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Design of stabilizing output feedback nonlinear model predictive controllers with an application to DC-DC converters

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Abstract— **This paper focuses on the synthesis of nonlinear Model Predictive Controllers that can guarantee robustness with respect to measurement noise. The input-to-state stability framework is employed to analyze the robustness of the resulting Model Predictive Control (MPC) closed-loop system. It is illustrated how the obtained robustness result can be employed to synthesize asymptotically stabilizing observer-based output-feedback nonlinear MPC controllers for a class of nonlinear discrete-time systems. The developed theory is illustrated by applying it to control a Buck-Boost DC-DC converter.**

I. INTRODUCTION

One of the problems in Nonlinear Model Predictive Control (NMPC) that receives increased attention and has reached a relatively mature stage, consists in guaranteeing closed-loop stability. The approach usually used to ensure nominal closed-loop stability in NMPC is to consider the value function of the NMPC cost as a candidate Lyapunov function, see the survey [1], for an overview. The stability results heavily rely on state space models of the system, and the assumption that the full state of the real system is available for feedback. However, in practice it is rarely the case that the full state of the system is available for feedback. A possible solution to this problem is the use of an observer. An observer can generate an estimate of the full state using knowledge of the output and input of the system. However, nominal stability results for NMPC usually do not guarantee closed-loop stability of an interconnected NMPC-observer combination. One of the potential approaches to guarantee closed-loop stability in the presence of estimation errors in the state, is to employ (inherent) robustness of the model predictive controller. In [2] asymptotic stability of state feedback NMPC is examined in face of asymptotically decaying disturbances. As stated by the authors of [2], their results are also useful for the solution of the output feedback problem, although a formal proof is missing. A stability result on observer based nonlinear model predictive control is reported in [3], under the standing assumption that the NMPC value function and the resulting NMPC control law are Lipschitz continuous. The stability problem of observer based nonlinear model predictive control is revisited in [4], where only continuity of the NMPC value function is assumed. In [4] robust global asymptotic stability is shown under the assumption that there are no state constraints present in the NMPC problem. Other related results on observer based nonlinear model predictive control can be found in [5], [6]. However, in [5], [6] a continuous-time perspective is taken, while we focus on discrete-time nonlinear systems.

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In this paper we present a framework for the design of asymptotic stabilizing observer-based output feedback nonlinear model predictive controllers for nonlinear discrete-time Lipschitz continuous systems in the presents of input and state constraints. The framework is based on an obtained result which enables to infer Input-to-State Stability (ISS), e.g. see [7], [8] and the references therein, with respect to *measurement noise* from ISS with respect to *additive disturbances* for input and state constrained systems. This result allows one to employ all existing NMPC synthesis techniques that can a priori guarantee ISS with respect to *additive disturbances*, in a scenario where the closed-loop system has to be rendered ISS with respect to measurement noise. The latter scenario is in particular interesting for certainty equivalence output feedback (NMPC) controller design for input and state constrained systems.

The paper is organized as follows. First, some basic definitions and notations are given in Section II, together with basic NMPC notions. In Section II-C we briefly explain the nonlinear observer-based output feedback problem for NMPC from which the problem set-up follows. In Section IV we point out how to design a state feedback NMPC controller which is robust (ISS) to state measurement noise or observation errors, present in, for example, state estimates generated by an observer. In Section V an illustrative example on an constrained output feedback control problem for a Buck-Boost DC-DC converter is given. Conclusions are summarized in Section VI.

II. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the set of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. A function $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a $\mathscr{K}$*-function* if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a $\mathscr{KL}$-*function* if, for each fixed $k \in \mathbb{R}_+$, the function $\beta(\cdot, k)$ is a $\mathscr{K}$*-function*, and for each fixed $s \in \mathbb{R}_+$, the function $\beta(s, \cdot)$ is non-increasing and $\beta(s,k) \to 0$ as $k \to \infty$. For any $x \in \mathbb{R}^n$, $|x|$ stands for its Euclidean norm. For any function $\phi : \mathbb{Z}_+ \mapsto \mathbb{R}^n$, we denote $\|\phi\| = \sup\{|\phi_k| \mid k \in \mathbb{Z}_+\}$, where we use the convention that $\phi_k \triangleq \phi(k)$. For a set $\mathscr{S} \subseteq \mathbb{R}^n$, we denote by $\mathrm{int}(\mathscr{S})$ its interior. For two arbitrary sets $\mathscr{S} \subseteq \mathbb{R}^n$ and $\mathscr{P} \subseteq \mathbb{R}^n$, let $\mathscr{S} \sim \mathscr{P} \triangleq \{x \in \mathbb{R}^n \mid x + \mathscr{P} \subseteq \mathscr{S}\}$ denote the *Pontryagin* difference. For a set $\mathscr{S}$, $\mathscr{S}^n$ denotes

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the *Cartesian product* $\mathscr{S}\times\mathscr{S}\times\ldots\times\mathscr{S}$, where $\mathscr{S}$ appears n times with $n\in\mathbb{Z}_{\geq 1}$. A function $g:\mathbb{X}\times\mathbb{S}\mapsto\mathbb{R}^n$ with $\mathbb{X}\subseteq\mathbb{R}^{n_x}$ and $\mathbb{S}\subseteq\mathbb{R}^{n_s}$ is (globally) *Lipschitz continuous* with respect to x in the domain $\mathbb{X}\times\mathbb{S}$, if there exists a constant $0\leq L_g<\infty$ such that for all x_1, $x_2\in\mathbb{X}$ and for all $s\in\mathbb{S}$, $|g(x_1,s)-g(x_2,s)|\leq L_g|x_1-x_2|$. The constant L_g is called the Lipschitz constant of g with respect to x. By the notation $\mathscr{F}:\mathbb{X}\hookrightarrow\mathbb{Y}$ for $\mathbb{X}\subseteq\mathbb{R}^{n_x}$ and $\mathbb{Y}\subseteq\mathbb{R}^{n_y}$, we mean that $\mathscr{F}$ is a set-valued function from $\mathbb{X}$ to $\mathbb{Y}$, i.e. $\mathscr{F}(x)\subseteq\mathbb{Y}$ for each $x\in\mathbb{X}$.

