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Design of Runs Rules Schemes

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ABSTRACT

Runs rules are often used to increase the sensitivity of a Shewhart control chart. In this work, plots of various runs rules schemes are given to simplify the determination of control limits based on a desired in-control average run length (ARL_0).

Key Words: Runs rules; Shewhart control chart; Average run length; In-control; Out-of-control; False alarm; Type-I error; Sensitivity analysis; Markov chain; States; Transition probabilities; Transient.

INTRODUCTION

The advantage of a Shewhart $\bar{X}$ control chart is its simplicity and its ability to detect large process average shifts quickly. However, the Shewhart chart, which signals an out-of-control condition when a single point falls beyond a 3-sigma limit, is also known for its insensitivity to small process average shifts. The average run length (ARL) is usually used to evaluate the performance of a Shewhart control chart. The ARL is the average number of points that must be plotted before a point indicates an out-of-control condition. For a Shewhart control chart, the ARL can be calculated easily from (Montgomery, 1996)

$$\mathrm{ARL} = \frac{1}{r} \quad (1)$$

where r is the probability that any single point exceeds the control limits. To illustrate, for the $\bar{X}$ chart with 3-sigma limits, $r = 0.0027$ is the probability that a single point falls outside the limits when the process is in-control. Therefore, the in-control ARL is:

$$\mathrm{ARL}_0 = \frac{1}{r} = \frac{1}{0.0027} \approx 370.4$$

Several proposals on runs rules have been made so far. Although these proposals increase the sensitivity of a Shewhart control chart in detecting small process average shifts, they do so at the expense of significant increases in the type-I error (Klein, 2000). In the same reference, Klein suggested two different schemes: the two-of-two and the two-of-three schemes. The designs of both schemes are based on the Markov chain approach. The two schemes perform better than the standard

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Shewhart chart and can be designed to have the same in-control ARL as that of the standard Shewhart chart. The design of the two-of-two scheme is simpler than the two-of-three scheme since its in-control ARL is given by Eq. (2) (Klein, 2000):

$$M_{14} = \frac{1 + p^*}{2(p^*)^2} \tag{2}$$

In the above equation, M_{14} is the in-control ARL and $p^* = pL = pU$, where pL denotes the probability of a single point falling in the lower region and pU in the upper region. However, the design of the two-of-three scheme is more complicated as it involves a linear system of seven equations shown below (Klein, 2000):

$$M_{18} = 1 + (p)M_{18} + (pU)M_{28} + (pL)M_{38}$$

$$M_{28} = 1 + (p)M_{58} + (pL)M_{48}$$

$$M_{38} = 1 + (p)M_{68} + (pU)M_{78}$$

$$M_{48} = 1 + (p)M_{68}$$

$$M_{58} = 1 + (p)M_{18} + (pL)M_{38}$$

$$M_{68} = 1 + (p)M_{18} + (pU)M_{28}$$

$$M_{78} = 1 + (p)M_{58} \tag{3}$$

Here, M_{18} is the in-control ARL. Finding the control limit, which is needed to obtain a desired in-control ARL, requires the additional constraints: $pL = pU$ and $pL + pU + p = 1$, where p denotes the probability of a single point falling in the center region. Solving the constraints in Eq. (3) plus these additional constraints for pL or pU based on a fixed value of M_{18} is extremely difficult and thus not practical for quality-control engineers in their daily work.

Therefore, a more user-friendly approach is proposed in this article. In addition, more superior runs rules schemes are proposed. The values of pU or pL based on a desired in-control ARL for all the proposed schemes are easily determined from the respective plots given. A sensitivity analysis can also be carried out to increase the accuracy of the values of pU or pL obtained from the plots. Thus, practitioners can determine the control limits for a selected scheme based on a desired in-control ARL without much difficulty.

DESIGN APPROACH USING PLOTS OF IN-CONTROL ARL VERSUS *pU* (OR *pL*)

Figures 1, 3, 5, and 7 give plots of ARL vs. pU (or pL) (since $pU = pL$) based on in-control ARL values between 100 and 1000 for the 2-of-3, 2-of-4, 3-of-3, and 3-of-4 schemes, respectively, while Figs. 2, 4, 6, and 8 give the corresponding plots for in-control ARL values from 1000 to 2000 for the above four schemes. The plots in Figs. 1 and 2 are constructed from points (ARL_0, pU [or pL]) obtained using Mathematica 4.0 based on the linear system in Eq. (3). Here, ARL_0 denotes the in-control ARL. As for plots in Figs. 3–8, they are constructed using the same method based on the corresponding linear systems for each scheme given in the Appendix.

To use the proposed procedure, the following steps are recommended:

(i) Choose the desired scheme based on the magnitude of shift that is important for a quick detection. If small shifts are deemed important for quick detection, then choose scheme 3-of-4. However, if bigger shifts are to be detected quickly, then select scheme 2-of-2. Finally, if shifts of very big magnitude are important to be detected quickly, then select the Shewhart control chart scheme instead.

(ii) Decide on the value of the in-control ARL for the case in which the process shift is zero. This corresponds to fixing the false alarm rate (i.e., the type-I error).

(iii) Based on steps (i) and (ii), determine the value of pU (or pL) from the corresponding plot.

(iv) Perform a sensitivity analysis by comparing the value of the in-control ARL obtained based on the value of pU (or pL) from step (iii) with the values of the in-control ARL for other values of pU (or pL) close to the above one. Then, choose the value of pU (or pL) whose in-control ARL best matches the desired ARL_0. Note that this step is optional since the value of pU (or pL) obtained from step (iii) gives an estimate of the in-control ARL that is almost similar to the desired in-control ARL. If this step is skipped, then proceed to step (v).

(v) Based on the value of pU (or pL) from step (iv), find the control limit by using the standard Normal tables or from a simple calculation using a programming language such as SAS (Statistical Analysis System).

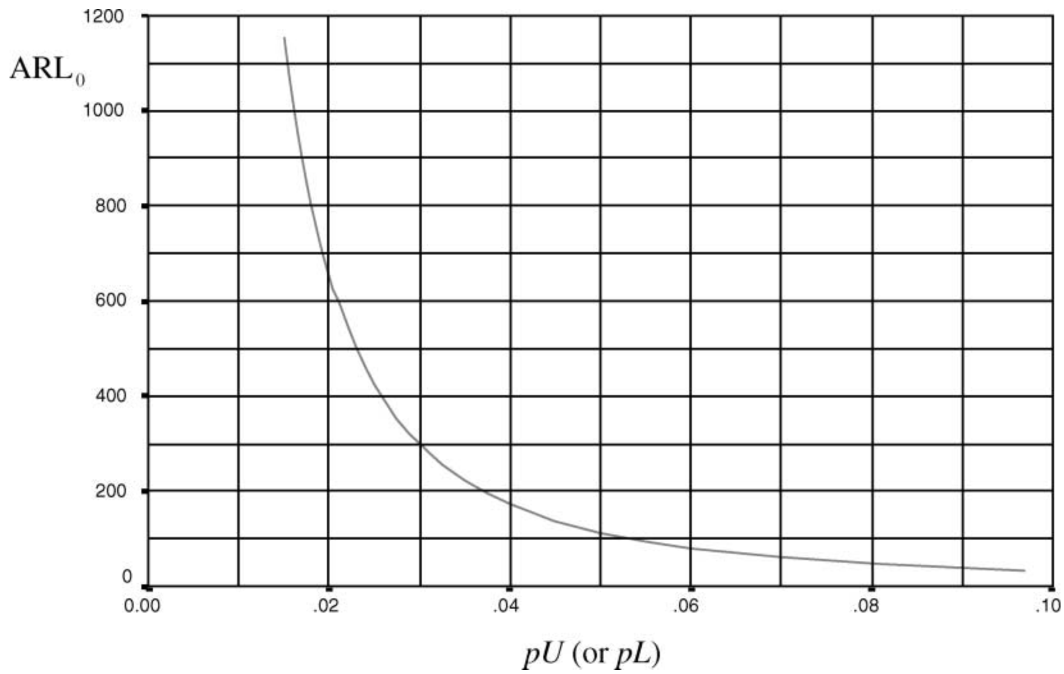

Figure 1. In-control ARL vs. *pU* (or *pL*) for rule 2-of-3 (in-control ARL values between 100 and 1000).

