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DESIGNS FOR REGRESSION PROBLEMS WITH CORRELATED ERRORS; MANY PARAMETERS

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1. Introduction. Suppose one may observe a stochastic process $Y(\cdot)$ having the form

$$Y(t) = \sum_{j=1}^{J} \beta_j f_j(t) + X(t), \qquad t \varepsilon [0, 1]$$

where the β_j's are unknown parameters, the f_j's are known functions and $X(\cdot)$ is a stochastic process with mean function zero and known covariance kernel R. Under minor conditions on the regression functions f_j and the kernel R, one may for a suitably large finite observation set T, give expression to the best linear estimate (BLE) $\hat{\beta}$ of $\beta = (\beta_1, \cdots, \beta_J)'$ and to the covariance matrix of this estimate, say A_T^{-1}.

In a previous paper [2], we treated the following design set-up for the case $J = 1$ above: if $D_n = \{T \mid T = \{t_1, \cdots, t_n\}, 0 \leqq t_1 < \cdots < t_n \leqq 1\}$, an optimum design in D_n is a set T^* which minimizes A_T^{-1} over D_n (here A_T^{-1} is simply the variance of the BLE of β_1 based on the observation set T). In [2], we discussed the question of existence of optimum designs and, under certain restrictions, we produced sequences of designs $\{T_n^*\}$, $T_n^* \varepsilon D_n$, asymptotically optimum as $n \to \infty$. The necessity of pursuing such an asymptotic theory is discussed at some length in the introduction to [2].

In the present paper, we consider the cases $J > 1$. The fundamental difference here is, of course, that A_T^{-1} is a $J \times J$ matrix. Since one cannot expect the minimum of A_T^{-1} to exist over D_n (minimum in the sense of the ordering of non-negative definite matrices), our current problems arise from the attempt to minimize certain one-dimensional measures of the size of A_T^{-1}. The criteria treated below include, for example, the variance of $\theta'\hat{\beta}$ (viz. $\theta' A_T^{-1} \theta$) where θ is a fixed vector, the maximum variance of $\theta'\hat{\beta}$ with the maximum taken over a compact set $\mathfrak{M}$ of vectors, and the generalized variance det A_T^{-1}. We also consider certain regret criteria where regret is measured relative to what could be achieved through use of the observation set [0, 1].

Our basic assumptions are stated in Section 2 together with various formulations of the optimality of designs and the asymptotic optimality of sequences of designs. The necessary asymptotic results for this study are presented in Section 3 and, in the final section, they are applied to find asymptotically optimum sequences of designs for a variety of criteria. The asymptotically optimum sequences found are all, what we call below, regular. That is, the $(n + 1)$st de-

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sign set in a given sequence is a set of n-tiles of some fixed absolutely continuous distribution function with support in [0, 1]. Thus, asymptotically optimum sequences are generated by suitable densities on [0, 1] and our results are phrased in this way.

It should be noted that we have not pursued the question of existence of optimum designs here. In fact, most questions of this nature are rather easily answered by reference to Section 2 of [2].

2. Assumptions and preliminaries. We suppose that observations may be taken on a stochastic process $Y(t)$ of the form

$$Y(t) = \sum_{j=1}^{J} \beta_j f_j(t) + X(t), \qquad t \in [0, 1], \tag{2.1}$$

where the β_j's are unknown parameters, the f_j's are known functions, and $X(\cdot)$ is a process with mean functions zero and covariance kernel R on the unit square. The restriction to [0, 1] in (2.1) is solely for convenience.

Associated with R is a reproducing kernel Hilbert space $\mathfrak{F}$ of functions on [0, 1]. Much of our analysis concerns the inner product in this space and we denote it by $\langle \, , \rangle$ with $\langle f, f \rangle = \|f\|^2$. The assumptions we make about R are the same as made in [2] and we repeat them here for convenience. These assumptions are discussed at length in [2].

ASSUMPTION. A. R has continuous partial derivatives up to order two at every (s, t) in the complement of the diagonal in the unit square. At the diagonal, R has right and left hand derivatives up to order two.

ASSUMPTION B. $\alpha(t) = R_2^-(t, t) - R_2^+(t, t)$ defines a continuous function α on $(0, 1)$ which can be extended to a strictly positive continuous function on [0, 1].

ASSUMPTION C. For each $t \in [0, 1]$, $R_{22}^{++}(\cdot, t)$ is in $\mathfrak{F}$ and

$$\sup_{0 \leq t \leq 1} \|R_{22}^{++}(\cdot, t)\| < \infty.$$

We will deal only with f_j's, which are of the form

$$f_j(t) = \int_0^1 R(s, t)\phi_j(s)\, ds, \qquad t \in [0, 1], \qquad j = 1, 2, \cdots, J, \tag{2.2}$$

where $\phi_1, \cdots, \phi_J$ are continuous and linearly independent functions on [0, 1]. From the assumption (2.2), it follows that each $f_j \in \mathfrak{F}$ and

$$\langle f_i, f_j \rangle = \int_0^1 f_i(t)\phi_j(t)\, dt, \qquad 1 \leqq i, j \leqq J. \tag{2.3}$$

REMARK 2.1. Our results will also apply when (2.2) is generalized to

$$f_j(t) = \int_0^1 R(s, t)\phi_j(s)\, ds + \sum_{k=1}^{K_j} C_{jk} R(\cdot, t_{jk}) \tag{2.4}$$

where the C_{jk}'s are real numbers, the t_{jk}'s are points in [0, 1], and the K_j's are integers. The method described in Remark 3.3 of [2] for such an extension is also applicable here.

LEMMA 2.1. *If Assumptions* A, B *and* C *are satisfied and* (2.4) *holds with* $\phi_1, \cdots, \phi_J$ *continuous and linearly independent, then* $f_1, \cdots, f_J$ *are linearly independent over* [0, 1].

PROOF. If $f_1, \cdots, f_J$ are not linearly independent, there is a continuous function ϕ which is not identically zero on [0, 1] such that

$$\int_0^1 R(s, t)\phi(s)\, ds = \sum_{k=1}^{K} C_k R(t, t_k) \tag{2.5}$$

for some real numbers $C_1, \cdots, C_K$ and numbers $t_1, \cdots, t_K$ all in [0, 1]. Since the left side of (2.5) is differentiable at every $t \in (0, 1)$ (Assumption A) and the right side of (2.5) is not differentiable at t_k if $t_k \in (0, 1)$ (Assumption B), it must be that

$$\int_0^1 R(s, t)\phi(s)\, ds = aR(t, 0) + bR(t, 1) = aR(0, t) + bR(1, t). \tag{2.6}$$

Let $f(t)$ denote the left side of (2.6). From Assumptions A and B we can differentiate f twice and obtain (see (3.6) of [2] and Assumption C)

$$f''(t) = -\alpha(t)\phi(t) + \langle R_{22}^{++}(\cdot, t), f\rangle . \tag{2.7}$$

Differentiating the right side of (2.6) twice yields

$$aR_{22}(0, t) + bR_{22}(1, t) = \langle R_{22}^{++}(\cdot, t),\, aR(\cdot, 0) + bR(\cdot, 1)\rangle = \langle R_{22}^{++}(\cdot, t), f\rangle .$$

This, together with (2.7), implies that $\alpha(t)\phi(t) = 0$ for all $t \in [0, 1]$ and so contradicts the fact that ϕ is not identically zero inasmuch as $\alpha > 0$. The lemma is thus proved.

COROLLARY. *Under the conditions of Lemma* 2.1, $A = \{\langle f_i, f_j\rangle, 1 \leqq i, j \leqq J\}$ *is a positive definite matrix.*

PROOF. A is non-negative definite. If $0 = \sum_{i,j} \lambda_i\lambda_j\langle f_i, f_j\rangle = \|\sum_j \lambda_j f_j\|^2$, then $\sum_j \lambda_j f_j$ is identically zero and $\lambda_1 = \cdots = \lambda_J = 0$.

Henceforth, for convenience, positive definite matrices will be called positive and non-negative definite matrices will be called non-negative. If B and C are non-negative, we will say $B > C$ if $B - C$ is positive and $B \geqq C$ if $B - C$ is non-negative.

Suppose now that we have n observations $Y(t_1), \cdots, Y(t_n)$ where $0 \leqq t_1 < t_2 < \cdots < t_n \leqq 1$ and suppose that $\beta_1, \cdots, \beta_J$ are all estimable. Let $T = \{t_1, \cdots, t_n\}$, let R_T be the $n \times n$ matrix $\{R(t_i, t_j)\}$ which we suppose is non-singular, let $R_T^{-1}(t_i, t_j)$ be the ijth element of R_T^{-1}, let A_T be the $J \times J$ matrix whose rsth element is $\sum_{i,j=1}^{n} f_r(t_i)R_T^{-1}(t_i, t_j)f_s(t_j)$, and let η be the J-vector whose rth coordinate is $\sum_{i,j=1}^{n} f_r(t_i)R_T^{-1}(t_i, t_j)Y(t_j)$. Then the best linear estimate of β is given by $A_T^{-1}\eta$ and has covariance matrix A_T^{-1}. Put into the language of reproducing kernel spaces we can say that, if T is a finite subset of [0, 1] and P_T is the projection operator from $\mathfrak{F}$ onto the subspace $\mathfrak{F}_T$ of $\mathfrak{F}$ which is generated by $\{R(\cdot, t); t \in T\}$, then $A_T = \{\langle P_T f_r, P_T f_s\rangle\}$. Of course, A_T as just defined makes sense when T is an arbitrary subset of [0, 1]. The best linear estimate of β is then obtainable as $A_T^{-1}\eta$ with the interpretation given following the proof of Theorem 2.3 in [2] and the covariance matrix of the BLE of β is A_T^{-1}. When $T = [0, 1]$, $P_T f_r = f_r$ and we denote $A_{[0,1]}$ by A. We note here that $A \geqq A_T$ for $T \subset [0, 1]$.

The design problem we are concerned with can be crudely stated as follows: if $D_n = \{T \mid T = \{t_1, \cdots, t_n\}, 0 \leqq t_1 < \cdots < t_n \leqq 1\}$, we wish to choose $T \varepsilon D_n$ so as to make A_T^{-1} "small." The first question is how to interpret "small." To this end we make the following definition:

DEFINITION. A criterion ψ is a continuous real-valued function defined on the non-negative matrices with $\psi(0) = 0$ and $\psi(B) \geqq \psi(C)$ if $B \geqq C$. If $\psi(B) > \psi(C)$ whenever $B > 0, B \geqq C$ and $B \neq C$, ψ is called strict.

Given a criterion ψ we can now formulate four ways of deciding when, for given n, a design makes A_T^{-1} "small."

(2.8) $T^* \varepsilon D_n$ is said to be ψ1-optimum in D_n if $\psi(A_{T^*}^{-1}) = \inf_{T \varepsilon D_n} \psi(A_T^{-1})$.

(2.9) $T^* \varepsilon D_n$ is said to be ψ2-optimum in D_n if $\psi(A_{T^*}) = \sup_{T \varepsilon D_n} \psi(A_T)$.

