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# DESIGNS WITH PARTIAL FACTORIAL BALANCE

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In this paper a class of multidimensional experimental designs said to have partial factorial balance is introduced. These designs are shown to belong to the more general class of multidimensional partially balanced designs. The analysis of designs with partial factorial balance is given in detail and several series of three, four and five dimensional designs are presented.

**1. Introduction.** An experimental design is said to be multidimensional if the design involves more than one factor; see e.g., Potthoff (1962a, b). For example, the ordinary balanced and partially balanced incomplete block designs are two dimensional. The Latin squares, Youden squares, and the designs of Shrikhande (1951) are three dimensional. Finally the Graeco-Latin square designs are four dimensional, and orthogonal arrays of strength two with $m$ constraints are $m$ dimensional designs. The usual analysis of each of the above designs assumes an additivity of the factorial effects; that is, all interaction effects are assumed to be zero.

Srivastava (1961) and Bose and Srivastava (1964) introduced the class of multidimensional partially balanced (MDPB) designs and the corresponding MDPB association schemes. These MDPB designs include as special cases the above mentioned designs, and have proved useful in further economizing on the number of observations to be taken while retaining a relative ease of analysis. Srivastava and Anderson (1970) establish some necessary conditions for the existence of MDPB designs and also consider the connectedness of such designs. Srivastava and Anderson (1971) introduce some new MDPB association schemes and consider procedures for the construction of MDPB designs.

The purpose of this paper is to introduce a special class of MDPB designs for the case where all factors have the same number of levels. These designs have additional properties which further ease their analysis and interpretation. This class of designs, termed to have partial factorial balance, is defined in Section 2 and the special features of their analysis are given. Series of three, four and five dimensional designs with partial factorial balance are given in Sections 3 and 4, which are economic in terms of number of observations required.

The mathematical model is expressed as:

$$E\{\mathbf{y}\} = X'\mathbf{p}\,, \qquad \mathrm{Cov}\,\{\mathbf{y}\} = \sigma^2 I_N\,, \tag{1.1}$$

where $\mathbf{y}$ denotes the $N \times 1$ vector of observations, $\mathbf{p}$ the $mn \times 1$ vector of unknown parameters, and $X'$ the design matrix. The normal equations are given by:

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(1.2) $$(XX')\hat{\mathbf{p}} = X\mathbf{y} .$$

For the basic definitions and properties of MDPB designs the reader is referred to Bose and Srivastava (1964) and Srivastava and Anderson (1970).

**2. Partial factorial balance.** Suppose that there is a partially balanced association scheme with $\eta$ associate classes defined on the set $\{0, 1, 2, \cdots, n-1\}$ with parameters $\eta^\alpha$, $\alpha = 0, 1, \cdots, \eta$, and $p(\alpha; \beta, \gamma)$, Bose and Mesner (1959). That is, if $i$ and $j$ are two integers in the set then there is a relation of association defined so that (a) $i$ and $j$ are either 0th (if $i = j$), 1st, $\cdots$, or $\eta$th associates, (b) each integer in the set has exactly $\eta^\alpha$ $\alpha$th associates, $\alpha = 0, 1, \cdots, \eta$, and the relation of association is symmetric, (c) if $i$ and $j$ are $\alpha$th associates the number of $\beta$th associates of $i$ which are also $\gamma$th associates of $j$ is a constant $p(\alpha; \beta, \gamma)$ independent of $i$ and $j$ so long as they are $\alpha$th associates. Denote by $B^0 = I_n$, $B^1, B^2, \cdots, B^\eta$ the $n \times n$ association matrices of the scheme.

Consider an experiment involving $m$ factors $F_1, F_2, \cdots, F_m$ each with $n$ levels, say $F_{u0}, F_{u1}, \cdots, F_{u,n-1}$, $u = 1, 2, \cdots, m$. We define a relation of association between the levels of factor $F_u$ and the levels of factor $F_v$ as follows: level $F_{uj_u}$ of factor $F_u$ is said to be an $\alpha$th associate of level $F_{vj_v}$ of factor $F_v$ if $j_u$ and $j_v$ are $\alpha$th associates in the scheme defined on the set $\{0, 1, 2, \cdots, n-1\}$, $u, v = 1, 2, \cdots, m$. Thus we have defined an association scheme on and between the $m$ sets of levels of the $m$ factors. It is easy to show that this scheme is MDPB. Note that all within and between set association relations are the same.