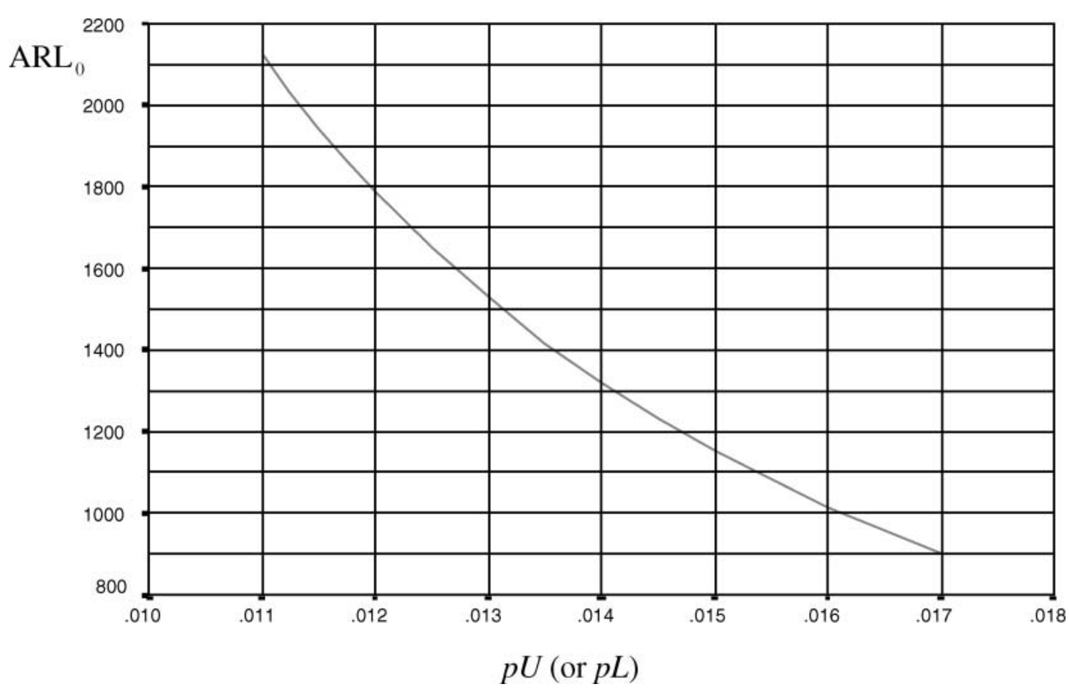

Figure 2. In-control ARL vs. *pU* (or *pL*) for rule 2-of-3 (in-control ARL values between 1000 and 2000).

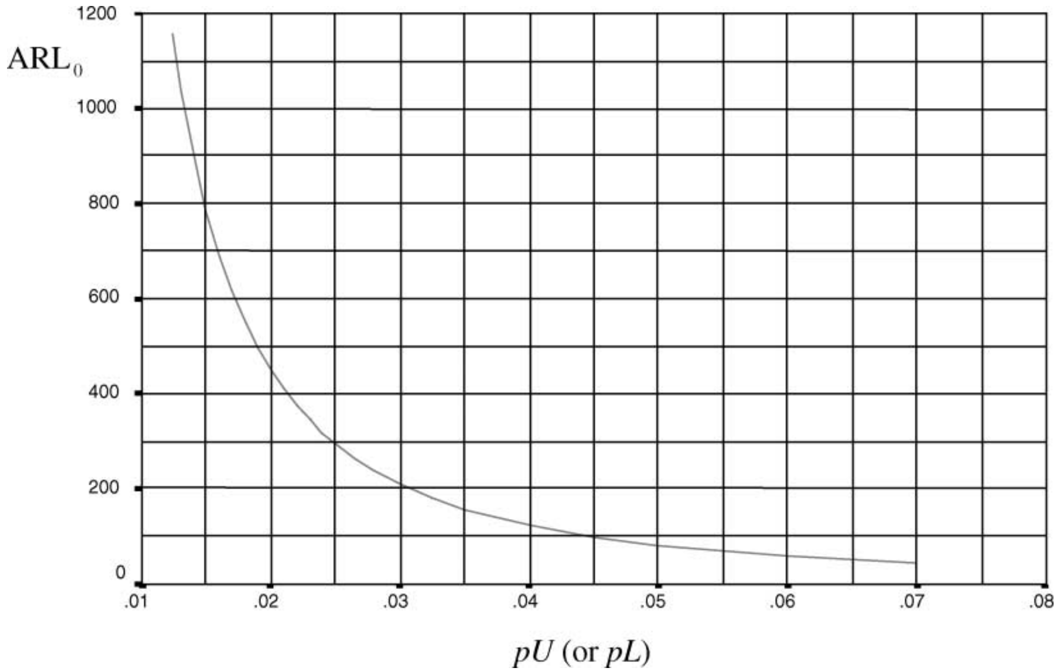

Figure 3. In-control ARL vs. pU (or pL) for rule 2-of-4 (in-control ARL values between 100 and 1000).

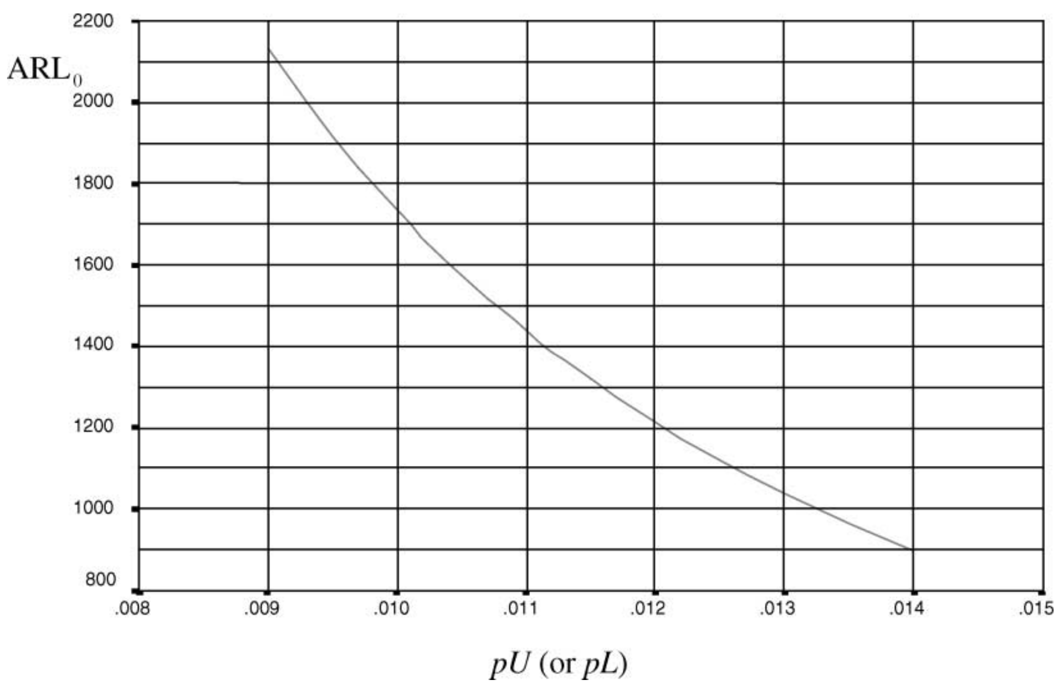

Figure 4. In-control ARL vs. pU (or pL) for rule 2-of-4 (in-control ARL values between 1000 and 2000).