A. Systems theory notions

Consider a non-autonomous system described by the discrete-time nonlinear difference inclusion

$$x_{k+1}\in\mathscr{F}(x_k,v_k),\qquad k\in\mathbb{Z}_+,\tag{1}$$

where $x_k\in\mathbb{R}^n$ is the state, $v_k\in\mathbb{V}\subseteq\mathbb{R}^{n_v}$ a disturbance at discrete-time $k\in\mathbb{Z}_+$, respectively. The set $\mathbb{V}$ is assumed to be a known set with $0\in\mathbb{V}$. Furthermore, $\mathscr{F}:\mathbb{R}^n\times\mathbb{V}\hookrightarrow\mathbb{R}^n$ is a set-valued mapping with $\mathscr{F}(0,0)=\{0\}$ and $\mathscr{F}(\xi,\upsilon)\neq\emptyset$ for all $\xi\in\mathbb{R}^n$ and all $\upsilon\in\mathbb{V}$. This guarantees that for each initial state x_0 at time $k=0$ and disturbance function $v:\mathbb{Z}_+\mapsto\mathbb{V}$ there exists a solution, not necessarily unique, to system (1). The set of corresponding solutions of the difference inclusion (1) is denoted by $\mathscr{S}_{\mathscr{F}}(x_0,v)$.

Definition II.1 For given sets $\mathscr{X}\subseteq\mathbb{R}^n$ and $\mathbb{V}\subseteq\mathbb{R}^{n_v}$, with $0\in\mathrm{int}(\mathscr{X})$ and $0\in\mathbb{V}$, we call system (1) *Input-to-state Stable* (ISS) with respect to disturbances $v:\mathbb{Z}_+\mapsto\mathbb{V}$ and initial states x_0 in $\mathscr{X}$, if there exist a $\mathscr{K}\mathscr{L}$*-function* β_x and a $\mathscr{K}$*-function* γ_x^v such that for each function $v:\mathbb{Z}_+\mapsto\mathbb{V}$ and each $x_0\in\mathscr{X}$ all solutions $x\in\mathscr{S}_{\mathscr{F}}(x_0,v)$ satisfy

$$|x_k|\leq\beta_x(|x_0|,k)+\gamma_x^v(\|v\|),\qquad\forall k\in\mathbb{Z}_+.\tag{2}$$

Definition II.2 Given a disturbance set $\mathbb{V}$, a set $\mathscr{P}\in\mathbb{R}^n$ is called *Robust Positively Invariant* (RPI) for system (1) if for all $\xi\in\mathscr{P}$ it holds that $\mathscr{F}(\xi,\upsilon)\subseteq\mathscr{P}$ for all $\upsilon\in\mathbb{V}$.

For sufficient conditions for the input-to-state stability property in Definition II.1 for system (1), we refer to [9], [8]. Note that ISS of system (1) implies Lyapunov asymptotic stability for the 0-disturbance system.

B. NMPC notions

Consider the following nominal and perturbed discrete-time nonlinear systems

$$x_{k+1}=f(x_k,u_k),\qquad k\in\mathbb{Z}_+,\tag{3a}$$
$$\widetilde{x}_{k+1}=f(\widetilde{x}_k,u_k)+w_k,\quad k\in\mathbb{Z}_+,\tag{3b}$$

where x_k, $\widetilde{x}_k\in\mathbb{R}^n$ and $u_k\in\mathbb{R}^m$ are the state and the input at discrete-time $k\in\mathbb{Z}_+$, respectively. Furthermore, $f:\mathbb{R}^n\times\mathbb{R}^m\mapsto\mathbb{R}^n$ and $f(0,0)=0$. The vector $w_k\in\mathbb{W}\subseteq\mathbb{R}^n$ denotes an unknown additive disturbance and $\mathbb{W}$ is assumed to be a known set with $0\in\mathbb{W}$. The nominal discrete-time nonlinear system (3a) will be used in an NMPC scheme to make an $N\in\mathbb{Z}_{\geq 1}$ time steps ahead prediction of the system behavior. The system given by (3b) represents a perturbed discrete-time system to which the NMPC controller based on the nominal model (3a) will be applied. Throughout the paper we assume that the state and the controls are constrained for both systems (3a) and (3b) to some *compact* sets $\mathbb{X}\subseteq\mathbb{R}^n$ with $0\in\mathrm{int}(\mathbb{X})$ and $\mathbb{U}\subseteq\mathbb{R}^m$ with $0\in\mathrm{int}(\mathbb{U})$.

For a fixed $N\in\mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(\tilde{x}_k,\mathbf{u}_k)\triangleq[x_{k+1|k}^\top,\ldots,x_{k+N|k}^\top]^\top$ denote the state sequence generated by the nominal system (3a) from initial state $x_{k|k}\triangleq\widetilde{x}_k$ at time k and by applying the input sequence $\mathbf{u}_k\triangleq[u_{k|k}^\top,\ldots,u_{k+N-1|k}^\top]^\top\in\mathbb{U}^N$. The class of *admissible input sequences* defined with respect to the state $x_k\in\mathbb{X}$ is $\mathscr{U}_N(\widetilde{x}_k)\triangleq\{\mathbf{u}_k\in\mathbb{U}^N\mid\mathbf{x}_k(\widetilde{x}_k,\mathbf{u}_k)\in\mathbb{X}^N\}$.

