarization.[9]

Historically, Chua found the Chua's double-scroll circuit.[16,17] Later, Suykens and Vandewalle proposed a family of n-double scroll chaotic attractors.[18] Yalcin and coworkers introduced a family of chaotic attractors using step functions, including one-directional (1-D) n-scroll, two-directional (2-D) $n \times m$-grid scroll, and three-directional (3-D) $n \times m \times l$-grid scroll chaotic attractors.[19,20] Lü and coworkers presented a switching manifold technique for creating chaotic attractors with multiple-merged basins of attraction.[21] They also proposed the hysteresis series[22] and saturated series[21,23] methods for generating 1-D n-scroll, 2-D $n \times m$-grid scroll, and 3-D $n \times m \times l$-grid scroll chaotic attractors with rigorously mathematical proofs and experimen-

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tal verifications.[23] Very recently, Lü and Chen reviewed the main advances in theories, approaches, and applications of multiscroll chaos generation over the last two decades.[24] Up to now, designing and realizing multiscroll chaotic attractors is no longer a difficult task in the classical differential systems.

Recently, many researchers surprisingly found that many nonlinear fractional differential systems also display complex bifurcation and chaos phenomena.[14,15,25–30] For example, the fractional Chua's circuit also has a double scroll chaotic attractor.[15] Moreover, Ahmad introduced a step function method for creating n-scroll chaotic attractors from fractional order systems.[25] However, the design of multidirectional multiscroll chaotic attractors is also a very challenging question. Therefore, it is very interesting to ask whether the fractional differential systems can also generate multidirectional multiscroll chaotic attractors as the classical differential systems. This paper will give a positive answer to this question.

This paper proposes a systematic design approach for creating multidirectional multiscroll chaotic attractors from a fractional differential system via saturated functions switching, including 1-D n-scroll, 2-D $n\times m$-grid scroll, and 3-D $n\times m\times l$-grid scroll chaotic attractors. Furthermore, we also investigate the dynamical mechanics of the fractional differential multiscroll systems. It should be especially pointed out that the fractional differential multiscroll systems are quite different from the classical differential multiscroll systems because the fractional derivative is a nonlocal operator. Also, we demonstrate that each saturated plateau of the controller corresponds to a unique equilibrium and also its unique scroll of the whole attractor. Therefore, the number of scrolls is equal to the number of saturated plateaus of the saturated function series controller.

The rest of this paper is organized as follows: In Sec. II, some preliminaries are introduced for the fractional differential systems. Then the saturated function series approach is proposed for generating multidirectional multiscroll chaotic attractors in Sec. III. The conclusions are finally given in Sec. IV.

II. FRACTIONAL DIFFERENTIAL SYSTEMS

This section briefly introduces some background knowledge for fractional differential systems.

As we know now, there are several different definitions for the fractional differential operator. Hereafter, the fractional differential operator is described by

$$D_*^{\alpha}y(x) = J^{m-\alpha}y^{(m)}(x), \quad \alpha > 0,$$

where $m=\lceil \alpha \rceil$, i.e., m is the first integer which is not less than α, $y^{(m)}$ is the general m-order derivative, and J^{β} is the β-order Riemann-Liouville integral operator which is given by

$$J^{\beta}z(x) = \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}z(t)d(t), \quad \beta > 0.$$

In general, the operator D_*^{α} is called "α-order Caputo differential operator" which is widely used in the engineering field. As a matter of fact, the fractional order derivative was first introduced earlier in the nineteenth century.[10]

A. Stability of the equilibria of fractional linear systems

This subsection presents several basic definitions and a Lemma for the stability of the equilibria of fractional linear autonomous systems.

It is well known that $D_*^{\alpha}c=0$ for any constant c and $\alpha \in \mathbb{R}^+$. In the following, one introduces several basic definitions for the fractional differential systems.

Definition 1: The roots of the equation $f(X)=0$ are called the equilibria of the fractional differential system $D_*^{\alpha}X = f(X)$, where $X=(x_1,x_2,\ldots,x_n)^T \in \mathbb{R}^n$, $f(X)\in R^n$ and $D_*^{\alpha}X = (D_*^{\alpha_1}x_1, D_*^{\alpha_2}x_2,\ldots,D_*^{\alpha_n}x_n)^T$, $\alpha_i \in \mathbb{R}^+$, $i=1,2,\ldots,n$.

Lemma 1: The equilibrium $X_0=-\mathbf{A}^{-1}\mathbf{b}$ of the fractional linear autonomous system $D_*^{\alpha}X=\mathbf{A}X+\mathbf{b}$ is asymptotically stable if and only if $|\arg(\lambda_i)|>\alpha_0\pi/2$ for $i=1,2,\ldots,n$, where λ_i are the eigenvalues of matrix $\mathbf{A}$, $X=(x_1,x_2,\ldots,x_n)^T\in\mathbb{R}^n$, $\mathbf{A}\in\mathbb{R}^n\times\mathbb{R}^n$, $|\mathbf{A}|\neq 0, \mathbf{b}\in R^n$, and $D_*^{\alpha}X=(D_*^{\alpha_1}x_1,D_*^{\alpha_2}x_2,\ldots,D_*^{\alpha_n}x_n)^T$, $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha_0\in\mathbb{R}^+$, $i=1,2,\ldots,n$.[26,27]