(2.10) $T^* \varepsilon D_n$ is said to be ψ3-optimum in D_n if

$$\psi(A_{T^*}^{-1} - A^{-1}) = \inf_{T \varepsilon D_n} \psi(A_T^{-1} - A^{-1}).$$

(2.11) $T^* \varepsilon D_n$ is said to be ψ4-optimum in D_n if

$$\psi(A - A_{T^*}) = \inf_{T \varepsilon D_n} \psi(A - A_T).$$

Generally these four problems will not be equivalent. There will be ψ's for which some of them are equivalent. For example, if $\psi(B) = \det B$ then ψ1-optimum is the same as ψ2-optimum. If M is a non-negative matrix and $\psi(B) = \operatorname{tr}(BM)$ (*tr* denotes trace) then ψ2-optimum is the same as ψ4-optimum and ψ1-optimum is the same as ψ3-optimum.

The difficulties encountered in [2] about finding exactly optimum designs are also present here. The existence of optimum designs can be handled pretty much the same as in Section 2 of [2] by use of the "simple present" condition imposed there. The difficulties in calculation of optimum designs leads us, as in [2], to introduce the notion of asymptotically optimum sequences of designs which we define as follows:

A sequence $\{T_n^*\}$ with $T_n^* \varepsilon D_n$ is said to be asymptotically ψ1-optimum if

(2.12) $$\lim_{n\to\infty} [\inf_{T \varepsilon D_n} \psi(A_T^{-1}) - \psi(A^{-1})][\psi(A_{T_n^*}^{-1}) - \psi(A^{-1})]^{-1} = 1.$$

When

(2.13) $$\lim_{n\to\infty} [\psi(A) - \sup_{T \varepsilon D_n} \psi(A_T)][\psi(A) - \psi(A_{T_n^*})]^{-1} = 1,$$

then $\{T_n^*\}$ is said to be asymptotically ψ2-optimum. When

(2.14) $$\lim_{n\to\infty} \inf_{T \varepsilon D_n} [\psi(A_T^{-1} - A^{-1})/\psi(A_{T_n^*}^{-1} - A^{-1})] = 1$$

then $\{T_n^*\}$ is asymptotically ψ3-optimum, and when

(2.15) $$\lim_{n\to\infty} \inf_{T \varepsilon D_n} [\psi(A - A_T)/\psi(A - A_{T_n^*})] = 1,$$

$\{T_n^*\}$ is asymptotically ψ4-optimum.

Let us now mention some specific criteria which will be of interest to us. If M is a fixed non-negative $J \times J$ matrix, consider ψ defined by $\psi(B) = \operatorname{tr}(BM)$.

M may be written as $M = \int_{S_J} \theta\theta' \, d\mu$ where μ is a finite measure with support on a finite number of points on the J-dimensional unit ball S_J. Hence, we can also write

$$\psi(B) = \int_{S_J} \theta' B \theta \, d\mu. \tag{2.16}$$

A particular case is $M = \theta\theta'$ for some fixed vector θ, so one optimization problem here is that of minimizing the variance of $\theta'\hat{\beta}$.

Our results concerning more general ψ will hinge on representations in terms of the trace functions above. For example, let ψ be a strict, continuously differentiable criterion. Let B and C be non-negative matrices with $B > 0$, $B \geqq C$ and denote the jkth elements of B and C by b_{jk} and c_{jk} respectively. Then

$$\psi(B - C) = \psi(B) - \textstyle\sum_{j,k=1}^{J} (\partial\psi/\partial b_{jk}) c_{jk} + o(|C|)$$

where $|C| = \max_{1 \leqq j,k \leqq J} |c_{jk}|$. If $M = \{\partial\psi/\partial b_{jk},\ 1 \leqq j,\ k \leqq J\}$,

$$\psi(B - C) = \psi(B) - \operatorname{tr}(MC) + o(|C|). \tag{2.17}$$

Now if B is fixed and (2.17) holds for all $0 \leqq C \leqq B$, the assumptions on ψ insure that $\operatorname{tr}(MC) > 0$ for all sufficiently small non-zero C and, consequently, for all non-zero $C \geqq 0$. Hence $M > 0$. For $B = A_T^{-1}$ and $C = A_T^{-1} - A^{-1}$, we have from (2.17)

$$\psi(A_T^{-1}) - \psi(A^{-1}) = \operatorname{tr} M(A_T^{-1} - A^{-1}) + o(|A_T^{-1} - A^{-1}|). \tag{2.18}$$

Use of (2.18) will allow us to show for instance, that a sequence $\{T_n^*\}$ which is $\psi 1$ asymptotically optimum for $\psi(B) = \operatorname{tr} AB$ is also $\psi 1$ asymptotically optimum for $\psi(B) = \det B$.

We shall also consider criteria of the form $\psi(B) = \sup_{M \in \mathfrak{M}} \operatorname{tr} MB$ where $\mathfrak{M}$ is a compact set of $J \times J$ matrices (i.e., $\mathfrak{M}$ is a compact set in the usual topology of $J(J + 1)/2$-dimensional Euclidean space). Several standard criteria arise by making suitable choices of $\mathfrak{M}$. As two examples, consider $\mathfrak{M} = \{f(t)f(t)' \mid 0 \leqq t \leqq 1\}$ with $f(t)' = (f_1(t), \cdots, f_J(t))$, and $\mathfrak{M} = \{\theta\theta' \mid \sum_{j=1}^{J} \theta_j^2 = 1\}$. For this kind of criterion also, we shall be able to relate asymptotic optimality to the asymptotic optimality with respect to an appropriate trace function.

3. Basic asymptotic results. We shall suppose that Assumptions A, B and C of Section 2 are in force and we shall only consider those f's of the form

$$f(t) = \int_0^1 R(s, t)\phi(s)\, ds; \qquad t \in [0, 1] \tag{3.1}$$

with ϕ continuous. If $T = \{t_0, \cdots, t_n;\ 0 \leqq t_0 < t_1 < \cdots < t_n \leqq 1\}$ then

$$\|f - P_T f\|^2 = \int_0^1 (f - P_T f)(t)\phi(t)\, dt. \tag{3.2}$$

For $i = 0, \cdots, n - 1$ let W_i be the symmetric kernel defined by

$$W_i(x, y) = (x - t_i)(t_{i+1} - y)/(t_{i+1} - t_i) \quad \text{for} \quad t_i \leqq x \leqq y \leqq t_{i+1}.$$

LEMMA 3.1. *If T is such that $t_0 = 0$ and $t_n = 1$ then*

$$(3.3)\quad \|f - P_T f\|^2 = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \alpha(x)\phi(x)\phi(y)W_i(x, y)\,dx\,dy - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \phi(x)\langle R_{22}(\cdot, y), f - P_T f\rangle W_i(x, y)\,dx\,dy.$$

PROOF. As noted in (3.42) of [2], $f - P_T f$ vanishes on T and is twice continuously differentiable on (t_i, t_{i+1}). An integration by parts yields

$$(3.4)\quad (f - P_T f)(x) = -\int_{t_i}^{t_{i+1}} (f - P_T f)''(y)W_i(x, y)\,dy.$$

Since, as noted at (2.7),

$$(3.5)\quad (f - P_T f)''(y) = -\alpha(y)\phi(y) + \langle R_{22}(\cdot, y), f - P_T f\rangle$$

we obtain the result of the lemma by using (3.4) and (3.5) in (3.2).

The essential properties of W_i that will be used below are as follows:

(3.6) (a) For $\theta \varepsilon L_2[t_i, t_{i+1}]$ define $W_i\theta = \int_{t_i}^{t_{i+1}} \theta(x)W_i(x, \cdot)\,dx$.

Then W_i is a positive, bounded, symmetric linear transformation from $L_2[t_i, t_{i+1}] \to L_2[t_i, t_{i+1}]$ and $[W_i\theta, \eta] = \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \theta(x)\eta(y)W_i(x, y)\,dx\,dy$ defines an inner product.

(b) $W_i(x, y) \leqq (t_{i+1} - t_i)/4$ for all $x, y \,\varepsilon\, [t_i, t_{i+1}]$.

(c) $\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} W_i(x, y)\,dx\,dy = (t_{i+1} - t_i)^3/12$.

We will be concerned with sequences $\{T_n ; n \geqq 1\}$ of designs with $T_n \,\varepsilon\, D_{n+1}$ (see Section 2), i.e., Card $T_n = n + 1$. The elements of T_n will depend on n but we shall usually suppress this dependence and write t_i instead of t_{in} for an element of T_n. Given a sequence $\{T_n\}$ we shall call $\{S_n\}$ an extension of $\{T_n\}$ if, for all n, $\{0, 1\} \cup T_n \subset S_n$. Note that S_n will often contain more than $n + 1$ elements. For convenience we set $f_n = P_{T_n} f$ and $\hat{f}_n = P_{S_n} f$. Note that, since $T_n \subset S_n$

$$(3.7)\quad \|f - f_n\|^2 \geqq \|f - \hat{f}_n\|^2.$$

If $\{T_n\}$ is a sequence of designs with $\{0, 1\} \subset T_n$ we put

$$(3.8)\quad A_{in}(\phi) = \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \alpha(x)\phi(x)\phi(y)W_i(x, y)\,dx\,dy,$$

$$A_n(\phi) = \sum_{i=0}^{n-1} A_{in}(\phi).$$

LEMMA 3.2. *If $\{0, 1\} \subset T_n$ for all n then there is a constant K, independent of ϕ, such that*

$$(3.9)\quad \|f - f_n\|^2 \leqq (1 + K\delta_n)A_n(|\phi|)$$

where $\delta_n = \sup_{0 \leqq i \leqq n-1} (t_{i+1} - t_i)$.

PROOF. Let $\rho_n(y) = \langle R_{22}(\cdot, y), f - f_n\rangle$. Then, from Assumptions A, B, and C, (3.6), and (3.8)

$$|[W_i\phi, \rho_n]| \leqq [W_i\phi, \phi]^{\frac{1}{2}}[W_i\rho_n, \rho_n]^{\frac{1}{2}} \leqq [W_i\phi, \phi]^{\frac{1}{2}}[W_i 1, 1]^{\frac{1}{2}} \|f - f_n\| \sup_y \|R_{22}(\cdot, y)\|$$

so that

$$
\begin{aligned}
(3.10)\qquad |\textstyle\sum_{i=0}^{n-1} [W_i\phi, \rho_n]| &\leqq c\,\|f - f_n\| \textstyle\sum_{i=0}^{n-1} [W_i\phi, \phi]^{\frac{1}{2}}[W_i1, 1]^{\frac{1}{2}} \\
&\leqq c\,\|f - f_n\|(\textstyle\sum_{i=0}^{n-1} [W_i\phi, \phi])^{\frac{1}{2}}(\sum [W_i1, 1])^{\frac{1}{2}} \\
&\leqq c'\,\|f - f_n\|\, A_n^{\frac{1}{2}}(|\phi|)(\textstyle\sum\{(t_{i+1} - t_i)^3\})^{\frac{1}{2}} \\
&\leqq c''\delta_n\,\|f - f_n\|\, A_n^{\frac{1}{2}}(|\phi|)
\end{aligned}
$$

where $A_n(|\phi|)$ is obtained from $\sum [W_i\phi, \phi]$ by first dominating ϕ by $|\phi|$ and then inserting $\alpha(x)$ which is bounded below. From (3.10) and Lemma 3.1 we obtain

$$(3.11)\qquad \|f - f_n\|^2 \leqq A_n(|\phi|) + c''\delta_n\,\|f - f_n\|\, A_n^{\frac{1}{2}}(|\phi|)$$

and (3.9) follows easily from this. Thus Lemma 3.2 is proved.