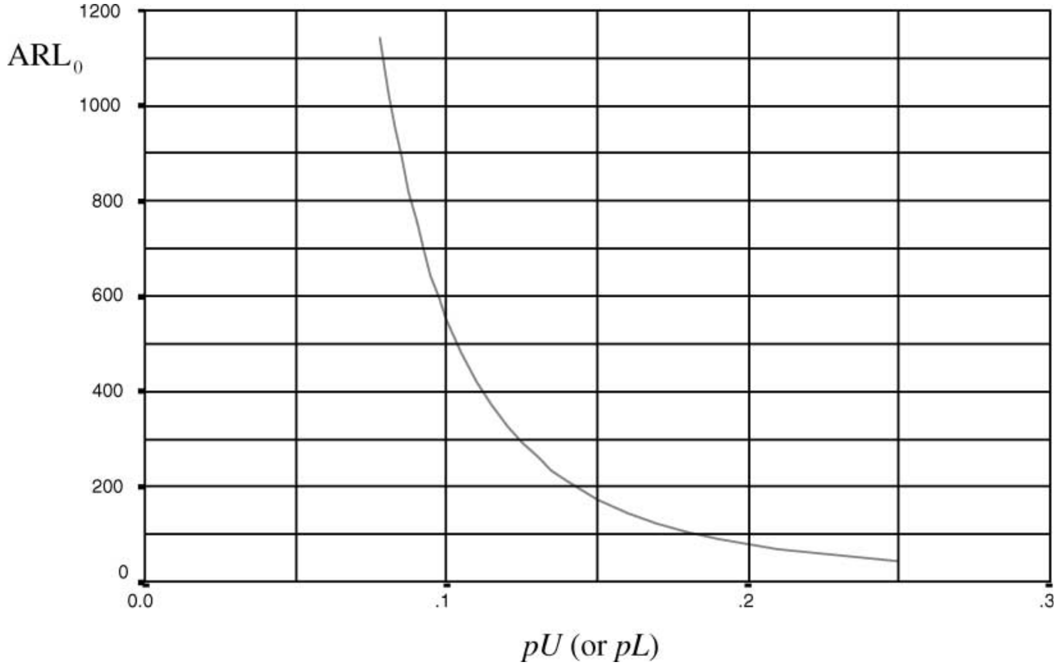

Figure 5. In-control ARL vs. *pU* (or *pL*) for rule 3-of-3 (in-control ARL values between 100 and 1000).

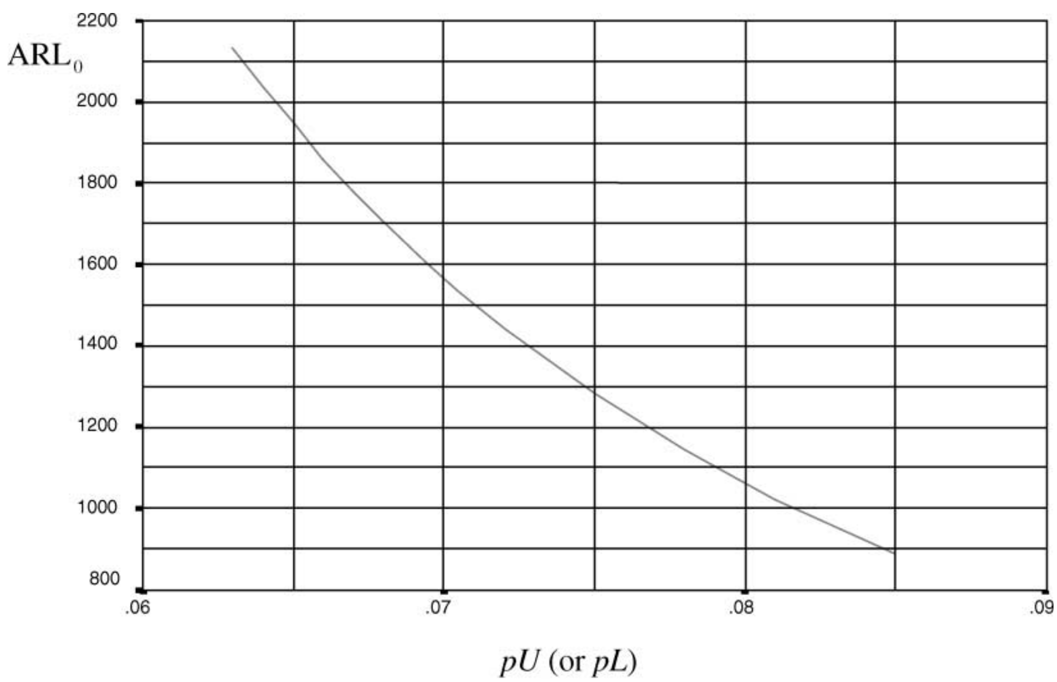

Figure 6. In-control ARL vs. *pU* (or *pL*) for rule 3-of-3 (in-control ARL values between 1000 and 2000).

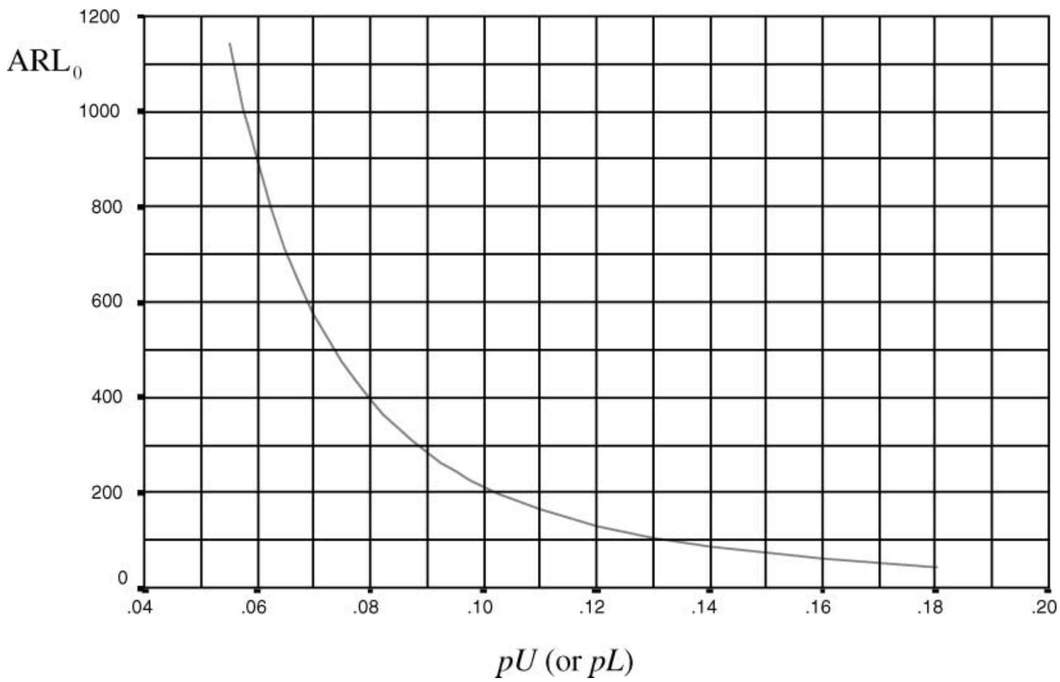

Figure 7. In-control ARL vs. *pU* (or *pL*) for rule 3-of-4 (in-control ARL values between 100 and 1000).