Definition 2: The equilibrium $X_0(\in\Omega)$ of $D_*^{\alpha}X=\mathbf{A}X+\mathbf{b}$ is called a saddle for $n_-n_+\neq 0$ and an antisaddle for $n_-n_+=0$, where n_- and n_+ are the number of eigenvalues λ_i, $i=1,2,\ldots,n$ of matrix $\mathbf{A}$ satisfying $|\arg(\lambda_i)|>\alpha_0\pi/2$ and $|\arg(\lambda_i)|<\alpha_0\pi/2$, respectively. Here $X=(x_1,x_2,\ldots,x_n)^T\in\Omega\subset\mathbb{R}^n$, Ω is a bounded open set, $\mathbf{A}\in\mathbb{R}^n\times\mathbb{R}^n$, $|\mathbf{A}|\neq 0$, $\mathbf{b}\in\mathbb{R}^n$, and $D_*^{\alpha}X=(D_*^{\alpha_1}x_1,D_*^{\alpha_2}x_2,\ldots,D_*^{\alpha_n}x_n)^T$, $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha_0\in R^+$, $i=1,2,\ldots,n$.

Definition 3: In Definition 2, for $n=3$, if one of the eigenvalues $\lambda_1<0$ and the other two eigenvalues $|\arg(\lambda_2)|=|\arg(\lambda_3)|<\alpha_0\pi/2$, then the equilibrium $X_0\in\Omega$ is called a saddle point of index 2; if one of the eigenvalues $\lambda_1>0$ and the other two eigenvalues $|\arg(\lambda_2)|=|\arg(\lambda_3)|>\alpha_0\pi/2$, then the equilibrium $X_0\in\Omega$ is called a saddle point of index 1.

B. Saturated function series

The simplest saturated function is described by

$$f_0(x;k)=\begin{cases} k, & \text{if } x>1 \\ kx, & \text{if } |x|\leq 1 \\ -k, & \text{if } x<-1 \end{cases} \quad (1)$$

where $k>0$ is the slope of the middle segment. The upper radial $\{f_0(x;k)=k|x\geq 1\}$ and lower radial $\{f_0(x;k)=-k|x\leq -1\}$ are called saturated plateaus, and the segment $\{f_0(x;k)=kx||x|\leq 1\}$ between the two saturated plateaus is called the saturated slope.

Definition 4: The piecewise linear (PWL) function

$$f(x;k,h,p,q)=\sum_{i=-p}^{q} f_i(x;k,h) \quad (2)$$

is called a saturated function series,[21,23] where $k>0$ and $h>2$ are the slope and saturated delay time of the saturated function series (2), respectively, p and q are positive integers, and

$$f_i(x;k,h)=\begin{cases}2k, & \text{if } x>ih+1\\ k(x-ih)+k, & \text{if } |x-ih|\leqslant 1\\ 0, & \text{if } x<ih-1\end{cases}$$

and

$$f_{-i}(x;k,h)=\begin{cases}0, & \text{if } x>-ih+1\\ k(x+ih)-k, & \text{if } |x+ih|\leqslant 1\\ -2k, & \text{if } x<-ih-1.\end{cases}$$

Moreover, the saturated function series $f(x;k,h,p,q)$ can be rewritten as follows:

$$f(x;k,h,p,q)=\begin{cases}(2q+1)k, & \text{if } x>qh+1\\ k(x-ih)+2ik, & \text{if } |x-ih|\leqslant 1,\ -p\leqslant i\leqslant q\\ (2i+1)k, & \text{if } ih+1<x<(i+1)h-1\\ & -p\leqslant i\leqslant q-1\\ -(2p+1)k, & \text{if } x<-ph-1.\end{cases} \tag{3}$$

C. The basic fractional linear autonomous system

This subsection will introduce a basic fractional linear autonomous system and further discuss its potential ability to generate the multidirectional multiscroll chaotic attractors via a suitable controller.

The fundamental fractional linear autonomous system is given by

$$\begin{pmatrix}D_*^{\alpha_1}x\\ D_*^{\alpha_2}y\\ D_*^{\alpha_3}z\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\-a&-b&-c\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\equiv\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix} \tag{4}$$

where x,y,z are state variables, a,b,c are positive real constants, and $\alpha_i\in(0,1)$, $i=1,2,3$.

When $\alpha_i\in(0,1)$ for $i=1,2,3$ are rational numbers, system (4) can be transformed into its equivalent system with the same fractional orders.[26] Therefore, we only need to discuss the fractional differential system

$$D_*^{\alpha}X=\mathbf{A}X, \tag{5}$$

where $X=(x,y,z)^T$, $D_*^{\alpha}X=(D_*^{\alpha_1}x,D_*^{\alpha_2}y,D_*^{\alpha_3}z)^T$, and $\alpha_1=\alpha_2=\alpha_3=\alpha_0\in(0,1)$.