Let h be a continuous density on [0, 1] and let T_n be defined by

$$(3.12)\qquad \int_0^{t_{in}} h(x)\,dx = i/n; \qquad i = 0, \cdots, n,$$

with the convention (in case of ambiguity) that $t_{0n} = 0$ and $t_{nn} = 1$. In other cases of ambiguity (which will arise if h is 0 on some intervals) take any t_{in} which satisfies (3.12) e.g., the smallest such. The sequence $\{T_n ; n \geqq 1\}$ so defined will be called a *Regular Sequence generated by* h(RS (h)).

LEMMA 3.3. *If* $\{T_n\}$ *is* RS(h) *then*

$$\liminf_{n\to\infty} n^2\,\|f - f_n\|^2 \geqq \tfrac{1}{12} \int_{h>0} \alpha(x)\phi^2(x)/h^2(x)\,dx.$$

PROOF. Let $\epsilon > 0$. Let $I_1 = \{i \mid |\phi(x)| > \epsilon$ and $h(x) > \epsilon$ for all $x \,\varepsilon\, [t_i, t_{i+1}]$ and let $I_2 = \{0, 1, \cdots, n - 1\} - I_1$. Let $S_n \supset T_n$ be such that

(3.13) (a) if $i \,\varepsilon\, \mathrm{I}_1$ then there is an r such that $s_r = t_i$, $s_{r+1} = t_{i+1}$,

(b) if $i \,\varepsilon\, \mathrm{I}_2$ then there are $s_r, s_{r+1}, \cdots, s_{r+m_i}$ such that $s_r = t_i$, $s_{r+m_i} = t_{i+1}$ and $s_{r+j+1} - s_{r+j} \leqq \gamma_n/n$ with $\gamma_n \to 0$.

Because of (3.7) it is enough to show that $n^2\,\|f - \hat{f}_n\|^2$ has the lower limit stated in the lemma.

Denote by $\hat{A}_{jn}(\phi)$ the A_j defined in (3.8) when S_n is the design. If $\nu_n + 1$ denotes the cardinality of S_n, let $J_1 = \{j \mid s_j = t_i, s_{j+1} = t_{i+1}$ for some $i \,\varepsilon\, I_1\}$ and let $J_2 = \{0, \cdots, \nu_n - 1\} - J_1$. For $j \,\varepsilon\, J_1$

$$(3.14)\qquad 1/n = \int_{t_i}^{t_{i+1}} h(x)\,dx = \int_{s_j}^{s_{j+1}} h(x)\,dx = h(\lambda_j)(s_{j+1} - s_j) \geqq \epsilon(s_{j+1} - s_j).$$

For $j \,\varepsilon\, J_2$, $s_{j+1} - s_j \leqq \gamma_n/n$. Thus $\hat{\delta}_n = \sup_j (s_{j+1} - s_j) \to 0$. For $j \,\varepsilon\, J_2$ there is a constant c such that

$$\hat{A}_{jn}(|\phi|) \leqq c(s_{j+1} - s_j)^3 \leqq c\gamma_n^2 n^{-2}(s_{j+1} - s_j)$$

so that

$$(3.15)\qquad n^2 \textstyle\sum_{j\varepsilon J_2} \hat{A}_{jn}(|\phi|) = o(1) \quad \text{as} \quad n \to \infty.$$

For $j \,\varepsilon\, J_1$, the continuity of ϕ guarantees that ϕ is of one sign on $[s_j, s_{j+1}]$ so that $\hat{A}_{jn}(\phi) = \hat{A}_{jn}(|\phi|)$. Thus, by Lemma 3.1, (3.10), and (3.15),

$$(3.16)\quad n^2\|f - \hat{f}_n\|^2 \geqq n^2\hat{A}_n(|\phi|) + o(1) - n^2 c_2 \hat{\delta}_n \|f - \hat{f}_n\| \hat{A}_n^{\frac{1}{2}}(|\phi|).$$

If $\liminf_{n\to\infty} n^2\hat{A}_n(|\phi|) > 0$ then (3.16) together with the fact that $\hat{\delta}_n \to 0$ implies that

$$(3.17)\qquad \liminf_{n\to\infty} n^2\|f - \hat{f}_n\|^2 \geqq \liminf_{n\to\infty} n^2\hat{A}_n(|\phi|).$$

Thus, (3.17) holds generally.

As already observed

$$n^2\hat{A}_n(|\phi|) = n^2 \textstyle\sum_{J_1} \hat{A}_{jn}(|\phi|) + o(1).$$

The mean-value theorem, (3.6(c)), and (3.14) give, for $j \in J_1$,

$$\begin{aligned}(3.18)\qquad \hat{A}_{jn}(|\phi|) &= \alpha(\mu_j)|\phi(\mu_j)|\cdot|\phi(\theta_j)|(s_{j+1} - s_j)^3/12 \\ &= \alpha(\mu_j)|\phi(\mu_j)|\cdot|\phi(\theta_j)|(h^2(\lambda_j))^{-1}(s_{j+1} - s_j)/12\end{aligned}$$

for some $\mu_j, \theta_j \in [s_j, s_{j+1}]$. Thus

$$(3.19)\qquad \liminf n^2\hat{A}_n(|\phi|) = \tfrac{1}{12}\textstyle\int_{|\phi|>\epsilon, h>\epsilon} \alpha(x)\phi^2(x)(h^2(x))^{-1}\,dx$$

which upon use of (3.17) and the last remark in the first paragraph of the proof yields

$$\liminf_{n\to\infty} n^2\|f - f_n\|^2 \geqq \tfrac{1}{12}\textstyle\int_{|\phi|>\epsilon, h>\epsilon} \alpha(x)\phi^2(x)(h^2(x))^{-1}\,dx.$$

Letting $\epsilon \to 0$ finishes the proof of Lemma 3.3

We can improve Lemma 3.3 by replacing the integral over the set $\{h > 0\}$ by the set $\{h > 0 \text{ or } |\phi| > 0\}$. This results from

LEMMA 3.4. *Let m denote Lebesgue measure, $B = \{x \mid h(x) = 0, \phi(x) \neq 0\}$, and let $\{T_n\}$ be* RS (h). *If $m(B) > 0$ then $n^2\|f - f_n\|^2 \to \infty$.*

PROOF. Let $\epsilon > 0$ and define $I_1 = \{i \mid |\phi(x)| > \epsilon \text{ for all } x \in [t_i, t_{i+1}]\}$ and let $I_2 = \{0, \cdots, n-1\} - I_1$. Let S_n, J_1, J_2 be as defined in (3.13) using the present definition of I_1. Even though δ_n may not $\to 0 (\sup_{J_1}(s_{j+1} - s_j) \nrightarrow 0)$ we can conclude from (3.16) (which is still valid) that, for some positive c_0,

$$(3.20)\qquad n^2\|f - \hat{f}_n\|^2 \geqq c_0 n^2 \hat{A}_n(|\phi|) + o(1).$$

If $\{n_m\}$ is a subsequence such that, along $\{n_m\}$, $\sup_{J_1}(s_{j+1} - s_j) \to \delta > 0$, then there is $\{j_m\}$ such that $(s_{j_m+1} - s_{j_m}) \to \delta$ and $\hat{A}_{n_m}(|\phi|) \geqq c'\epsilon^2(s_{j_m+1} - s_{j_m})^3 \geqq c'' > 0$ for all m large enough. Thus, by (3.20), if $\delta_{n_m} \to \delta > 0$,

$$(3.21)\qquad \lim_{m\to\infty} n_m^2\|f - \hat{f}_{n_m}\|^2 = \infty.$$

Suppose that $\{n_m\}$ is a subsequence such that $\hat{\delta}_{n_m} \to 0$. Assume $\{n_m\} = \{n\}$ in what follows. Let $\epsilon > 0$ be such that $m(h = 0, |\phi| > \epsilon) > 0$. Define, for $\eta > 0$, $J_3 = \{j \mid h(x) < \eta, |\phi(x)| > \epsilon \text{ all } x \in [s_j, s_{j+1}]\}$ and observe that, with $\alpha_0 = \inf \alpha(x)$,

$$(3.22)\qquad \textstyle\sum_{j\in J_3} \hat{A}_{jn}(|\phi|) \geqq \alpha_0\epsilon^2 \sum_{j\in J_3}(s_{j+1} - s_j)^3/12$$

and, since $s_j = t_i$, $s_{j+1} = t_{i+1}$ for some i if $j \in J_3 \subset J_1$,

$$(3.23)\qquad 1/n = \textstyle\int_{s_j}^{s_{j+1}} h(x)\,dx < \eta(s_{j+1} - s_j)$$

for all $j \varepsilon J_3$. Thus ((3.22) and (3.23))

$$n^2\hat{A}_n(|\phi|) \geqq n^2 \textstyle\sum_{j\varepsilon J_3} \hat{A}_{jn}(|\phi|) \geqq \frac{1}{12}\alpha_0\epsilon^2\eta^{-2} \sum_{j\varepsilon J_3} (s_{j+1} - s_j) \tag{3.24}$$

and, since $\hat{\delta}_n \to 0$,

$$\textstyle\sum_{j\varepsilon J_3} (s_{j+1} - s_j) \to m(h < \eta, |\phi| > \epsilon) > m(h = 0, |\phi| > \epsilon) > 0.$$

Using this in (3.24) and then letting $\eta \to 0$ we obtain $\lim_{m\to\infty} n_m{}^2\hat{A}_{n_m}(|\phi|) = \infty$ if $\hat{\delta}_{n_m} \to 0$ and (3.20) and (3.21) then give the conclusion of the lemma.

The major fact that we would like to prove is that, if $\{T_n\}$ is RS (h) then

$$\lim_{n\to\infty} n^2 \|f - f_n\|^2 = \tfrac{1}{12} \int \alpha(x)\phi^2(x)(h^2(x))^{-1}\, dx \tag{3.25}$$

for all continuous ϕ for which the right side of (3.25) makes sense where $\phi(x)/h(x) = 0$ if $\phi(x) = h(x) = 0$. We already know from Lemmas 3.3 and 3.4 that, if the right side of (3.25) is ∞ then so is the left side. Unfortunately we are unable to obtain such a result for all ϕ and any h. We have, however,

Theorem 3.1. *Let* $\{T_n\}$ *be* RS (h).

(A). *If* $\int_0^1 (h^2(x))^{-1} < \infty$ *then* (3.25) *holds for all continuous* ϕ.

(B). *If* ϕ/h *is continuous* ($\phi(x)/h(x) = 0$ *if* $\phi(x) = h(x) = 0$) *then* (3.25) *holds.*

(C). *If there is a* K *such that*

$$(b - a) \int_a^b h^2(x)\, dx \leqq K\left(\int_a^b h(x)\, dx\right)^2$$

for all $[a, b] \subset [0, 1]$, *then* (3.25) *holds.*

Remark. (A), (B), and (C) all give (3.25) if h is never 0. This case was adequately treated in [2]. (B) is true if $h(x) = \gamma\alpha^{1/3}(x)\phi^{2/3}(x)$ which is the example of principal importance for the context of [2]. (A) and (C) will be adequate for handling many different examples of h's which may have zeroes but are regularly behaved near their zeroes.