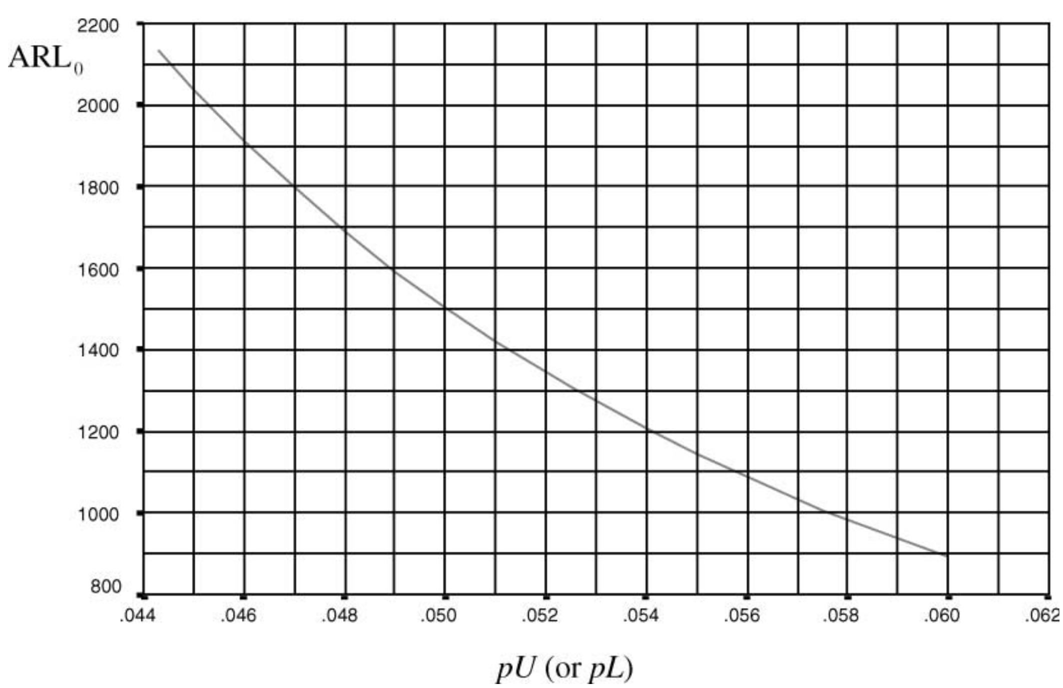

Figure 8. In-control ARL vs. *pU* (or *pL*) for rule 3-of-4 (in-control ARL values between 1000 and 2000).

Table 1. Sensitivity analysis for the 3-of-4 scheme based on pU (or pL) = 0.069.

pU/pL	In-control ARL
0.069	599.098
0.06899	599.343
0.06898	599.589
0.06897	599.835
0.06896	600.082

EXAMPLE

The following steps are taken to design a 3-of-4 scheme with an average false alarm rate of not greater than 1 in 600:

Step 1: The desired in-control ARL is 600.

Step 2: From the plot in Fig. 7 and the curve associated with it, the value of pU (or pL) corresponding to an in-control ARL value of 600 is determined to be approximately 0.069. This approximation can be made from the curve using a pencil and a ruler.

Step 3: A sensitivity analysis is then performed, and its results are given in Table 1. Note that a sensitivity analysis can be carried out using Mathematica 4.0 by finding the other in-control ARL values from the corresponding matrix algebra, m (see Appendix) based on values of pU (or pL) that are close to 0.069. The matrix algebra, m represents the linear system in Eq. (A.3), is also given in the Appendix. Note that in the Appendix, pU is denoted by u, pL by k, and p by the same letter. An example of the output obtained for the case $u = k = 0.06896$ and $p = 1 - k - u$ is shown in Fig. 9. Here, the value of the in-control ARL that we are interested in is shown as the last entry in line 1 of the output. Its value is 600.082, which means an average of one false out-of-control signal in approximately 600 observations. This is the ARL value from the starting state 1 to the absorbing (out-of-control) state 26.

Step 4: From Table 1, the value of pU (or pL) = 0.06896 is found to have the closest in-control ARL to our desired value 600. Based on this value of pU (or pL), the control limits are ± 1.4836 from the standard Normal tables.

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	600.082
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	592.831
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	597.603
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	600.082
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	592.831
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	597.603
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	600.082
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	521.149
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	556.893
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	559.2
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	521.149
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	556.893
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	559.2
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	590.353
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	592.831
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	592.831
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	590.353
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	597.603
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	597.603
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	552.12
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	554.414
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	521.149
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	552.12
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	554.414
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	521.149

Figure 9. The ARL values for the 3-of-4 scheme for the various states based on $k = u = 0.06896$ and $p = 1 - k - u$.

Table 2. ARL profiles based on in-control ARL 370.

	Shewhart	2-of-2	2-of-3	2-of-4	3-of-3	3-of-4
	Control limits					
Shift	±3	±1.781	±1.929	±2.011	±1.2	±1.392
0	375.6942	368.9999	370.9544	367.9462	370.9259	375.6688
0.2	311.2865	**278.0977**	**267.8767**	**265.0058**	**260.1067**	**249.6439**
0.4	200.8238	**152.4105**	**141.8001**	**139.2316**	**130.8848**	**116.8499**
0.6	118.8520	**79.7316**	**72.8887**	**70.3019**	**65.3161**	**57.1153**
0.8	71.4727	**43.7537**	**40.0956**	**38.5194**	**35.9541**	**30.9231**
1	44.0847	**26.0346**	**23.3690**	**22.5946**	**21.4363**	**18.6225**
1.2	27.6007	**16.4331**	**14.6154**	**14.2008**	**13.9222**	**12.1751**
1.4	18.1069	**10.8979**	**9.9428**	**9.7403**	**9.9001**	**8.6923**
1.6	12.3450	**7.7931**	**7.1649**	**7.0558**	**7.3666**	**6.6233**
1.8	8.6633	**5.8888**	**5.4140**	**5.4016**	**5.8625**	**5.3295**
2	6.3096	**4.5900**	**4.3037**	**4.3295**	**4.9011**	**4.5377**
2.2	4.7255	**3.7564**	**3.5948**	**3.6231**	**4.2796**	**4.0285**
2.4	3.6656	**3.2135**	**3.1043**	**3.1244**	3.8685	3.6707
2.6	2.9129	**2.8370**	**2.7673**	**2.7961**	3.5695	3.4457
2.8	2.3826	2.5749	2.5192	2.5635	3.3781	3.2933
3	2.0069	2.3923	2.3571	2.3911	3.2413	3.1913
3.5	1.4421	2.1495	2.1249	2.1534	3.0631	3.0558
4	1.1880	2.0416	2.0395	2.0520	3.0146	3.0147
4.5	1.0726	2.0112	2.0088	2.0123	3.0027	3.0022
5	1.0245	2.0015	2.0024	2.0029	3.0002	3.0004
5.5	1.0057	2.0001	2.0002	2.0005	3.0000	3.0000
6	1.0014	2.0000	2.0000	2.0000	3.0000	3.0000

PERFORMANCES OF THE RUNS RULES SCHEMES

A simulation study is conducted to evaluate the performances of the various schemes. Computer programs are written using the SAS (Version 6.12) programming language. Results are given in Tables 2–4 for in-control ARLs of 370, 500, and 1000, respectively. The determination of the control limits for a standard Shewhart control chart (i.e., the 1-of-1 scheme) based on the above three in-control ARL values can be done easily using standard Normal tables. For example, if the in-control ARL is 500, then the probability that a point falls outside the center region is $pU + pL = \frac{1}{500} = 0.002$. Hence, pU (or pL) $= 0.001$ and, from the standard Normal tables, the control limits are ± 3.09. Note that the control limits for the other schemes are determined based on the method discussed in the previous section. The ARL profiles for all the schemes in Tables 2–4 are based on process average shifts from zero (in-control) to out-of-control values of up to six-sigma.