Let $N\in\mathbb{Z}_{\geq 1}$ be given and let $F:\mathbb{R}^n\mapsto\mathbb{R}_+$ with $F(0)=0$ and $L:\mathbb{R}^n\times\mathbb{R}^m\mapsto\mathbb{R}_+$ with $L(0,0)=0$ be continuous bounded mappings. At time $k\in\mathbb{Z}_+$, let $\widetilde{x}_k\in\mathbb{X}$ be given. The basic model predictive control scenario consists in minimizing, via *optimization*, at each time $k\in\mathbb{Z}_+$ a finite horizon cost function of the form

$$J(\widetilde{x}_k,\mathbf{u}_k)\triangleq F(x_{k+N|k})+\sum_{i=0}^{N-1}L(x_{k+i|k},u_{k+i|k}),\tag{4}$$

with prediction model (3a), over all sequences $\mathbf{u}_k$ in $\mathscr{U}_N(\widetilde{x}_k)$. We call a state $\tilde{x}_k\in\mathbb{X}$ *feasible* if $\mathscr{U}_N(\tilde{x}_k)\neq\emptyset$. Let $\mathscr{X}_f(N)\subseteq\mathbb{X}$ denote the set of *feasible initial states* with respect to the mentioned optimization problem. Then $V_{\text{MPC}}:\mathscr{X}_f(N)\to\mathbb{R}_+$,

$$V_{\text{MPC}}(\widetilde{x}_k)\triangleq\inf_{\mathbf{u}_k^{[0,N-1]}\in\mathscr{U}_N(\widetilde{x}_k)}J(\widetilde{x}_k,\mathbf{u}_k^{[0,N-1]})\tag{5}$$

is the nonlinear model predictive control value function corresponding to the cost (4). If there exists an optimal sequence of controls $\mathbf{u}_k^\star\triangleq[u_{k|k}^{\star\top},u_{k+1|k}^{\star\top},\ldots,u_{k+N-1|k}^{\star\top}]^\top$ that minimizes (5), see [10], the infimum in (5) is a minimum and $V_{\text{MPC}}(\widetilde{x}_k)=J(\widetilde{x}_k,\mathbf{u}_k^\star)$. However, in practice numerical solvers usually provide a feasible (non-unique), *sub-optimal* sequence $\overline{\mathbf{u}}_k\triangleq[\overline{u}_{k|k}^\top,\overline{u}_{k+1|k}^\top,\ldots,\overline{u}_{k+N-1|k}^\top]^\top$ to the MPC optimization problem with resulting value function $\overline{V}_{\text{MPC}}(\widetilde{x})\triangleq J(\widetilde{x},\overline{\mathbf{u}}_k)$. Then, an NMPC control law is denoted as

$$u_k\in\kappa^{\text{MPC}}(\widetilde{x}_k)\triangleq\overline{u}_{k|k},\quad k\in\mathbb{Z}_+.\tag{6}$$

The NMPC control law either optimal or sub-optimal can be substituted in (3b) and yields the closed-loop system

$$\widetilde{x}_{k+1}=f(\widetilde{x}_k,\kappa^{\text{MPC}}(\widetilde{x}_k))+w_k\triangleq\mathscr{F}_w(\widetilde{x}_k,w_k),\quad k\in\mathbb{Z}_+,\tag{7}$$

with $w_k\in\mathbb{W}\subseteq\mathbb{R}^n$.

C. Observer-based output feedback: A summary

Consider the following system

$$x_{k+1}=f(x_k,u_k),\quad y_k=g(x_k),\quad k\in\mathbb{Z}_+,\tag{8}$$

where $x_k\in\mathbb{R}^n$, $u_k\in\mathbb{R}^m$ and $y_k\in\mathbb{R}^\ell$ is the state, the control and the output at discrete-time $k\in\mathbb{Z}_+$, respectively. Furthermore, f is defined as in (3a) and $g:\mathbb{R}^n\mapsto\mathbb{R}^\ell$ with $g(0)=0$. The observer problem for (8) deals with the question how to reconstruct the state trajectory $x(\cdot,x_0,u)$ on the basis of the knowledge of the input u and the output y of the system. The observer design problem in its full generality is a problem that is not yet fully solved for nonlinear systems of the form (8). Loosely speaking a full order observer (observer for brevity) for system (8) is, for example, a dynamical system of the form

$$\hat{x}_{k+1} \triangleq \hat{f}(\hat{x}, y_k, u_k), \quad k \in \mathbb{Z}_+ \tag{9}$$

where $\hat{x} \in \mathbb{R}^n$ is an estimate of the state x, and $\hat{f} : \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is designed such that the estimation error $e_k \triangleq x_k - \hat{x}_k$ (at least) asymptotically converges to zero as $k \to \infty$ for all initial conditions x_0 and $\hat{x}_0$ in some subset of $\mathbb{R}^n$. In this paper we will however not deal with the observer design problem. The focus is on how to synthesize a state feedback NMPC controller which can handle the presence of state estimation errors in the state used for feedback. This is of great importance if a certainty equivalence output feedback control approach is employed. That is, by lack of knowledge of the real state x_k an estimate of the state $\hat{x}_k$ is injected to a state feedback NMPC controller instead, i.e. $u_k \in \kappa^{\text{MPC}}(\hat{x}_k)$. The state estimate $\hat{x}_k$ is obtained by an observer of which the *error dynamics* (i.e. the dynamics which describes the error signal e) is assumed to be *asymptotically stable*.

III. Problem Formulation

In order to obtain an asymptotically stable closed-loop system, resulting from employing the certainty equivalence output feedback control approach, we will synthesis an NMPC controller which is robust, i.e. ISS, with respect to the estimation errors e induced by the observer. We assume that the observer, with its asymptotically stable error dynamics, is initialized in such a way that $e_k \in \mathbb{E} \subseteq \mathbb{R}^n$ for all $k \in \mathbb{Z}_+$, i.e. $\hat{x}_0 \in \mathbb{X}$ such that $e_k = (\hat{x} - x_k) \in \mathbb{E}$ for all $k \in \mathbb{Z}_+$. Then, if the controller renders the following system

$$x_{k+1} = f(x_k, \kappa^{\text{MPC}}(x_k + e_k)) \triangleq \mathscr{F}_e(x_k, e_k), \quad k \in \mathbb{Z}_+, \tag{10}$$

ISS with respect to the estimation errors (measurement noise) $e : \mathbb{Z}_+ \mapsto \mathbb{E}$, it is known that if the estimation error vanishes, e.g. $e_k \to 0$ for $k \to \infty$ also $x_k \to 0$ for $k \to \infty$. This follows directly from the ISS system property given in Definition II.1. Hence, an asymptotically stable closed-loop system, resulting from employing the certainty equivalence output feedback control approach, is obtained. In the next section we will show how to render (10) ISS with respect to $e : \mathbb{Z}_+ \mapsto \mathbb{E}$.