Proof of Theorem 3.1. Let us first prove (A). First observe that the condition in (A) implies that $m(h(x) = 0) = 0$ so that, using a Hölder inequality,

$$(t_{i+1} - t_i) = \int_{t_i}^{t_{i+1}} 1\, dx = \int_{t_i}^{t_{i+1}} h^{2/3}(x)(h^{2/3}(x))^{-1}\, dx$$
$$\leqq \left(\int_{t_i}^{t_{i+1}} h(x)\, dx\right)^{2/3}\left(\int_{t_i}^{t_{i+i}} dx/h^2(x)\right)^{1/3}$$

or

$$(t_{i+1} - t_i)^3 \leqq n^{-2} \int_{t_i}^{t_{i+1}} (h^2(x))^{-1}\, dx. \tag{3.26}$$

Also since $m(h(x) = 0) = 0$ we must have $\delta_n = \sup_{0\leqq i\leqq n-1} (t_{i+1} - t_i) \to 0$. Define $I_1 = \{i \mid h(x) > \epsilon \text{ for all } x\, \varepsilon\, [t_i, t_{i+1}]\}$ and $I_2 = \{0, \cdots, n - 1\} - I_1$. In the same way that (3.18) was obtained we get

$$n^2 \textstyle\sum_{i\varepsilon I_1} A_{in}(|\phi|) \to \frac{1}{12} \int_{h>\epsilon} \alpha(x)\phi^2(x)(h^2(x))^{-1}\, dx. \tag{3.27}$$

The uniform continuity of h and the fact that $\delta_n \to 0$ implies that, for n large enough, $h(x) < 2\epsilon$ for $x\, \varepsilon\, [t_i, t_{i+1}]$ for all $i\, \varepsilon\, I_2$. Then, using (3.26), we have, for

$i \varepsilon I_2$,

$$(3.28)\quad \sum_{i\varepsilon I_2} n^2 A_{in}(|\phi|) \leqq \tfrac{1}{12} n^2 \sup_x \alpha(x) \sup_x \phi^2(x) \sum_{i\varepsilon I_2} (t_{i+1} - t_i)^3 \leqq c \sum_{i\varepsilon I_2} \int_{t_i}^{t_{i+1}} (h^2(x))^{-1}\,dx \leqq c \int_{h<2\epsilon} (h^2(x))^{-1}\,dx.$$

Thus ((3.27) and (3.28))

$$\limsup_{n\to\infty} n^2 A_n(|\phi|) \leqq \tfrac{1}{12} \int_{h>\epsilon} \alpha(x)\phi^2(x)(h^2(x))^{-1}\,dx + c \int_{h<2\epsilon} (h^2(x))^{-1}\,dx$$

and, letting $\epsilon \to 0$, this yields

$$(3.29)\quad \limsup_{n\to\infty} n^2 A_n(|\phi|) \leqq \tfrac{1}{12} \int \alpha(x)\phi^2(x)(h^2(x))^{-1}\,dx.$$

Since $\delta_n \to 0$, we can now use (3.29), Lemma 3.2, and Lemma 3.3 to conclude (A).

For the proof of (B) let $\mathfrak{C}_1 = \{\theta \mid \theta$ is continuous on $[0, 1]$ and, for some $\epsilon > 0$, $\theta = 0$ on the set $\{h(x) \leqq 2\epsilon\}\}$ and let $\mathfrak{C}_2 = \{\theta \mid \theta$ is continuous and $\theta = 0$ on the set $\{h = 0\}\}$. $\mathfrak{C}_1 \subset \mathfrak{C}_2$ and if we topologize $\mathfrak{C}_2$ by $\|\theta_1 - \theta_2\| = \sup_{0\leqq x\leqq 1} |\theta_1(x) - \theta_2(x)|$ it is easy to see that $\mathfrak{C}_1$ is dense in $\mathfrak{C}_2$.

For $\theta \varepsilon \mathfrak{C}_1$ with associated ϵ, let $\phi = \theta h$ and put $I_1 = \{i \mid h(x) > \epsilon$ for some $x \varepsilon [t_i, t_{i+1}]\}$, $I_2 = \{0, \cdots, n-1\} - I_1$, $I_3 = \{i \mid h(x) > \epsilon$ for all $x \varepsilon [t_i, t_{i+1}]\}$, $I_4 = I_1 - I_3$. Since $1/n \geqq \int_{[t_i,t_{i+1}]\cap\{h>\epsilon/2\}} h(x)\,dx$ it follows from the uniform continuity of h that $\sup_{i\varepsilon I_1} (t_{i+1} - t_i) = \delta_{1n} \to 0$ as $n \to \infty$. Since ϕ vanishes on $[t_i, t_{i+1}]$ if $i \varepsilon I_2$, the sums in (3.3) need only extend over $i \varepsilon I_1$. In the same way that Lemma 3.2 was obtained we can conclude that

$$\sum_{i\varepsilon I_1} [W_i\,\phi, \rho_n] \leqq c\delta_{1n} \|f - f_n\|\, A_n(|\phi|)$$

so that, since $\delta_{1n} \to 0$,

$$(3.30)\quad \limsup_{n\to\infty} n^2 \|f - f_n\|^2 \leqq \limsup_{n\to\infty} n^2 A_n(|\phi|) = \limsup_{n\to\infty} n^2 \sum_{I_1} A_{in}(|\phi|).$$

Now, just as we obtained (3.27), we can obtain

$$(3.31)\quad \lim_{n\to\infty} n^2 \sum_{i\varepsilon I_3} A_{in}(|\phi|) = \tfrac{1}{12} \int_{h>\epsilon} \alpha(x)\phi^2(x)(h^2(x))^{-1}\,dx = \tfrac{1}{12} \int_0^1 \alpha(x)\phi^2(x)(h^2(x))^{-1}\,dx.$$

(The last equality follows because $\phi = 0$ on $\{h \leqq \epsilon\}$.) For n large enough we have, for all $i \varepsilon I_4$, $h(x) \leqq 2\epsilon$ for all $x \varepsilon [t_i, t_{i+1}]$ so that $\phi = 0$ on $[t_i, t_{i+1}]$. Consequently $A_{in}(|\phi|) = 0$ for $i \varepsilon I_4$. This together with (3.30),(3.31) and Lemma 3.3 gives the result that, for $\phi = \theta h$ with $\theta \varepsilon \mathfrak{C}_1$, (3.25) holds.

To extend the result to $\phi = \theta h$ with $\theta \varepsilon \mathfrak{C}_2$, let $\theta_\epsilon \varepsilon \mathfrak{C}_1$ with $\|\theta_\epsilon - \theta\| \leqq \epsilon$ and put $\phi_\epsilon = \theta_\epsilon h$ and let f_ϵ be the associated f (3.1). Then, according to the triangle inequality,

$$(3.32)\quad \|f - f_n\| \leqq \|f - f_\epsilon - (f - f_\epsilon)_n\| + \|f_\epsilon - f_{\epsilon n}\|.$$

Since $\theta_\epsilon \varepsilon \mathfrak{C}_1$ we know that

$$(3.33)\quad \lim_{n\to\infty} n^2 \|f_\epsilon - f_{\epsilon n}\|^2 = \tfrac{1}{12} \int_0^1 \alpha(x)\theta_\epsilon^2(x)\,dx = \tfrac{1}{12} \int_0^1 \alpha(x)\theta^2(x)\,dx + r_\epsilon$$

where $r_\epsilon \to 0$ as $\epsilon \to 0$. Since $\theta - \theta_\epsilon \varepsilon \mathfrak{C}_2$ we have

$$A_n(|\phi - \phi_\epsilon|) \leqq c \sum_{i=0}^{n-1} (t_{i+1} - t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} (\theta - \theta_\epsilon)(x) \cdot (\theta - \theta_\epsilon)(y) h(x) h(y)\, dx\, dy \tag{3.34}$$
$$\leqq c \|\theta - \theta_\epsilon\|^2 n^{-2} \sum_{i=0}^{n-1} (t_{i+1} - t_i) = cn^{-2} \|\theta - \theta_\epsilon\|^2$$

so that by Lemma 3.2

$$\limsup_{n\to\infty} n^2 \|f - f_\epsilon - (f - f_\epsilon)_n\|^2 \leqq c_1 \|\theta - \theta_\epsilon\|^2. \tag{3.35}$$

(3.32), (3.33), and (3.35) yield, upon letting $\epsilon \to 0$,

$$\limsup_{n\to\infty} n^2 \|f - f_n\|^2 \leqq \tfrac{1}{12} \int_0^1 \alpha(x) \phi^2(x) (h^2(x))^{-1}\, dx$$

for ϕ satisfying the condition in (B). Now apply Lemma 3.3 to obtain (B).

For the proof of (C), let $\mathfrak{C}_3 = \{\theta \mid \theta = 0 \text{ if } h = 0 \text{ and } \int_0^1 \theta^2(x)\, dx < \infty\}$. $\mathfrak{C}_2$ is dense in $\mathfrak{C}_3$ with the topology on $\mathfrak{C}_3$ being the L_2-topology. The proof just given for extending the result of (B) from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ can now be employed provided we can verify something like (3.34). The condition stated in (C) yields, for $\phi = \theta h$,

$$\begin{aligned} n^2 A_n(|\phi|) &\leqq c_0 \sum_{i=0}^{n-1} (t_{i+1} - t_i) \Big[\int_{t_i}^{t_{i+1}} |\theta(x)| h(x)\, dx\Big]^2 \\ &\leqq n^2 c_0 \sum_{i=0}^{n-1} (t_{i+1} - t_i) \int_{t_i}^{t_{i+1}} h^2(x)\, dx \int_{t_i}^{t_{i+1}} \theta^2(x)\, dx \\ &\leqq n^2 K c_0 \sum_{i=0}^{n-1} \Big[\int_{t_i}^{t_{i+1}} h(x)\, dx\Big]^2 \int_{t_i}^{t_{i+1}} \theta^2(x)\, dx \\ &= K c_0 \|\theta\|^2 \end{aligned}$$

where $\| \;\|$ is now the L_2 norm. But this is exactly the version of (3.34) needed and this gives (C) and finishes the proof of the theorem.

THEOREM 3.2. *Let $f_1, \cdots, f_p$ be p functions of the form* (3.1) *with associated $\phi_1, \cdots, \phi_p$. Let $a_1, \cdots, a_p$ be positive numbers. If $\{T_n\}$ is any sequence of designs then*

$$\liminf_{n\to\infty} n^2 \sum_{k=1}^{p} a_k \|f_k - f_{kn}\|^2 \geqq \tfrac{1}{12} \Big(\int_0^1 \alpha^{\frac{1}{3}}(x) \Big(\sum_1^p a_k \phi_k^2(x)\Big)^{\frac{1}{3}} dx\Big)^3 = (12\eta^3)^{-1} (\text{say}). \tag{3.36}$$

If $h(x) = \eta \alpha^{\frac{1}{3}}(x) (\sum_1^p a_k \phi_k^2(x))^{\frac{1}{3}}$ and if $\{T_n^*\}$ is RS(h) *then*

$$\lim_{n\to\infty} n^2 \sum_1^p a_k \|f_k - f_{kn}^*\|^2 = (12\eta_3)^{-1} \tag{3.37}$$

(f_{kn}^* denotes $P_{T_n^*} f_k$).