As expected, all the schemes performed better than the standard Shewhart control chart for small to moderate process average shifts (see the bold faced ARL values in Tables 2–4). The 3-of-4 scheme has the best overall performance, followed by the 3-of-3 and the 2-of-4 schemes for small to moderate process average shifts. Note also that the 2-of-4, 3-of-3, and 3-of-4 schemes perform better than the other two schemes suggested by Klein (2000). However, for large process average shifts, the standard Shewhart performs slightly better than the other schemes. This is only a small concern since the standard Shewhart can detect large shifts, the earliest by one observation ahead of the 2-of-2, 2-of-3, and 2-of-4 schemes and by two observations ahead of the 3-of-3 and 3-of-4 schemes. On the contrary, for small process average shifts, the difference in the time of detecting an out-of-control signal between the standard Shewhart and the other schemes are quite significant.

APPLICATION

To illustrate the proposed procedure in real working situations, simulated data with known properties are used. Thirty observations are generated, as shown in

Table 3. ARL profiles based on in-control ARL 500.

	Shewhart	2-of-2	2-of-3	2-of-4	3-of-3	3-of-4
	Control limits					
Shift	± 3.09	± 1.851	± 1.995	± 2.076	± 1.2608	± 1.449
0	504.4192	498.3582	501.0706	493.7532	507.1547	507.4956
0.2	415.4434	**364.5605**	**355.1243**	**350.7271**	**341.5392**	**329.0058**
0.4	263.5408	**195.3699**	**181.9076**	**176.2584**	**163.8272**	**146.8434**
0.6	154.2100	**99.7721**	**90.3505**	**87.3447**	**80.3150**	**69.5863**
0.8	89.7071	**53.7457**	**48.7604**	**46.4805**	**42.3397**	**36.6257**
1	54.7631	**30.8248**	**27.7530**	**26.5227**	**24.9177**	**21.4054**
1.2	33.9694	**19.2224**	**17.1589**	**16.3689**	**15.7250**	**13.5938**
1.4	21.9063	**12.4159**	**11.2158**	**10.9519**	**10.9468**	**9.5045**
1.6	14.6785	**8.6895**	**7.8598**	**7.7976**	**7.9982**	**7.1335**
1.8	10.1128	**6.4491**	**5.9194**	**5.8785**	**6.2368**	**5.6356**
2	7.2587	**5.0009**	**4.6593**	**4.6268**	**5.1470**	**4.7240**
2.2	5.3632	**4.0089**	**3.7927**	**3.8315**	**4.4373**	**4.1534**
2.4	4.0956	**3.3714**	**3.2482**	**3.2652**	**3.9710**	**3.7616**
2.6	3.2133	**2.9551**	**2.8810**	**2.8892**	3.6502	3.5021
2.8	2.5826	2.6553	2.5989	2.6287	3.4301	3.3326
3	2.1611	2.4521	2.4111	2.4408	3.2797	3.2170
3.5	1.5138	2.1715	2.1484	2.1745	3.0778	3.0660
4	1.2227	2.0494	2.0437	2.0597	3.0197	3.0166
4.5	1.0879	2.0134	2.0112	2.0149	3.0030	3.0026
5	1.0299	2.0025	2.0024	2.0035	3.0002	3.0004
5.5	1.0071	2.0003	2.0004	2.0008	3.0000	3.0000
6	1.0019	2.0000	2.0000	2.0000	3.0000	3.0000

Table 4. ARL profiles based on in-control ARL 1000.

	Shewhart	2-of-2	2-of-3	2-of-4	3-of-3	3-of-4
	Control limits					
Shift	± 3.29	± 2.0026	± 2.141	± 2.219	± 1.394	± 1.575
0	999.6642	1008.3300	999.4830	990.6290	1009.91	1014.27
0.2	814.0900	**700.1159**	**693.9343**	**675.6902**	**651.6131**	**628.4106**
0.4	491.9208	**349.0751**	**321.9579**	**312.3122**	**285.7709**	**255.5132**
0.6	276.6994	**169.0419**	**150.5389**	**146.1836**	**130.3835**	**111.2154**
0.8	156.4401	**85.8652**	**75.7744**	**72.3462**	**64.3844**	**54.1093**
1	90.0255	**46.8022**	**41.7294**	**39.2269**	**35.3305**	**29.5563**
1.2	54.8066	**27.5242**	**24.0643**	**23.0240**	**21.0889**	**17.9095**
1.4	33.9794	**17.1546**	**15.0865**	**14.4417**	**13.7476**	**11.7889**
1.6	21.9081	**11.3147**	**10.1256**	**9.9006**	**9.8184**	**8.4850**
1.8	14.6785	**8.0366**	**7.2603**	**7.1452**	**7.3125**	**6.4852**
2	10.1128	**6.0480**	**5.4493**	**5.4622**	**5.8317**	**5.2471**
2.2	7.2587	**4.7045**	**4.0855**	**4.3680**	**4.8771**	**4.4835**
2.4	5.3632	**3.8252**	**3.5881**	**3.6432**	**4.2668**	**3.9909**
2.6	4.0956	**3.2601**	**3.0698**	**3.1418**	**3.8593**	**3.6488**
2.8	3.2133	**2.8703**	**2.7817**	**2.8069**	3.5660	3.4265
3	2.5826	2.5997	**2.5286**	**2.5720**	3.3701	3.2797
3.5	1.7164	2.2296	2.1943	2.2284	3.1142	3.0907
4	1.3127	2.0771	2.0677	2.0835	3.0291	3.0240
4.5	1.1286	2.0186	2.0176	2.0233	3.0051	3.0044
5	1.0483	2.0047	2.0045	2.0058	3.0008	3.0008
5.5	1.0136	2.0007	2.0008	2.0009	3.0000	3.0000
6	1.0032	2.0000	2.0000	2.0001	3.0000	3.0000

Table 5. Simulated values for 30 observations.

Observation No., i	X_i	Observation No., i	X_i
1	1.40419	16	−0.91188
2	−1.42641	17	−1.24454
3	0.62378	18	−0.93233
4	1.48897	19	−0.97555
5	0.45396	20	0.73460
6	0.65258	21	2.40419
7	−1.76759	22	−0.42641
8	−0.81356	23	1.62378
9	−0.22403	24	2.48897
10	0.53134	25	1.45396
11	−0.081560	26	1.65258
12	1.33234	27	−0.76759
13	1.14637	28	0.18644
14	−0.72117	29	0.77597
15	1.81601	30	1.53134

Table 5. Observations 1 to 20 are generated from a $N(0, 1)$ distribution to simulate an in-control process with a mean zero. Observations 21 to 30 are generated from a $N(1, 1)$ distribution to simulate an out-of-control process with a permanent shift in the process mean. It is clearly shown that the mean has increased by one standard deviation for each observation from 21 to 30.