IV. ISS NMPC controller design

As explained in the previous section, we seek for NMPC schemes that renders (10) ISS with respect to the estimation error e. Rendering system (10) ISS with respect to the estimation error e using NMPC is however difficult. The problem was considered in [3], where robustness to estimation errors is shown under the assumption of Lipschitz continuity of the NMPC value function and control law. A similar result was obtained more recently in [4], under the milder assumption of continuity of the NMPC value function and not necessarily of the NMPC control law. However, in [4] state constraints are not considered. To the best of the authors' knowledge, besides the global results of [4] which holds under the condition that there are no state constraints considered, no general practically applicable NMPC schemes are available in literature that can a priori guarantee ISS of (10) with respect to the estimation error e as input in the presents of state constraints. However, due to the result obtained in this section we can infer ISS of (10) with respect to e from ISS of (7) with respect to additive disturbances w. This result then allows us to employ all existing NMPC schemes, e.g. [2], [9], [11], [12], that can a priori guarantee ISS of (7) to also establish a priori ISS of (10). To give an example of an NMPC scheme that is ISS with respect to additive disturbances, we briefly recall the ISS MPC scheme of [12]. This scheme will also be employed later in Section V to control a DC- DC converter. Let $P_V \in \mathbb{R}^{p_v \times n}$ and $Q_V \in \mathbb{R}^{q_v \times n}$ be full-column rank matrices.

Algorithm IV.1

Step 1) Given the state $\widetilde{x}_k$ at time $k \in \mathbb{Z}_+$, let $x_{k|k} \triangleq \widetilde{x}_k$ and find a control sequence that satisfies

$$|P_V(f(x_{k|k}, u_{k|k}))| - |P_V x_{k|k}| \leq -|Q_V x_{k|k}| \tag{11a}$$

$$\mathbf{u}_k \in \mathscr{U}_N(\widetilde{x}_k) \tag{11b}$$

and optionally also minimize the cost $J(\widetilde{x}_k, \mathbf{u}_k)$ in (5).

Step 2) Let

$$\kappa^{\text{MPC}}(\widetilde{x}_k) \triangleq \left\{ u_{k|k} \in \mathbb{U} \;\middle|\; \mathbf{u}_k \text{ satisfies } (11) \right\}.$$

Furthermore, let $\overline{\mathbf{u}}_k$ with $\overline{u}_k \in \kappa^{\text{MPC}}(\widetilde{x}_k)$ denote a feasible sequence of controls with respect to the optimization problem formulated at Step 1. Apply a control $u_k = \overline{u}_{k|k} \in \kappa^{\text{MPC}}(\widetilde{x}_k)$ to the perturbed system (3b), increment k be one and go to Step 1.

The following result is proven in [12] for the nonlinear system (3b) in closed-loop with Algorithm IV.1 forming closed-loop system (7).

Theorem IV.2 [12] *Let $\mathscr{X}_f(N)$ be the set of states $\widetilde{x}_k \in \mathbb{X}$ for which the optimization problem in Step 1 of Algorithm IV.1 is feasible and let $\widetilde{\mathscr{X}}(N) \subseteq \mathscr{X}_f(N)$ be an RPI set with $0 \in \text{int}(\widetilde{\mathscr{X}}(N))$ for closed-loop system* (7) *perturbed by additive disturbances $w : \mathbb{Z}_+ \mapsto \mathbb{W}$. Then, system* (7) *is input-to-state stable with respect to disturbances $w : \mathbb{Z}_+ \mapsto \mathbb{W}$ and initial states $\widetilde{x}_0$ in $\widetilde{\mathscr{X}}(N)$.*

For ways to compute matrices P_V and Q_V *off-line*, we refer the reader to [12]. Now we are ready to state the main result of this section, which enables one to obtain an NMPC controller that is ISS with respect to *estimation errors e* from an NMPC controller, like the one just presented, that is ISS with respect to *additive disturbances w*.

Assumption IV.3 Let the function $f(\cdot,\cdot)$ be Lipschitz continuous with respect to its first argument in the domain $\mathbb{X} \times \mathbb{U}$ with Lipschitz constant L_f.

Assumption IV.4 Let $\mathbb{W} \triangleq \left\{ \omega \in \mathbb{R}^n \mid |\omega| \leq \lambda \right\}$, for some $\lambda \in \mathbb{R}_{>0}$. Suppose that system (7) is ISS with respect to *additive disturbances* in $\mathbb{W}$ and initial states $\widetilde{x}_0$ in $\widetilde{\mathscr{X}}(N) \subseteq \mathbb{X}$ with $0 \in \text{int}(\widetilde{\mathscr{X}}(N))$, i.e. there exist a *$\mathscr{K}\mathscr{L}$-function* $\beta_{\tilde{x}}$ and a *$\mathscr{K}$-function* $\gamma^w_{\tilde{x}}$ such that for all $\widetilde{x}_0 \in \widetilde{\mathscr{X}}(N)$ and $w : \mathbb{Z}_+ \mapsto \mathbb{W}$ all solutions $\widetilde{x} \in \mathscr{S}_{\mathscr{F}_w}(\widetilde{x}_0, w)$ satisfy

$$|\widetilde{x}_k| \leq \beta_{\tilde{x}}(|\widetilde{x}_0|, k) + \gamma^w_{\tilde{x}}(\|w\|), \qquad \forall k \in \mathbb{Z}_+. \tag{12}$$

Furthermore, assume that $\widetilde{\mathscr{X}}(N)$ is RPI for system (7) perturbed by *additive disturbances* in $\mathbb{W}$.