If $T$ denotes a design for this $m$ dimensional experiment let $\lambda_{1,2,\cdots,m}^{j_1,j_2,\cdots,j_m}(T)$ denote the number of times the assembly $(F_{1j_1}, F_{2j_2}, \cdots, F_{mj_m})'$ appears in $T$. Similarly $\lambda_u^{j_u}(T)$ denotes the number of assemblies in $T$ in which level $F_{uj_u}$ of factor $F_u$ appears, and $\lambda_{uv}^{j_uj_v}(T)$ the number of assemblies in which $F_{uj_u}$ and $F_{vj_v}$ both appear.

DEFINITION 2.1. The design $T$ is said to have partial factorial balance if

(i) $\lambda_u^{j_u} = \mu$, a constant independent of $u$ and $j_u$.

(ii) $\lambda_{uv}^{j_uj_v} = d^\alpha$, $u \neq v = 1, 2, \cdots, m$, a constant depending on $\alpha$ but independent of $u, v, j_u$, and $j_v$ so long as $F_{uj_u}$ and $F_{vj_v}$ are $\alpha$th associates.

In the remainder of this section we consider the analysis of designs having partial factorial balance. It follows directly, Bose and Srivastava (1964), that each diagonal block of $XX'$ is $\mu I_n$ and each off diagonal block is $B = \sum_{\alpha=0}^{\eta} d^\alpha B^\alpha$. Hence,

(2.1) $$(XX') = (I_m \otimes \mu I_n) + (J_{mm} - I_m) \otimes B , \qquad B = \textstyle\sum_{\alpha=0}^{\eta} d^\alpha B^\alpha ,$$

where $J_{pq}$ denotes the $p \times q$ matrix with every element unity and $\otimes$ denotes the usual left Kronecker product, i.e.,

(2.2) $$A \otimes B = ((a_{ij})) \otimes B = ((a_{ij} B)) .$$

If the design $T$ is completely connected it follows that the matrix

(2.3) $$M = (XX') + (I_m \otimes \theta J_{nn}) , \qquad \theta \neq 0$$

is nonsingular and that $M^{-1}$ is a conditional inverse of $(XX')$. Hence, a solution to the normal equations is given by

$$\hat{\mathbf{p}} = M^{-1}X\mathbf{y} . \tag{2.4}$$

Let $A = \mu I_n + \theta J_{nn}$, then the matrix $M$ may be expressed as

$$M = [I_m \otimes (A - B)] + [J_{mm} \otimes B] \tag{2.5}$$

and

$$M^{-1} = [I_m \otimes (V - W)] + [J_{mm} \otimes W]$$

where $(V - W) = (A - B)^{-1}$,

$$W = -[A + (m - 1)B]^{-1}B[A - B]^{-1} = \sum_{i=0}^{\eta} w_i B^i , \tag{2.6}$$
$$V = [A + (m - 1)B]^{-1}[A + (m - 2)B][A - B]^{-1} = \sum_{i=0}^{\eta} v_i B^i .$$

Thus the calculation of $M^{-1}$ involves the inversion of two matrices, $(A - B)$ and $[A + (m - 1)B]$, where $A$ and $B$ are known from the parameters of the design. The properties of the linear associative algebra generated by the association matrices $B^0, B^1, \cdots, B^\eta$ and its regular representation may be employed to simplify the calculations. For example, if $\eta = 2$ the problem reduces to the inversion of two $3 \times 3$ matrices. In general there will be two $(\eta + 1) \times (\eta + 1)$ matrices. A detailed example is given in the next section.

It is obvious from (2.6) that the variance of a simple contrast of $\alpha$th associates is $2\sigma^2(v_0 - v_\alpha)$ and there are $\eta$ accuracies. Suppose there did exist an orthogonal design (usually there does not) with the same value of $N$ and $\mu$. For such a design the variance of a simple contrast would be $2\sigma^2/\mu$. The "efficiencies" of a design are obtained by considering the ratio of these two variances, that is

$$E_\alpha = \frac{2\sigma^2/\mu}{2\sigma^2(v_0 - v_\alpha)} = \frac{1}{\mu(v_0 - v_\alpha)} , \qquad \alpha = 1, 2, \cdots, \eta , \tag{2.7}$$