These values are plotted on a standard Shewhart control chart (see Fig. 10) with 3-sigma control limits (i.e., ± 3 since $\sigma = 1$) with a false alarm rate of 0.0027 for a point falling outside the $\pm 3\sigma$ limits when the process is in-control. Thus, the in-control ARL (ARL_0) is $\frac{1}{0.0027} \approx 370.4$. The control limits for the 2-of-2, 2-of-3, 2-of-4, 3-of-3, and 3-of-4 schemes are determined based on this in-control ARL value to be ± 1.78, ± 1.93, ± 2.01, ± 1.20, and ± 1.39, respectively (see Fig. 10). Note that the control limits for the 2-of-2 scheme are determined using Eq. (1), while the control limits for the other schemes are determined based on the proposed procedure.

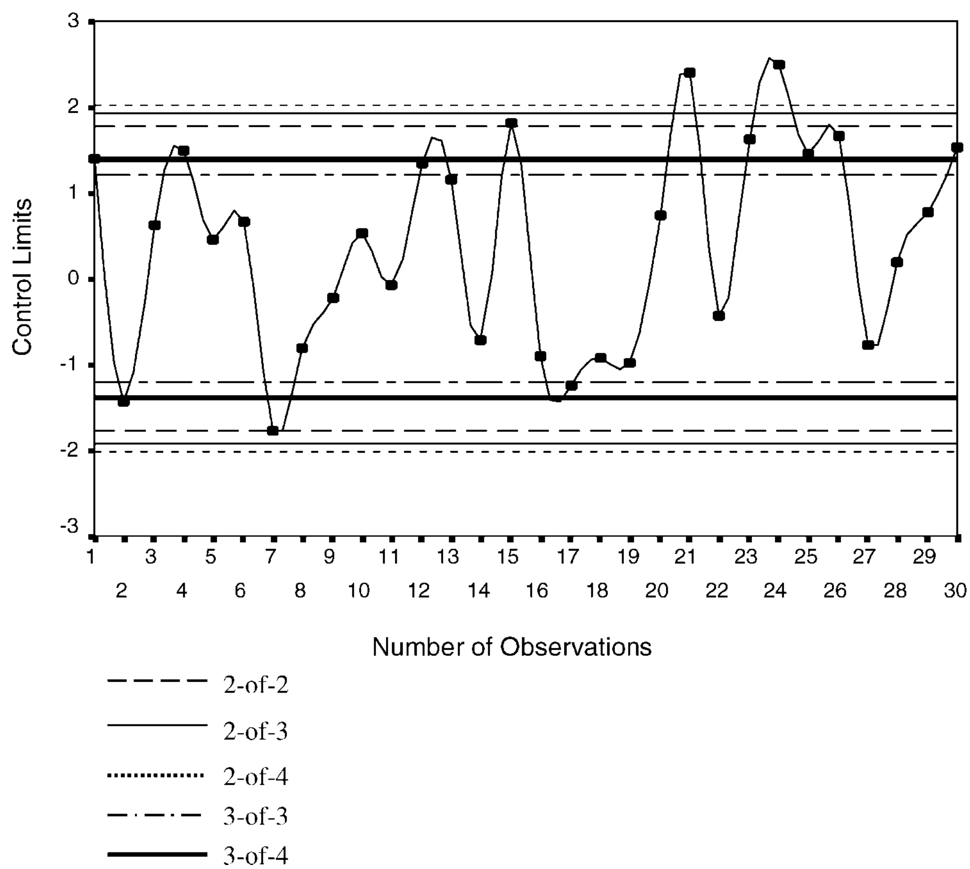

Figure 10. Control chart for the plotted 30 observations.

From Fig. 10, it is noticed that no out-of-control signal is detected based on the control limits of the standard Shewhart and the 2-of-2 and 2-of-3 schemes. However, both the 2-of-4 and 3-of-4 schemes signal an out-of-control at observation 24, while the 3-of-3 scheme detects an out-of-control signal at observation 25. This simple example shows that runs rules schemes can significantly increase the speed and probability of detecting permanent shifts in the process mean, which may otherwise be left undetected by the standard Shewhart scheme. Furthermore, the control limits of any of the schemes discussed in this article can be easily determined based on a desired false alarm rate.

CONCLUSION

In this article, a simple and practical approach for designing runs rules schemes based on a desired type-I error is proposed. Due to its simplicity, quality-control engineers may find this approach user-friendly and hence useful in industrial settings. Practitioners who have been using the standard Shewhart control chart may find the 3-of-4 scheme attractive due to its favorable ARL properties.

APPENDIX

The notations used here are $pU = u$, denoting the probability of a single point falling in the upper region; $pL = k$ in the lower region; and p in the center region. Note that the additional constraints required to solve the in-control ARL of each of the following schemes are $pL = pU$ and $pL + pU + p = 1$. All Mathematica 4.0 programs that come with the following schemes can be obtained from the author upon request.

The 2-of-3 Scheme

The expected number of transitions from each of the states to the absorbing state are represented by the linear system in Eq. (3). This linear system can also be represented by the matrix algebra in Fig. 11.

Note that the above matrix is typed using Mathematica 4.0, and running this program for a $k = u$ value of our choice gives the corresponding in-control ARL value.

```
1-1
0
u=k=
  p=1-k-u
Needs["LinearAlgebra`MatrixManipu`"]

      (1-p  -u  -k   0   0   0   0  1)
      ( 0    1   0  -k  -p   0   0  1)
      ( 0    0   1   0   0  -p  -u  1)
m:=   ( 0    0   0   1   0  -p   0  1)
      (-p    0  -k   0   1   0   0  1)
      (-p   -u   0   0   0   1   0  1)
      ( 0    0   0   0  -p   0   1  1)

MatrixForm[RowReduce[m]]
```

Figure 11. Matrix algebra for the linear system in Eq. (3).

The 2-of-4 Scheme

The Markov chain for this scheme consists of 14 states with the first 13 of them as transient. The states are:

State (OOO): has three successive points between both control limits;

State (OOU): has the first two points between both control limits and the third above the upper control limit (UCL);

State (OUO): has the first and third points between both control limits and the second above the UCL;

State (UOO): has the first point above the UCL and the second and third between both control limits;

State (OOL): has the first two points between both control limits and the third below the lower control limit (LCL);

State (OLO): has the first and third points between both control limits and the second below the LCL;

State (LOO): has the first point below the LCL and the second and third between both control limits;

State (OUL): has the first point between both control limits, the second above the UCL, and the third below the LCL;

Table 6. Transition probabilities for Markov chain with 13 transient states.

	States at time $t+1$													
States at time t	1 (OOO)	2 (OOU)	3 (OUO)	4 (UOO)	5 (OOL)	6 (OLO)	7 (LOO)	8 (OUL)	9 (OLU)	10 (UOL)	11 (ULO)	12 (LOU)	13 (LUO)	14 (OOC)
1 (OOO)	p	pU			pL									
2 (OOU)			p					pL						pU
3 (OUO)				P						pL				pU
4 (UOO)	p				pL									pU
5 (OOL)						p			pU					pL
6 (OLO)							p					pU		pL
7 (LOO)	p	pU												pL
8 (OUL)											p			$pU + pL$
9 (OLU)													p	$pU + ;pL$
10 (UOL)						p								$pU + pL$
11 (ULO)							p							$pU + pL$
12 (LOU)			p											$pU + pL$
13 (LUO)				P										$pU + pL$
14 (OOC)														1

State (OLU): has the first point between both control limits, the second below the LCL, and the third above the UCL;

State (UOL): has the first point above the UCL, the second between both control limits, and the third below the LCL;

State (ULO): has the first point above the UCL, the second below the LCL, and the third between both control limits;

State (LOU): has the first point below the LCL, the second between both control limits, and the third above the UCL;

State (LUO): has the first point below the LCL, the second above the UCL, and the third between both control limits;

State (OOC): the absorbing state, has two of four points either below the LCL or above the UCL.