Theorem IV.5 *Suppose Assumptions IV.3 and IV.4 hold. Let* $\mathbb{E} \triangleq \{\varepsilon \in \mathbb{R}^n \mid |\varepsilon| \leq \lambda/(L_f+1)\}$*,* $\mathscr{X}(N) \triangleq \widetilde{\mathscr{X}}(N) \sim \mathbb{E}$ *and suppose* $0 \in \text{int}(\mathscr{X}(N))$*. Then, the following statements hold.*
i) *The set* $\mathscr{X}(N) \subset \mathbb{X}$ *is an RPI set for closed-loop system* (10) *perturbed by state measurement errors in* $\mathbb{E}$*;*
ii) *The state and input constraints are satisfied for all trajectories of* (10) *with initial states* x_0 *in* $\mathscr{X}(N)$ *and measurement errors in* $\mathbb{E}$*, i.e. for all* $x \in \mathscr{S}_{\mathscr{F}_e}(x_0, e)$ *with* $x_0 \in \mathscr{X}(N)$ *and* $e : \mathbb{Z}_+ \mapsto \mathbb{E}$ *it holds that* $x_k \in \mathbb{X}(N)$ *and* $\kappa(x_k + e_k) \subseteq \mathbb{U}$ *for all* $k \in \mathbb{Z}_+$*;*
iii) *The closed-loop system* (10) *is ISS with respect to state measurement errors in* $\mathbb{E}$ *and initial states* x_0 *in* $\mathscr{X}(N)$*. In particular, we have that for all* $x_0 \in \mathscr{X}(N)$ *and* $e : \mathbb{Z}_+ \mapsto \mathbb{E}$ *all solutions* $x \in \mathscr{S}_{\mathscr{F}_e}(x_0, e)$ *satisfy*

$$|x_k| \leq \beta_x(|x_0|, k) + \gamma_x^e(\|e\|), \qquad \forall k \in \mathbb{Z}_+, \tag{13}$$

with $\beta_x(|x_0|, k) \triangleq \beta_{\tilde{x}}(2|x_0|, k)$ *and*

$$\gamma_x^e(\|e\|) \triangleq \beta_{\tilde{x}}(2\|e\|, 0) + \gamma_{\tilde{x}}^w((L_f+1)\|e\|) + \|e\|.$$

Proof:

i) Let $\xi \in \mathscr{X}(N)$ and $\varepsilon \in \mathbb{E}$. We will show that for all $\overline{\varepsilon} \in \mathbb{E}$,

$$f(\xi, \kappa(\xi + \varepsilon)) + \overline{\varepsilon} \subseteq \widetilde{\mathscr{X}}(N) \tag{14}$$

as this yields $f(\xi, \kappa(\xi + \varepsilon)) \subseteq \widetilde{\mathscr{X}}(N) \sim \mathbb{E} = \mathscr{X}(N)$. Then, since $\xi \in \mathscr{X}(N)$ and $\varepsilon \in \mathbb{E}$ are arbitrary, this would prove that $\mathscr{X}(N)$ is RPI. We proceed by observing that

$$f(\xi, \mu) + \overline{\varepsilon} = f(\widetilde{\xi}, \mu) + \omega, \quad \forall \mu \in \kappa(\widetilde{\xi}) \subseteq \mathbb{U} \tag{15}$$

with $\widetilde{\xi} \triangleq \xi + \varepsilon$ and $\omega \triangleq f(\xi, \mu) - f(\widetilde{\xi}, \mu) + \overline{\varepsilon}$. Using the Lipschitz property of f yields $|f(\widetilde{\xi} - \varepsilon, \mu) - f(\widetilde{\xi}, \mu)| \leq L_f|\varepsilon|$. Therefore, it holds that for all $\varepsilon, \overline{\varepsilon} \in \mathbb{E}$ and $\widetilde{\xi} \in \widetilde{\mathscr{X}}(N)$

$$\begin{aligned} |\omega| &= |f(\widetilde{\xi} - \varepsilon, \mu) - f(\widetilde{\xi}, \mu) + \overline{\varepsilon}|, \ \forall \mu \in \kappa(\widetilde{\xi}) \subseteq \mathbb{U}, \\ &\leq L_f|\varepsilon| + |\overline{\varepsilon}| \leq L_f|\varepsilon| + |\varepsilon| \leq \lambda. \end{aligned} \tag{16}$$

The last inequality in (16) shows that $\omega \in \mathbb{W}$. Together with the hypothesis of Theorem IV.5, i.e. RPI of $\widetilde{\mathscr{X}}(N)$ for system (7) under additive disturbances in $\mathbb{W}$, (15) yields that for all $\overline{\varepsilon} \in \mathbb{E}$ and $\widetilde{\xi} \in \widetilde{\mathscr{X}}(N)$,

$$f(\xi, \mu) + \overline{\varepsilon} \in \widetilde{\mathscr{X}}(N), \quad \forall \mu \in \kappa(\widetilde{\xi}) \subseteq \mathbb{U}.$$

Hence, we obtain that for all $\overline{\varepsilon} \in \mathbb{E}$ (14) holds.

ii) Due to i), it holds that for any $x_0 \in \mathscr{X}(N)$ and any $e : \mathbb{Z}_+ \mapsto \mathbb{E}$ all trajectories $x \in \mathscr{S}_{\mathscr{F}_e}(x_0, e)$ satisfy $x_k \in \mathscr{X}(N) \subseteq \mathbb{X}$, $x_k + e_k \in \widetilde{\mathscr{X}}(N) \subseteq \mathbb{X}$ for all $k \in \mathbb{Z}_+$ and thus $u_k \in \kappa(x_k + e_k) \subseteq \mathbb{U}$ for all $k \in \mathbb{Z}_+$.