**3. Example from triangular association scheme.** In this section a series of three dimensional designs with partial factorial balance is constructed from the triangular association scheme. The number of levels of each factor is $n = t(t - 1)/2$ where $t$ is a positive integer. We take a $t \times t$ square, and fill the $t(t - 1)/2$ positions above the main diagonal with the $n$ integers $0, 1, 2, \cdots, n - 1$ taken in order (see Fig. 3.1). The positions on the main diagonal are left blank, while the positions below the main diagonal are filled so that the $t \times t$ matrix is symmetric. Then $F_{ui}$ and $F_{vj}$ are said to be 0th associates if $i = j$, 1st associates if $i$ and $j$, $i \neq j$, lie in a common row (or column), and 2nd associates if they do not lie in a common row.

| | | | | |
|---|---|---|---|---|
| × | 0 | 1 | 2 | 3 |
| 0 | × | 4 | 5 | 6 |
| 1 | 4 | × | 7 | 8 |
| 2 | 5 | 7 | × | 9 |
| 3 | 6 | 8 | 9 | × |

FIG. 3.1. $t = 5, n = 10$.

The parameters of the triangular association scheme are given in matrices

$$(3.1)\qquad P^1 = \begin{bmatrix} 0 & 1 & 0 \\ 2t-4 & t-2 & 4 \\ 0 & t-3 & 2t-8 \end{bmatrix},$$

$$(3.2)\qquad P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & t-3 & 2t-8 \\ \frac{(t-2)(t-3)}{2} & \frac{(t-3)(t-4)}{2} & \frac{(t-4)(t-5)}{2} \end{bmatrix},$$

where in general $P^\gamma = ((c_{\beta\alpha}))$ and $c_{\beta\alpha} = p(\alpha; \beta, \gamma)$.

It is well known (see Bose and Mesner (1959)), that the mappings

$$I_n \to I_3\,, \qquad B^1 \to P^1\,, \qquad B^2 \to P^2$$

generate the regular representation of the linear associative algebra generated by the association matrices. Then if $B = b_0 I_n + b_1 B^1 + b_2 B^2$, its representation is $P = b_0 I_3 + b_1 P^1 + b_2 P^2$ where $P$ is a $3 \times 3$ matrix. If $B$ is nonsingular so is $P$, and if $P^{-1} = c_0 I_3 + c_1 P^1 + c_2 P^2$ it follows that $B^{-1} = c_0 I_n + c_1 B^1 + c_2 B^2$. Thus the inversion of the $n \times n$ matrix $B$ reduces to the inversion of the $3 \times 3$ matrix $P$.

Consider the set of assemblies $(F_{1x}, F_{2y}, F_{3z})'$ such that all pairs of levels are first associates and such that $x$, $y$, and $z$ do not lie in a common row of the array. For example, with $t = 5$ and the array of Fig. 3.1, $F_{10}$ and $F_{21}$ are a pair of first associates. The levels of $F_3$ which are first associates of both $F_{10}$ and $F_{21}$ are $F_{32}$, $F_{33}$ and $F_{34}$. The only level of $F_3$ such that there is no row containing all three is $F_{34}$, hence the corresponding assembly is $(F_{10}, F_{21}, F_{34})'$ or more compactly $(0, 1, 4)'$ It is easy to see that for all $t$ and for any pair of first associates $x, y$ there is a unique $z$ satisfying the above condition.

The parameters of this series are

$$(3.3)\qquad N = t(t-1)(t-2)\,, \qquad \mu = 2(t-2)\,, \qquad d^0 = d^2 = 0\,, \qquad d^1 = 1\,.$$

The design may be increased in size by taking $\rho$ replications of the $n$ assemblies $(F_{1i}, F_{2i}, F_{3i})'$, $i = 0, 1, \cdots, n-1$. In this case

$$(3.4)\qquad \begin{aligned} &N = t(t-1)(t-2) + \rho n\,, \qquad \mu = 2(t-2) + \rho\,, \\ &d^0 = \rho\,, \quad d^1 = 1\,, \quad d^2 = 0\,. \end{aligned}$$

We shall now consider the analysis of these latter designs with parameters as in (3.4). From (2.1) we have

$$(3.5)\qquad (XX') = (I_3 \otimes \mu I_n) + (J_{33} - I_3) \otimes B\,, \qquad B = \rho I_n + B^1\,.$$

In (2.3) let $\theta = 1$, then $M = I_3 \otimes (A - B) + (J_{33} \otimes B)$ and the two matrices to be inverted are

$$(3.6)\qquad (A - B) = [2(t-2) + \rho]I_n + J_{nn} - B = (2t-3)I_n + B^2$$

$$(3.7)\qquad \begin{aligned}(A + 2B) &= [2(t-2) + \rho]I_n + J_{nn} + 2\rho I_n + 2B^1\\ &= [2t - 3 + 3\rho]I_n + 3B^1 + B^2\,.\end{aligned}$$