The characteristic equation of the coefficient matrix $\mathbf{A}$ of system (5) is described by

$$\lambda^3+c\lambda^2+b\lambda+a=0. \tag{6}$$

Note that (6) is not the characteristic equation of (5) because the characteristic equation of the fractional differential system has a different meaning.[26]

According to Lemma 1, the stability of the equilibrium (0, 0, 0) of the fractional differential system (5) is completely determined by the eigenvalues of (6). Denote $\hat{q}=(2/27)c^3-(1/3)bc+a$, and $\Delta=(ac^3/27)-(b^2c^2/108)-(abc/6)+(b^3/27)+(a^2/4)$. From (6), one has

$$\lambda_1=-\frac{c}{3}+\sqrt[3]{-\frac{\hat{q}}{2}+\sqrt{\Delta}}+\sqrt[3]{-\frac{\hat{q}}{2}-\sqrt{\Delta}} \tag{7}$$

and

$$\lambda_{2,3}=-\frac{c}{3}-\frac{1}{2}\left(\sqrt[3]{-\frac{\hat{q}}{2}+\sqrt{\Delta}}+\sqrt[3]{-\frac{\hat{q}}{2}-\sqrt{\Delta}}\right)\pm\frac{\sqrt{3}}{2}i\left(\sqrt[3]{-\frac{\hat{q}}{2}+\sqrt{\Delta}}-\sqrt[3]{-\frac{\hat{q}}{2}-\sqrt{\Delta}}\right)=\beta\pm\gamma i. \tag{8}$$

Our theoretical and numerical analysis show that system (5) with a saturated function series switching controller (3) may generate chaotic behavior under the condition of $\lambda_1<0$, $\beta>0$, $\gamma\neq 0$ and $|\arctan(\gamma/\beta)|<\alpha_0\pi/2$. Here (0, 0, 0) is a saddle point of index 2 of (5).

Hereafter, one always supposes that

$$\begin{aligned}&\Delta=\frac{ac^3}{27}-\frac{b^2c^2}{108}-\frac{abc}{6}+\frac{b^3}{27}+\frac{a^2}{4}>0,\\ &\lambda_1=-\frac{c}{3}+\sqrt[3]{-\frac{\hat{q}}{2}+\sqrt{\Delta}}+\sqrt[3]{-\frac{\hat{q}}{2}-\sqrt{\Delta}}<0,\\ &\beta=-\frac{c}{3}-\frac{1}{2}\left(\sqrt[3]{-\frac{\hat{q}}{2}+\sqrt{\Delta}}+\sqrt[3]{-\frac{\hat{q}}{2}-\sqrt{\Delta}}\right)>0,\\ &|\arctan(\gamma/\beta)|<\alpha_0\pi/2.\end{aligned} \tag{9}$$

D. The numerical computational schemes

In the following, the predictor-corrector scheme is used to numerically solve the fractional differential equation. Notice that this scheme is a natural generalization of the known Adams-Bashforth-Moulton scheme.[28]

The fractional differential equation is given by

$$\begin{cases}D_*^{\alpha_1}x(t)=l_1(x,y,z)\\ D_*^{\alpha_2}y(t)=l_2(x,y,z)\\ D_*^{\alpha_3}z(t)=l_3(x,y,z),\end{cases}$$

where the initial values $x(0)=x_0$, $y(0)=y_0$, $z(0)=z_0$, $\alpha_i\in(0,1)$, $i=1$, 2, 3, and $0\leqslant t\leqslant T$. Thus it is equivalent to the Volterra integral equation

$$\begin{cases}x(t)=x(0)+\dfrac{1}{\Gamma(\alpha_1)}\displaystyle\int_0^t(t-s)^{\alpha_1-1}l_1(x(s),y(s),z(s))ds\\ y(t)=y(0)+\dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_0^t(t-s)^{\alpha_2-1}l_2(x(s),y(s),z(s))ds\\ z(t)=z(0)+\dfrac{1}{\Gamma(\alpha_3)}\displaystyle\int_0^t(t-s)^{\alpha_3-1}l_3(x(s),y(s),z(s))ds.\end{cases}$$

Let $h=T/N$, $t_n=nh$, $n=0,1,\ldots,N\in Z^+$. Discretizing the Volterra integral equation as above yields

$$\begin{cases} x_h(t_{n+1}) = x(0) + \dfrac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} \times (x_h^p(t_{n+1}) + (2^{(\alpha_1+1)} - 2) \times l_1(x_h(t_n), y_h(t_n), z_h(t_n)) + \text{temp1}) \\ y_h(t_{n+1}) = y(0) + \dfrac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} \times (y_h^p(t_{n+1}) + (2^{(\alpha_2+1)} - 2) \times l_2(x_h(t_n), y_h(t_n), z_h(t_n)) + \text{temp2}) \\ z_h(t_{n+1}) = z(0) + \dfrac{h^{\alpha_3}}{\Gamma(\alpha_3+2)} \times (z_h^p(t_{n+1}) + (2^{(\alpha_3+1)} - 2) \times l_3(x_h(t_n), y_h(t_n), z_h(t_n)) + \text{temp3}), \end{cases}$$