iii) Let x_0 in $\mathscr{X}(N)$, $e : \mathbb{Z}_+ \mapsto \mathbb{E}$ and $x \in \mathscr{S}_{\mathscr{F}_e}(x_0, e)$. We perform the following coordinate change on (10)

$$x_k = \widetilde{x}_k - e_k, \qquad \forall k \in \mathbb{Z}_+, \tag{17}$$

which gives

$$\widetilde{x}_{k+1} \in f(\widetilde{x}_k - e_k, \kappa(\widetilde{x}_k)) + e_{k+1}, \qquad k \in \mathbb{Z}_+, \tag{18}$$

or

$$\widetilde{x}_{k+1} \in f(\widetilde{x}_k, \kappa(\widetilde{x}_k)) + w_k, \qquad k \in \mathbb{Z}_+, \tag{19}$$

where

$$\begin{aligned} w_k \triangleq & f(\widetilde{x}_k - e_k, u_k) - f(\widetilde{x}_k, u_k) + e_{k+1}, \ \text{for some} \\ & u_k \in \kappa(\widetilde{x}_k) \subseteq \mathbb{U}, \ e_k, e_{k+1} \in \mathbb{E} \ \text{ and } \ \widetilde{x}_k \in \widetilde{\mathscr{X}}(N). \end{aligned} \tag{20}$$

Hence,

$$w_k \in \overline{\mathbb{W}} \triangleq \Big\{ f(\widetilde{\xi} - \varepsilon, \mu) - f(\widetilde{\xi}, \mu) + \overline{\varepsilon} \mid \mu \in \kappa(\widetilde{\xi}) \subseteq \mathbb{U}, \\ \varepsilon, \overline{\varepsilon} \in \mathbb{E}, \ \widetilde{\xi} \in \widetilde{\mathscr{X}}(N) \Big\}.$$

We claim that $\overline{\mathbb{W}} \subseteq \mathbb{W}$. Indeed, if $\omega \in \overline{\mathbb{W}}$, then we can utilize the Lipschitz property of f to obtain that for all $\varepsilon, \overline{\varepsilon} \in \mathbb{E}$ and $\widetilde{\xi} \in \widetilde{\mathscr{X}}(N)$ (16) holds, which implies that $\overline{\mathbb{W}} \subseteq \mathbb{W}$ and therefore $w_k \in \mathbb{W}$ for all $k \in \mathbb{Z}_+$. Due to the fact that $w_k \in \mathbb{W}$ for all $k \in \mathbb{Z}_+$ and that item ii) in the proof, i.e. $x_k + e_k \in \widetilde{\mathscr{X}}$ for all $k \in \mathbb{Z}_+$, holds, we obtain that $\widetilde{x}_k \in \widetilde{\mathscr{X}}(N)$ for all $k \in \mathbb{Z}_+$. As a consequence, the hypothesis in Theorem IV.5 shows that (19) is ISS w.r.t. *additive disturbance* $w : \mathbb{Z}_+ \mapsto \mathbb{W}$ and initial conditions $\widetilde{x}_0 \in \widetilde{\mathscr{X}}$. Hence, we have that (12) holds true. Via (20) and utilizing the Lipschitz property of f in a similar manner as in (16), we obtain that for all $u_k \in \kappa(\widetilde{x}_k) \subseteq \mathbb{U}$, $e_k, e_{k+1} \in \mathbb{E}$, $\widetilde{x}_k \in \widetilde{\mathscr{X}}(N)$ and $k \in \mathbb{Z}_+$

$$\begin{aligned} |w_k| &\leq |f(\widetilde{x}_k - e_k, u_k) - f(\widetilde{x}_k, u_k) + e_{k+1}| \\ &\leq |f(\widetilde{x}_k - e_k, u_k) - f(\widetilde{x}_k, u_k)| + \|e\| \\ &\leq (L_f + 1)\|e\|. \end{aligned} \tag{21}$$

Substituting the last inequality in (21) into (12) yields for all $k \in \mathbb{Z}_+$

$$|\widetilde{x}_k| \leq \beta_{\tilde{x}}(|\widetilde{x}_0|, k) + \gamma_{\tilde{x}}^e(\|e\|), \tag{22}$$

where $\gamma_{\tilde{x}}^e(\|e\|) = \gamma_{\tilde{x}}^w((L_f+1)\|e\|)$. Applying (17) and property (22) yields

$$\begin{aligned} |x_k| &= |\widetilde{x}_k - e_k| \leq |\widetilde{x}_k| + |e_k| \leq \\ &\leq \beta_{\tilde{x}}(|x_0 + e_0|, k) + \gamma_{\tilde{x}}^e(\|e\|) + |e_k| \\ &\leq \beta_{\tilde{x}}(|x_0| + |e_0|, k) + \gamma_{\tilde{x}}^{e_x}(\|e\|) + \|e\| \\ &\leq \beta_{\tilde{x}}(2|x_0|, k) + \beta_{\tilde{x}}(2|e_0|, k) + \gamma_{\tilde{x}}^e(\|e_x\|) + \|e\| \\ &\leq \beta_{\tilde{x}}(2|x_0|, k) + \beta_{\tilde{x}}(2\|e\|, 0) + \gamma_{\tilde{x}}^e(\|e\|) + \|e\| \\ &= \beta_x(|x_0|, k) + \gamma_x^e(\|e\|). \end{aligned}$$

■

V. Output feedback control of a Buck-Boost DC-DC converter

In this section we illustrate Algorithm IV.1, where instead of x_k we use $\hat{x}_k$ for feedback. The estimate $\hat{x}_k$ is generated by an observer having asymptotically stable error dynamics. The resulting output-based NMPC scheme is employed to control a Buck-Boost DC-DC converter power circuit. Buck-Boost circuits are very important and they are widely used in relevant applications, ranging from hybrid and electric vehicles to solar plants, etc. An schematic view of the circuit is given in Fig. 1. A discrete-time nonlinear averaged model of the converter has been developed in [13], i.e.

$$x_{k+1}^m = \begin{bmatrix} x_{1,k}^m + \frac{T}{L}x_{2,k}^m - \frac{T}{L}(x_{2,k}^m - V_{in})u_k^m \\ -\frac{T}{C}x_{1,k}^m + \frac{T}{C}x_{1,k}^m u_k^m + (1 - \frac{T}{RC})x_{2,k}^m \end{bmatrix}, \ \ y_k^m = x_{2,k}^m, \tag{23}$$