The calculations are easily made by taking the same linear combinations of the $3 \times 3$ regular representation matrices $P^0$, $P^1$, and $P^2$, that is $P_1 = (2t-3)I_3 + P^2$, $P_2 = [2t - 3 + 3\rho]I_3 + 3P^1 + P^2$, and $P_3 = \rho I_3 + P^1$. Now calculate $P_1^{-1}$, $P_2^{-1}$ and

$$-P_2^{-1}P_3P_1^{-1} = w_0I_3 + w_1P^1 + w_2P^2$$
$$P_1^{-1} - P_2^{-1}P_3P_1^{-1} = v_0I_3 + v_1P^1 + v_2P^2\,.$$

Then we have

$$W = w_0I_n + w_1B^1 + w_2B^2\,, \qquad V = v_0I_n + v_1B^1 + v_2B^2$$

and the inverse is complete.

The values of $v_0$, $v_1$, $v_2$; $w_0$, $w_1$, $w_2$ and the efficiencies for $t = 4, 5, 6, 7$ and $\rho = 0, 1, 2, 3$, except for the design with $t = 4$ and $\rho = 0$ which is not completely connected, are given in Table 3.1. The design with $t = 4$ and $\rho = 1$ may be regarded as a Latin square with the diagonal deleted, and with $\rho = 2$ there is a

TABLE 3.1

*Analysis of designs from triangular association scheme*

| $n$ | $\rho$ | $N$ | $v_0$<br>$w_0$ | $v_1$<br>$w_1$ | $v_2$<br>$w_2$ | $\mathscr{E}_1$ | $\mathscr{E}_2$ |
|---|---|---|---|---|---|---|---|
| 6 | 1 | 30 | .2024<br>−.0060 | −.0159<br>−.0159 | −.0119<br>.0298 | .91 | .93 |
| | 2 | 36 | .1764<br>−.0319 | −.0069<br>−.0069 | −.0236<br>.0181 | .91 | .83 |
| | 3 | 42 | .1661<br>−.0422 | −.0041<br>−.0041 | −.0262<br>.0155 | .84 | .74 |
| 10 | 0 | 60 | .2029<br>.0504 | −.0221<br>−.0246 | .0029<br>.0254 | .74 | .83 |
| | 1 | 70 | .1482<br>−.0043 | −.0063<br>−.0088 | −.0109<br>.0116 | .92 | .90 |
| | 2 | 80 | .1330<br>−.0195 | −.0027<br>−.0052 | −.0134<br>.0091 | .92 | .85 |
| | 3 | 90 | .1256<br>−.0269 | −.0012<br>−.0037 | −.0143<br>.0082 | .88 | .79 |
| 15 | 0 | 120 | .1398<br>.0198 | −.0074<br>−.0107 | −.0046<br>.0087 | .85 | .87 |
| | 1 | 135 | .1165<br>−.0035 | −.0025<br>−.0059 | −.0073<br>.0060 | .93 | .90 |
| | 2 | 150 | .1067<br>−.0133 | −.0007<br>−.0041 | −.0081<br>.0052 | .93 | .87 |
| | 3 | 165 | .1011<br>−.0189 | .0002<br>−.0032 | −.0085<br>.0048 | .90 | .83 |
| 21 | 0 | 210 | .1091<br>.0104 | −.0030<br>−.0064 | −.0040<br>.0045 | .89 | .89 |
| | 1 | 231 | .0958<br>−.0029 | −.0009<br>−.0043 | −.0049<br>.0036 | .94 | .90 |
| | 2 | 252 | .0889<br>−.0098 | .0001<br>−.0033 | −.0053<br>.0032 | .94 | .88 |
| | 3 | 273 | .0846<br>−.0140 | .0007<br>−.0027 | −.0054<br>.0031 | .91 | .85 |

Latin square with $N = 36$. For higher values of $t$ these designs become more attractive in terms of decreasing the number of observations and maintaining reasonable efficiency.

**4. Cyclic association scheme.** In this section we consider a cyclic association scheme and some three, four, and five dimensional designs obtained from this scheme. As before let $S_1, S_2, \cdots, S_m$ denote the $m$ sets of factor levels, $S_u = \{F_{u0}, F_{u1}, \cdots, F_{u,n-1}\}$, $u = 1, 2, \cdots, m$.