where

$$\begin{cases} x_h^p(t_{n+1}) = x(0) + \dfrac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} \times ((2^{(\alpha_1+1)} - 1) \times l_1(x_h(t_n), y_h(t_n), z_h(t_n)) + \text{temp1}) \\ y_h^p(t_{n+1}) = y(0) + \dfrac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} \times ((2^{(\alpha_2+1)} - 1) \times l_2(x_h(t_n), y_h(t_n), z_h(t_n)) + \text{temp2}) \\ z_h^p(t_{n+1}) = z(0) + \dfrac{h^{\alpha_3}}{\Gamma(\alpha_3+2)} \times ((2^{(\alpha_3+1)} - 1) \times l_3(x_h(t_n), y_h(t_n), z_h(t_n)) + \text{temp3}), \end{cases}$$

$$\begin{cases} \text{temp1} = \dfrac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} \sum_{j=0}^{n-1} a_{1,j,n+1} l_1(x_h(t_j), y_h(t_j), z_h(t_j)) \\ \text{temp2} = \dfrac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} \sum_{j=0}^{n-1} a_{2,j,n+1} l_2(x_h(t_j), y_h(t_j), z_h(t_j)) \\ \text{temp3} = \dfrac{h^{\alpha_3}}{\Gamma(\alpha_3+2)} \sum_{j=0}^{n-1} a_{3,j,n+1} l_3(x_h(t_j), y_h(t_j), z_h(t_j)), \end{cases}$$

$$a_{i,j,n+1} = \begin{cases} n^{\alpha_i+1} - (n-\alpha_i)(n+1)^{\alpha_i}, & j=0 \\ (n-j+2)^{\alpha_i+1} + (n-j)^{\alpha_i+1} & \\ -2(n-j+1)^{\alpha_i+1}, & 1 \leqslant j \leqslant n-1 \end{cases}$$

where $i=1,2,3$.

Moreover, the error estimate is described by

$$\max\Big\{ \max_{j=0,1,\ldots,N} |x(t_j) - x_h(t_j)|, \max_{j=0,1,\ldots,N} |y(t_j) - y_h(t_j)|, \max_{j=0,1,\ldots,N} |z(t_j) - z_h(t_j)| \Big\} = O(h^q),$$

where $q=\min\{1+\alpha_1, 1+\alpha_2, 1+\alpha_3\}$.

III. DESIGN OF MULTIDIRECTIONAL MULTISCROLL CHAOTIC ATTRACTORS

A systematic switching control approach is proposed for creating the multidirectional multiscroll chaotic attractors from the fundamental fractional differential linear system (4) via saturated function series controller (3). It includes 1-D n-scroll, 2-D $n\times m$-grid scroll, and 3-D $n\times m\times l$-grid scroll chaotic attractors. The controlled fractional differential system is described by

$$D_*^{\alpha} X = \mathbf{A}X + \mathbf{B}U(X), \tag{10}$$

where

$$\mathbf{B} = \begin{pmatrix} 0 & -\dfrac{d_2}{b} & 0 \\ 0 & 0 & -\dfrac{d_3}{c} \\ d_1 & d_2 & d_3 \end{pmatrix},$$

$X=(x,y,z)^T$, $D_*^{\alpha}X=(D_*^{\alpha_1}x, D_*^{\alpha_2}y, D_*^{\alpha_3}z)^T$, and $U(X)$ is the saturated functions series switching controller.

A. 1-D *n*-scroll attractors

To generate n-scroll ($n\geqslant 3$) chaotic attractors from the controlled system (10), the saturated function series switching controller is designed as follows:

$$\mathbf{U}(\mathbf{X}) = \begin{pmatrix} f(x;k_1,h_1,p_1,q_1) \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

where $f(x;k_1,h_1,p_1,q_1)$ is defined by (3), and a, b, c, d_1 are positive constants.

Suppose that

$$d_1k_1 > a,\ 2d_1k_1 \geqslant ah_1,\ \max\{p_1,q_1\}\frac{|ah_1 - 2d_1k_1|}{d_1k_1 - a} \leqslant 1,$$
$$(2d_1k_1 - ah_1)(q_1 - 1) < ah_1 - d_1k_1 - a. \tag{12}$$

Obviously, all $(2(p_1+q_1)+3)$ equilibria of system (10) with (11) are located along the x axis, and can be classified into two different sets

$$A_x = \left\{ -\frac{(2p_1+1)d_1k_1}{a}, \frac{(-2p_1+1)d_1k_1}{a}, \cdots, \frac{(2q_1+1)d_1k_1}{a} \right\} \tag{13}$$

and

$$B_x = \left\{ -\frac{p_1d_1k_1(h_1-2)}{d_1k_1-a}, \frac{(-p_1+1)d_1k_1(h_1-2)}{d_1k_1-a}, \cdots, \frac{q_1d_1k_1(h_1-2)}{d_1k_1-a} \right\}. \tag{14}$$