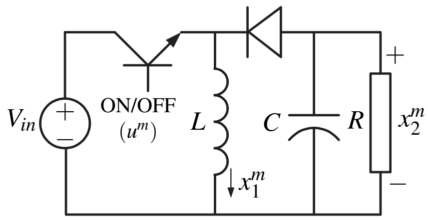

Fig. 1. A schematic representation of a Buck-Boost converter.

where $x_k^m = [x_{1,k}^m \; x_{2,k}^m]^\top \in \mathbb{R}^2$, $y_k^m \in \mathbb{R}$ and $u_k^m \in \mathbb{R}$ are the state the output and control, respectively. x_1^m represents the current flowing through the inductor, x_2^m the voltage across the capacitor and u^m represents the duty cycle (i.e. the fraction of the sampling period during which the transistor is kept ON). The sampling period T corresponds to a sampling frequency of $10[kHz]$, the inductance $L = 4.2 \times 10^{-3}[H]$, the capacitance $C = 2200[\mu F]$, the load resistance $R = 165[\Omega]$ and the source input voltage is equal to $V_{in} = 15[V]$. The control objective is to reach a desired steady state value of the output voltage, i.e. x_2^{ss}, as fast as possible and with minimum overshoot. In practice only the output, i.e. output voltage $x_{2,k}^m$, is available for feedback, so output based controllers with a stability guarantee are needed. At the same time, constraints on the current $x_{1,k}^m$ must be fulfilled. The output based MPC framework, as proposed in this paper, is used to design a controller for this task. From x_2^{ss} one can obtain the steady state duty cycle and inductor current as follows:

$$u^{ss} = \frac{x_2^{ss}}{x_2^{ss} - V_{in}}, \quad x_1^{ss} = \frac{x_2^{ss}}{R(u^{ss}-1)}. \tag{24}$$

Furthermore, the following physical constraints must be fulfilled at all times $k \in \mathbb{Z}_+$:

$$x_{1,k}^m \in [0.01, 5], \quad x_{2,k}^m \in [-20, 0], \quad u_k^m \in [0.1, 0.6]. \tag{25}$$

To implement the proposed MPC scheme we perform first the following coordinate transformation on (23)

$$x_{1,k} = x_{1,k}^m - x_1^{ss}, \quad x_{2,k} = x_{2,k}^m - x_2^{ss}, \quad u_k = u_k^m - u^{ss}. \tag{26}$$

This is done in order to obtain a model of the form (8) with $f(0,0) = 0$. We obtain then the following vector fields f and g, defining a nonlinear model of form (8):

$$f(x_k, u_k) = \begin{bmatrix} x_{1,k} + \alpha x_{2,k} + (\beta - \frac{T}{L} x_{2,k}) u_k \\ (\frac{T}{C} x_{1,k} + \gamma) u_k + (1 - \frac{T}{RC}) x_{2,k} + \delta x_{1,k} \end{bmatrix}, \tag{27}$$
$$g(x_k) = x_{2,k}.$$

The constants α, β, γ and δ depend on the fixed steady state value x_2^{ss} as follows

$$\alpha = \frac{T}{L}\left(1 - \frac{x_2^{ss}}{x_2^{ss} - V_{in}}\right), \; \beta = \frac{T}{L}(V_{in} - x_2^{ss}),$$

$$\gamma = \frac{T}{RCV_{in}} x_2^{ss}(x_2^{ss} - V_{in}) \quad \text{and} \quad \delta = \frac{T}{C}\left(\frac{x_2^{ss}}{x_2^{ss} - V_{in}} - 1\right).$$

Using (26) and (24), the constraints given in (25) can be converted to:

$$\begin{aligned} \mathbb{X} &= \left\{ x \in \mathbb{R}^2 \,\middle|\, x_1 \in [\underline{b}^{x_1}, \overline{b}^{x_1}], \, x_2 \in [\underline{b}^{x_2}, \overline{b}^{x_2}] \right\} \\ \mathbb{U} &= \left\{ u \in \mathbb{R} \,\middle|\, u \in [\underline{b}^{u}, \overline{b}^{u}] \right\}, \end{aligned} \tag{28}$$

where

$$\underline{b}^{x_1} = 0.01 - \frac{1}{RV_{in}} x_2^{ss}(x_2^{ss} - V_{in}), \; \overline{b}^{x_1} = 5 - \frac{1}{RV_{in}} x_2^{ss}(x_2^{ss} - V_{in}),$$
$$\underline{b}^{x_2} = -20 - x_2^{ss}, \quad \overline{b}^{x_2} = -x_2^{ss},$$
$$\underline{b}^{u} = 0.1 - \frac{x_2^{ss}}{x_2^{ss} - V_{in}} \quad \text{and} \quad \overline{b}^{u} = 0.6 - \frac{x_2^{ss}}{x_2^{ss} - V_{in}}.$$

The control objective can now be formulated as to stabilize (27) around the equilibrium $(0,0)$ while fulfilling the constraints given in (28).

1) Controller: The ISS (w.r.t. additive disturbances) NMPC scheme from Section IV is employed to design the NMPC controller. The method in [12] is applied to find a matrix P_V which defines the proposed ISS Lyapunov function $V(x) = |P_V x|$. For $Q_V = \left[\begin{smallmatrix} 0.1 & 0 \\ 0 & 0.1 \end{smallmatrix}\right]$ we have obtained $P_V = \left[\begin{smallmatrix} 2.4545 & 4.9275 \\ 5.6292 & -6.0353 \end{smallmatrix}\right]$. The employed NMPC costs defined by F and L are given as $F = |Px_{k+N|k}|$ and $L = |Qx_{k+i|k}| + |R_u u_{k+i|k}|$. To achieve good performance, the NMPC cost matrices have been chosen as follows: $P = \left[\begin{smallmatrix} 5 & 0 \\ 0 & 10 \end{smallmatrix}\right]$, $Q = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 6 \end{smallmatrix}\right]$ and $R_u = 0.001$. Further we chose the prediction horizon to be $N = 5$. To conclude about ISS w.r.t. e_x (possibly induced by an observer) we rely on Theorem IV.5.