The transition probabilities of the Markov chain for this scheme are given in Table 6. The expected number of transitions from each of the states to the absorbing state can be determined by solving the linear system in Eq. (A.1). Here, $M_{1(14)}$ is the in-control ARL for this scheme.

$$M_{1(14)} = 1 + (p)M_{1(14)} + (pU)M_{2(14)} + (pL)M_{5(14)}$$

$$M_{2(14)} = 1 + (p)M_{3(14)} + (pL)M_{8(14)}$$

$$M_{3(14)} = 1 + (p)M_{4(14)} + (pL)M_{10(14)}$$

$$M_{4(14)} = 1 + (p)M_{1(14)} + (pL)M_{5(14)}$$

$$M_{5(14)} = 1 + (p)M_{6(14)} + (pU)M_{9(14)}$$

$$M_{6(14)} = 1 + (p)M_{7(14)} + (pU)M_{12(14)}$$

$$M_{7(14)} = 1 + (p)M_{1(14)} + (pU)M_{2(14)}$$

$$M_{8(14)} = 1 + (p)M_{11(14)}$$

$$M_{9(14)} = 1 + (p)M_{13(14)}$$

$$M_{10(14)} = 1 + (p)M_{6(14)}$$

$$M_{11(14)} = 1 + (p)M_{7(14)}$$

$$M_{12(14)} = 1 + (p)M_{3(14)}$$

$$M_{13(14)} = 1 + (p)M_{4(14)} \qquad \text{(A.1)}$$

This linear system can also be represented by the matrix algebra in Fig. 12 using Mathematica 4.0.

```
1-1
0
u=k=
  p=1-k-u
0
Needs["LinearAlgebra`MatrixManipu`"]
```

$$m := \begin{pmatrix} 1-p & -u & 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -p & 0 & 0 & 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -p & 0 & 0 & 0 & 0 & 0 & -k & 0 & 0 & 0 & 1 \\ -p & 0 & 0 & 1 & -k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -p & 0 & 0 & -u & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -p & 0 & 0 & 0 & 0 & -u & 0 & 1 \\ -p & -u & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -p & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -p & 1 \\ 0 & 0 & 0 & 0 & 0 & -p & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -p & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

```
MatrixForm[RowReduce[m]]
```

Figure 12. Matrix algebra for the linear system in Eq. (A.1).

The 3-of-3 Scheme

The Markov chain for this scheme consists of 10 states with the first 9 states as transient. The states are:

State (OO): has two successive points between both control limits;

State (OU): has a first point between both control limits and the second above the UCL;

State (OL): has a first point between both control limits and the second below the LCL;

State (UL): has a first point above the UCL and the second below the LCL;

Table 7. Transition probabilities for Markov chain with 10 transient states.

	States at time $t+1$									
States at time t	1 (OO)	2 (OU)	3 (OL)	4 (UL)	5 (LU)	6 (UO)	7 (LO)	8 (UU)	9 (LL)	10 (OOC)
1 (OO)	p	pU	pL							
2 (OU)				pL		p		pU		
3 (OL)					pU		p		pL	
4 (UL)					pU		p		pL	
5 (LU)				pL		p		pU		
6 (UO)	p	pU	pL							
7 (LO)	p	pU	pL							
8 (UU)				pL		p				pU
9 (LL)					pU		p			pL
10 (OOC)										1

State (LU): has a first point below the LCL and the second above the UCL;

State (UO): has a first point above the UCL and the second between both control limits;

State (LO): has a first point below the LCL and the second between both control limits;

State (UU): has two successive points above the UCL;

State (LL): has two successive points below the LCL;

State (OOC): the absorbing state, has three successive points either below the LCL or above the UCL.The transition probabilities of the Markov chain for this scheme are given in Table 7.

Equation (A.2) gives the expected number of transitions from each of the states to the absorbing state. Here, $M_{1(10)}$ is the in-control ARL for this scheme.

$$M_{1(10)} = 1 + (p)M_{1(10)} + (pU)M_{2(10)} + (pL)M_{3(10)}$$

$$M_{2(10)} = 1 + (p)M_{6(10)} + (pU)M_{8(10)} + (pL)M_{4(10)}$$

$$M_{3(10)} = 1 + (p)M_{7(10)} + (pU)M_{5(10)} + (pL)M_{9(10)}$$

$$M_{4(10)} = 1 + (p)M_{7(10)} + (pU)M_{5(10)} + (pL)M_{9(10)}$$

$$M_{5(10)} = 1 + (p)M_{6(10)} + (pU)M_{8(10)} + (pL)M_{4(10)}$$

$$M_{6(10)} = 1 + (p)M_{1(10)} + (pU)M_{2(10)} + (pL)M_{3(10)}$$

$$M_{7(10)} = 1 + (p)M_{1(10)} + (pU)M_{2(10)} + (pL)M_{3(10)}$$

$$M_{8(10)} = 1 + (p)M_{6(10)} + (pL)M_{4(10)}$$

$$M_{9(10)} = 1 + (p)M_{7(10)} + (pU)M_{5(10)} \tag{A.2}$$

The linear system in Eq. (A.2) can be represented by the matrix algebra in Fig. 13 using Mathematica 4.0.

```
1-1
0
u=k=
  p=1-k-u
Needs["LinearAlgebra`MatrixManipu`"]

     1-p -u -k  0  0  0  0  0  0  1
      0   1  0 -k  0 -p  0 -u  0  1
      0   0  1  0 -u  0 -p  0 -k  1
      0   0  0  1 -u  0 -p  0 -k  1
m:=   0   0  0 -k  1 -p  0 -u  0  1
     -p  -u -k  0  0  1  0  0  0  1
     -p  -u -k  0  0  0  1  0  0  1
      0   0  0 -k  0 -p  0  1  0  1
      0   0  0  0 -u  0 -p  0  1  1

MatrixForm[RowReduce[m]]
```

Figure 13. Matrix algebra for the linear system in Eq. (A.2).

The 3-of-4 Scheme

The Markov chain for this scheme consists of 26 states. Fourteen of the state descriptions are similar to that of the 2-of-4 scheme given earlier. The remaining 12 states are:

Table 8. Transition probabilities for Markov chain with 26 transient states.