DEFINITION 4.1. The element $F_{ui} \in S_u$ is said to be an $\alpha$th associate of $F_{vj} \in S_v$ if $i - j \equiv \alpha \bmod (n)$ or $i - j \equiv -\alpha \bmod (n)$. The sets $S_u$ and $S_v$ are not necessarily distinct.

It follows directly from the definition of the association scheme that,

$$\begin{aligned} \eta &= (n+1)/2 \quad \text{if } n \text{ is odd;} \\ &= (n+2)/2 \quad \text{if } n \text{ is even,} \end{aligned} \tag{4.1}$$

$$\begin{aligned} \eta^\alpha &= 1 \quad \alpha = 0 \\ &= 2 \quad \alpha = 1, 2, \cdots, (n-1)/2 \qquad n \text{ odd;} \end{aligned} \tag{4.2}$$

$$\begin{aligned} \eta^\alpha &= 1 \quad \alpha = 0 \text{ or } n/2 \\ &= 2 \quad \alpha = 1, 2, \cdots, n/2 - 1 \qquad n \text{ even.} \end{aligned} \tag{4.3}$$

The association matrices $B^\alpha$ are most easily expressed in terms of the powers of the $(n \times n)$ matrix $P$, where

$$P = \begin{bmatrix} 0 & 0 & 0 \cdots 0 & 1 \\ 1 & 0 & 0 \cdots 0 & 0 \\ 0 & 1 & 0 \cdots 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 \cdots 1 & 0 \end{bmatrix}. \tag{4.4}$$

It is well known that the powers of $P$,

$$P, P^2, P^3 \cdots, P^{n-1}, P^n = I_n,$$

form a basis for the class of circulants. Then from Definition 4.1 we see that if $n$ is even

$$\begin{aligned} B^\alpha &= P^\alpha + P^{n-\alpha}, \qquad \alpha = 1, 2, \cdots, n/2 - 1 \\ &= P^{n/2}, \qquad \alpha = n/2 \\ &= I_n, \qquad \alpha = 0, \end{aligned} \tag{4.5}$$

and if $n$ is odd

$$\begin{aligned} B^\alpha &= P^\alpha + P^{n-\alpha} \qquad \alpha = 1, 2, \cdots, (n-1)/2 \\ &= I_n, \qquad \alpha = 0. \end{aligned} \tag{4.6}$$

This representation of the $B^\alpha$ in terms of the powers of $P$ simplifies multiplication of association matrices.

Series of designs with partial factorial balance will now be given which correspond to this cyclic association scheme. For compactness an assembly $(F_{1j_1}, F_{2j_2}, \cdots, F_{mj_m})'$ will be denoted by $(j_1, j_2, \cdots, j_m)'$. All integers are assumed to be mod $(n)$.

DESIGN 4.1. $n \times n \times n$ design $n \geqq 3$, $N = 3n$, $n_e = 2$.

$$T = \begin{bmatrix} k & k & k & \\ k-1 & k & k+1: & k = 0, 1, \cdots, n-1 \\ k & k-1 & k+1 & \end{bmatrix}.$$

Parameters: $\mu = 3$; $d^0 = d^1 = 1$; $d^\alpha = 0$, $\alpha > 1$

$$(XX') = [I_3 \otimes (3I_n - B)] + [J_{33} \otimes B]$$

where $B = I_n + B^1$ and $B^1$ is the $(n \times n)$ matrix corresponding to first associates as defined in (4.5).

DESIGN 4.2. $n \times n \times n$ design $n \geqq 4$, $N = 4n$, $n_e = n + 2$.

$$T = \begin{bmatrix} k & k & k & k & \\ k-1 & k & k+1 & k: & k = 0, 1, \cdots, n-1 \\ k & k-1 & k+1 & k & \end{bmatrix}.$$

Parameters: $\mu = 4$, $d^0 = 2$, $d^1 = 1$, $d^\alpha = 0$, $\alpha > 1$,

$$(XX') = [I_3 \otimes (4I_n - B)] + [J_{33} \otimes B], \qquad B = 2I_n + B^1.$$

DESIGN 4.3. $n \times n \times n$ design $n \geqq 5$, $N = 5n$, $n_e = 2n + 2$.