PROOF. For the proof of (3.36) we may as well suppose that $\{0, 1\} \subset T_n$ since, otherwise, we can adjoin $\{0, 1\}$ and reduce the left side of (3.36). We will follow the ideas in the proofs of Lemmas 3.2, 3.2, 3.4. First put $\psi(x, y) = \sum_1^p a_k \phi_k(x) \phi_k(y)$ and let $I_1 = \{i \mid \psi(x, y) > \epsilon \text{ for all } x, y \in [t_i, t_{i+1}]\}$ and let $I_2 = \{0, \cdots, n - 1\} - I_1$. Define S_n, J_1, and J_2 as in (3.13) using the present I_1. Let

$$\hat{B}_{jn}(\phi) = \int_{s_j}^{s_{j+1}} \int_{s_j}^{s_{j+1}} \alpha(x) \psi(x, y) W_j(x, y)\, dx\, dy,$$
$$\hat{C}_{jn}(\phi) = \int_{s_j}^{s_{j+1}} \int_{s_j}^{s_{j+1}} \psi(x, y) W_j(x, y)\, dx\, dy,$$
$$\hat{B}_n(\phi) = \sum_j \hat{B}_{jn}(\phi).$$

As in the verification of (3.15) we have

$$(3.38) \qquad (a)\ n^2 \textstyle\sum_{j\varepsilon J_2} \hat{B}_{jn}(\phi) = o(1), \qquad (b)\ n^2 \sum_{j\varepsilon J_2} \hat{C}_{jn}(\phi) = o(1).$$

When $j \varepsilon J_1$, $\psi(x, y) > 0$ for all $x, y \varepsilon [s_j, s_{j+1}]$ so that

$$(3.39) \qquad \alpha_0 \hat{C}_{jn}(\phi) \leqq \hat{B}_{jn}(\phi) \leqq \alpha_1 \hat{C}_{jn}(\phi); \qquad j \varepsilon J_1,$$

where $\alpha_0 = \inf \alpha(x)$, $\alpha_1 = \sup \alpha(x)$. Put $Q_n^2 = \sum a_k \|f_k - \hat{f}_{kn}\|^2$ and use the methods in the proof of Lemma 3.2, (3.38) and (3.39) to conclude that

$$(3.40) \qquad n^2 \hat{B}_n(\phi) - C_1 \hat{\delta}_n n Q_n [n^2 \hat{B}_n(\phi) + o(1)]^{\frac{1}{2}} \leqq n^2 Q_n^2$$

where $\hat{\delta}_n = \sup_j (s_{j+1} - s_j)$. From (3.40) we can conclude that whether or not $\hat{\delta}_n \to 0$

$$(3.41) \qquad n^2 Q_n^2 \geqq c_0 n^2 \hat{B}_n(\phi) + o(1).$$

Now just as was done in the paragraphs following (3.20) we can conclude that if $\{n_m\}$ is a subsequence such that $\hat{\delta}_{n_m} \to \delta > 0$ then

$$(3.42) \qquad \lim_{m\to\infty} n_m^2 Q_{n_m}^2 = \infty.$$

If $\hat{\delta}_{n_m} \to 0$ then (3.40) can be used to obtain

$$(3.43) \qquad n_m^2 Q_{n_m}^2 \geqq n_m^2 \hat{B}_{n_m}(\phi) + o(1) \quad \text{as} \quad m \to \infty.$$

Let us work as if $\{n_m\} = \{n\}$. Since, for $j \varepsilon J_1$, $\psi(x, y) > 0$ for $x, y \varepsilon [s_j, s_{j+1}]$ we have, using the mean-value theorem and a Hölder inequality,

$$(3.44) \qquad n^2 \textstyle\sum_{J_1} \hat{B}_{jn}(\phi) = \frac{1}{12} n^2 \sum_{j\varepsilon J_1} \alpha(\theta_j)\psi(\theta_j, \mu_j)(s_{j+1} - s_j)^3$$
$$\geqq \tfrac{1}{12} n^2 \left(\textstyle\sum_{j\varepsilon J_1} \alpha^{\frac{1}{3}}(\theta_j)\psi^{\frac{1}{3}}(\theta_j, \mu_j)(s_{j+1} - s_j)\right)^3 / \left(\sum_{J_1} 1\right)^2$$

where $\theta_j, \mu_j \varepsilon [s_j, s_{j+1}]$. Since $\sup (s_{j+1} - s_j) \to 0$, $\psi(\theta_j, \mu_j) = \psi(\theta_j, \theta_j) + o(1) = \sum_1^p a_k \phi_k^2(\theta_j) + o(1)$. Also, $\sum_{J_1} 1 \leqq n$. It follows then from (3.44) that, if $\hat{\delta}_{n_m} \to 0$

$$(3.45) \qquad n_m^2 \textstyle\sum_{J_n} \hat{B}_{jn_m}(\phi) \geqq \frac{1}{12}\left(\int_{\Sigma a_k \phi_k^2(x) > \epsilon} \alpha^{\frac{1}{3}}(x)\left(\sum a_k \phi_k^2(x)\right)^{\frac{1}{3}} dx\right)^3 + o(1)$$

as $m \to \infty$. (3.42), (3.43), (3.45), and the arbitrariness of ϵ yield (3.36).

(3.37) is obtained from (B) of Theorem 3.1 by observing that $\phi_k(x)/h(x)$ is continuous since $h^2(x) \geqq c_0 \phi_k^{4/3}(x)$ for a positive constant c_0. This complete the proof of Theorem 3.2.

REMARK. When $p = 1$ Theorem 3.2 is the same as Theorem 3.1 of [2]. There is a gap in the proof in [2] when ϕ_1 can have zeros. Theorem 3.2 fills this gap.

4. Asymptotic optimality. In this section we will deal with the criteria of optimality discussed in Section 2. In Section 3 we have developed the necessary machinery for obtaining asymptotically optimum sequences of designs for several different criteria. We suppose, as usual, that we are in the context of Section 2 so that (2.1), (2.2), etc. are in force.

Let us put $\phi'(t) = (\phi_1(t), \cdots, \phi_J(t))$ so that $\phi(t)$ is a column vector. We shall also let $f'(t) = (f_1(t), \cdots, f_J(t))$.

THEOREM 4.1. *Let $\psi(B) = \text{tr}(BM)$ with M a non-negative $J \times J$ matrix. Let $h^*(t) = \eta\alpha^{\frac{1}{3}}(t)(\phi'(t)M\phi(t))^{\frac{1}{3}}$ where η is such that $\int_0^1 h^*(t)\,dt = 1$. If $\{T_n^*\}$ is RS(h^*) then $\{T_n^*\}$ is asymptotically $\psi 2$ and $\psi 4$ optimal.*

PROOF. *For this choice of ψ, $\psi 2$ and $\psi 4$* optimality are the same. From (2.16) and (3.36) we can write, for some vectors $\theta_1, \cdots, \theta_p$ in S_j, $\lambda_1, \cdots, \lambda_p$ positive numbers, and any sequence $\{T_n\}$ of designs,

$$n^2 \text{tr}(A - A_{T_n})M$$

$$(4.1) \qquad = n^2 \textstyle\sum_1^p \lambda_i\theta_i'(A - A_{T_n})\theta_i = n^2 \sum_1^p \lambda_i\|\theta_i'f - (\theta_i'f)_n\|^2$$

$$\geq \tfrac{1}{12}\left(\int_0^1 \alpha^{\frac{1}{3}}(t)\left[\textstyle\sum_1^p \lambda_i(\theta_i'\phi(t))^2\right]^{\frac{1}{3}} dt\right)^3 + o(1)$$

$$= \tfrac{1}{12}\left(\int_0^1 \alpha^{\frac{1}{3}}(t)(\phi'(t)M\phi(t))^{\frac{1}{3}} dt\right)^3 + o(1).$$

Applying the other half of Theorem 3.2, namely (3.37), finishes the proof of Theorem 4.1.

We note in passing that if h^* is as defined in Theorem 4.1, and $\theta \,\varepsilon$ Range (M) then

$$(4.2) \qquad \lim_{n\to\infty} n^2\theta'(A - A_{T_n^*})\theta = \tfrac{1}{12}\int_0^1 \alpha(t)(\theta'\phi(t))^2(h^{*2}(t))^{-1}\,dt.$$

This is a consequence of Theorem 3.1 (B) which is applicable because

$$|\theta'\phi(t)|/h^*(t) \leqq |(M\rho, \phi)|/h^* \leqq c(M\phi, \phi)^{\frac{1}{2}}(M\rho, \rho)^{\frac{1}{2}}/(M\phi, \phi)^{\frac{1}{3}}$$

which approaches 0 as $(M\phi, \phi)$ approaches 0 so that $|\theta'\phi|/h^*$ is continuous. We also have, if $N \geqq 0$ and Range $(N) \subset$ Range (M),

$$(4.3) \qquad \lim_{n\to\infty} n^2 \text{tr}(A - A_{T_n^*})N = \tfrac{1}{12}\int_0^1 \alpha(t)\phi'(t)N\phi(t)(h^{*2}(t))^{-1}\,dt.$$

THEOREM 4.2. *Suppose ψ is a strict and continuously differentiable criterion so that $\psi(A) - \psi(A_T) = \text{tr}((A - A_T)M) + o(|A - A_T|)$ with $M > 0$. Then $h^*(t) = \eta\alpha^{\frac{1}{3}}(t)(\phi'(t)M\phi(t))^{\frac{1}{3}}$ generates a RS (h^*), $\{T_n^*\}$, which is asymptotically ψ 2-optimal.*

PROOF. Since ψ is strict and $M > 0$ we need only consider sequences $\{T_n\}$ such that $|A - A_{T_n}| \to 0$. If $n^2|A - A_{T_n}|$ is not $O(1)$ as $n \to \infty$ then

$$\limsup_{n\to\infty} n^2[\text{tr}((A - A_{T_n})M) + o(|A - A_{T_n}|)]$$
$$\geqq c_0 \limsup_{n\to\infty} n^2 \text{tr}(A - A_{T_n}) = \infty$$

so that $\limsup_{n\to\infty} n^2[\psi(A) - \psi(A_{T_n})] = \infty$ and such sequences $\{T_n\}$ can be ignored since, as we will see, $n^2[\psi(A) - \psi(A_{T_n^*})]$ has a finite limit. In fact, from Theorem 4.1 we have $\lim_{n\to\infty} n^2 \text{tr}(A - A_{T_n^*})M = (12\eta^3)^{-1} < \infty$ and from (4.2) we obtain $|A - A_{T_n^*}| = O(n^{-2})$. Hence $\lim_{n\to\infty} n^2[\psi(A) - \psi(A_{T_n^*})] = (12\eta_3)^{-1} < \infty$. For sequences $\{T_n\}$ with $|A - A_{T_n}| = O(n^{-2})$, we have with the aid of Theorem 4.1, that

$$(4.4) \qquad \liminf_{n\to\infty} n^2[\psi(A) - \psi(A_{T_n})] \geqq (12\eta^3)^{-1}.$$

Since there is equality in (4.4) when $\{T_n\}$ is replaced by $\{T_n^*\}$ we have finished the proof of Theorem 4.2.

COROLLARY. *If* $\psi(B) = \det B$ *then* $h^*(t) = \eta\alpha^{\frac{1}{3}}(t)[\phi'(t)A^{-1}\phi(t)]^{\frac{1}{3}}$ *generates a* RS, $\{T_n^*\}$, *which is asymptotically* $\psi 2$*-optimum.*

PROOF. Theorem 4.2 and (2.17) with $\psi(B) = \det B$.