Assume that (9) holds and $\alpha_1=\alpha_2=\alpha_3=\alpha_0$. Then the stability of the equilibria in A_x of system (10) with (11) is completely determined by the eigenvalues of (6) and these equilibria are saddle points of index 2. However, the stability of the equilibria in B_x of system (10) with (11) is completely determined by the eigenvalues of the following equation:

$$\lambda^3 + c\lambda^2 + b\lambda + a - d_1k_1 = 0. \tag{15}$$

Since $\lambda_1+\lambda_2+\lambda_3=-c<0$ and $\lambda_1\lambda_2\lambda_3=-(a-d_1k_1)>0$, (15) has one positive eigenvalue and two negative eigenvalues, or one positive eigenvalue and a pair of complex conjugate eigenvalues with negative real parts. Figures 1(a) and 1(b) show a 5-scroll chaotic attractor with fractional order (0.9, 0.9, 0.9) and a 6-scroll chaotic attractor with fractional order (0.85, 0.9, 0.9), respectively. The corresponding Lyapunov exponents spectrums are $LE_1=0.2328$, $LE_2=0$, $LE_3=-1.3306$ and $LE_1=0.2251$, $LE_2=0$, $LE_3=-1.2143$, respectively. The parameters are given by $a=2$, $b=1$, $c=0.6$, $d_1=2$, and $k_1=10$. Since $\lambda_1=-1.1836<0$, $\beta=0.2981$, $\gamma=1.2667$, $|\arg(\lambda_2)|=|\arg(\lambda_3)|=|\arctan(\gamma/\beta)|=1.3362<(0.9\pi)/2=1.4137$, then the equilibria in A_x are saddle points of index 2. Since $\lambda_1=2.3180>0$, $\beta=-1.4590$, $\gamma=0.6409$, $|\arg(\lambda_2)|=|\arg(\lambda_3)|=|\pi+\arctan(\gamma/\beta)|=2.7277>(0.9\pi)/2=1.4137$, then the equilibria in B_x are saddle points of index 1.

It should be pointed out that the (p_1+q_1+2) equilibria in A_x are responsible for generating the (p_1+q_1+2) scrolls of attractor and the (p_1+q_1+1) equilibria in B_x are responsible for connecting these (p_1+q_1+2) scrolls to form into a whole attractor. In particular, each equilibrium in A_x corresponds to a unique saturated plateau of the saturated function series controller (3) and also corresponds to a unique scroll of the whole attractor. However, each equilibrium in B_x corresponds to a unique saturated slope of the saturated function series controller (3) and also corresponds to a unique connection between two neighboring scrolls. Here parameters p_1, q_1 can control the numbers of scrolls in negative and positive x directions, respectively.

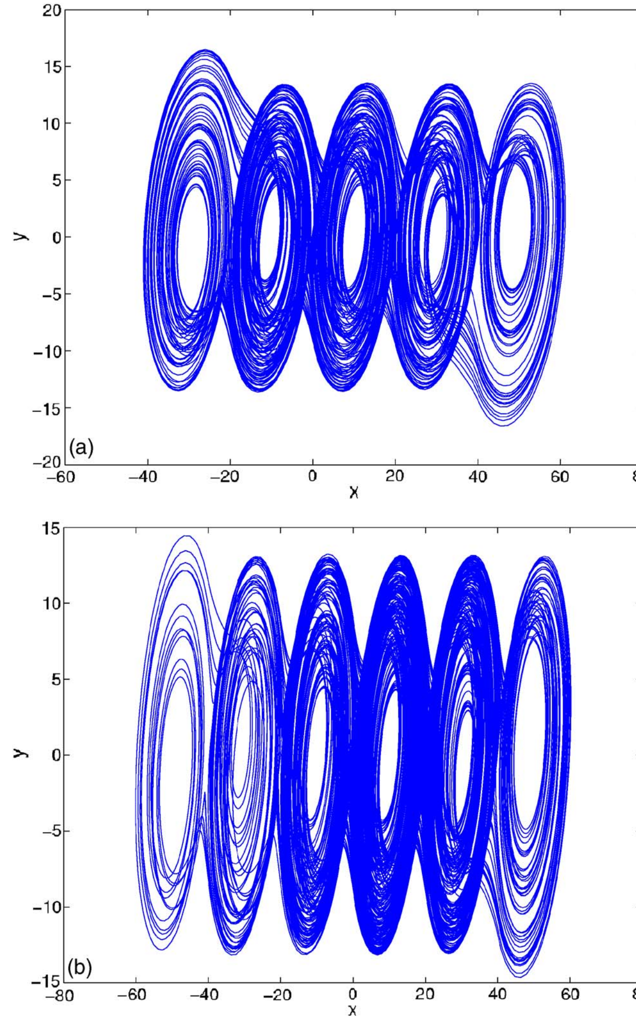

FIG. 1. Two 1-D n-scroll attractors. (a) 5-scroll with fractional order (0.9, 0.9, 0.9); and (b) 6-scroll with fractional order (0.85, 0.9, 0.9).

B. 2-D $n\times m$-grid scroll attractors

To create 2-D $n\times m$-grid scroll chaotic attractors from the controlled system (10), the saturated function series switching controller is then recasted as follows:

$$U(X) = \begin{pmatrix} f(x;k_1,h_1,p_1,q_1) \\ f(y;k_2,h_2,p_2,q_2) \\ 0 \end{pmatrix} \tag{16}$$

where $f(x;k_1,h_1,p_1,q_1)$ and $f(y;k_2,h_2,p_2,q_2)$ are defined by (3), and a, b, c, d_1, d_2 are positive constants.