$$T = \begin{bmatrix} k & k & k & k & k & \\ k-2 & k-1 & k & k+1 & k+2: & k = 0, 1, \cdots, n-1 \\ k-1 & k-2 & k+2 & k+1 & k & \end{bmatrix}.$$

Parameters: $\mu = 5$; $d^0 = d^1 = d^2 = 1$; $d^\alpha = 0$, $\alpha > 2$

$$(XX') = [I_3 \otimes (5I_n - B)] + [J_{33} \otimes B], \qquad B = I_n + B^1 + B^2.$$

DESIGN 4.4. $n \times n \times n \times n$ design $n \geqq 5$, $N = 5n$, $n_e = n + 3$.

$$T = \begin{bmatrix} k & k & k & k & k & \\ k-2 & k-1 & k & k+1 & k+2: & k = 0, 1, \cdots, n-1 \\ k-1 & k-2 & k+2 & k+1 & k & \\ k & k-2 & k+1 & k-1 & k+2 & \end{bmatrix}.$$

Parameters: $\mu = 5$; $d^0 = d^1 = d^2 = 1$; $d^\alpha = 0$, $\alpha > 0$.

$$(XX') = [I_4 \otimes (5I_n - B)] + [J_{44} \otimes B], \qquad B = I_n + B^1 + B^2.$$

A second four-dimensional design may be obtained from the preceding by adjoining the assemblies $(k, k, k, k)'$, $k = 0, 1, \cdots, n-1$. For this design we have $N = 6n$, $n_e = 2n + 3$, $\mu = 6$, $d^0 = 2$, $d^1 = d^2 = 1$, and $d^\alpha = 0$, $\alpha > 2$.

DESIGN 4.5. $n \times n \times n \times n$ design $n \geqq 7$, $N = 7n$, $n_e = 3n + 3$.

$$T = \begin{bmatrix} k & k & k & k & k & k & k \\ k-3 & k-2 & k-1 & k & k+1 & k+2 & k+3 : \\ k-1 & k-3 & k+2 & k & k-2 & k+3 & k+1 \\ k-2 & k & k+2 & k-3 & k-1 & k+1 & k+3 \\ & & & & \multicolumn{3}{r}{k = 0, 1, \cdots, n-1} \end{bmatrix}.$$

Parameters: $\mu = 7$; $d^\alpha = 1$, $\alpha = 0, 1, 2, 3$; $d^\alpha = 0$, $\alpha > 3$.

$$(XX') = [I_4 \otimes (7I_n - B)] + [J_{44} \otimes B], \qquad B = I_n + B^1 + B^2 + B^3.$$

It should be noted that an $n \times n \times n$ design with $N = 6n$ may be obtained from Design 4.5 by deleting the assemblies $(k, k, k, k-3)'$ and then disregarding factor $F_4$. Also a $n \times n \times n$ design with $N = 7n$ is obtained by simply disregarding factor $F_4$ from Design 4.5.

DESIGN 4.6. $n \times n \times n \times n \times n$ design $n \geqq 5$, $N = 5n$, $n_e = 4$.

$$T = \begin{bmatrix} k & k & k & k & k & & \\ k-2 & k-1 & k & k+1 & k-2 & & \\ k-1 & k-2 & k+2 & k+1 & k & : & k = 0, 1, \cdots, n-1 \\ k & k-2 & k+1 & k-1 & k+2 & & \\ k-2 & k & k+2 & k-1 & k+1 & & \end{bmatrix}.$$

Parameters: $\mu = 5$; $d^0 = d^1 = d^2 = 1$; $d^\alpha = 0$, $\alpha > 2$.

$$(XX') = [I_5 \otimes (5I_n - B)] + [J_{55} \otimes B], \qquad B = I_n = B^1 + B^2.$$

Design 4.6 has only four degrees of freedom for error. If more degrees of freedom are required the assemblies $(k, k, k, k, k)'$ $k = 0, 1, \cdots, n-1$ may be adjoined. In this case we have $N = 6n$ and $n_e = n + 4$.

Several other designs may be constructed which have the properties of those given. The examples given above should be sufficient to illustrate the general structure of the cyclic designs.

Bruner (1967) has tabled the matrices $V$ and $W$ for each of the designs given in this section with $n \leqq 15$ and all matrices required for testing hypotheses of the usual type. It has been observed from these tables that for $n \leqq 15$ these designs possess the property of contiguity. That is, the variance of a simple contrast of $\alpha$th associates increases as $\alpha$ increases. Further results on this property will appear in a later publication.

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