THEOREM 4.3. *Let* $\mathfrak{M}$ *be a compact set of non-negative matrices and put* $\psi(B) = \min_{M \varepsilon \mathfrak{M}} \operatorname{tr}(BM)$. *Let* M^* *be such that the minimum over* $\mathfrak{M}$ *of* $\int_0^1 \alpha^{\frac{1}{3}}(t) \cdot [\phi'(t)M\phi(t)]^{\frac{1}{3}}\,dt$ *is attained at* M^*. *Then*

$$h^*(t) = \eta\alpha^{\frac{1}{3}}(t)[\phi'(t)M^*\phi(t)]^{\frac{1}{3}}$$

generates a RS $\{T_n^*\}$ *which is asymptotically* $\psi 4$*-optimal.*

PROOF. For any $\{T_n\}$, we obtain from Theorem 4.1,

$$(4.5) \quad n^2 \operatorname{tr}(A - A_{T_n})M \geqq \tfrac{1}{12}\left(\int_0^1 \alpha^{\frac{1}{3}}(t)[\phi'(t)M\phi(t)]^{\frac{1}{3}}\,dt\right)^3 + o(1) \geqq (12\eta^3)^{-1} + o(1)$$

for each $M \varepsilon \mathfrak{M}$. It is not hard to show that the $o(1)$ term is uniform for $M \varepsilon \mathfrak{M}$. Consequently,

$$(4.6) \quad n^2 \inf_{M \varepsilon \mathfrak{M}} \operatorname{tr}(A - A_{T_n})M \geqq (12\eta^3)^{-1} + o(1).$$

On the other hand, by Theorem 4.1,

$$(4.7) \quad n^2 \inf_{M \varepsilon \mathfrak{M}} \operatorname{tr}(A - A_{T_n^*})M \leqq n^2 \operatorname{tr}(A - A_{T_n^*})M^* = (12\eta^3)^{-1} + o(1).$$

(4.7) and (4.6) imply the conclusion of Theorem 4.3.

We will use Theorem 4.3 to establish asymptotic $\psi 4$-optimality when $\psi(B) = \det B$. First we need the following lemma.

LEMMA 4.1. *Let* $\{T_n\}$ *be any sequence of designs and let* λ_n *denote the smallest eigenvalue of* $n^2(A - A_{T_n})$. *Then* $\liminf_{n\to\infty} \lambda_n > 0$. *Furthermore, there is a sequence* $\{\tilde{T}_n\}$ *such that the largest eigenvalue* $\tilde{\mu}_n$ *of* $n^2(A - A_{\tilde{T}_n})$ *satisfies*

$$\limsup_{n\to\infty} \tilde{\mu}_n < \infty.$$

PROOF. Let Q_n be an orthogonal matrix such that, putting $g_n(t) =$ first coordinate of $Q_n(f(t) - f_n(t))$, $n^2\|g_n\|^2 = \lambda_n$. If $\liminf_{n\to\infty} \lambda_n = 0$ then there is a subsequence $\{n_k\}$ and an orthogonal matrix Q such that $\lambda_{n_k} \to 0$ and $|Q_{n_k} - Q| \to 0$. Then, by Theorem 3.2 (with $p = 1$),

$$0 = \lim_{k\to\infty} n_k^2\|g_{n_k}\|^2 \geqq \tfrac{1}{12}\left[\int_0^1 \alpha^{\frac{1}{3}}(t)(Q\phi(t))_1^{\frac{2}{3}}\,dt\right]^3.$$

Thus $0 = (Q\phi(t))_1 = \sum_1^J q_{1j}\phi_j(t)$ for all $t \varepsilon [0, 1]$ which contradicts (2.2). Thus $\liminf_{n\to\infty} \lambda_n > 0$.

For the other part of Lemma 4.1 note that $\mu_n \leqq n^2 \operatorname{tr}(A - A_{T_n})$ and by Theorem 4.1 (with $M = I$) there is a $\{\tilde{T}_n\}$ with $n^2 \operatorname{tr}(A - A_{\tilde{T}_n})$ bounded for all n. This finishes the proof of Lemma 4.1.

THEOREM 4.4. *Let* $\psi(B) = \det B$. *Let* $\mathfrak{M} = \{M \mid M > 0, \det M = 1\}$ *and let* M^* *be such that* $\int_0^1 \alpha^{\frac{1}{3}}(t)[\phi'(t)M\phi(t)]^{\frac{1}{3}}\,dt$ *is minimized over* $\mathfrak{M}$ *by* M^* *(it is shown in the course of the proof that such a minimum exists even though* $\mathfrak{M}$ *is not compact). Then* $h^*(t) = \eta\alpha^{\frac{1}{3}}(t)[\phi'(t)M^*\phi(t)]^{\frac{1}{3}}$ *generates a* RS $\{T_n^*\}$ *which is asymptotically* $\psi 4$*-optimal.*

PROOF. If $B > 0$ then

$$J(\det B)^{1/J} = \min_{\mathfrak{M}} \operatorname{tr}(B.M) \tag{4.8}$$

(cf [1], pg. 131). Let λ_B, μ_B denote the smallest and largest eigenvalues of B. It is easy to verify that if $0 < \lambda_0 \leqq \lambda_B \leqq \mu_B \leqq \mu_0 < \infty$ then the M which minimizes the right side of (4.8) must satisfy $\lambda_M \geqq \lambda_0/\mu_0$.

For the sequence $\{\tilde{T}_n\}$ of Lemma 4.1 we have $\tilde{\mu}_n \leqq c_1$ so that $\det(n^2(A - A_{\tilde{T}_n})) \leqq c_1^J < \infty$ and we may therefore limit ourselves, in establishing asymptotic $\psi 4$-optimality, to sequences $\{T_n\}$ with

$$\limsup_{n\to\infty} \det(n^2(A - A_{T_n})) \leqq c_1^J. \tag{4.9}$$

For any sequence $\{T_n\}$, $\lambda_n \geqq c_0$(say) for all n large enough so that, for any sequence $\{T_n\}$ satisfying (4.9), $\mu_n \leqq C_3$ for all n large enough. Hence by (4.8) and the discussion following (4.8) we can write, for $\{T_n\}$ satisfying (4.9) and some $\epsilon > 0$,

$$J[\det n^2(A - A_{T_n})]^{1/J} = \min_{\mathfrak{M}_\epsilon} \operatorname{tr}(n^2(A - A_{T_n})M) \tag{4.10}$$

where $\mathfrak{M}_\epsilon = \{M \mid M > 0, \det M = 1, \lambda_M \geqq \epsilon\}$. Applying Theorem 4.3 with $\mathfrak{M}_\epsilon$ (which is compact) we obtain M_ϵ^* such that

$$\begin{aligned} \liminf_{n\to\infty} J[\det n^2(A - A_{T_n})]^{1/J} & \\ &\geqq \lim_{n\to\infty} J[\det(n^2(A - A_{T_n^{*\epsilon}}))]^{1/J} \\ &= \tfrac{1}{12}\left[\int_0^1 \alpha^{\frac{1}{3}}(t)[\phi'(t)M_\epsilon^*\phi(t)]^{\frac{1}{3}}\,dt\right]^3 \end{aligned} \tag{4.11}$$

where $T_n^{*\epsilon}$ is generated by M_ϵ^* in the same way T_n^* is generated by M^* in Theorem 4.3.

Now, for $M \in \mathfrak{M} - \mathfrak{M}_\epsilon$ and $\{T_n\}$ a RS generated by $h = \eta\alpha^{\frac{1}{3}}(\phi' M\phi)^{\frac{1}{3}}$ we have, using (4.10), (4.11) and Theorem 4.1,

$$\tfrac{1}{12}\left[\int \alpha^{\frac{1}{3}}(\phi' M\phi)^{\frac{1}{3}}\right]^3 = \lim_{n\to\infty} \operatorname{tr}(n^2(A - A_{T_n})M) \geqq \tfrac{1}{12}\left[\int_0^1 \alpha^{\frac{1}{3}}(\phi' M_\epsilon^*\phi)^{\frac{1}{3}}\right]^3$$

so that M_ϵ^* minimizes $\int \alpha^{\frac{1}{3}}(\phi' M\phi)^{\frac{1}{3}}$ over $\mathfrak{M}$. This and (4.11) gives the conclusion of Theorem 4.4 and finishes the proof.

THEOREM 4.5. *Let $\psi(B) = \operatorname{tr}(BM)$, M non-negative. Let $h^*(t) = \eta\alpha^{\frac{1}{3}}(t)\cdot[\phi'(t)A^{-1}MA^{-1}\phi(t)]^{\frac{1}{3}}$ and let $\{T_n^*\}$ be RS (h^*). If T^* = support of h and $A_* = A_{T^*}$ is non-singular then $\{T_n^*\}$ is asymptotically $\psi 1$ and $\psi 3$ optimum.*

REMARK 4.1. The non-singularity of A_{T^*} is clearly essential in order for Theorem 4.5 to have meaning. If M is positive then $A_* = A$ (which is non-singular) because the h^* of Theorem 4.5 is the same as the h^* of Theorem 4.1 with M there replaced by $A^{-1}MA^{-1}$ and since $n^2 \operatorname{tr}(A - A_{T_n^*})A^{-1}MA^{-1} = n^2 \operatorname{tr}(M^{\frac{1}{2}}A^{-1}(A - A_{T_n^*})A^{-1}M^{\frac{1}{2}})$ is then bounded, it must be that $A_* = \lim_{n\to\infty} A_{T_n^*} = A$.

If support of $h^* = [0, 1]$ then, obviously, $A_* = A$ which is non-singular. If $f_1, \cdots, f_J$ are linearly independent over T^* (e.g., if they are linearly independent over every interval of positive length) then A_* is non-singular.

PROOF OF THEOREM 4.5. *If* A_{T_n} is non-singular for n large enough then, for such n,

$$A_{T_n}^{-1} - A^{-1} = A^{-1}(A - A_{T_n})A^{-1} + A^{-1}(A - A_{T_n})A_{T_n}^{-1}(A - A_{T_n})A^{-1} \geqq A^{-1}(A - A_{T_n})A^{-1}. \tag{4.12}$$

Hence, by Theorem 4.1,

$$\begin{aligned} & n^2 \operatorname{tr} (A_{T_n}^{-1} - A^{-1})M \\ & \geqq n^2 \operatorname{tr} (A^{-1}(A - A_{T_n})A^{-1}M) = n^2 \operatorname{tr} ((A - A_{T_n})A^{-1}MA^{-1}) \\ & \geqq (12\eta^3)^{-1} + o(1) \end{aligned} \tag{4.13}$$

and

$$n^2 \operatorname{tr} (A - A_{T_n^*})A^{-1}MA^{-1} = (12\eta^3)^{-1} + o(1). \tag{4.14}$$

Observe now that if B is positive then

$$\operatorname{tr} CBC' \leqq \mu_B \operatorname{tr} C'C \tag{4.15}$$

where μ_B is the largest eigenvalue of B. Also, if $B > 0$,

$$(B\theta, B\theta) \leqq (B\theta, \theta) \operatorname{tr} (B) \tag{4.16}$$

where (,) denotes the usual Euclidean inner product. Let $A^{-1}MA^{-1} = \sum \theta_i\theta_i'$ with $\theta_i \varepsilon$ Range $(A^{-1}MA^{-1})$ so that by (4.14) and (4.2)

$$\theta_i'(A - A_{T_n^*})\theta_i = O(n^2). \tag{4.17}$$

Now apply (4.15) with $C = M^{\frac{1}{2}}A^{-1}(A - A_{T_n^*})$, $B = A_*^{-1}$ and then (4.16) with $B = A - A_{T_n^*}$ to obtain, with the use of (4.17),

$$\begin{aligned} & \operatorname{tr} [M^{\frac{1}{2}}A^{-1}(A - A_{T_n^*})A_*^{-1}(A - A_{T_n^*})A^{-1}M^{\frac{1}{2}}] \\ & \leqq \mu_{A_*^{-1}} \textstyle\sum_i ((A - A_{T_n^*})\theta_i, (A - A_{T_n^*})\theta_i) \\ & \leqq \mu_{A_*^{-1}} \operatorname{tr} (A - A_{T_n^*})O(n^{-2}) \end{aligned} \tag{4.18}$$

so that, letting $n \to \infty$ in (4.18) we obtain $M^{\frac{1}{2}}A^{-1}(A - A_*) = 0$ or $A_*^{-1}M = A^{-1}M$. Then

$$\operatorname{tr} (A_{T_n^*}^{-1} - A^{-1})M = \operatorname{tr} (A_{T_n^*}^{-1} - A_*^{-1})M$$

and we can replace A by A_* everywhere above. We may as well assume then that $A = A_*$ so that $A_{T_n^*} \to A$. It follows that $\operatorname{tr} (A - A_{T_n^*}) \to 0$ so that the left side of (4.18) is $o(n^{-2})$ and remains so even with $A_{T_n^*}^{-1}$ in place of A_*^{-1}. But this together with (4.12) and (4.14) yields

$$n^2 \operatorname{tr} (A_{T_n^*}^{-1} - A^{-1})M = (12\eta^3)^{-1} + o(1)$$

so that by (4.13) $\{T_n^*\}$ is asymptotically $\psi1$-optimal. Since, for the criterion of this theorem, $\psi1$ and $\psi3$ optimality are the same, we have finished the proof of Theorem 4.5.