In addition to (13) and (14), denote

$$A_y = \left\{ -\frac{(2p_2+1)d_2k_2}{b}, \frac{(-2p_2+1)d_2k_2}{b}, \cdots, \frac{(2q_2+1)d_2k_2}{b} \right\} \tag{17}$$

and

$$B_y = \left\{ -\frac{p_2 d_2 k_2 (h_2-2)}{d_2 k_2 - b}, \frac{(-p_2+1) d_2 k_2 (h_2-2)}{d_2 k_2 - b}, \cdots, \frac{q_2 d_2 k_2 (h_2-2)}{d_2 k_2 - b} \right\}. \quad (18)$$

Assume that (12) and

$$d_2k_2 > b,\ 2d_2k_2 > bh_2,\ \max\{p_2, q_2\}\frac{|bh_2 - 2d_2k_2|}{d_2k_2 - b} \leqslant 1,$$
$$(2d_2k_2 - bh_2)(q_2-1) < bh_2 - d_2k_2 - b \quad (19)$$

hold. Then system (10) with (16) has $(2p_1+2q_1+3)\times(2p_2+2q_2+3)$ equilibria, which are located on the x-y plane and given by

$$O_{xy} = \{(x^*, y^*) | x^* \in A_x \cup B_x,\ y^* \in A_y \cup B_y\}. \quad (20)$$

Clearly, all equilibria in (20) can be classified into four different sets:

$$A_1 = \{(x^*, y^*) | x^* \in A_x,\ y^* \in A_y\}$$

$$A_2 = \{(x^*, y^*) | x^* \in A_x,\ y^* \in B_y\}$$

$$A_3 = \{(x^*, y^*) | x^* \in B_x,\ y^* \in A_y\}$$

$$A_4 = \{(x^*, y^*) | x^* \in B_x,\ y^* \in B_y\}.$$

Suppose that (9) holds and $\alpha_1=\alpha_2=\alpha_3=\alpha_0$. Figure 2 shows a 2-D 5×5-grid scroll chaotic attractor with fractional order (0.9, 0.9, 0.9) and a 2-D 6×6-grid scroll chaotic attractor with fractional order (0.8, 0.9, 1.0), where the parameters are given by $a=2$, $b=1$, $c=0.5$, $d_1=2$, $d_2=1$, $k_1=50$, and $k_2=50$. The corresponding Lyapunov exponents spectrums are $LE_1=0.2655$, $LE_2=0$, $LE_3=-1.4531$ and $LE_1=0.2131$, $LE_2=0$, $LE_3=-1.5718$, respectively. Since $\lambda_1=-1.1480<0$, $\beta=0.3240$, $\gamma=1.2794$, $|\arg(\lambda_2)|=|\arg(\lambda_3)|=|\arctan(\gamma/\beta)|=1.3228<(0.9\pi)/2=1.4137$, then the equilibria in A_1 are saddle points of index 2. Similarly, the equilibria in A_3 are saddle points of index 1; the equilibria in A_4 are saddle points of index 2; and the equilibria in A_2 are not saddle points of index 1 or 2.

Our numerical simulations demonstrate that only the equilibria in A_1 can create scrolls. Therefore, the condition of saddle point of index 2 is only a necessary condition but not a sufficient condition for generating scrolls. System (10) with (16) has the potential ability to create a maximum of 2-D $(p_1+q_1+2)\times(p_2+q_2+2)$-grid scroll chaotic attractors for some suitable parameters. Moreover, each equilibrium in A_1 corresponds to a unique 2-D saturated plateau and also corresponds to a unique scroll in the whole attractor. However, the other equilibria in A_2, A_3, A_4 correspond to the saturated slopes and are responsible for connecting these $(p_1+q_1+2)\times(p_2+q_2+2)$ scrolls. Parameters p_1, q_1 control the numbers of scrolls in negative and positive x directions, respectively. Parameters p_2, q_2 control the numbers of scrolls in negative and positive y directions, respectively.