2) Observer: For the observer design we employ the proposed extended observer design methodology proposed in [14]. For the considered system, defined by f and g, we can obtain an observer with asymptotically stable error dynamics. Due to the scope of the paper and space limitations we will not go into further details concerning the observer design. For more details on the employed observer design methodology in relation to NMPC we refer the reader to [15], [16]. However, in order to provide the reader with results that can be reproduced, we give the functions that define an observer from the used observer design methodology, and refer to [15] for more details on this issue.

The following functions f_z and h_z define the employed observer on the domain $\mathbb{X} \times \mathbb{U}$

$$f_z(y_{k-1}, y_k, u_{k-1}, u_k) =$$
$$\begin{bmatrix} 0 \\ \left(\left(\frac{T}{C}\left(\varpi + \alpha y_{k-1} + \left(\beta - \frac{T}{L} y_{k-1}\right) u_{k-1}\right) + \gamma\right) u_k \\ + \left(1 - \frac{T}{RC}\right)\left(\left(\frac{T}{C}\varpi + \gamma\right) u_{k-1} + \left(1 - \frac{T}{RC}\right) y_{k-1} + \delta\varpi\right) + \delta(\varpi + \alpha y_{k-1} \\ + \left(\beta - \frac{T}{L} y_{k-1}\right) u_{k-1})\right) \end{bmatrix},$$

with

$$\varpi = \frac{RC(y_k - \gamma u_{k-1} - y_{k-1}) + T y_{k-1}}{R(T u_{k-1} + \delta C)},$$

and

$$h_z(z_{n,k}) = z_{n,k}. \tag{29}$$

For observer gains $\ell_1 = 0.1$ and $\ell_2 = 0.1$ we have an observer with asymptotically stable error dynamics.

3) Simulation: Simulation results for the closed-loop system, i.e. NMPC controller interconnected with the observer and the system, are given in Fig. 2 and Fig. **??**. Note that although the NMPC controller and observer computations were performed for the transformed system, defined by (27), we chose to present all variables in Fig. 2 in original coordinates (corresponding to (23)), in order to preserve the physical meaning of the results. For the obtained simulation the system was initialized with $x_0^m = [0.01\ 0]^T$ and the observer was initialized with $\hat{z}_0 = [0.1\ 0.2]^T$, $u_{-1} = 0$ and $y_{-1} = 0$ (see [14]). Note the estimation errors $e_{x_1,k}$ and $e_{x_2,k}$ converge to zero and the desired steady state value x_2^{ss} of system (23) is reached within reasonable time without any overshoot. Moreover, the constraints (25) are satisfied at all times.

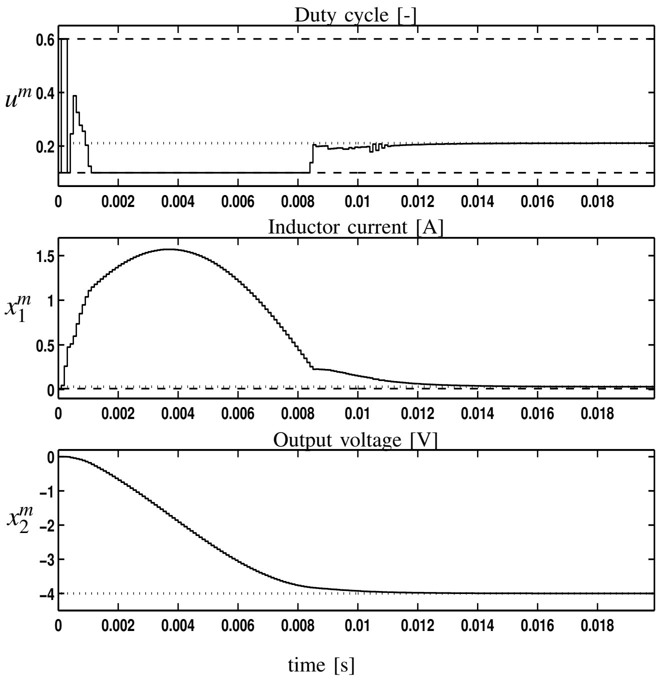

Fig. 2. The state and control trajectories are represented by the solid lines. The dashed and dotted lines represent the constraints and the desired steady state values ($x_1^{ss} = 0.0307[A]$, $x_2^{ss} = -4[V]$, $u^{ss} = 0.2105[-]$), respectively.

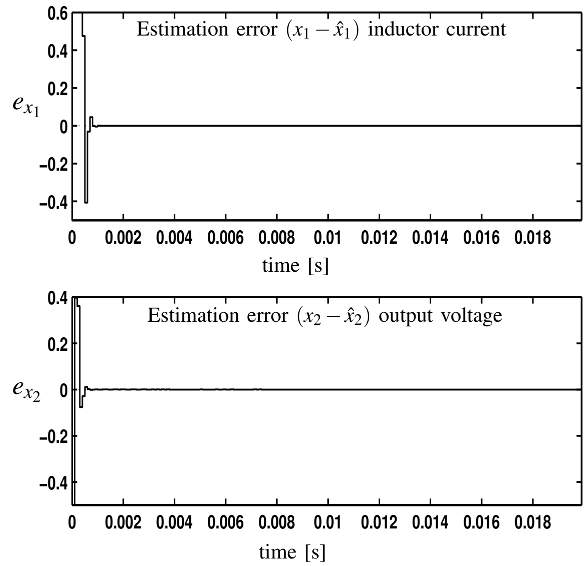

Fig. 3. Estimation error trajectories (in x-coordinates) of the observer error dynamics.

VI. Conclusions

We propose a framework for the design of (local) asymptotically stabilizing observer-based output feedback model predictive controllers for nonlinear discrete time Lipschitz continuous systems with input and state constraints. The main result on which the framework is based, is the ability to infer ISS with respect to *measurement noise* (e.g. presence of observer errors in the state) from ISS with respect to *additive disturbances* for the considered class of systems. Furthermore, a case study on the output-based control of a Buck-Boost DC-DC power converter is presented.

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