THEOREM 4.6. *If ψ is strict and continuously differentiable so that* $\psi(A_T^{-1}) - \psi(A^{-1}) = \operatorname{tr}(A_T^{-1} - A^{-1})M + o(|A_T^{-1} - A^{-1}|)$ *for some positive M then* $h^*(t) = \eta\alpha^{\frac{1}{3}}(t)[\phi'(t)A^{-1}MA^{-1}\phi(t)]^{\frac{1}{3}}$ *generates a RS $\{T_n^*\}$ which is asymptotically ψ1-optimal.*

PROOF. M is positive so that, by Remark 4.1, $\lim_{n\to\infty} A_{T_n^*} = A$. Using Theorem 4.5 and following the proof of Theorem 4.2 we obtain Theorem 4.6.

THEOREM 4.7. (A) *Let $\mathfrak{M}$ be a compact set of non-negative matrices and* $\psi(B) = \operatorname{tr}(BM)$. *Let M^* minimize* (*over* $\mathfrak{M}$) $\int_0^1 \alpha^{\frac{1}{3}}[\phi'A^{-1}MA^{-1}\phi]^{\frac{1}{3}}$. *Then* $h^*(t) = \eta\alpha^{\frac{1}{3}}(t)[\phi'(t)A^{-1}M^*A^{-1}\phi(t)]^{\frac{1}{3}}$ *generates a RS $\{T_n^*\}$ and if* $\lim A_{T_n^*} = A_*$ *is non-singular then $\{T_n^*\}$ is asymptotically ψ3-optimal.*

(B) *If* $\mathfrak{M} = \{M \mid M > 0, \det M = 1\}$ *and* $\psi(B) = \det B$, *then there is an* $M^* \varepsilon \mathfrak{M}$ *which minimizes* $\int_0^1 \alpha^{\frac{1}{3}}[\phi'A^{-1}MA^{-1}\phi]^{\frac{1}{3}}$ *over* $\mathfrak{M}$ *and* $h^*(t) = \eta\alpha^{\frac{1}{3}}(t)\cdot[\phi'(t)A^{-1}M^*A^{-1}\phi(t)]^{\frac{1}{3}}$ *generates a RS $\{T_n^*\}$ which is asymptotically ψ3-optimal (and, comparing with Theorem 4.4, ψ4-optimal).*

PROOF. Follow the arguments of Theorems 4.3 and 4.4, and use Theorem 4.5.

THEOREM 4.8. *Let $\mathfrak{M}$ be a compact set of non-negative $J \times J$ matrices and let $\mathfrak{M}^+$ denote its convex hull. Let* $\psi(B) = \max_{M\varepsilon\mathfrak{M}} \operatorname{tr}(BM) = \max_{\mathfrak{M}^+} \operatorname{tr}(BM)$. *Let M^* maximize* (over $\mathfrak{M}^+$) $\int_0^1 \alpha^{\frac{1}{3}}(\phi'M\phi)^{\frac{1}{3}}$ *and let* $h^*(t) = \eta\alpha^{\frac{1}{3}}(t)(\phi'(t)M^*\phi(t))^{\frac{1}{3}}$. *If, for all $M \varepsilon \mathfrak{M}$,*

$$n^2 \operatorname{tr}(A - A_{T_n^*})M \leqq \tfrac{1}{12}\int_0^1 \alpha(t)\phi'(t)M\phi(t)(h^{*2}(t))^{-1}\,dt + o(1) \tag{4.19}$$

where the $o(1)$ term is uniform in M then $\{T_n^*\}$ is asymptotically ψ4-optimal.

A comment about (4.19) will follow the proof of Theorem 4.8).

PROOF. From Theorem 4.1 we have, for any sequence $\{T_n\}$,

$$\begin{aligned} n^2 \max_{\mathfrak{M}^+} \operatorname{tr}(A - A_{T_n})M &\geqq \max_{\mathfrak{M}^+} \tfrac{1}{12}[\textstyle\int_0^1 \alpha^{\frac{1}{3}}(\phi'M\phi)^{\frac{1}{3}}]^3 + o(1) \\ &= (12\eta^3)^{-1} + o(1). \end{aligned} \tag{4.20}$$

If $M \neq M^*$ let $g(\lambda) = \int_0^1 \alpha^{\frac{1}{3}}[\phi'(\lambda M + (1-\lambda)M^*)\phi]^{\frac{1}{3}}$ for $\lambda \varepsilon [0, 1]$. Then g is continuous and has a maximum at $\lambda = 0$. For $0 < \lambda < 1$

$$g'(\lambda) = \tfrac{1}{3}\int \alpha^{\frac{1}{3}}[\phi'M\phi - \phi'M^*\phi][\phi'(\lambda M + (1-\lambda)M^*)\phi]^{-\frac{2}{3}}\,dt$$

and

$$\liminf_{\lambda\to 0} g'(\lambda) \geqq \tfrac{1}{3}\int \alpha\phi'M\phi/h^{*2} - \tfrac{1}{3}\int \alpha\phi'M^*\phi/h^{*2}.$$

Since g has a maximum at 0, $\liminf_{\lambda\to 0} g'(\lambda) \leqq 0$ and, therefore,

$$\int \alpha\phi'M\phi/h^{*2} \leqq \int \alpha\phi'M^*\phi/h^{*2} = \eta^{-3} \tag{4.21}$$

so the right hand side of (4.19) is finite for all $M \varepsilon \mathfrak{M}^+$. Now (4.19) and (4.21) imply that

$$\limsup_{n\to\infty} n^2 \sup_{\mathfrak{M}} \operatorname{tr}(A - A_{T_n^*})M \leqq (12\eta^3)^{-1}$$

and use of (4.20) yields the theorem.

REMARK 4.2. Conditions guaranteeing the validity of (4.19) can be obtained from Theorem 3.1. Thus, if M^* is positive, then $\phi' M\phi/h^{*2}$ is continuous and (B) of Theorem 3.1 will give (4.19). Also, if h^* satisfies (A) or (C) of Theorem 3.1 then (4.19) will hold.

COROLLARY. *If Theorem* 4.8 *holds for* $\bar{\mathfrak{M}} = \{A^{-1}MA^{-1} \mid M \in \mathfrak{M}\}$ *then* $\bar{T}_n{}^*$ *is asymptotically* $\psi 3$*-optimal for the criterion* $\psi(B) = \sup_{\mathfrak{M}} \operatorname{tr}(BM)$ *provided* A_{T^*} *is non-singular.*

PROOF. Use Theorems 4.5 and 5.8.

For the criterion of Theorem 4.8 we wish to establish asymptotic $\psi 1$ and $\psi 2$ optimality. Let $\mathfrak{M}^+$ be the convex hull of the compact set $\mathfrak{M}$ and let $\mathfrak{M}_1 = \{M' \in \mathfrak{M}^+ \mid \sup_{\mathfrak{M}^+} \operatorname{tr}(AM) = \operatorname{tr}(AM')\}$. Let $\{T_n\}$ be any sequence of designs and let M_n be any element in $\mathfrak{M}^+$ such that $\sup_{\mathfrak{M}^+} \operatorname{tr} A_{T_n}M = \operatorname{tr} A_{T_n}M_n$. Then, if $M_0 \in \mathfrak{M}_1$,

$$\begin{aligned} &\sup_{\mathfrak{M}} \operatorname{tr}(AM) - \sup_{\mathfrak{M}} \operatorname{tr}(A_{T_n}M) \\ &\qquad = \sup_{\mathfrak{M}^+} \operatorname{tr} AM - \sup_{\mathfrak{M}^+} \operatorname{tr} A_{T_n}M \\ &\qquad = \operatorname{tr} A(M_0 - M_n) + \operatorname{tr}(A - A_{T_n})M_n \\ &\qquad \geqq \operatorname{tr}(A - A_{T_n})M_n \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} &\sup_{\mathfrak{M}} \operatorname{tr} AM - \sup_{\mathfrak{M}} \operatorname{tr} A_{T_n}M \\ &\qquad = \inf_{M_0 \in \mathfrak{M}_1} [\operatorname{tr}(A - A_{T_n})M_0 + \operatorname{tr} A_{T_n}(M_0 - M_n)] \\ &\qquad \leqq \inf_{M_0 \in \mathfrak{M}_1} \operatorname{tr}(A - A_{T_n})M_0 . \end{aligned} \tag{4.23}$$

As we shall see below there are sequences, $\{T_n\}$ for which the left side of (4.22) is $O(n^{-2})$ so that we can limit ourselves to such sequences and conclude from (4.22) that $\operatorname{tr} A(M_0 - M_n) \to 0$. But this means that $\inf_{M \in \mathfrak{M}_1} |M_n - M| \to 0$. For $\epsilon > 0$ let $\mathfrak{M}(\epsilon) = \{M \in \mathfrak{M}^+ | \inf_{N \in \mathfrak{M}_1} |M - N| \leqq \epsilon\}$. Then, for n large enough using Theorem 4.3,

$$\begin{aligned} n^2 \operatorname{tr}(A - A_{T_n})M_n &\geqq \inf_{\mathfrak{M}(\epsilon)} n^2 \operatorname{tr}(A - A_{T_n})M \\ &\geqq \tfrac{1}{12} \inf_{M \in \mathfrak{M}(\epsilon)} \left(\int_0^1 \alpha^{\frac{1}{3}}(t)(\phi'(t)M\phi(t))^{\frac{1}{3}}\, dt\right)^3 + o(1). \end{aligned} \tag{4.24}$$

First let $n \to \infty$ and then let $\epsilon \to 0$ and obtain from (4.24) that

$$\liminf_{n \to \infty} n^2 \operatorname{tr}(A - A_{T_n})M_n \geqq \tfrac{1}{12} \inf_{\mathfrak{M}_1} \left(\int_0^1 \alpha^{\frac{1}{3}}(\phi' M\phi)^{\frac{1}{3}}\right)^3. \tag{4.25}$$

Now let M^* be any element of $\mathfrak{M}_1$ (which is compact) which minimizes the right side of (4.25), let $h^* = \eta\alpha^{\frac{1}{3}}(\phi' M^* \phi)^{\frac{1}{3}}$, and let $\{T_n{}^*\}$ be RS (h^*), and apply Theorem 4.3 to the right side of (4.23) and obtain

$$n^2 \inf_{\mathfrak{M}_1} \operatorname{tr}(A - A_{T_n{}^*})M_0 = (12\eta^3)^{-1} + o(1). \tag{4.26}$$

(4.26), (4.25), (4.22) and (4.23) yield

THEOREM 4.9. $\{T_n{}^*\}$ *as defined following* (4.25) *is asymptotically* $\psi 2$*-optimal for the criterion* $\psi(B) = \sup_{\mathfrak{M}} \operatorname{tr}(BM)$, *where* $\mathfrak{M}$ *is a compact set of non-negative* $J \times J$ *matrices.*

By use of Theorem 4.5 we can obtain a corresponding result for $\psi 1$ optimality. We omit the details.