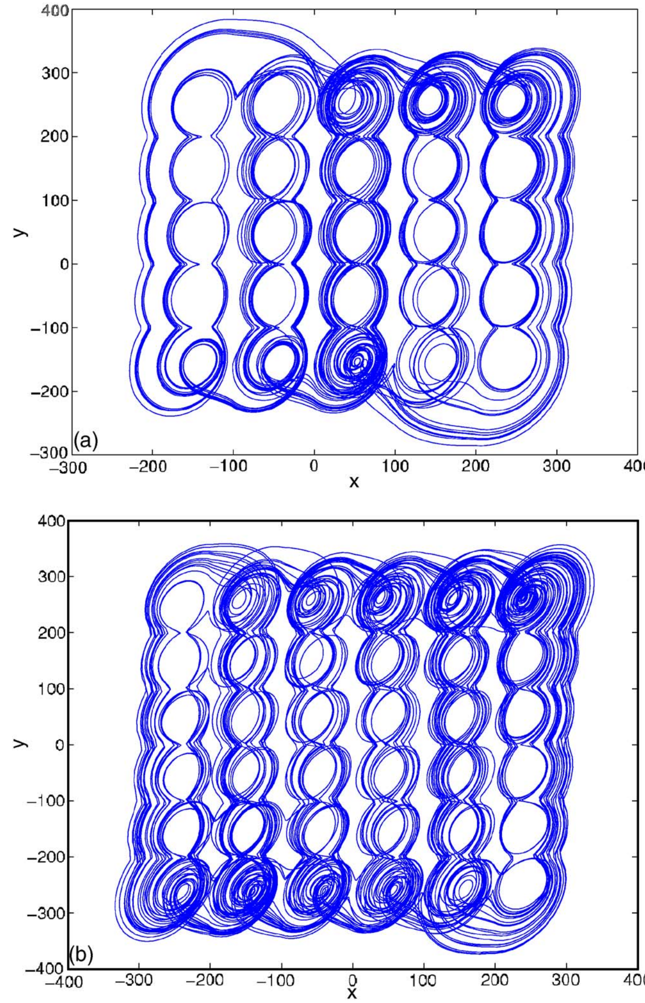

FIG. 2. Two 2-D $n\times m$-grid scroll attractors. (a) 5×5-grid scroll with fractional order (0.9, 0.9, 0.9); and (b) 6×6-grid scroll with fractional order (0.8, 0.9, 1.0).

C. 3-D $n\times m\times l$-grid scroll attractors

To generate 3-D $n\times m\times l$-grid scroll chaotic attractors from the controlled system (10), the saturated function series switching controller is then designed as follows:

$$U(X) = \begin{pmatrix} f(x; k_1, h_1, p_1, q_1) \\ f(y; k_2, h_2, p_2, q_2) \\ f(z; k_3, h_3, p_3, q_3) \end{pmatrix} \quad (21)$$

where $f(x;k_1,h_1,p_1,q_1)$, $f(y;k_2,h_2,p_2,q_2)$, $f(z;k_3,h_3,p_3,q_3)$ are defined by (3), and a, b, c, d_1, d_2, d_3 are positive constants.

In addition to (13), (13), (17), and (18), denote

$$A_z = \left\{ -\frac{(2p_3+1)d_3k_3}{c}, \frac{(-2p_3+1)d_3k_3}{c}, \cdots, \frac{(2q_3+1)d_3k_3}{c} \right\} \quad (22)$$

and

$$B_z = \left\{ -\frac{p_3 d_3 k_3 (h_3 - 2)}{d_3 k_3 - c}, \frac{(-p_3 + 1) d_3 k_3 (h_3 - 2)}{d_3 k_3 - c}, \cdots, \frac{q_3 d_3 k_3 (h_3 - 2)}{d_3 k_3 - c} \right\}. \quad (23)$$

Assume that (12) and (19) hold and

$$d_3 k_3 > c, \; 2 d_3 k_3 > c h_3, \; \max\{p_3, q_3\} \frac{|c h_3 - 2 d_3 k_3|}{d_3 k_3 - c} \leqslant 1, \quad (24)$$
$$(2 d_3 k_3 - c h_3)(q_3 - 1) < c h_3 - d_3 k_3 - c.$$

Thus system (10) with (21) has $(2p_1+2q_1+3)\times(2p_2+2q_2+3)\times(2p_3+2q_3+3)$ equilibria, described by

$$O_{xyz} = \{(x^*, y^*, z^*) | x^* \in A_x \cup B_x, y^* \in A_y \cup B_y, z^* \in A_z \cup B_z\}. \quad (25)$$

Similarly, the equilibria as above can be classified into eight different sets as follows:

$$\bar{A}_1 = \{(x^*, y^*, z^*) | x^* \in A_x, \; y^* \in A_y, \; z^* \in A_z\}$$

$$\bar{A}_2 = \{(x^*, y^*, z^*) | x^* \in A_x, \; y^* \in A_y, \; z^* \in B_z\}$$

$$\bar{A}_3 = \{(x^*, y^*, z^*) | x^* \in A_x, \; y^* \in B_y, \; z^* \in A_z\}$$

$$\bar{A}_4 = \{(x^*, y^*, z^*) | x^* \in A_x, \; y^* \in B_y, \; z^* \in B_z\}$$

$$\bar{A}_5 = \{(x^*, y^*, z^*) | x^* \in B_x, \; y^* \in A_y, \; z^* \in A_z\}$$

$$\bar{A}_6 = \{(x^*, y^*, z^*) | x^* \in B_x, \; y^* \in A_y, \; z^* \in B_z\}$$

$$\bar{A}_7 = \{(x^*, y^*, z^*) | x^* \in B_x, \; y^* \in B_y, \; z^* \in A_z\}$$

$$\bar{A}_8 = \{(x^*, y^*, z^*) | x^* \in B_x, \; y^* \in B_y, \; z^* \in B_z\}.$$

Suppose that (9) holds and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_0$. Figure 3 shows a 3-D $6\times6\times6$-grid scroll chaotic attractor with fractional order (0.9, 0.9, 0.9), where the parameters are given by $a=2.2$, $b=1.3$, $c=0.6$, $d_1=2.2$, $d_2=1.3$, $d_3=0.6$, $k_1=100$, $k_2=40$, and $k_3=40$. The corresponding Lyapunov exponents spectrum is $LE_1=0.3629$, $LE_2=0$, $LE_3=-1.2742$.