	States at time $t + 1$																									
States at time t	1 (OOO)	2 (OOU)	3 (OUO)	4 (UOO)	5 (OOL)	6 (OLO)	7 (LOO)	8 (OUU)	9 (UOU)	10 (UUO)	11 (OLL)	12 (LOL)	13 (LLO)	14 (OUL)	15 (UOL)	16 (LOU)	17 (OLU)	18 (ULO)	19 (LUO)	20 (UUL)	21 (ULU)	22 (LUU)	23 (LLU)	24 (LUL)	25 (ULL)	26 (OOC)
1 (OOO)	p	pU			pL																					
2 (OOU)			p					pU						pL												
3 (OUO)				p					pU						pL											
4 (UOO)	p	pU			pL																					
5 (OOL)						p					pL						pU									
6 (OLO)							p					pL				pU										
7 (LOO)	p	pU			pL																					
8 (OUU)										p										pL						pU
9 (UOU)			p											pL												pU
10 (UUO)				p											pL											pU
11 (OLL)													p										pU			pL
12 (LOL)						p											pU									pL
13 (LLO)							p									pU										pL
14 (OUL)																		p			pU				pL	
15 (UOL)						p					pL						pU									
16 (LOU)			p					pU						pL												
17 (OLU)																			p			pU		pL		
18 (ULO)							p					pL				pU										
19 (LUO)				p					pU						pL											
20 (UUL)																		p							pL	pU
21 (ULU)																			p					pL		pU
22 (LUU)										p										pL						pU
23 (LLU)																			p			pU				pL
24 (LUL)																		p			pU					pL
25 (ULL)													p										pU			pL
26 (OOC)																										1

State (OUU): has the first point between both control limits and the second and third above the UCL;

State (UOU): has the first and third points above the UCL and the second between both control limits;

State (UUO): has the first two points above the UCL and the third between both control limits;

State (OLL): has the first point between both control limits and the second and third below the LCL;

State (LOL): has the first and third points below the LCL and the second between both control limits;

State (LLO): has the first two points below the LCL and the third between both control limits;

State (UUL): has the first two points above the UCL and the third below the LCL;

State (ULU): has the first and third points above the UCL and the second below the LCL;

State (LUU): has the first point below the LCL and the second and third above the UCL;

State (LLU): has the first two points below the LCL and the third above the UCL;

State (LUL): has the first and third points below the LCL and the second above the UCL;

State (ULL): has the first point above the UCL and the second and third below the LCL.The transition probabilities of the Markov chain for this scheme are given in Table 8.

The expected number of transitions from each of the states to the absorbing state is given in Eq. (A.3).

$$M_{1(26)} = 1 + (p)M_{1(26)} + (pU)M_{2(26)} + (pL)M_{5(26)}$$

$$M_{2(26)} = 1 + (p)M_{3(26)} + (pU)M_{8(26)} + (pL)M_{14(26)}$$

$$M_{3(26)} = 1 + (p)M_{4(26)} + (pU)M_{9(26)} + (pL)M_{15(26)}$$

```
1-1
0
u= k=
  p= 1- k- u
Needs ["LinearAlgebra`MatrixManipulation` "]
m :=

 1-p -u  0  0 -k  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1
  0   1 -p  0  0  0  0 -u  0  0  0  0  0 -k  0  0  0  0  0  0  0  0  0  0  0  1
  0   0  1 -p  0  0  0  0 -u  0  0  0  0  0 -k  0  0  0  0  0  0  0  0  0  0  1
 -p  -u  0  1 -k  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1
  0   0  0  0  1 -p  0  0  0  0 -k  0  0  0  0  0 -u  0  0  0  0  0  0  0  0  1
  0   0  0  0  0  1 -p  0  0  0  0 -k  0  0  0 -u  0  0  0  0  0  0  0  0  0  1
 -p  -u  0  0 -k  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1
  0   0  0  0  0  0  0  1  0 -p  0  0  0  0  0  0  0  0  0 -k  0  0  0  0  0  1
  0   0 -p  0  0  0  0  0  1  0  0  0  0 -k  0  0  0  0  0  0  0  0  0  0  0  1
  0   0  0 -p  0  0  0  0  0  1  0  0  0  0 -k  0  0  0  0  0  0  0  0  0  0  1
  0   0  0  0  0  0  0  0  0  0  1  0 -p  0  0  0  0  0  0  0  0  0 -u  0  0  1
  0   0  0  0  0 -p  0  0  0  0  0  1  0  0  0  0 -u  0  0  0  0  0  0  0  0  1
  0   0  0  0  0  0 -p  0  0  0  0  0  1  0  0 -u  0  0  0  0  0  0  0  0  0  1
  0   0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0 -p  0  0 -u  0  0  0 -k  1
  0   0  0  0  0 -p  0  0  0  0 -k  0  0  0  1  0 -u  0  0  0  0  0  0  0  0  1
  0   0 -p  0  0  0  0 -u  0  0  0  0  0 -k  0  1  0  0  0  0  0  0  0  0  0  1
  0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0 -p  0  0 -u  0 -k  0  1
  0   0  0  0  0  0 -p  0  0  0  0 -k  0  0  0 -u  0  1  0  0  0  0  0  0  0  1
  0   0  0 -p  0  0  0  0 -u  0  0  0  0  0 -k  0  0  0  1  0  0  0  0  0  0  1
  0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -p  0  1  0  0  0  0 -k  1
  0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -p  0  1  0  0 -k  0  1
  0   0  0  0  0  0  0  0  0 -p  0  0  0  0  0  0  0  0  0 -k  0  1  0  0  0  1
  0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -p  0  0 -u  1  0  0  1
  0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -p  0  0 -u  0  0  1  0  1
  0   0  0  0  0  0  0  0  0  0  0  0 -p  0  0  0  0  0  0  0  0  0 -u  0  1  1

MatrixForm [RowReduce[m]]
```

Figure 14. Matrix algebra for the linear system in Eq. (A.3).

$$M_{4(26)} = 1 + (p)M_{1(26)} + (pU)M_{2(26)} + (pL)M_{5(26)}$$

$$M_{5(26)} = 1 + (p)M_{6(26)} + (pU)M_{17(26)} + (pL)M_{11(26)}$$

$$M_{6(26)} = 1 + (p)M_{7(26)} + (pU)M_{16(26)} + (pL)M_{12(26)}$$

$$M_{7(26)} = 1 + (p)M_{1(26)} + (pU)M_{2(26)} + (pL)M_{5(26)}$$

$$M_{8(26)} = 1 + (p)M_{10(26)} + (pL)M_{20(26)}$$

$$M_{9(26)} = 1 + (p)M_{3(26)} + (pL)M_{14(26)}$$

$$M_{10(26)} = 1 + (p)M_{4(26)} + (pL)M_{15(26)}$$

$$M_{11(26)} = 1 + (p)M_{13(26)} + (pU)M_{23(26)}$$

$$M_{12(26)} = 1 + (p)M_{6(26)} + (pU)M_{17(26)}$$

$$M_{13(26)} = 1 + (p)M_{7(26)} + (pU)M_{16(26)}$$

$$M_{14(26)} = 1 + (p)M_{18(26)} + (pU)M_{21(26)} + (pL)M_{25(26)}$$

$$M_{15(26)} = 1 + (p)M_{6(26)} + (pU)M_{17(26)} + (pL)M_{11(26)}$$

$$M_{16(26)} = 1 + (p)M_{3(26)} + (pU)M_{8(26)} + (pL)M_{14(26)}$$

$$M_{17(26)} = 1 + (p)M_{19(26)} + (pU)M_{22(26)} + (pL)M_{24(26)}$$

$$M_{18(26)} = 1 + (p)M_{7(26)} + (pU)M_{16(26)} + (pL)M_{12(26)}$$

$$M_{19(26)} = 1 + (p)M_{4(26)} + (pU)M_{9(26)} + (pL)M_{15(26)}$$

$$M_{20(26)} = 1 + (p)M_{18(26)} + (pL)M_{25(26)}$$

$$M_{21(26)} = 1 + (p)M_{19(26)} + (pL)M_{24(26)}$$

$$M_{22(26)} = 1 + (p)M_{10(26)} + (pL)M_{20(26)}$$

$$M_{23(26)} = 1 + (p)M_{19(26)} + (pU)M_{22(26)}$$

$$M_{24(26)} = 1 + (p)M_{18(26)} + (pU)M_{21(26)}$$

$$M_{25(26)} = 1 + (p)M_{13(26)} + (pU)M_{23(26)} \qquad \text{(A.3)}$$

The matrix algebra typed using Mathematica 4.0 that corresponds to the above linear system is given in Fig. 14.

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ABOUT THE AUTHOR

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