The conclusions of Theorems 4.3, 4.4, 4.7, 4.8 and, to a lesser extent, 4.9 are not as satisfactory as those of the remaining theorems of this section. To apply these theorems, one must find a matrix M^* which maximizes or minimizes $\int \alpha^{\frac{1}{3}}(\phi' M\phi)^{\frac{1}{3}}$ (or $\int \alpha^{\frac{1}{3}}(\phi' A^{-1} MA^{-1}\phi)^{\frac{1}{3}}$) over some compact set of non-negative matrices M. Even more, some of these problems require a knowledge of A^{-1} when A is more readily available (see (2.3)). We give two examples here to indicate some of the difficulties which are present.

Suppose $R(s, t) = \exp[-|s - t|]$, $f_j(t) = 2t^j$, $j = 0, 1, \cdots, J$, and, without loss of generality, suppose the interval of observation is $[-b, b]$. We then find α is constant, $\phi_j(t) = t^j - j(j-1)t^{j-2}$, $j = 0, 1, \cdots, J$, and

$$A_{ij} = b^{i+j+1}/(i+j+1) + b^{i+j} + (ij/(i+j-1))b^{i+j-1} \qquad \text{if } i+j \text{ is even} \tag{4.27}$$

$$= 0 \qquad \text{if } i+j \text{ is odd, } 0 \leqq i, j \leqq J.$$

Consider the problem of minimizing the maximum variance of the BLE of an estimable function with limiting variance 1, i.e., consider $\psi 1$ (or $\psi 3$) optimality for $\psi(B) = \sup_{x'A^{-1}x=1} x'Bx$. We would expect to find an asymptotically optimum sequence of designs by maximizing $\int (\phi' A^{-1}MA^{-1}\phi)^{\frac{1}{3}}$ over the set $\mathfrak{M}^+ = \{M \geqq 0 \mid \operatorname{tr} A^{-1}M = 1\}$. Equivalently, we could try to maximize $\int (\phi' M\phi)^{\frac{1}{3}}$ over the set $\bar{\mathfrak{M}}^+ = \{M \geqq 0 \mid \operatorname{tr} AM = 1\}$. For the special cases $J = 0$ and $J = 1$, the uniform density generates asymptotically optimum sequences for all $b > 0$, but this is no longer true when $J = 2$. We show below that if b is large, J is arbitrary and $M^*(b)$ is a maximizing matrix, then $(\phi' M^*(b)\phi)^{\frac{1}{3}}$ is approximately uniform on $[-b, b]$. Since for large b the corresponding density will be positive, we may appeal to Theorem 4.7 and say that the sequence of designs it generates is asymptotically $\psi 1$ optimum for the problem posed above.

First let us note that if $m_{ij}^*(b)$ denotes the ijth entry of $M^*(b)$, $0 \leqq i, j \leqq J$, then $m_{ij}^*(b) = 0$ for $i + j$ odd. If this were not the case, we could find $M^{**}(b)$ in $\bar{\mathfrak{M}}^+$ with $m_{ij}^{**}(b) = m_{ij}^*(b)$, $i + j$ even, and $m_{ij}^{**}(b) = -m_{ij}^*(b)$, $i + j$ odd. But then $\frac{1}{2}M^*(b) + \frac{1}{2}M^{**}(b)$ is also in $\bar{\mathfrak{M}}^+$ and

$$\int (\phi'(\tfrac{1}{2}M^*(b) + \tfrac{1}{2}M^{**}(b))\phi)^{\frac{1}{3}} > \tfrac{1}{2}\int (\phi' M^*(b)\phi)^{\frac{1}{3}} + \tfrac{1}{2}\int (\phi' M^{**}(b)\phi)^{\frac{1}{3}}$$
$$= \int (\phi' M^*(b)\phi)^{\frac{1}{3}}$$

which contradicts the maximizing property of $M^*(b)$. The side condition on $M^*(b)$ is

$$1 = \operatorname{tr} AM^*(b) \tag{4.28}$$
$$= \sum{}' m_{ij}^*(b)[b^{i+j+1}/(i+j+1) + b^{i+j} + (ij/(i+j-1))b^{i+j-1}]$$

where $\sum'$ denotes the sum over $i + j$ even. It follows from (4.28) that $m_{ij}^*(b)$

$= O(b^{-i-j-1})$ as $b \to \infty$ so let $\hat{m}_{ij}(b) = b^{i+j+1}m_{ij}^*(b)$. Suppose along a sequence of b's, we have $\hat{m}_{ij}(b) \to \hat{m}_{ij}$. Then

$$1 = \operatorname{tr} MH = \sum' \hat{m}_{ij}(i + j + 1)^{-1}. \tag{4.29}$$

Along the sequence of b's, let us now compute as follows:

$$\begin{aligned} \tfrac{1}{2}b^{-\frac{1}{2}} \int_{-b}^{b} (\phi' M^*(b)\phi)^{\frac{1}{2}} &= \int_0^1 [\textstyle\sum' u^{i+j}\hat{m}_{ij}]^{\frac{1}{2}}\,du + o(1) \\ &\leqq (\textstyle\int_0^1 [\sum' u^{i+j}\hat{m}_{ij}]\,du)^{\frac{1}{2}} + o(1) = 1 + o(1) \end{aligned} \tag{4.30}$$

where inequality will hold in (4.30) unless $\sum' u^{i+j}\hat{m}_{ij}$ is constant. On the other hand, if $M(b)$ is given by $m_{00}(b) = (b + 1)^{-1}$, $m_{ij}(b) = 0$ for $(i, j) \neq (0, 0)$. $\operatorname{tr} AM(b) = 1$ and

$$\tfrac{1}{2}b^{-\frac{1}{2}} \int_{-b}^{b} [\phi' M(b)\phi]^{\frac{1}{2}} = 1 + o(1). \tag{4.31}$$

A comparison of (4.30) and (4.31) shows that $\hat{m}_{00} = 1$ and $\hat{m}_{ij} = 0$ for $(i, j) \neq (0, 0)$. Thus, $m_{00}^*(b) = b^{-1}(1 + o(1))$ and $m_{ij}^*(b) = o(b^{-i-j-1})$ and from this it follows easily that $(\phi M^*(b)\phi)^{\frac{1}{2}}$ is approximately uniform on $[-b, b]$ when b is large.

We continue with the same set-up and investigate $\psi 3$ asymptotic optimality under the criterion $\psi(B) = \sup_{-b \leqq t \leqq b} f(t)'Bf(t)$. This criterion differs only by a constant multiple (recall that $f(t) = 2t^i$, $i = 0, 1, \cdots, J$) from the criterion $\psi(B) = \sup_{\mathfrak{M}^+} \operatorname{tr} BM$ where $\mathfrak{M}^+$ consists of the moment matrices to order $2J$ of probability measures ξ having support in $[-b, b]$. That is, the matrices in question have the form $M = (\int_{-b}^{b} x^{i+j}\,d\xi(x);\ 0 \leqq i, j \leqq J)$.

We seek M^* (or ξ^*) so as to maximize $\int (\phi' A^{-1}MA^{-1}\phi)^{\frac{1}{2}}$ over $\mathfrak{M}^+$. As before, it suffices to consider just those M for which $m_{ij} = 0$ if $i + j$ is odd, or, to consider symmetric measures ξ. Now if $\phi' A^{-1}$ is denoted by $(\lambda_0(t), \cdots, \lambda_J(t))$, ξ^* is the symmetric probability measure on $[-b, b]$ which maximizes

$$\begin{aligned} &\int_{-b}^{b} [\textstyle\sum_{i,j} \lambda_i(t)\lambda_j(t) \int_{-b}^{b} x^{i+j}\,d\xi(x)]^{\frac{1}{2}}\,dt \\ &\qquad = \int_{-b}^{b} [\textstyle\int_{-b}^{b} (\sum_{i=0}^{J} \lambda_i(t)x^i)^2\,d\xi(x)]^{\frac{1}{2}}\,dt. \end{aligned} \tag{4.32}$$

If we set $t = bs$ and $x = bu$ in the above, the problem can also be viewed as that of choosing ξ^* on $[-1, 1]$ so as to maximize

$$\int_{-1}^{1} [\textstyle\int_{-1}^{1} (\sum_{i=0}^{J} \lambda_i^b(s)u^i)^2\,d\xi(u)]^{\frac{1}{2}}\,ds \tag{4.33}$$

where $\lambda_i^b(s) = b^i\lambda_i(bs)$, $i = 0, 1, \cdots, J$. Inasmuch as explicit use of A^{-1} cannot be avoided here, we will look only at the first few values of J.

For the one parameter problem $J = 0$, the appropriate density is, trivially, uniform on $[-b, b]$. When $J = 1$, reference to (4.27) shows that $(\lambda_0(t), \lambda_1(t))$ is proportional to $(b + b^2 + \frac{1}{3}b^3)$, $(1 + b)t)$. It is then easy to see in (4.32) that ξ^* should concentrate mass $\frac{1}{2}$ at each of $\pm b$. Accordingly, the (positive) density on $[-b, b]$ which provides an asymptotically optimal sequence of designs is proportional to $((b + b^2 + \frac{1}{3}b^3)^2 + b^2(1 + b)^2t^2)^{\frac{1}{2}}$.

When $J = 2$, the situation worsens to the extent that $(\lambda_0^b(s), \lambda_1^b(s), \lambda_2^b(s))$ is now proportional to

$$((1 + b + \tfrac{1}{3}b^2)[\tfrac{1}{5}b^3 + b^2 + 2b + 2 - s^2b^2(1 + \tfrac{1}{3}b)], b^2(b + 1)(\tfrac{1}{5}b^2 + b + \tfrac{4}{3})s,$$
$$(1 + b + \tfrac{1}{3}b^2)[s^2b^2(b + 1) - (\tfrac{1}{3}b^3 + b^2 + 2b + 2)]).$$

As $b \to 0$, we find that $(\sum \lambda_i^b(s)u^i)^2$ is proportional to $(1 + u^2)^2 + o(1)$ so a maximizing ξ^* at (4.33) should concentrate mass $\frac{1}{2}$ at each of ± 1. As $b \to \infty$, the same argument leaves us with the non-trivial problem of choosing ξ^* so as to maximize

$$\int_{-1}^{1} [\int_{-1}^{1} ((\tfrac{1}{15} - \tfrac{1}{9}s^2) + \tfrac{1}{5}su + (\tfrac{1}{3}s^2 - \tfrac{1}{9})u^2)^2 \, d\xi(u)]^{\frac{1}{3}} \, ds.$$

We will not pursue these matters further here.

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