Since $\lambda_1=-1.4430<0$, $\beta=0.2722$, $\gamma=1.3596$, $|\arg(\lambda_2)| = |\arg(\lambda_3)| = |\arctan(\gamma/\beta)| = 1.3732 < (0.9\pi)/2 = 1.4137$, then the equilibria in $\bar{A}_1$ are saddle points of index 2. Similarly, the equilibria in $\bar{A}_2$, $\bar{A}_5$, $\bar{A}_8$ are saddle points of index 1; the equilibria in $\bar{A}_4$, $\bar{A}_6$, $\bar{A}_7$ are saddle points of index 2; and the equilibria in $\bar{A}_3$ are not saddle points of index 1 or 2. Numerical observations reveal that only the equilibria in $\bar{A}_1$ can create scrolls. Furthermore, system (10) with (21) has the potential ability to create a maximum of 3-D $(p_1+q_1+2)\times(p_2+q_2+2)\times(p_3+q_3+2)$-grid scroll chaotic attractor for some suitable parameters. In particular, each equilibrium in $\bar{A}_1$ corresponds to a unique 3-D saturated plateau and also corresponds to a unique scroll in the whole attractor. However, the other equilibria in $\bar{A}_i$ $(2 \leqslant i \leqslant 8)$ correspond to the saturated slopes and are responsible for connecting these $(p_1+q_1+2)\times(p_2+q_2+2)\times(p_3+q_3+2)$ scrolls. Parameters p_1, q_1 control the numbers of scrolls in negative and positive x directions, respectively. Parameters p_2, q_2 control the numbers of scrolls in negative and positive y directions, respectively. Parameters p_3, q_3 control the numbers of scrolls in negative and positive z directions, respectively.

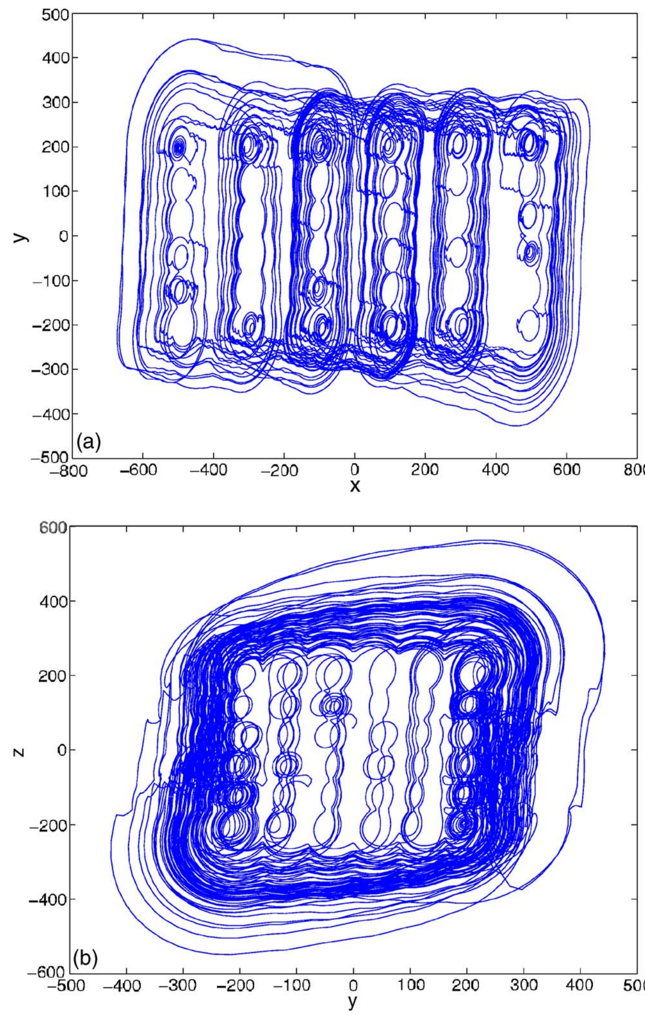

FIG. 3. A 3-D $6\times6\times6$-grid scroll attractor with fractional order (0.9, 0.9, 0.9): (a) x-y plane; and (b) y-z plane.

IV. CONCLUSIONS

This paper has presented a systematic approach for generating the multidirectional multiscroll chaotic attractors from the fractional differential systems. It includes the 1-D n-scroll, 2-D $n\times m$-grid scroll, and 3-D $n\times m\times l$-grid scroll attractors. In particular, it is the first time in the literature to report the multidirectional multiscroll chaotic attractors from a fractional differential systems. Moreover, some underlying dynamical mechanics are further explored for the generation of multidirectional multiscroll chaotic attractors in the fractional differential systems. We also discover that each saturated plateau corresponds to a unique equilibrium and also corresponds to the unique scroll of the whole attractor. Last but not least, one can arbitrarily design a multidirectional multiscroll chaotic attractor with the desired number of scrolls, the desired spatial positions and orientations by adjusting the parameters p_i and q_i $(i=1,2,3)$ of the saturated function series switching controller.

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